

The space of Lelek fans in the Cantor fan is homeomorphic to Hilbert space

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Abstract. We show that the space of all Lelek fans in a Cantor fan, equipped with the Hausdorff metric, is homeomorphic to the separable Hilbert space. This result is a special case of a general theorem we prove about spaces of upper semicontinuous functions on compact metric spaces that are strongly discontinuous.

1. Introduction.

All topological spaces in this paper are assumed to be separable metric. A real-valued function f defined on a topological space X is called *upper semicontinuous* (USC) if $f_{\geq t} = \{x \in X : f(x) \geq t\}$ is closed in X for every $t \in \mathbf{R}$. For a space X and a function $f : X \rightarrow [0, \infty)$, let

$$\downarrow f = \{(x, t) \in X \times \mathbf{R} : 0 \leq t \leq f(x)\},$$

$$\mathcal{G}_f = \{(x, f(x)) \in X \times \mathbf{R} : x \in X\},$$

and

$$\mathcal{G}_f^+ = \{(x, f(x)) \in X \times \mathbf{R} : x \in X \text{ and } f(x) > 0\},$$

as subspaces of the product space $X \times \mathbf{R}$. Then $\mathcal{G}_f^+ \subset \mathcal{G}_f \subset \downarrow f$ and $\downarrow f$ is closed in $X \times \mathbf{R}$ if and only if f is USC. An upper semicontinuous map $f : X \rightarrow [0, \infty)$ is said to be *strongly discontinuous* if \mathcal{G}_f^+ is dense in $\downarrow f$. So the strongly discontinuous maps are USC maps that are in some sense maximally discontinuous. Well-known examples of strongly discontinuous functions are Lelek functions (arc

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length functions for Lelek fans) and hair length functions for hairy arcs; see [12] and [1]. For a compact space X let $\text{USC}(X)$ denote the set of all USC functions from X into $\mathbf{I} = [0, 1]$. We topologize $\text{USC}(X)$ by letting the *hypograph distance* $D(f, g)$ for $f, g \in \text{USC}(X)$ be equal to the Hausdorff distance between $\downarrow f$ and $\downarrow g$ as subsets of $X \times \mathbf{I}$. A space is called a *Hilbert cube* if it is homeomorphic to $Q = \mathbf{I}^{\mathbf{N}}$. Yang and Zhou [16] have shown that $\text{USC}(X)$ is a Hilbert cube if and only if X is infinite. We now define the subspace

$$\text{SDC}(X) = \{f \in \text{USC}(X) : f \text{ is strongly discontinuous}\}$$

of $\text{USC}(X)$. A subset A of a Hilbert cube M is called a *pseudointerior* if the pair (M, A) is homeomorphic to (Q, s) , that is, there is a homeomorphism $h: M \rightarrow Q$ such that $h(A) = s = (0, 1)^{\mathbf{N}}$. The Anderson Theorem [2] states that pseudointeriors are homeomorphic to the separable Hilbert space ℓ^2 . We shall prove the following result.

THEOREM 1. *For a compact space X , $\text{SDC}(X)$ is a pseudointerior in $\text{USC}(X)$ if and only if X is dense in itself.*

If X is a compact space then the *cone* ΔX over X is the quotient space $(X \times \mathbf{I})/(X \times \{0\})$. If \mathfrak{C} is the Cantor set then $\Delta \mathfrak{C}$ is called a *Cantor fan*. Let $q: \mathfrak{C} \times \mathbf{I} \rightarrow \Delta \mathfrak{C}$ be the quotient mapping. If $f \in \text{SDC}(\mathfrak{C})$ then $q(\downarrow f)$ is called a *standard Lelek fan*, see Lelek [12]. A *Lelek fan* is any space that is homeomorphic to a standard Lelek fan. According to Bula and Oversteegen [3] and Charatonik [4] Lelek fans are topologically unique and can be characterized as the only smooth fans with a dense set of endpoints. Let \mathfrak{L} stand for the space of all Lelek fans that are contained in the Cantor fan, equipped with the Hausdorff metric. We derive the following result from Theorem 1.

THEOREM 2. *The space \mathfrak{L} is homeomorphic to Hilbert space.*

Lelek fans have received a measure of attention in recent years because of the proof of Kawamura, Oversteegen, and Tymchatyn [11] that their endpoint sets are homeomorphic to complete Erdős space [10]. The uniqueness of the Lelek fan makes that the space plays a central role in characterizing complete Erdős space and similar spaces; see Dijkstra and van Mill [7], [9], [8] and Dijkstra [6].

2. Preliminaries.

For a metric space (X, d) , an $x \in X$, a subset A of X , and an $\varepsilon > 0$ we define the open sets $U_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ and $U(A, \varepsilon) = \bigcup \{U(a, \varepsilon) : a \in A\}$.

For a compact metric space (X, d) , let 2^X be the family of all nonempty closed subsets in X . For $A, B \in 2^X$, their *Hausdorff distance* d_H is defined by

$$d_H(A, B) = \inf\{\varepsilon : A \subset U_d(B, \varepsilon) \text{ and } B \subset U_d(A, \varepsilon)\}.$$

Then $(2^X, d_H)$ is a compact metric space. The following characterization of the topology that corresponds to the Hausdorff metric is well-known.

LEMMA 3. *Let X be a compact metric space. Then a sequence $(A_n)_n$ in 2^X converges to an element $A \in 2^X$ if and only if the following conditions hold:*

- (1) *if $a_n \in A_n$ for every n , then every limit point of the sequence $(a_n)_n$ is contained in A and*
- (2) *for every $a \in A$, there exists a sequence $(a_n)_n$ in X such that $a_n \in A_n$ for each n and $\lim_{n \rightarrow \infty} a_n = a$.*

For every pair $(x, \lambda), (y, \mu) \in X \times \mathbf{I}$, we define $d'((x, \lambda), (y, \mu)) = \max\{d(x, y), |\lambda - \mu|\}$. Then d' is a metric on $X \times \mathbf{I}$. The *hypograph metric* D on $\text{USC}(X)$ is defined by

$$D(f, g) = d'_H(\downarrow f, \downarrow g).$$

Unless stated otherwise we will assume that $\text{USC}(X)$ and its subspaces are equipped with the *hypograph topology* that is generated by this metric D . An upper semicontinuous map $f : X \rightarrow [0, \infty)$ is said to be *strongly* discontinuous* if \mathcal{G}_f is dense in $\downarrow f$. We define

$$\text{SDC}^*(X) = \{f \in \text{USC}(X) : f \text{ is strongly}^* \text{ discontinuous}\}.$$

A map $f : X \rightarrow \mathbf{R}$ is said to be *Lipschitz* if there exists some $k \geq 0$ such that $|f(x) - f(x')| \leq kd(x, x')$ for all $x, x' \in X$. The smallest such k is called the *Lipschitz constant* of f and denoted by $\text{lip } f$. If $\text{lip } f \leq k$, then f is said to be *k-Lipschitz*. Define

$$\text{LIP}(X) = \{f \in \text{USC}(X) : f \text{ is Lipschitz}\}$$

and

$$\text{LIP}_k(X) = \{f \in \text{USC}(X) : f \text{ is } k\text{-Lipschitz}\}.$$

The following result can be found in [17].

LEMMA 4. *For each k the hypograph topology on $\text{LIP}_k(X)$ coincides with the topology of uniform convergence.*

As mentioned in the introduction a subset A of a Hilbert cube M is called a pseudointerior if $(M, A) \approx (Q, s)$. The complement of a pseudointerior in a Hilbert cube is called a *pseudoboundary*, $B(Q) = Q \setminus s$ being the standard example. A closed subset A of a Hilbert cube (M, d) is called a *Z-set* if for every $\varepsilon > 0$ there is a continuous map $f: M \rightarrow M \setminus A$ such that $d(x, f(x)) < \varepsilon$ for every $x \in M$. A σZ -set is a countable union of Z-sets. A subset A of a space X is called *homotopy dense* if there is a homotopy $H: X \times I \rightarrow X$ such that H_0 is the identity and $H_t(X) \subset A$ for each $t \in (0, 1]$. Clearly, a closed set in M whose complement is homotopy dense is a Z-set. The pseudoboundaries of a Hilbert cube M can be characterized as *capsets* or *Z-absorbers* in M . An immediate consequence of this characterization is the following useful property; see [13, Theorems 5.4.3 and 5.4.12].

PROPOSITION 5. *If A is a σZ -set in a Hilbert cube that contains a pseudoboundary and B is a Z-set then $A \setminus B$ is also a pseudoboundary.*

We shall use the following result of Zhang and Yang [17].

THEOREM 6. *$\text{LIP}(X)$ is a pseudoboundary in $\text{USC}(X)$ provided that X is an infinite compact metric space.*

We conclude this section with a few basic results about strongly discontinuous functions.

LEMMA 7. *If $f: X \rightarrow (0, \infty)$ is continuous and $g: X \rightarrow [0, \infty)$ is strongly discontinuous, then $\min\{f, g\}$ is strongly discontinuous.*

PROOF. It is evident that $h = \min\{f, g\}$ is USC. Let $(x, t) \in \downarrow h$ and note that $t \leq f(x)$ and $(x, t) \in \downarrow g$. Since g is strongly discontinuous we can select a sequence $(x_n)_n$ that converges to x such that $\lim_{n \rightarrow \infty} g(x_n) = t$ and $g(x_n) > 0$ for all n . Note that $h(x_n) > 0$ for all n . Then by the continuity of f we have

$$\lim_{n \rightarrow \infty} h(x_n) = \lim_{n \rightarrow \infty} \min\{f(x_n), g(x_n)\} = \min\{f(x), t\} = t.$$

Thus \mathcal{G}_h^+ is dense in $\downarrow h$. □

If $f: X \rightarrow [0, \infty)$ and $t \geq 0$, then $f_{>t} = \{x \in X : f(x) > t\}$ and $\text{supp } f$ is the closure of $f_{>0}$ in X .

LEMMA 8. *A map $f: X \rightarrow [0, \infty)$ has the following properties:*

- (a) *f is strongly* discontinuous if and only if $f \upharpoonright \text{supp } f$ is strongly discontinuous.*
- (b) *f is strongly discontinuous if and only if f is strongly* discontinuous and $\text{supp } f = X$.*
- (c) *If f is strongly* discontinuous then $\text{supp } f$ is dense in itself.*

PROOF.

(a). Let f be strongly* discontinuous and put $S = \text{supp } f$. Then we have that \mathcal{G}_f is dense in $\downarrow f$ and hence $\mathcal{G}_f^+ = \mathcal{G}_{f \upharpoonright S}^+$ is dense in $(\downarrow f) \cap (X \times (0, \infty))$. Let $y \in f_{>0}$ and note that then $\{y\} \times (0, f(y)]$ is a nonempty interval that is contained in the closure of \mathcal{G}_f^+ and hence $(y, 0) \in \overline{\mathcal{G}_f^+}$. Since $f_{>0}$ is dense in S we have that $S \times \{0\}$ is contained in $\overline{\mathcal{G}_f^+}$ and that $f \upharpoonright S$ is strongly discontinuous. The converse implication is a triviality.

(b). Let f be strongly discontinuous and let $p: X \times \mathbf{R} \rightarrow X$ be the projection map. Then G_f^+ is dense in $\downarrow f$ and hence $f_{>0} = p(G_f^+)$ is dense in $X = p(\downarrow f)$ and $\text{supp } f = X$. Now (b) follows if we combine this result with part (a).

(c). Let f be strongly* discontinuous. It suffices to show that $f_{>0}$ is dense in itself. Let $x \in f_{>0}$ and let O be a neighbourhood of x in X . Since \mathcal{G}_f is dense in $\downarrow f$ there is a $y \in O$ such that $0 < f(y) < f(x)$. Note that $y \neq x$ and $y \in f_{>0}$. \square

If $\varepsilon > 0$ then a subset A of (X, d) is called an ε -net if $B(A, \varepsilon) = X$. The metric d is called *totally bounded* if there is a finite ε -net for every $\varepsilon > 0$.

LEMMA 9. *If X is dense in itself and d is a totally bounded metric on X , then there exists a strongly discontinuous function $g: X \rightarrow \mathbf{I}$ such that $g_{\geq t}$ is a t^2 -net in X for each $t \in (0, 1]$.*

PROOF. Strongly discontinuous functions on the Cantor set \mathfrak{C} exist and are called Lelek functions; see [12]. A simple construction of such a $\varphi \in \text{SDC}(\mathfrak{C})$ can be found in Dijkstra [5]: represent \mathfrak{C} by the product space $\{0, 1\}^{\mathbf{N}}$ and put $\varphi(x_1, x_2, \dots) = (1 + \sum_{n=1}^{\infty} x_n/n)^{-1}$. Choose a countable subset A_1 of \mathfrak{C} such that $\varphi(x) > 0$ for each $x \in A_1$ and the set $\{(x, \varphi(x)) \in \mathfrak{C} \times \mathbf{I} : x \in A_1\}$ is dense in $\downarrow \varphi$. Observe that $\varphi \upharpoonright A_1 \in \text{SDC}(A_1)$ and that A_1 is dense in itself by Lemma 8(c). Let A_2 be a countable dense subset of X . Then both A_1 and A_2 are homeomorphic to the space of rational numbers \mathbf{Q} . Let $h: A_2 \rightarrow A_1$ be a homeomorphism and put $\psi = \varphi \circ h$. Then $\psi \in \text{SDC}(A_2)$. We define the USC-extension $f: X \rightarrow \mathbf{I}$ of ψ by

$$f(x) = \lim_{\varepsilon \searrow 0} (\sup\{\psi(y) : y \in A_2 \cap U_d(x, \varepsilon)\}),$$

for $x \in X$. By the same argument as used in the proof of [9, Lemma 4.8], we can

show that $f \in \text{SDC}(X)$.

For $n \in \mathbf{N}$ use the fact that d is totally bounded to select a finite $1/n^2$ -net B_n for X in the dense set A_2 . Since $f(x) = \psi(x) > 0$ for each $x \in A_2$ we can choose a $t_n > 0$ such that $t_n < f(x)$ for each $x \in B_n$. We can clearly arrange that $t_n > t_{n+1}$ for each n . Select a homeomorphism $\alpha: [0, \infty) \rightarrow [0, \infty)$ such that $\alpha(t_{n+1}) = 1/n$ for each $n \in \mathbf{N}$. Then $\alpha \circ f$ is obviously also strongly discontinuous. Define $g = \min\{1, \alpha \circ f\}$ and note that $g \in \text{SDC}(X)$ by Lemma 7. Let $t \in (0, 1]$. Then there is an $n \in \mathbf{N}$ such that $1/(n+1) < t \leq (1/n)$. Observe that

$$B_{n+1} \subset f_{\geq t_{n+1}} = g_{\geq \frac{1}{n}} \subset g_{\geq t}.$$

Since B_{n+1} is a $1/(n+1)^2$ -net it is a t^2 -net and hence $g_{\geq t}$ is also a t^2 -net. \square

3. Proofs of the main theorems.

Throughout this section we assume that the space (X, d) is a compact metric space. We define the maps $G, G' : \text{USC}(X) \times \mathbf{I} \rightarrow \text{USC}(X)$ as follows:

$$G(f, t)(x) = G_t(f)(x) = \begin{cases} \max \{f(z) - \frac{1}{t}d(x, z) : z \in X\}, & \text{if } t > 0; \\ f(x), & \text{if } t = 0; \end{cases}$$

and

$$G'(f, t)(x) = G'_t(f)(x) = \min\{1, G(f, t)(x) + t\},$$

for every pair $(f, t) \in \text{USC}(X) \times \mathbf{I}$ and $x \in X$. For the definition we used the fact that a USC function assumes a maximum on a compact domain. Observe that always $G'(f, t)(x) \geq G(f, t)(x) \geq f(x)$ so $\downarrow f \subset \downarrow G(f, t) \subset \downarrow G'(f, t)$. Note that for $f \in \text{USC}(X)$ and $t \in (0, 1]$ the map $G'(f, t)$ is strictly positive, and that also $G'_0(f) = f$. The following result shows that $\text{LIP}(X)$ is homotopy dense in $\text{USC}(X)$.

CLAIM 10. *The maps G and G' are homotopies on $\text{USC}(X)$ such that for each $t \in (0, 1]$ we have $G_t(\text{USC}(X)) \cup G'_t(\text{USC}(X)) \subset \text{LIP}_{1/t}(X)$.*

PROOF. It suffices to verify that the map G satisfies the requirements of Claim 10 as the result for G' follows then immediately.

Let $f \in \text{USC}(X)$, $t \in (0, 1]$, and $x, y \in X$. Select a $z \in X$ such that $G(f, t)(x) = f(z) - d(x, z)/t$. Then $G(f, t)(x) \leq f(z) - d(y, z)/t + d(x, y)/t \leq G(f, t)(y) + d(x, y)/t$ and hence $G_t(f)$ is $1/t$ -Lipschitz.

We proceed to show that G is continuous. Suppose that $\lim_{n \rightarrow \infty} (f_n, t_n) = (f, t)$ in $\text{USC}(X) \times \mathbf{I}$. With the help of Lemma 3 we show that $\lim_{n \rightarrow \infty} G(f_n, t_n) = G(f, t)$.

For condition (1) of Lemma 3 we consider an arbitrary sequence $(x_n, r_n)_n$ such that $(x_n, r_n) \in \downarrow G(f_n, t_n)$ and (x, r) is a limit point of the sequence. We need to show that $(x, r) \in \downarrow G(f, t)$, or equivalently, that $r \leq G(f, t)(x)$. We consider the following two cases:

Case a: $t > 0$. We may assume without loss of generality that $t_n > 0$ for every $n \in \mathbf{N}$. Then $r_n \leq G(f_n, t_n)(x_n) = f_n(z_n) - (1/t_n)d(x_n, z_n)$ for some $z_n \in X$. Since $X \times \mathbf{I}$ is compact we can arrange by passing to a subsequence that $\lim_{i \rightarrow \infty} (z_{n(i)}, f_{n(i)}(z_{n(i)})) = (z, a)$ while moreover $\lim_{i \rightarrow \infty} (x_{n(i)}, r_{n(i)}) = (x, r)$. Then by Lemma 3 we have that $a \leq f(z)$. Thus

$$\begin{aligned} r &= \lim_{i \rightarrow \infty} r_{n(i)} \leq \lim_{i \rightarrow \infty} f_{n(i)}(z_{n(i)}) - \frac{d(x_{n(i)}, z_{n(i)})}{t_{n(i)}} \\ &= a - \frac{d(x, z)}{t} \leq f(z) - \frac{d(x, z)}{t} \leq G(f, t)(x). \end{aligned}$$

Case b: $t = 0$. Since $G(f_n, 0) = f_n$ and $G(f, t) = f$ we may assume that every t_n is positive. Again we have $r_n \leq f_n(z_n) - (1/t_n)d(x_n, z_n)$ for some $z_n \in X$ with $\lim_{i \rightarrow \infty} (z_{n(i)}, f_{n(i)}(z_{n(i)})) = (z, a) \in \downarrow f$ and $\lim_{i \rightarrow \infty} (x_{n(i)}, r_{n(i)}) = (x, r)$. Note that

$$0 \leq \lim_{i \rightarrow \infty} d(x_{n(i)}, z_{n(i)}) \leq \lim_{i \rightarrow \infty} t_{n(i)}(f_{n(i)}(z_{n(i)}) - r_{n(i)}) = t(a - r) = 0$$

and hence $x = z$. Thus we have $r = \lim_{i \rightarrow \infty} r_{n(i)} \leq \lim_{i \rightarrow \infty} f_{n(i)}(z_{n(i)}) = a \leq f(z) = f(x) = G(f, t)(x)$.

We now consider two cases for condition (2) of Lemma 3.

Case a: $t > 0$. For any $(x, r) \in \downarrow G_0(f, t)$ we show that for every $n \in \mathbf{N}$ there exists an $r_n \in \mathbf{I}$ such that $(x, r_n) \in \downarrow G(f_n, t_n)$ and $\lim_{n \rightarrow \infty} (x, r_n) = (x, r)$. Select a $z \in X$ such that $r \leq f(z) - (1/t)d(x, z)$. This means that $(z, r + (1/t)d(x, z)) \in \downarrow f$. Since $\lim_{n \rightarrow \infty} f_n = f$ there exists a sequence $(z_n, s_n)_n$ in $X \times \mathbf{I}$ such that $(z_n, s_n) \in \downarrow f_n$ and $\lim_{n \rightarrow \infty} (z_n, s_n) = (z, r + (1/t)d(x, z))$. Moreover, since $G(f_n, t_n)(x) \geq f_n(z_n) - (1/t_n)d(x, z_n) \geq s_n - (1/t_n)d(x, z_n)$, we have that $(x, s_n - (1/t_n)d(x, z_n)) \in \downarrow G(f_n, t_n)$. Put $r_n = s_n - (1/t_n)d(x, z_n)$. Then we have $(x, r_n) \in \downarrow G(f_n, t_n)$ and $\lim_{n \rightarrow \infty} r_n = r + (1/t)d(x, z) - (1/t)d(x, z) = r$.

Case b: $t = 0$. Let $(x, r) \in \downarrow G(f, t) = \downarrow f$. Since $\lim_{n \rightarrow \infty} f_n = f$, there exists a sequence $(x_n, r_n)_n$ in $X \times \mathbf{I}$ such that $(x_n, r_n) \in \downarrow f_n$ and $\lim_{n \rightarrow \infty} (x_n, r_n) = (x, r)$. Since $\downarrow f_n \subset \downarrow (G(f_n, t_n))$, we have $(x_n, r_n) \in \downarrow G(f_n, t_n)$. \square

LEMMA 11. *If X is dense in itself, then $\text{SDC}(X)$ is homotopy dense in $\text{USC}(X)$.*

PROOF. Let $g \in \text{SDC}(X)$ be a map as in Lemma 9. We define the map $H : \text{USC}(X) \times \mathbf{I} \rightarrow \text{USC}(X)$ as follows:

$$H(f, t) = \begin{cases} \min \{G'(f, t), \frac{1}{t}g\} & \text{if } t > 0, \\ f & \text{if } t = 0, \end{cases}$$

where $G' : \text{USC}(X) \times \mathbf{I} \rightarrow \text{USC}(X)$ is the homotopy of Claim 10. By Claim 10 and Lemma 7, $H(\text{USC}(X) \times (0, 1]) \subset \text{SDC}(X)$. It remains to verify that the map H is continuous. To this end we first prove the following inequality:

$$D(H(f, t), G'(f, t)) \leq t \quad \text{for } f \in \text{USC}(X) \text{ and } t \in \mathbf{I}.$$

For $t = 0$, $H(f, t) = G'(f, t) = f$ so we may assume that $t > 0$. One direction is trivial because $\downarrow H(f, t) \subset \downarrow G'(f, t)$. Consider now an $(x, r) \in \downarrow G'(f, t)$. Since $g_{\geq t}$ is a t^2 -net there is a $y \in X$ such that $d(x, y) < t^2$ and $g(y) \geq t$. This means that $H(f, t)(y) = G'(f, t)(y)$. Since $G'(f, t)$ is $1/t$ -Lipschitz we have that

$$H(f, t)(y) = G'(f, t)(y) \geq G'(f, t)(x) - \frac{d(x, y)}{t} \geq G'(f, t)(x) - t \geq r - t.$$

If $s = \max\{r - t, 0\}$ then $(y, s) \in \downarrow H(f, t)$. Since $d'((x, r), (y, s)) < \max\{t^2, t\} = t$ the inequality is proved. It is now obvious that H is continuous in all points of the form $(f, 0)$.

It remains to show that H is continuous on the open set $\text{USC}(X) \times (0, 1]$. Let $\varepsilon > 0$ and note that $H(\text{USC}(X) \times (\varepsilon, 1]) \subset \text{LIP}_{1/\varepsilon}(X)$. The operation $\min\{f_1, f_2\}$ is not continuous with respect to the hypograph topology but it is continuous with respect to the topology of uniform convergence. Using Lemma 4 and the obvious fact that the function $t \mapsto (1/t)g$ is continuous with respect to the topology of uniform convergence we find that $H \upharpoonright (\text{USC}(X) \times (\varepsilon, 1])$ is continuous. Thus H is continuous. \square

Let $\mathcal{B} = \{B_1, B_2, \dots\}$ be a countable (open) basis for X . Take $i, n \in \mathbf{N}$ and $r \in (0, 1] \cap \mathbf{Q}$. Let $B_j \in \mathcal{B}$ be such that $B_j \subset U_d(B_i, 1/n)$. For any $f \in \text{USC}(X)$ and $B_k \in \mathcal{B}$, define $M_k(f) = \sup\{f(x) : x \in B_k\}$. Note that M_k is not necessarily continuous with respect to the hypograph topology but it is continuous with respect to the topology of uniform convergence on $\text{USC}(X)$. Put

$$A_{inrj} = \left\{ f \in \text{USC}(X) : |rM_i(G_{\frac{1}{n}}(f)) - M_j(f)| < \frac{1}{n} \right\},$$

where G is the homotopy of Claim 10. We now define

$$O_{inr} = \bigcup \left\{ A_{inrj} : B_j \subset U_d\left(B_i, \frac{1}{n}\right) \right\}.$$

CLAIM 12. *The set O_{inr} is open in $\text{USC}(X)$.*

PROOF. Let $f \in O_{inr}$. Then there exists a $j \in \mathbf{N}$ such that $f \in A_{inrj}$. Put

$$\varepsilon = \frac{1}{n} - |rM_i(G_{\frac{1}{n}}(f)) - M_j(f)| > 0.$$

By Claim 10 the map $G_{1/n} : \text{USC}(X) \rightarrow \text{LIP}_n(X)$ is continuous for every $n \in \mathbf{N}$. By Lemma 4 the composition $M_i \circ G_{1/n}$ is continuous thus there exists a neighbourhood V of f in $\text{USC}(X)$ such that

$$|M_i(G_{\frac{1}{n}}(f)) - M_i(G_{\frac{1}{n}}(g))| < \frac{\varepsilon}{2}$$

for every $g \in V$. By the definition of $M_j(f)$, there exists a $z \in B_j$ such that $M_j(f) - f(z) < \varepsilon/4$. Select a $k \in \mathbf{N}$ and a $\delta > 0$ such that

$$U(z, \delta) \subset B_k \subset U(B_k, \delta) \subset B_j \subset U\left(B_i, \frac{1}{n}\right).$$

Put $\mu = \min\{\delta, \varepsilon/4\}$ and let $g \in U_D(f, \mu)$. There exists an $(x, t) \in \downarrow g$ such that $\max\{d(z, x), |f(z) - t|\} < \mu$ and hence $x \in B_k$ and

$$g(x) \geq t > f(z) - \mu > f(z) - \frac{\varepsilon}{4} > M_j(f) - \frac{\varepsilon}{2}.$$

So we have $M_k(g) > M_j(f) - \varepsilon/2$. On the other hand, if $y \in B_k$ then there is a $(p, s) \in \downarrow f$ such that $\max\{d(y, p), |g(y) - s|\} < \mu$ and hence $x \in U_d(B_k, \delta) \subset B_j$ and $g(y) < s + \mu \leq f(p) + \mu < M_j(f) + \mu < M_j(f) + \varepsilon/2$. In conclusion, we have

$$|M_j(f) - M_k(g)| < \frac{\varepsilon}{2}.$$

If $g \in V \cap U_D(f, \mu)$ then

$$\begin{aligned} |rM_i(G_{\frac{1}{n}}(g)) - M_k(g)| &\leq |rM_i(G_{\frac{1}{n}}(g)) - rM_i(G_{\frac{1}{n}}(f))| \\ &\quad + |rM_i(G_{\frac{1}{n}}(f)) - M_j(f)| + |M_j(f) - M_k(g)| \\ &< \frac{r\varepsilon}{2} + \left(\frac{1}{n} - \varepsilon\right) + \frac{\varepsilon}{2} \\ &\leq \frac{1}{n}. \end{aligned}$$

Hence $g \in A_{inrk} \subset O_{inr}$ and we may conclude that O_{inr} is open. \square

LEMMA 13. *The space $\text{SDC}^*(X)$ is a G_δ -set in $\text{USC}(X)$.*

PROOF. Define

$$A = \bigcap_{i=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcap_{r \in (0,1] \cap \mathbf{Q}} O_{inr}$$

By Lemma 12 it suffices to prove that $A = \text{SDC}^*(X)$.

Let $f \in \text{SDC}^*(X)$. Fix $B_i \in \mathcal{B}$, $r \in (0, 1] \cap \mathbf{Q}$, and $n \in \mathbf{N}$. Then there exists an $x \in B_i$ such that $|M_i(G_{1/n}(f)) - G_{1/n}(f)(x)| < 1/n$. From the definition of G it follows that there exists a $z \in X$ such that $G_{1/n}(f)(x) = f(z) - nd(x, z)$. We may assume that $d(x, z) < 1/n$: if $G_{1/n}(f)(x) = 0$ then we can choose $z = x$ and if $G_{1/n}(f)(x) > 0$ then $nd(x, z) = f(z) - G_{1/n}(f)(x) < 1$. In any case we have $z \in U(B_i, 1/n)$ and $rG_{1/n}(f)(x) \leq G_{1/n}(f)(x) \leq f(z)$ thus $(z, G_{1/n}(f)(x)) \in \downarrow f$. Let $\delta = 1/n - |M_i(G_{1/n}(f)) - G_{1/n}(f)(x)|$. Since \mathcal{G}_f is dense in $\downarrow f$, there exists $a \in U_d(B_i, 1/n)$ such that $|f(a) - rG_{1/n}(f)(x)| < \delta/2$. Since f is upper semicontinuous, there exists an $\epsilon > 0$ such that $U_d(a, \epsilon) \subset U_d(B_i, 1/n)$ and $f(y) < f(a) + \delta/2$ for every $y \in U_d(a, \epsilon)$. Choose a $B_j \in \mathcal{B}$ satisfying $a \in B_j \subset U_d(a, \epsilon)$. Then we have

$$rG_{\frac{1}{n}}(f)(x) < f(a) + \frac{\delta}{2} \leq M_j(f) + \frac{\delta}{2}$$

and

$$M_j(f) \leq f(a) + \frac{\delta}{2} < rG_{\frac{1}{n}}(f)(x) + \frac{\delta}{2} + \frac{\delta}{2}.$$

Hence $|M_j(f) - rG_{1/n}(f)(x)| < \delta$. Therefore, we have

$$\begin{aligned}
& |rM_i(G_{\frac{1}{n}}(f)) - M_j(f)| \\
& \leq r|M_i(G_{\frac{1}{n}}(f)) - G_{\frac{1}{n}}(f)(x)| + |rG_{\frac{1}{n}}(f)(x) - M_j(f)| < \frac{1}{n}.
\end{aligned}$$

This means that $f \in A_{inrj} \subset O_{inr}$. Since B_i , r and n were chosen arbitrarily, we have $f \in A$.

On the other hand, let $f \in A$. For any $(x, t) \in \downarrow f$, let O be an open set of $X \times \mathbf{I}$ such that $(x, t) \in O$. Then there are $B_i \in \mathcal{B}$ and $n \in \mathbf{N}$ such that $(x, t) \in U_d(B_i, 1/n) \times (t - 2/n, t + 2/n) \subset O$. Since $G_{1/n}(f)(x) \geq f(x) \geq t$ we can choose an $r \in (0, 1] \cap \mathbf{Q}$ such that $|rM_i(G_{1/n}(f)) - t| < 1/n$. It follows from $f \in O_{inr}$ that there exists a $B_j \subset U_d(B_i, 1/n)$ such that $|rM_i(G_{1/n}(f)) - M_j(f)| < 1/n$. Now, choose a $y \in B_j$ satisfying $M_j(f) - f(y) < 1/n - |rM_i(G_{1/n}(f)) - M_j(f)|$. Then

$$|t - f(y)| \leq |t - rM_i(G_{\frac{1}{n}}(f))| + |rM_i(G_{\frac{1}{n}}(f)) - M_j(f)| + M_j(f) - f(y) < \frac{2}{n}.$$

Hence we have

$$(y, f(y)) \in U_d\left(B_i, \frac{1}{n}\right) \times \left(t - \frac{2}{n}, t + \frac{2}{n}\right) \subset O.$$

Therefore, \mathcal{G}_f is dense in $\downarrow f$, that is, $f \in \text{SDC}^*(X)$. □

LEMMA 14. *The space $\text{SDC}(X)$ is a G_δ -set in $\text{USC}(X)$.*

PROOF. Let \mathcal{B} be a countable open basis for X consisting of nonempty sets. Observe that Lemma 8(b) shows that

$$\text{SDC}(X) = \text{SDC}^*(X) \setminus \bigcup_{B \in \mathcal{B}} \{f \in \text{USC}(X) : f \upharpoonright B = 0\}.$$

To complete the proof of the lemma, we need to show that the set $\{f \in \text{USC}(X) : f \upharpoonright B = 0\}$ is closed in $\text{USC}(X)$ for each $B \in \mathcal{B}$. Let $g \in \text{USC}(X)$ be such that there is an $x \in B$ with $g(x) > 0$. Then for any f with $f \upharpoonright B = 0$ we have $D(f, g) \geq \min\{g(x), d(x, X \setminus B)\} > 0$. Thus we have that the complement of $\{f \in \text{USC}(X) : f \upharpoonright B = 0\}$ is open. □

The following theorem includes Theorem 1.

THEOREM 15. *If X is a compact metric space then the following statements*

are equivalent.

- (1) X is dense in itself.
- (2) $\text{SDC}(X)$ is not empty.
- (3) $\text{SDC}(X)$ is a pseudointerior in $\text{USC}(X)$.
- (4) $\text{SDC}^*(X)$ is a pseudointerior in $\text{USC}(X)$.
- (5) $\text{SDC}^*(X) \setminus \{0\}$ is a pseudointerior in $\text{USC}(X)$.

PROOF. The equivalence of (1) and (2) follows immediately from Lemmas 8 and 9. The implication (3) \Rightarrow (2) is trivial.

(4) \vee (5) \Rightarrow (1). Note that pseudointeriors are dense. Assuming (4) or (5) we find for every $\varepsilon > 0$ an $f_\varepsilon \in \text{SDC}(X)$ with $D(1, f_\varepsilon) < \varepsilon$. Note that $\text{supp } f_\varepsilon$ is an ε -net in X that is dense in itself by Lemma 8. So $\bigcup_{\varepsilon > 0} \text{supp } f_\varepsilon$ is dense in itself and dense in X . We have that X is dense in itself.

(1) \Rightarrow (3)&(4)&(5). Assume that X is dense in itself. By Lemmas 13 and 14 the complements of $\text{SDC}(X)$, $\text{SDC}^*(X)$ and $\text{SDC}^*(X) \setminus \{0\}$ are F_σ -sets in $\text{USC}(X)$ that are σZ -sets because of Lemma 11. Note that $\text{SDC}^*(X) \cap \text{LIP}(X) = \{0\}$ and hence it follows from Proposition 5 and Theorem 6 that $\text{SDC}(X)$, $\text{SDC}^*(X)$ and $\text{SDC}(X)^* \setminus \{0\}$ are all pseudointeriors in $\text{USC}(X)$. \square

A space is called *scattered* if every nonempty subspace has isolated points.

THEOREM 16. *If X is a compact metric space then the following statements are equivalent.*

- (1) X is not scattered.
- (2) $\text{SDC}^*(X) \setminus \{0\}$ is not empty.
- (3) $\text{SDC}^*(X) \approx \text{SDC}^*(X) \setminus \{0\} \approx \ell^2$.

PROOF. (3) \Rightarrow (2) is trivial and for (2) \Rightarrow (1) note that if $f \in \text{SDC}^*(X) \setminus \{0\}$ then $\text{supp } f$ is not empty and dense in itself by Lemma 8(c).

(1) \Rightarrow (3). If X is not scattered then $B = \bigcup \{A \subset X : A \text{ dense in itself}\}$ is a nonempty compactum that is dense in itself. By Theorem 15 we have that $\text{SDC}^*(B)$ and $\text{SDC}^*(B) \setminus \{0\}$ are homeomorphic to ℓ^2 . Note that by Lemma 8(c) we have $\text{supp } f \subset B$ for every $f \in \text{SDC}^*(X)$. Define the map $\psi: \text{SDC}^*(X) \rightarrow \text{SDC}^*(B)$ by $\psi(f) = f \upharpoonright B$ and note that it is an isometric bijection. Thus we have statement (3). \square

We finish by proving the result about Lelek fans. An *endpoint* of a space is a point that is not an internal point of some arc in the space. As in the introduction we let $q: \mathfrak{C} \times \mathbf{I} \rightarrow \Delta \mathfrak{C}$ be the quotient map. Consider a standard Lelek fan $q(f)$ where $f \in \text{SDC}(\mathfrak{C})$. Then clearly $q(\mathcal{G}_f^+)$ is the (dense) set of endpoints of the Lelek fan.

Let $C(\Delta X, b)$ consist of all subcontinua of ΔX that contain the base point $b = X \times \{0\}$ of the cone, equipped with the Hausdorff metric. The following result implies Theorem 2.

THEOREM 17. *The space of Lelek fans \mathfrak{L} is a pseudointerior of $C(\Delta \mathfrak{C}, b)$.*

PROOF. We will prove this theorem by showing that the pair $(C(\Delta \mathfrak{C}, b), \mathfrak{L})$ is homeomorphic to $(\text{USC}(\mathfrak{C}), \text{SDC}^*(\mathfrak{C}) \setminus \{0\})$.

We define the map $\chi: \text{USC}(C) \rightarrow C(\Delta \mathfrak{C}, b)$ by $\chi(f) = q(\downarrow f)$ for $f \in \text{USC}(\mathfrak{C})$. It is clear that χ is well-defined, one-to-one, and continuous. If $K \in C(\Delta \mathfrak{C}, b)$ then by the Boundary Bumping Theorem [14, Theorem 5.6] every component of $K \setminus \{b\}$ is an interval of the form $\{x\} \times (0, t]$. Thus there exists a function $f: X \rightarrow \mathbf{I}$ such that $q(\downarrow f) = K$. Since K is compact we have that $f \in \text{USC}(\mathfrak{C})$ and thus χ is surjective and a homeomorphism by the compactness of $\text{USC}(\mathfrak{C})$.

Next we show that $\chi(\text{SDC}^*(C) \setminus \{0\}) = \mathfrak{L}$. Let $f \in \text{SDC}^*(C) \setminus \{0\}$. Consider $A = \text{supp } f$ and note that $\chi(f) = q(\downarrow(f \upharpoonright A))$. By Lemma 8(c) A is a nonempty, zero-dimensional compactum that is dense in itself thus $A \approx \mathfrak{C}$. By Lemma 8(a) we have $f \upharpoonright A \in \text{SDC}(A)$ and $\chi(f)$ is a (standard) Lelek fan.

Let $L \in \mathfrak{L}$. If $b \notin L$ then $L \subset \mathfrak{C} \times (0, 1]$ and hence the Lelek fan is contained in an arc, which is impossible. So we have $b \in L$ and hence $L = \chi(f)$ for some $f \in \text{USC}(\mathfrak{C})$. Note that $f \neq 0$ and that we need to show that \mathcal{G}_f is dense in $\downarrow f$. Let (x, t) be an arbitrary element of $\downarrow f$. If $f(x) = 0$ then $(x, t) = (x, 0) \in \mathcal{G}_f$. Let $f(x) > 0$ and assume that $t > 0$. If F is a closed subset of $\downarrow f$ such that $(x, t) \notin F$ then $q(F)$ is a closed set in L that does not contain $q(x, t)$ so there exists an endpoint $q(a, f(a)) \in L \setminus F$. Then $(a, f(a)) \in \mathcal{G}_f \setminus F$ and we have that $\{x\} \times (0, f(x)]$ is contained in the closure of \mathcal{G}_f . Consequently, also $\{x\} \times [0, f(x)]$ is contained in \mathcal{G}_f and we may conclude that $f \in \text{SDC}^*(\mathfrak{C})$. \square

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