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A new proof of the Baker-Campbell-Hausdorff formula

Dedicated to Professor Shôkichi Iyanaga on his 60th birthday

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This formula states

$$e^{A}\cdot e^{B}=e^{Z}, \qquad Z=\sum_{n=1}^{\infty}F_{n}(A, B) \tag{1}$$

for noncommuting indeterminates A, B with homogeneous polynomials $F_{n}(A, B)$ of degree n which have the essential property that they are formed from A, B by Lie multiplication, except for $F_{1}(A, B)=A+B$. We shall briefly speak of Lie polynomials. The usual proofs (e. g. [**1**], [**2**]) employ preliminary theorems by Finkelstein or Friedrichs characterizing Lie polynomials by formal properties (see also [**3**]). In the following lines I give a short proof which needs no preparations.

It is evident that polynomials $F_{n}(A, B)$ exist satisfying (1). We only have to prove that they are Lie polynomials. The first two are

$$F_{1}(A, B)=A+B, \qquad F_{2}(A, B)=\frac{1}{2}(AB-BA).$$

Now let $n>2$ and assume that all $F_{\nu}(A, B)$ with $\nu<n$ are Lie polynomials. With 3 indeterminates we express

$$(e^{A}e^{B})e^{C}=e^{A}(e^{B}e^{C}):$$

$$W=\sum_{i=1}^{\infty}F_{i}\Big(\sum_{j=1}^{\infty}F_{j}(A, B), C\Big)=\sum_{i=1}^{\infty}F_{i}\Big(A, \sum_{j=1}^{\infty}F_{j}(B, C)\Big)$$

and compare the homogeneous terms of degree n on both sides, using the following 2 facts: 1) If $F(A, B, \cdots)$, $X(A, B, \cdots)$, $Y(A, B, \cdots), \cdots$ are Lie polynomials then also $G(A, B, \cdots)=F(X(A, B, \cdots), Y(A, B, \cdots), \cdots)$ is one. 2) If $F(A, B, \cdots)$ is a Lie polynomial then the homogeneous summands into which F splits up are Lie polynomials. The induction assumption implies that all homogeneous terms of degree n in both expressions for W are Lie polynomials with the possible exceptions of $F_{n}(A, B)+F_{n}(A+B, C)$ on the left side and $F_{n}(A, B+C)+F_{n}(B, C)$ on the right. In other words, the difference is a Lie polynomial. We can abbreviate this as

$$F(A, B)+F(A+B, C)\sim F(A, B+C)+F(B, C) \tag{2}$$

with $F=F_n$ (for sake of simplicity we drop the subscript n from now on). A second property is evident:

$$F(\lambda A, \mu A)=0 \tag{3}$$

where λ, μ are commuting variables. The properties (2) and (3) suffice to show $F(A, B)\sim 0$, and the proof yields a recursive scheme for their computation.

First we insert $C=-B$ in (2) and observe (3):

$$F(A, B)\sim -F(A+B, -B)\,. \tag{4}$$

Similarly we insert $A=-B$, but write A, B instead of B, C:

$$F(A, B)\sim -F(-A, A+B)\,. \tag{5}$$

Applying in order (5), (4), (5) we get

$$F(A, B)\sim -(-1)^n F(B, A)\,, \tag{6}$$

because $F(A, B)$ is homogeneous of degree n.

Secondly we insert $C=-\frac{1}{2}B$ in (2):

$$F(A, B)\sim F\Big(A, \frac{1}{2}B\Big)-F\Big(A+B, -\frac{1}{2}B\Big) \tag{7}$$

and similarly with $A=-\frac{1}{2}B$ and A, B instead of B, C:

$$F(A, B)\sim F\Big(\frac{1}{2}A, B\Big)-F\Big(-\frac{1}{2}A, A+B\Big)\,. \tag{8}$$

Application of (7) to both terms on the right of (8) yields

$$\begin{aligned} F(A, B)\sim F\Big(\frac{1}{2}A, \frac{1}{2}B\Big)-F\Big(-\frac{1}{2}A, \frac{1}{2}A+\frac{1}{2}B\Big) \\ -F\Big(\frac{1}{2}A+B, -\frac{1}{2}B\Big)+F\Big(\frac{1}{2}A+B, -\frac{1}{2}A-\frac{1}{2}B\Big)\,. \end{aligned}$$

Here we employ (5) in the 2nd term on the right and (4) in the 3rd and 4th, remembering the homogeneity:

$$F(A, B)\sim 2^{1-n}F(A, B)+2^{-n}F(A+B, B)-2^{-n}F(B, A+B)\,.$$

and by (6)

$$(1-2^{1-n})F(A, B)\sim 2^{-n}(1+(-1)^n)F(A+B, B)\,.$$

For odd n this is already the contention. For even n we insert $A-B$ for A and apply (4) for a last time:

$$-F(A, -B)\sim F(A-B, B)\sim 2^{-n}(1+(-1)^n)(1-2^{1-n})^{-1}F(A, B)\,.$$

Iteration of this formula gives

$$F(A, B)\sim 2^{-2n}(1+(-1)^n)^2(1-2^{1-n})^{-2}F(A, B)\,,$$

and the factor on the right is $\neq 1$ because $n > 2$. This finishes the proof.

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References

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[2] W. Magnus, On the exponential solutions of differential equations for a linear operator, Comm. Pure Appl. Math., 7 (1954), 649–673.
[3] W. v. Waldenfels, Zur Charakterisierung Liescher Elemente in freien Algebren, Arch. Math., 12 (1966), 44–48.