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A classification of weighted homogeneous Saito free divisors

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Abstract. We describe an approach to classification of weighted homogeneous Saito free divisors in $\boldsymbol{C}^3$. This approach is mainly based on properties of Lie algebras of vector fields tangent to reduced hypersurfaces at their non-singular points. In fact we also obtain a classification of such Lie algebras having similar properties as ones for discriminants associated with irreducible real reflection groups of rank 3. Among other things we briefly discuss some applications to the theory of discriminants of irreducible reflection groups of rank 3, some interesting relationships with root systems of types E_6, E_7, E_8, and few examples in higher dimensional cases.

Introduction.

It is well-known that the discriminant D associated with an arbitrary irreducible real reflection group W_n of rank n can be defined in $\boldsymbol{C}^n$ as zero-set of a polynomial, the determinant of a square matrix of order n whose entries are coefficients of vector fields tangent to D at its non-singular points (cf. [**15**], [**16**], [**21**]). Furthermore, the discriminant is, in fact, a weighted homogeneous reduced hypersurface of special kind. It is possible to prove [loc. cite] that tangent vector fields generate a free module over the polynomial ring $\boldsymbol{C}[z_1, \ldots, z_n]$. Following A. G. Aleksandrov (cf. [**3**]), a reduced hypersurface is called a Saito free divisor if the tangent vector fields generate a free module over the polynomial ring. In this sense, discriminants are Saito free divisors.

The purpose of this paper is to obtain a complete list of weighted homogeneous Saito free divisors in $\boldsymbol{C}^3$ whose Lie algebras of tangent vector fields have similar properties as ones for discriminants associated with irreducible real reflection groups of rank 3, that is, with groups A_3, B_3 and H_3. The key idea of our approach is to compute the required list in parallel with enumeration of certain Lie algebras of rank 3 making use of properties of vector fields tangent to discriminants.

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Let us briefly describe the content of the paper. In the first three sections we discuss basic notions and results from the theory of Saito free divisors related with discriminants of finite reflection groups. Then we describe all Lie algebras of three variables satisfied certain conditions. As a consequence we obtain a total of 17 non-isomorphic Lie algebras associated with different weighted homogeneous polynomials which define Saito free divisors in $\boldsymbol{C}^3$. In the next two sections we prove the main result of the paper. It states that the set $\mathscr{D}$ of all these polynomials is the union of 3 subsets $\mathscr{D}_A$, $\mathscr{D}_B$ and $\mathscr{D}_H$, containing 2, 7 and 8 elements with types of homogeneity $\pi_A = (12; 2, 3, 4)$, $\pi_B = (9; 1, 2, 3)$ and $\pi_H = (15; 1, 3, 5)$, respectively. In its turn, they correspond to types of homogeneity of discriminants associated with reflection groups A_3, B_3 and H_3, respectively. In Section 5 we prove that any polynomial from $\mathscr{D}$ determines an affine hypersurface which can be regarded as an affine deformation of a singular plane curve having simple singularities of types E_6, E_7 or E_8. In a more general context (see [**5**]) such hypersurfaces can be also considered as affine quasicones over projective weighted plane curves. In Section 6 we establish close relationships between polynomials from $\mathscr{D}$ and subroot systems of root systems of types E_6, E_7 or E_8. To be more precise, our result is the following.

There are natural bijections between $\mathscr{D}_B$ and $\mathscr{D}_H$ and two sets of corank one subdiagrams of Dynkin diagram of type E_7 and E_8, respectively. On the other hand, there is a natural bijection between $\mathscr{D}_A$ and the set of corank one subdiagrams of Dynkin diagram of type E_6 invariant under its non-trivial symmetry.

In section 7 we discuss some related topics while in Appendix we give explicit representations for coefficient matrices associated with generators of Lie algebras of tangent vector fields for 17 hypersurfaces determined by polynomials from $\mathscr{D}$.

It should be noted that E. Brieskorn, K. Saito [**7**] and P. Deligne [**10**] proved that the complement $\mathscr{C}$ of the discriminant associated with an arbitrary finite irreducible reflection group is a $K(\pi, 1)$-space and the fundamental group $\pi(\mathscr{C})$ is an Artin group. That is why it is quite interesting to analyze fundamental groups of the complement of hypersurfaces determined by the polynomials from $\mathscr{D}$. Recently T. Ishibe and K. Saito computed these groups explicitly (cf. [**12**]).

In conclusion we also remark that there are few works devoted to the problem of classification of Saito free divisors in the context of the theory of arrangements (see [**13**]) or non-isolated singularities (see [**5**]); there are also many other studies devoted to similar problematic from other points of view (for example, see [**9**], [**11**], [**20**]).

1. Affine Saito free divisors.

First let us consider an algebraic version of the original definition of free divisors; the latter appeared in the context of the theory of unfoldings of functions with isolated critical points initiated by K. Saito (see [**15**]). Recall his basic definition. Let S be an n-dimensional complex manifold, let D be a reduced hypersurface of S, and let $o \in D$. Then locally D is defined by an equation $h(z) = 0$, where $h(z) = h(z_1, \ldots, z_n)$ is the germ of a holomorphic function in an open neighbourhood U of o, $h(z)$ has no multiple factors, and $\langle z_1, \ldots, z_n\rangle$ is a system of local coordinates in U.

Following K. Saito we denote the coherent sheaf of $\mathscr{O}_S$-modules of vector fields *logarithmic* along D by $\mathrm{Der}_S(\log D)$. This $\mathscr{O}_S$-module is often called the *module of tangent vector fields*; it consists of germs of holomorphic vector fields $\eta \in \mathrm{Der}(\mathscr{O}_S)$ on S such that $\eta(h)$ belongs to the principal ideal $(h)\cdot\mathscr{O}_S$. In particular, the vector field η is tangent to D at its non-singular points. It should be remarked that $\mathrm{Der}_S(\log D)$ is naturally endowed with structure of Lie algebra denoted by $\mathscr{L}_D$. It is usually called the Lie algebra of vector fields tangent to the hypersurface D.

The next statement is due to K. Saito [**15**] and it gives a criterion of freeness for reduced hypersurfaces in the local situation.

PROPOSITION 1 (Saito's Criterion). *The stalk* $\mathrm{Der}_{S,o}(\log D)$ *is a free* $\mathscr{O}_{S,o}$*-module if and only if there are* n *germs of logarithmic vector fields* $V^0, \ldots, V^{n-1} \in \mathrm{Der}_{S,o}(\log D)$ *such that the determinant of the square matrix* $M = \|v_{ij}\|$ *of order* n *whose entries are coefficients of* V^i*,* $i = 0, \ldots, n-1$*, is equal to* αh*, where* α *is a unit. These vector fields form a basis of* $\mathrm{Der}_{S,o}(\log D)$*.*

The hypersurface D is called a *Saito free* divisor when $\mathrm{Der}_S(\log D)$ is a *locally free* $\mathscr{O}_S$-module (see [**8**]). For example, $\mathrm{Der}_S(\log D)$ as well as its $\mathscr{O}_S$-dual $\Omega^1_S(\log D)$, the module of logarithmic differential forms with poles along D, are locally free if D is a smooth hypersurface, a plane curve (see [**15**], Corollary (1.7)), or a divisor with *strict* normal crossings (see [**3**]). It is not difficult to see that the system of vector fields $V = \langle V^0, \ldots, V^{n-1}\rangle$ is involutive; the corresponding Lie algebra $\mathscr{L}(V)$ of rank n is isomorphic to $\mathscr{L}_D$.

One can get an affine globalization of this notion as follows. Let now $S = \boldsymbol{A}^n$ be the n-dimensional affine space over $\boldsymbol{C}$, and let $R = \boldsymbol{C}[z_1, \ldots, z_n]$ be the polynomial algebra of n variables. In such a case, a local freeness of $\mathrm{Der}_S(\log D)$ over R means that this $\mathscr{O}_S$-module is *stable free* over R, that is, for any large enough N there is a R-module isomorphism

$$\vartheta_N : \mathrm{Der}_S(\log D) \oplus R^N \to R^{n+N}.$$

Of course, the requirement of freeness over the polynomial ring R is much stronger than the condition of local freeness: any affine hypersurface with free module $\mathrm{Der}_S(\log D)$ of logarithmic vector fields is a Saito free divisor in the original sense. However, the converse is not true in general. Of course, Saito's Criterion remains still valid for affine hypersurfaces.

2. Discriminants and determinants.

Now we restrict ourselves by the case of affine space of dimension three, that is, $S \cong \boldsymbol{C}^3$. Let now p, q and r be positive integers such that $p < q < r$. We assume that p, q, r have no common factors > 1. Let $R = \boldsymbol{C}[x, y, z]$ be the graded polynomial algebra generated by weighted variables x, y and z with entire positive weights equal to p, q and r, respectively. Further, denote by ∂_x, ∂_y, ∂_z the partial derivatives with respect to x, y, z, respectively.

Let $E = px\partial_x + qy\partial_y + rz\partial_z$ be a linear vector field; it is usually called the Euler vector field. If $f \in R$ is a polynomial such that $E(f) = df$, then f is called weighted homogeneous of type $(d; p, q, r)$. The number d is called the degree of f.

Let us now define the following triple of regular vector fields on S:

$$
\begin{aligned}
V^0 &= px\partial_x + qy\partial_y + rz\partial_z, \\
V^1 &= qy\partial_x + h_{22}\partial_y + h_{23}\partial_z, \\
V^2 &= rz\partial_x + h_{32}\partial_y + h_{33}\partial_z,
\end{aligned}
$$

where V^0 is the Euler field and $h_{ij} \in R$ are weighted homogeneous polynomials. Analogously to notations of Saito's Criterion one can associate with this triple a square matrix of the third order whose entries are coefficients of these vector fields:

$$
M = \begin{pmatrix} px & qy & rz \\ qy & h_{22} & h_{23} \\ rz & h_{32} & h_{33} \end{pmatrix}.
$$

Let us consider the following requirements on triples of vector fields defined above.

CONDITION 1.
(i) $[V^0, V^1] = (q-p)V^1$, $[V^0, V^2] = (r-p)V^2$;
(ii) $[V^1, V^2] = f_0V^0 + f_1V^1 + f_2V^2$, where $f_j \in R$ are polynomials for all $j = 0, 1, 2$;
(iii) $h_{22} = az + g(x, y)$, where $a \in \boldsymbol{C}^*$ is a non-zero constant and $g(x, y)$ is a

polynomial depending on x and y only;
(iv) the polynomial $F=\det(M)$ is not equivalent to the monomial z^3 under weighted changes of variables.

REMARK 1. The first two requirements of Condition 1 imply that the system of vector fields $V=\langle V^0,\ V^1,\ V^2\rangle$ is involutive. That is, these vector fields generate Lie algebra $\mathscr{L}(V)$ of rank 3 over R. The third requirement yields that $p+r=2q$ and $\det(M)$ contains the monomial z^3 with a non-zero coefficient. That is, the degree of the polynomial $\det(M)$ is equal to $3r=3(2q-p)$. If $\det(M)=z^3$, then $F=0$ defines a non-reduced affine hyperplane D and it is clear that $\mathscr{L}_D$ is generated by ∂_x, ∂_y, $z\partial_z$, that is, $\mathscr{L}(V)\ncong\mathscr{L}_D$; it is reasonable to exclude such case from further considerations with the help of the fourth requirement.

3. Basic examples.

Let W be a finite irreducible reflection group acting on a real vector space of dimension 3. Let x,y,z be the basic W-invariant polynomials and let F be the discriminant of W. Then F is a polynomial containing in the graded ring R; it determines an affine reduced hypersurface $D\subset \boldsymbol{C}^3$. All vector fields tangent to D satisfy Condition 1; they generate Lie algebra $\mathscr{L}_D$ isomorphic to $\mathscr{L}(V)$. The following three basic examples can be found in [**15**] or [**22**].

A_3-case: $\pi_A=(12;2,3,4)$, and

$$M=\begin{pmatrix} 2x & 3y & 4z\\ 3y & -x^2+4z & -\frac{1}{2}xy\\ 4z & -\frac{1}{2}xy & \frac{1}{4}(8xz-3y^2)\end{pmatrix}.$$

Then $\det(M)=F_{A,DSC}=\frac{1}{4}(-16x^4z+4x^3y^2+128x^2z^2-144xy^2z+27y^4-256z^3)$. Further, $[V^0,V^1]=V^1$, $[V^0,V^2]=V^2$ and $[V^1,V^2]=\frac{1}{2}V^0-\frac{1}{2}xV^1$ (cf. [**2**], (6.1)).

B_3-case: $\pi_B=(9;1,2,3)$, and

$$M=\begin{pmatrix} x & 2y & 3z\\ 2y & xy+3z & 2xz\\ 3z & 2xz & yz\end{pmatrix}.$$

Then $\det(M)=F_{B,DSC}=z(-4x^3z+x^2y^2+18xyz-4y^3-27z^2)$, $[V^1,V^2]=xV^2-zV^0$ (cf. [**2**], (6.4)).

H_3-case: $\pi_H=(15;1,3,5)$, and

$$M = \begin{pmatrix} x & 3y & 5z \\ 3y & 2x^2y+2z & 7xy^2+2x^4y \\ 5z & 7xy^2+2x^4y & \frac{1}{2}(15y^3+4x^4z+18x^3y^2) \end{pmatrix}.$$

Then $\det(M) = F_{H,DSC} = -50z^3 + (4x^5 - 50x^2y)z^2 + (4x^7y + 60x^4y^2 + 225xy^3)z - \frac{135}{2}y^5 - 115x^3y^4 - 10x^6y^3 - 4x^9y^2$. In this case $[V^0, V^1] = 2V^1$, $[V^0, V^2] = 2V^2$ and $[V^1, V^2] = (4x^3y + 2y^2)V^0 + 4xyV^1$.

The aim of the paper is to describe a class of hypersurfaces which can be regarded as analogues of the discriminants associated with finite irreducible reflection groups. In particular, they are to be Saito free divisors.

PROBLEM 1. How to describe all triples $\langle V^0, V^1, V^2\rangle$ of vector fields satisfying Condition 1 up to weighted changes of variables and the corresponding Lie algebras $\mathscr{L}(V)$ up to weighted algebraic isomorphisms?

4. The main result.

Our solution of Problem 1 can be summarized as follows.

THEOREM 1. *In notations of Section* 2 *the following assertions hold.*

(i) *If* $(p,q,r) \neq (2,3,4), (1,2,3), (1,3,5)$, *then there are no triples* $\langle V^0, V^1, V^2\rangle$ *of vector fields satisfying Condition* 1.

(ii) *The remaining cases are described as follows.*

($\mathscr{D}_A$) *if* $(p,q,r) = (2,3,4)$, *then up to weighted changes of variables there are two triples of vector fields satisfying Condition* 1; *the corresponding polynomials* $F = \det(M)$ *of degree* 12 *are the following*:

$$F_{A,1} = 16x^4z - 4x^3y^2 - 128x^2z^2 + 144xy^2z - 27y^4 + 256z^3;$$
$$F_{A,2} = 2x^6 - 3x^4z + 18x^3y^2 - 18xy^2z + 27y^4 + z^3.$$

($\mathscr{D}_B$) *if* $(p,q,r) = (1,2,3)$, *then up to weighted changes of variables there are seven triples of vector fields satisfying Condition* 1; *the corresponding polynomials* $F = \det(M)$ *of degree* 9 *are the following*:

$$F_{B,1} = z(x^2y^2 - 4y^3 - 4x^3z + 18xyz - 27z^2);$$
$$F_{B,2} = z(-2y^3 + 4x^3z + 18xyz + 27z^2);$$
$$F_{B,3} = z(-2y^3 + 9xyz + 45z^2);$$
$$F_{B,4} = z(9x^2y^2 - 4y^3 + 18xyz + 9z^2);$$

$$F_{B,5} = xy^4 + y^3z + z^3;$$
$$F_{B,6} = 9xy^4 + 6x^2y^2z - 4y^3z + x^3z^2 - 12xyz^2 + 4z^3;$$
$$F_{B,7} = \tfrac{1}{2}xy^4 - 2x^2y^2z - y^3z + 2x^3z^2 + 2xyz^2 + z^3.$$

($\mathscr{D}_H$) *if* $(p, q, r) = (1, 3, 5)$, *then up to weighted changes of variables there are eight triples of vector fields satisfying Condition* 1; *the corresponding polynomials* $F = \det(M)$ *of degree* 15 *are the following*:

$$\begin{aligned}
F_{H,1} &= -50z^3 + (4x^5 - 50x^2y)z^2 + (4x^7y + 60x^4y^2 + 225xy^3)z - \tfrac{135}{2}y^5 - 115x^3y^4 \\
&\quad - 10x^6y^3 - 4x^9y^2; \\
F_{H,2} &= 100x^3y^4 + y^5 + 40x^4y^2z - 10xy^3z + 4x^5z^2 - 15x^2yz^2 + z^3; \\
F_{H,3} &= 8x^3y^4 + 108y^5 - 36xy^3z - x^2yz^2 + 4z^3; \\
F_{H,4} &= y^5 - 2xy^3z + x^2yz^2 + z^3; \\
F_{H,5} &= x^3y^4 - y^5 + 3xy^3z + z^3; \\
F_{H,6} &= x^3y^4 + y^5 - 2x^4y^2z - 4xy^3z + x^5z^2 + 3x^2yz^2 + z^3; \\
F_{H,7} &= xy^3z + y^5 + z^3; \\
F_{H,8} &= x^3y^4 + y^5 - 8x^4y^2z - 7xy^3z + 16x^5z^2 + 12x^2yz^2 + z^3.
\end{aligned}$$

In all cases ($\mathscr{D}_A$), ($\mathscr{D}_B$), ($\mathscr{D}_H$), *there are isomorphisms* $\mathscr{L}_D \cong \mathscr{L}(V)$, *where the zero-set of the corresponding determinant polynomial* $F = \det(M)$ *is denoted by* D.

REMARK 2.

1. The first variant of Theorem 1 has been proved by the author in 1992 (see [**17**]).

2. The polynomials $F_{A,1}$, $F_{B,1}$ and $F_{H,1}$ are equal to the discriminants $F_{A,DSC}$, $F_{B,DSC}$ and $F_{H,DSC}$ associated with irreducible reflection groups of types A_3, B_3 and H_3, respectively (see Section 3).

3. The polynomial $F_{A,2}$ was found by M. Sato (cf. [**23**], [**17**]).

4. Let $F(x, y, z)$ be any polynomial from Theorem 1. If the weights of variables x, y, z are equal to $(2, 3, 4)$ (or to $(1, 2, 3)$, $(1, 3, 5)$), then the zero-set $\mathscr{E}$ of the polynomial $F(0, y, z)$ is a plane curve with a simple singularity of type E_6 (or E_7, E_8, respectively) (cf. [**19**]). It should be noted here that the curve defined by $F(0, y, z) = 0$ does not depend on the choice of the weighted homogeneous coordinates (x, y, z) since $p < q$, r. Moreover, the polynomial $F(x, y, z)$ defines a family of plane curves $\mathscr{E}_x$ in yz-space; this family can be considered as an affine deformation of the curve $\mathscr{E} = \mathscr{E}_0$ or an affine quasicone over $\mathscr{E}$.

5. As was already remarked in Introduction, the complement $\mathscr{C}$ of the discriminants associated with an arbitrary finite irreducible reflection group is a $K(\pi, 1)$-space and the fundamental group $\pi(\mathscr{C})$ is an Artin group. In this connection it should be remarked that recently T. Ishibe acquainted the author with his master thesis where all these fundamental groups are computed (see [**12**]).

5. Proof of Theorem 1.

For the proof we need the following technical statement.

LEMMA 1. *Assume that m, n, l are positive integers and i, j, k are non-negative integers satisfied the following conditions*:

(a) $l \in \{2, 3, 4\}$;

(b) $m(i+j+k-1) = n(l-j-2k)$;

(c) *if $m > 1$ and $n > 1$, then m and n are mutually prime.*

Then the following conclusions are valid.

(i) *Suppose that $m > 2$.*

If $l = 2$, then $(i, j, k) = (0, 0, 1)$.

If $l = 3$, then $(j, k) = (0, 0)$ and $m = 3, i = n + 1$.

If $l = 4$, then $m = 3, i = n, j = 1, k = 0$ or $m = 4, i = n + 1, j = k = 0$.

(ii) *Suppose that $m = 2$.*

If $l = 2$, then $(i, j, k) \in \{(0, 0, 1), (n + 1, 0, 0)\}$.

If $l = 3$, then $(i, j, k) = (n, 1, 0)$.

If $l = 4$, then $(i, j, k) \in \{(n, 0, 1), (n - 1, 2, 0)\}$.

(iii) *Suppose that $m = 1$.*

If $l = 2$, then $(i, j, k) \in \{(0, 0, 1), (2n + 1, 0, 0), (n, 1, 0)\}$.

If $l = 3$, then $(i, j, k) \in \{(n, 0, 1), (n - 1, 2, 0), (2n, 1, 0), (3n + 1, 0, 0)\}$.

If $n > 1$ and $l = 4$, then $(i, j, k) \in \{(2n, 0, 1), (n - 2, 3, 0), (2n - 1, 2, 0), (3n, 1, 0), (4n + 1, 0, 0)\}$.

If $n = 1$ and $l = 4$, then $(i, j, k) \in \{(2, 0, 1), (1, 2, 0), (3, 1, 0), (5, 0, 0)\}$.

We omit the proof of this lemma since it is quite elementary.

Now we are able to prove Theorem 1. At first we are going to determine the polynomials h_{ij}, the entries of the matrix M from Section 2. The proof is divided into few computational steps.

STEP 1. As was also remarked Condition 1 implies that $2q = p + r$. Set $p = m$, $q = m + n$, $r = m + 2n$, where m, n are positive integers. In addition, if $m > 1$, then one can assume that m and n have no common factors. Under our assumptions one obtains

$$\deg h_{22} = m + 2n, \quad \deg h_{23} = \deg h_{32} = m + 3n, \quad \deg h_{33} = m + 4n,$$
$$\deg x^i y^j z^k = im + j(m+n) + k(m+2n),$$

and the following relations:

(c.1) $\deg x^i y^j z^k = \deg h_{22}$ if and only if $m(i+j+k-1) = n(2-j-2k)$.
(c.2) $\deg x^i y^j z^k = \deg h_{23}$ if and only if $m(i+j+k-1) = n(3-j-2k)$.
(c.3) $\deg x^i y^j z^k = \deg h_{32}$ if and only if $m(i+j+k-1) = n(3-j-2k)$.
(c.4) $\deg x^i y^j z^k = \deg h_{33}$ if and only if $m(i+j+k-1) = n(4-j-2k)$.

All the statements below are easy consequences of relations (c.1)–(c.4) and Lemma 1.

(d.i) The case $m > 4$: there is a non-zero constant a such that

$$h_{22} = az, \quad h_{23} = h_{32} = h_{33} = 0.$$

(d.ii) The case $m = 4$: there is a non-zero constant a and a constant b such that

$$h_{22} = az, \quad h_{23} = h_{32} = 0, \quad h_{33} = bx^{n+1}.$$

(d.iii) The case $m = 3$: there is a non-zero constant a and constants b_1, b_2, b_3 such that

$$h_{22} = az, \quad h_{23} = b_1 x^{n+1}, \quad h_{32} = b_2 x^{n+1}, \quad h_{33} = b_3 x^n y.$$

(d.iv) The case $m = 2$: the integer n is odd and there is a non-zero constant a_1 and constants a_2, a_3, b_1, b_2, b_3 such that

$$\begin{aligned}
h_{22} &= a_1 z + a_2 x^{n+1},\\
h_{23} &= a_3 x^n y,\\
h_{32} &= b_1 x^n y,\\
h_{33} &= x^{n-1}(b_2 xz + b_3 y^2 + b_4 x^{n+2}).
\end{aligned}$$

(d.v.1) The case $m = 1$ and $n > 1$: there is a non-zero constant a_1 and constants $a_2, \ldots, a_7, b_1, \ldots, b_{10}$ such that

$$\begin{aligned}
h_{22} &= a_1 z + a_2 x^n y + a_3 x^{2n+1},\\
h_{23} &= a_4 x^n z + a_5 x^{n-1} y^2 + a_6 x^{2n} y + a_7 x^{3n+1},\\
h_{32} &= b_1 x^n z + b_2 x^{n-1} y^2 + b_3 x^{2n} y + b_4 x^{3n+1},\\
h_{33} &= b_5 x^{n-1} yz + b_6 x^{2n} z + b_7 x^{n-2} y^3 + b_8 x^{2n-1} y^2 + b_9 x^{3n} y + b_{10} x^{4n+1}.
\end{aligned} \tag{1}$$

(d.v.2) The case $m=1$ and $n=1$: there is a non-zero constant a_1 and constants $a_2,\dots,a_7,b_1,\dots,b_6,b_8,b_9,b_{10}$ such that

$$\begin{aligned}
h_{22} &= a_1z + a_2x^2y + a_3x^3,\\
h_{23} &= a_4xz + a_5y^2 + a_6x^2y + a_7x^4,\\
h_{32} &= b_1xz + b_2y^2 + b_3x^2y + b_4x^4,\\
h_{33} &= b_5yz + b_6x^2z + b_8xy^2 + b_9x^3y + b_{10}x^5.
\end{aligned}$$

Cases (d.v.1) and (d.v.2) differ by the term $x^{n-2}y^3$ of h_{23}.

STEP 2. Let us analyze three cases (d.i), (d.ii) and (d.iii), subsequently.

Case (d.i). In this case $F=\det(M)$ is equal to the monomial z^3 up to a non-zero constant; it should be excluded in view of the requirement (iv) of Condition 1.

Case (d.ii). One gets $V^1=(n+4)y\partial_x+az\partial_y$, $V^2=(2n+4)z\partial_x+bx^{n+1}\partial_z$. Hence, $[V^1,V^2]=bx^n(-ax\partial_y+(n+1)(n+4)y\partial_z)$. The involutivity implies that $b=0$ and $\det(M)$ can be transformed to the monomial z^3. It contradicts the requirement (iv) of Condition 1 again.

Case (d.iii). If $n>1$, then n is not divided by 3. In view of the arguments above one has

$$\begin{aligned}
V^0 &= 3x\partial_x+(n+3)y\partial_y+(2n+3)z\partial_z,\\
V^1 &= (n+3)y\partial_x+az\partial_y+b_1x^{n+1}\partial_z,\\
V^2 &= (2n+3)z\partial_x+b_2x^{n+1}\partial_y+b_3x^ny\partial_z.
\end{aligned}$$

Hence,

$$\begin{aligned}
[V^1,V^2] = {}&\{(2n+3)b_1-(n+3)b_2\}x^{n+1}\partial_x+\{(n+1)(n+3)b_2-ab_3\}x^ny\partial_y\\
&+[n(n+3)b_3x^{n-1}y^2+\{ab_3-(n+1)(2n+3)b_1\}x^nz]\partial_z.
\end{aligned}$$

Therefore one obtains $3[V^1,V^2]=\{(2n+3)b_1-(n+3)b_2\}x^nV^0$. This implies the following relations:

$$\begin{aligned}
(n+3)(2n+3)b_1-2(n+3)(2n+3)b_2+3a_1b_3&=0,\\
3n(n+3)b_3&=0,\\
-(2n+3)(5n+6)b_1+(n+3)(2n+3)b_2+3a_1b_3&=0.
\end{aligned}$$

Since n is a positive integer, it follows that $b_1 = b_2 = b_3 = 0$. That is, $\det(M)$ can be transformed to the monomial z^3; it is the same contradiction as above.

STEP 3. Case (d.iv). Analogously to the above arguments one gets

$$\begin{aligned} V^0 &= 2x\partial_x + (n+2)y\partial_y + 2(n+1)z\partial_z, \\ V^1 &= (n+2)y\partial_x + (a_1 z + a_2 x^{n+1})\partial_y + a_3 x^n y\partial_z, \\ V^2 &= 2(n+1)z\partial_x + b_1 x^n y\partial_y + x^{n-1}(b_2 xz + b_3 y^2 + b_4 x^{n+2})\partial_z. \end{aligned}$$

Since $m = 2$ one may assume in considerations below that n is odd. Set $V^3 = [V^1, V^2]$ and let us find conditions on a_j and b_k guaranteed the inclusion $V^3 \in \mathscr{L}(V)$. Set

$$U = V^3 - \frac{2(n+1)a_3 - (n+2)b_1}{n+2} V^1.$$

It is sufficient to find conditions under which the relation $U = 0$ holds. The definition implies that there are polynomials $Q_1,\ Q_2 \in R$ such that $U = Q_1\partial_y + Q_2\partial_z$. Let us define a vector field $W \in \mathscr{L}(V)$ as follows:

$$W = (n+2)yV^0 - 2xV^1.$$

Since the both fields U and W don't contain the differentiation ∂_x, the weight condition implies that U coincides with $x^{n-1}W$ up to a constant multiple. Comparing the coefficients of ∂_y of U and W, one gets

$$U = \frac{n(n+2)b_1 - a_1 b_3}{(n+2)^2} W. \tag{2}$$

Now we are going to describe conditions on $a_1, a_2, a_3, b_1, b_2, b_3, b_4$ under which relation (2) holds. We treat two cases $n = 1$ and $n > 1$, separately. Let us first assume that $n = 1$. In this case relation (2) implies the following system of equations:

$$\begin{aligned} 12a_2a_3 - 24a_2b_1 + 2a_1a_2b_3 + 9a_1b_4 &= 0, \\ 72a_2 + 12a_1a_3 - 24a_1b_1 + 9a_1b_2 + 2a_1^2b_3 &= 0, \\ 12a_3^2 - 6a_3b_1 - 9a_3b_2 - 18a_2b_3 + 2a_1a_3b_3 - 81b_4 &= 0, \\ 12a_3 + 12b_1 - 9b_2 - 10a_1b_3 &= 0. \end{aligned}$$

Solving this system, one obtains

$$\begin{aligned}
a_2 &= -(1/12)a_1(-6b_1 + 3b_2 + 2a_1b_3), \\
a_3 &= (1/12)(-12b_1 + 9b_2 + 10a_1b_3), \\
b_4 &= (1/162)(36b_1^2 - 27b_1b_2 - 72a_1b_1b_3 + 27a_1b_2b_3 + 26a_1^2b_3^2), \\
c &= (1/9)(3b_1 - a_1b_3),
\end{aligned}$$

and

$$a_1(24b_1 - 9b_2 - 10a_1b_3)(24b_1 - 9b_2 - 2a_1b_3) = 0.$$

Since $a_1 \neq 0$, one gets the following two possibilities:

(e.1) $24b_1 - 9b_2 - 10a_1b_3 = 0$, or (e.2) $24b_1 - 9b_2 - 2a_1b_3 = 0$.

Let us first treat Case (e.1), that is, $b_2 = \frac{2}{9}(12b_1 - 5a_1b_3)$. If $b_3 \neq 0$, then, making use of the following change of variables

$$z = -z' + \frac{6b_1 - a_1b_3}{36}x^2,\ x = \left(\frac{3}{a_1b_3}\right)^{1/2}x',\ y = \left(\frac{3a_1}{16b_3}\right)^{1/4}y',$$

one obtains that $\det(M)$ can be transformed up to a constant multiple to the following form:

$$16x'4z' - 4x'^3y'^2 - 128x'^2z'^2 + 144x'y'^2z' - 27y'^4 + 256z'^3.$$

If $b_3 = 0$, then it is not difficult to verify that $\det(M) = \frac{2}{27}a_1(b_1x^2 - 6z)^3$, that is, $\det(M)$ can be transformed to z^3; it contradicts the requirement (iv) of Condition 1.

Let us consider Case (e.2), that is, $b_2 = \frac{2}{9}(12b_1 - a_1b_3)$. If $b_3 \neq 0$, then, making use of the following change of variables

$$z = z' - \frac{6b_1 - a_1b_3}{36}x^2,\ x = \left(\frac{12}{a_1b_3}\right)^{1/2}x',\ y = \left(\frac{48a_1}{b_3}\right)^{1/4}y',$$

one obtains that $\det(M)$ can be transformed up to a constant multiple to the following form:

$$2x'^6 - 12x'z' + 18x'^3y'^2 - 18x'y'^2z' + 27y'^4 + z'^3.$$

This is, in fact, the polynomial $F_{A,2}$. If $b_3 = 0$, then it is easy to see that $\det(M)$ is exactly the same as in Case (e.1).

At last let us consider the case $n > 1$. Since the coefficient of the term $x^{n-2}y^3$ is equal to $(n-1)(n+2)b_3$, one obtains that $b_3 = 0$. Then relation (2) implies

$$\begin{aligned}(n+2)a_1b_4 - 4(n+1)a_2b_1 + 2(n+1)a_2a_3 &= 0,\\ 4(n+1)a_1b_1 - (n+2)a_1b_2 - 2(n+1)a_1a_3 - 2(n+1)^2(n+2)a_2 &= 0,\\ 2(n+1)a_3^2 - 2na_3b_1 - (n+2)a_3b_2 - (n+2)^2(2n+1)b_4 &= 0,\\ 2(n+1)a_3 + 2(n+1)b_1 - (n+2)b_2 &= 0.\end{aligned}$$

Solving this system of equations, one gets

$$a_2 = -\frac{a_1b_1}{(n+1)(n+2)},\ a_3 = b_1,\ b_2 = \frac{4(n+1)b_1}{n+2},\ b_4 = -\frac{2b_1^2}{(n+2)^2}.$$

As a result one obtains

$$\det(M) = \frac{4a_1\{b_1x^{n+1} - (n+1)(n+2)z\}^3}{(n+1)(n+2)^3},$$

that is, $\det(M)$ can be transformed to z^3. It contradicts the requirement (iv) of Condition 1.

Step 4. Case (d.v.1). Assume now that $n = 2$ and $h_{22}, h_{23}, h_{32}, h_{33}$ are the polynomials given by (1). Then

$$\begin{aligned}V^0 &= x\partial_x + 3y\partial_y + 5z\partial_z,\\ V^1 &= 3y\partial_x + (a_1z + a_2x^2y + a_3x^5)\partial_y + (a_4x^2z + a_5xy^2 + a_6x^4y + a_7x^7)\partial_z,\\ V^2 &= 3z\partial_x + (b_1x^2z + b_2xy^2 + b_3x^4y + b_4x^7)\partial_y\\ &\quad + (b_5xyz + b_6x^4z + b_7y^3 + b_8x^3y^2 + b_9x^6y + b_{10}x^9)\partial_z.\end{aligned}$$

We are able to compute relations between the constants $a_1, a_2, \ldots, b_{10}$ which provide that $\langle V^0, V^1, V^2\rangle$ satisfy Condition 1. In virtue of the requirement (iii) one can assume that $a_1 \neq 0$ in considerations below. Taking a suitable weighted changes of variables

$$(x, y, z) = (x', y' + c_1x'^3, z' + c_2x'^2y' + c_3x'^5)$$

with some constants c_1, c_2, c_3, one may assume that $a_4 = a_5 = a_6 = 0$. Moreover, replacing V^2 by $V^2+c_4x^2(2yV^0-xV^1)$ with a suitable constant c_4, one may also assume that $b_2=0$. Let us determine conditions under which the commutator $[V^1,V^2]$ is contained in $\mathscr{L}_D$. Set

$$W=[V^1,V^2]-(p_1xz+p_2y^2+p_3x^3y+p_3x^6)V^0-(q_1xy+q_2x^4)V^1-r_1x^2V^2,$$

where $p_1,\dots,q_2,r_1$ are constants. Let us now determine a condition under which the commutator W vanishes. Direct computations show that

$$\begin{gathered}p_1=-\tfrac{1}{5}(a_1b_5),\ p_2=\tfrac{1}{3}a_1b_7,\ p_3=\tfrac{1}{27}(-108b_3+a_1a_2b_7+9a_1b_8),\ p_4=-3(a_7-b_4),\\ q_1=-\tfrac{1}{9}a_1b_7,\ q_2=\tfrac{1}{81}(189b_3-a_1a_2b_7-9a_1b_8),\ r_1=\tfrac{1}{15}(15b_1+a_1b_5).\end{gathered}$$

As a result one gets that

$$\begin{aligned}405W=&(9C_1xyz+C_2x^4z-C_3x^6y+C_4x^9)\partial_y\\&+(-135C_5y^2z-27C_6x^2y^3-3C_7x^3yz-27C_8x^5y^2+27C_9x^6z-9C_{10}x^8y\\&+C_{11}x^{11})\partial_z,\end{aligned}$$

where

$$\begin{aligned}C_1&=270a_2-270b_1+72a_1b_5+5a_1^2b_7,\\C_2&=6075a_3+405a_2b_1-405b_1^2-1350a_1b_3-27a_1b_1b_5+405a_1b_6+5a_1^2a_2b_7\\&\quad+45a_1^2b_8,\\C_3&=-3645a_7+945a_2b_3+405b_1b_3+12150b_4+27a_1b_3b_5-5a_1a_2^2b_7-45a_1a_3b_7\\&\quad-45a_1a_2b_8-405a_1b_9,\\C_4&=-405a_7b_1+405a_1b_{10}-1350a_3b_3+405a_2b_4-405b_1b_4-27a_1b_4b_5+5a_1a_2a_3b_7\\&\quad+45a_1a_3b_8,\\C_5&=9b_5+14a_1b_7,\\C_6&=45a_2b_7+15b_1b_7+a_1b_5b_7+135b_8,\\C_7&=-2700b_3+135a_2b_5+135b_1b_5+9a_1b_5^2+1620b_6+25a_1a_2b_7+495a_1b_8,\\C_8&=45a_3b_7+30a_2b_8+15b_1b_8+a_1b_5b_8+270b_9,\\C_9&=540a_7-225b_4-15a_3b_5-15b_1b_6-a_1b_5b_6-15a_1b_9,\\C_{10}&=1215b_{10}+45a_7b_5-5a_1a_7b_7+90a_3b_8+45a_2b_9+45b_1b_9+3a_1b_5b_9,\\C_{11}&=-405b_1b_{10}-945a_7b_3-27a_1b_{10}b_5-405a_7b_6+5a_1a_2a_7b_7+45a_1a_7b_8-405a_3b_9.\end{aligned}$$

Hence, $W = 0$ if and only if $C_j = 0$ for all $j = 1, 2, \ldots, 11$. One can verify that nine equations $C_j = 0$ for $1 \le j \le 11$, $j \ne 4, 11$, imply that

$$\begin{aligned}
a_2 &= (270b_1 + 107a_1^2b_7)/270,\\
a_3 &= -a_1(-2430b_6 + 2970a_1b_1b_7 + 313a_1^3b_7^2)/36450,\\
a_7 &= a_1b_7(1239300b_1^2 - 1224720a_1b_6 + 915435a_1^2b_1b_7 + 97559a_1^4b_7^2)/82668600,\\
b_3 &= (14580b_6 - 5535a_1b_1b_7 - 1001a_1^3b_7^2)/24300,\\
b_4 &= (-2755620b_1b_6 + 1283040a_1b_1^2b_7 - 867510a_1^2b_6b_7\\
&\quad + 609579a_1^3b_1b_7^2 + 63602a_1^5b_7^3)/41334300,\\
b_5 &= -14a_1b_7/9,\\
b_8 &= -b_7(1080b_1 + 293a_1^2b_7)/2430,\\
b_9 &= b_7(29160b_1^2 - 4374a_1b_6 + 19953a_1^2b_1b_7 + 2380a_1^4b_7^2)/393660,\\
b_{10} &= -b_7(551124000b_1^3 - 303118200a_1b_1b_6 + 634230000a_1^2b_1^2b_7\\
&\quad + 19952730a_1^3b_6b_7 + 132987420a_1^4b_1b_7^2 + 6967331a_1^6b_7^3)/100442349000.
\end{aligned}$$

As a result the remaining two equations $C_4 = C_{11} = 0$ yield the following two relations

$$\begin{aligned}
&a_1(13392313200b_6^2 + 2056873500b_1^3b_7 - 19647570600a_1b_1b_6b_7\\
&\qquad + 6098394825a_1^2b_1^2b_7^2 - 1513549800a_1^3b_6b_7^2\\
&\qquad + 1187984070a_1^4b_1b_7^3 + 40787747a_1^6b_7^4) \quad = \quad 0,\\
&b_7(297606960000b_1^4 - 2382981444000a_1b_1^2b_6 + 1968670040400a_1^2b_6^2\\
&\qquad + 1041024028500a_1^2b_1^3b_7 - 2043418201200a_1^3b_1b_6b_7\\
&\qquad + 663011412075a_1^4b_1^2b_7^2 - 252174542760a_1^5b_6b_7^2\\
&\qquad + 130912141140a_1^6b_1b_7^3 + 8033686661a_1^8b_7^4) \quad = \quad 0.
\end{aligned}$$

It is clear that $b_7 = 0$ satisfies the second equation. Hence, since $a_1 \ne 0$ one gets $b_6 = 0$. Direct computations show that $\det(M)$ can be transformed to z^3; it contradicts the requirement (iv) of Condition 1. Let us analyze the case $b_7 \ne 0$. Since the above two relations can be considered as polynomials depending on b_6 one can compute their resultant:

$$\begin{aligned}
G &= b_1(46305b_1 - 2048a_1^2b_7)(8640b_1 - 329a_1^2b_7)(540b_1 - 119a_1^2b_7)(20b_1 + 3a_1^2b_7)\\
&\quad \times (108b_1 + 5a_1^2b_7)(135b_1 + 64a_1^2b_7)(135b_1 + 364a_1^2b_7).
\end{aligned}$$

When $b_7 \neq 0$, then it is not difficult to verify that b_6 is uniquely determined for each solution b_1 of the equation $G = 0$. As a result one gets that (b_1, b_6) is equal to one of the following eight couples:

$$\begin{array}{ll}
(0, 1001a_1^3b_7^2/14580), & (-364a_1^2b_7/135, -(1/60)a_1^3b_7^2), \\
(-64a_1^2b_7/135, -4229a_1^3b_7^2/43740), & (-3a_1^2b_7/20, 139a_1^3b_7^2/29160), \\
(-5a_1^2b_7/108, 17a_1^3b_7^2/360), & (329a_1^2b_7/8640, 3607a_1^3b_7^2/46080), \\
(2048a_1^2b_7/46305, 288079a_1^3b_7^2/3025260), & (119a_1^2b_7/540, 821a_1^3b_7^2/3240).
\end{array}$$

It should be remarked that in Case (d.v.1) for $n > 2$ it is possible to verify by similar considerations that $\det(M)$ can be transformed to z^3. However, in this case direct computations are very complicated and we omit them. In conclusion, it remains to analyze Case (d.v.2). In fact, one obtains seven polynomials $F_{B,1}, \dots, F_{B,7}$ by similar considerations as in Case (d.v.1), $n = 2$. For this reason we omit again computational details.

Thus, this completes the proof of Theorem 1.

It should be also underlined that there is the following relation between solutions of the equation $G = 0$ and the set of polynomials $F_{H,j}$ for $j = 1, 2, \dots, 8$. To be more precise, one has the following statement.

PROPOSITION 2. *If b_1 is equal to $0, -\frac{364}{135}a_1^2b_7, -\frac{64}{135}a_1^2b_7, -\frac{3}{20}a_1^2b_7, -\frac{5}{108}a_1^2b_7, \frac{329}{8640}a_1^2b_7, \frac{2048}{46305}a_1^2b_7$, or $\frac{119}{540}a_1^2b_7$, then $\det(M)$ can be transformed by suitable weighted changes of variables to the polynomials $F_{H,1}$, $F_{H,5}$, $F_{H,4}$, $F_{H,8}$, $F_{H,3}$, $F_{H,6}$, $F_{H,2}$ or $F_{H,7}$, respectively.*

6. Singular loci of hypersurfaces.

As was remarked in Section 1 an arbitrary Saito free divisor has singular locus of codimension one (see [**4**]). Let us show how it is possible to analyze properties of hyperplane sections of singular loci of polynomials from Theorem 1. Thus, let $F(x, y, z)$ be such a polynomial, let Z_F be the affine hypersurface in $\boldsymbol{C}^3$ defined by $F = 0$, and let S_F be the set of singular points of Z_F. Direct computations show that the intersection $S_F \cap \{x \neq 0\}$ is a smooth manifold. Let S_F^j, $j = 1, 2, \dots, k$, be the set of irreducible components of S_F. It is not difficult to see that each irreducible component S_F^j is a plane curve. Take a point $\mathrm{P} \in S_F^j \cap \{x \neq 0\}$. The purpose of this section is to study the hypersurface S_F near P.

First let us recall the well-known list of plane curves with simple singularities at the origin:

$$A_n : G_{A_n}(u, v) = u^{n+1} + v^2 = 0, n \geq 1;$$

$$D_n : G_{D_n}(u,v) = u(u^{n-2} + v^2) = 0, n \geq 4;$$
$$E_6 : G_{E_6}(u,v) = u^4 + v^3 = 0;$$
$$E_7 : G_{E_7}(u,v) = u(u^2 + v^3) = 0;$$
$$E_8 : G_{E_8}(u,v) = u^5 + v^3 = 0.$$

We say that the type of singularity of Z_F along S_F^j is A_n if there is an open neighbourhood U of P and a biholomorphic map φ of U to an open neighbourhood U' of the origin in $\boldsymbol{C}^3$ such that $\varphi(Z_F \cap U)$ coincides with $\{(u,v,w) \in \boldsymbol{C}^3 : G_{A_n}(u,v) = 0\} \cap U'$ and $\varphi(\mathrm{P}) = (0,0,0)$. Similarly we say that the type of singularity of Z_F along S_F^j is D_n (or $E_k (k = 6,7,8)$) if there is an open neighbourhood U of P and a biholomorphic map φ of U to an open neighbourhood U' of the origin in $\boldsymbol{C}^3$ such that $\varphi(Z_F \cap U)$ coincies with $\{(u,v,w) \in \boldsymbol{C}^3 : G_{D_n}(u,v) = 0\} \cap U'$ and $\varphi(\mathrm{P}) = (0,0,0)$ (or to $\{(u,v,w) \in \boldsymbol{C}^3 : G_{E_k}(u,v) = 0\} \cap U'$ and $\varphi(\mathrm{P}) = (0,0,0)$).

We state the main result of this section as follows.

THEOREM 2.
(i) *There is a natural bijection between the set of polynomials $\mathscr{D}_A$ and the set of corank one subdiagrams of Dynkin diagram of type E_6 invariant under its non-trivial involution.*
(ii) *There are natural bijections between two sets of polynomials $\mathscr{D}_B$, $\mathscr{D}_H$ and two sets of corank one subdiagrams of Dynkin diagrams of type E_7, E_8, respectively.*

This is a direct consequence of the following proposition. In fact,

$$A_2 + A_2 + A_1, A_5$$

are types of corank one subdiagrams of Dynkin diagram of type E_6 invariant under its non-trivial involution. Moreover,

$$A_3 + A_2 + A_1, \quad A_5 + A_1, \quad D_6, \quad D_5 + A_1, \quad E_6, \quad A_4 + A_2, \quad A_6$$

are types of corank one subdiagrams of Dynkin diagram of type E_7, while

$$A_4 + A_2 + A_1, \quad A_4 + A_3 + A_1, \quad D_5 + A_2, \quad D_7, \quad E_6 + A_1, \quad A_7, \quad E_7, \quad A_6 + A_1$$

are types of corank one subdiagrams of Dynkin diagram of type E_8.

PROPOSITION 3. *The types of singularities of hypersurfaces defined by* *seventeen polynomials of Theorem* 1 *are given in the following table.*

F	S_F^j	*Type*
$F_{A,1}$	$y=0, z=\frac{1}{4}x^2$	A_1
	$y=\sqrt{-\frac{8}{27}}x^{3/2},\ z=-\frac{1}{12}x^2$	A_2
	$y=-\sqrt{-\frac{8}{27}}x^{3/2},\ z=-\frac{1}{12}x^2$	A_2
$F_{A,2}$	$y=0,\ z=x^2$	A_5
$F_{B,1}$	$y=0,\ z=0$	A_3
	$y=\frac{1}{4}x^2,\ z=0$	A_1
	$y=\frac{1}{3}x^2,\ z=\frac{1}{27}x^3$	A_2
$F_{B,2}$	$y=0,\ z=0$	A_5
	$y=-\frac{2}{3}x^2,\ z=\frac{4}{27}x^3$	A_1
$F_{B,3}$	$y=0,\ z=0$	D_6
$F_{B,4}$	$y=0,\ z=0$	D_5
	$y=\frac{9}{4}x^2,\ z=0$	A_1
$F_{B,5}$	$y=0,\ z=0$	E_6
$F_{B,6}$	$y=0,\ z=0$	A_4
	$y=2x^2,\ z=4x^3$	A_2
$F_{B,7}$	$y=0,\ z=0$	A_6

F	S_F^j	*Type*
$F_{H,1}$	$y=0, z=0$	A_1
	$y=\frac{2}{5}x^3,\ z=\frac{8}{25}x^5$	A_4
	$y=-\frac{2}{27}x^3,\ z=\frac{8}{81}x^5$	A_2
$F_{H,2}$	$y=0,\ z=0$	A_4
	$y=12x^3,\ z=144x^5$	A_3
$F_{H,3}$	$y=0,\ z=0$	D_5
	$y=\frac{1}{54}x^3,\ z=\frac{1}{162}x^5$	A_2
$F_{H,4}$	$y=0,\ z=0$	D_7
$F_{H,5}$	$y=0,\ z=0$	E_6
	$y=-x^3,\ z=x^5$	A_1
$F_{H,6}$	$y=0,\ z=0$	A_7
$F_{H,7}$	$y=0,\ z=0$	E_7
$F_{H,8}$	$y=0,\ z=0$	A_6
	$y=-3x^3,\ z=9x^5$	A_1

Remark 3.
1. The author proved this proposition by direct computations. However, somewhat later he found that it can be proved by a method of V. I. Arnol'd (see [**6**]) which allows one to reduce functions with isolated critical points to normal forms.
2. The author develops this approach (see [**18**]) and obtains a list of Lie algebras depending on three variables satisfied certain conditions related with eight exceptional singularities from the list of Arnol'd. This result produces a number of Saito free divisors which can be considered as one-parameter deformations of these singularities. (See also [**5**] for related topics).

7. Discriminants associated with reflection groups of A_k-type.

Let $x_2,\dots,x_n$ be independent variables considered as coefficients of the polynomial of degree n

$$P_n(t)=t^n+x_2t^{n-2}+\cdots\cdots+x_{n-1}t+x_n.$$

Each x_k is the k-th elementary symmetric function of roots of the polynomial; this function is usually called Viète polynomial of weight k. In fact, the polynomial $P_n(t)$ defines a versal unfolding of an A_{n-1}-singularity.

Let ∂_k be the partial derivative with respect to x_k, $k = 2, \dots, n$, and let $E = 2x_2\partial_2 + 3x_3\partial_3 + \dots + nx_n\partial_n$ be the Euler vector field with respect to weights $2, 3, \dots, n$, so that $Ex_k = kx_k$, $k = 2, 3, \dots, n$. Let us consider the following vector fields whose coefficients are contained in the polynomial ring $R = \boldsymbol{C}[x_2, x_3, \dots, x_n]$:

$$V^i = \sum_{j=1}^{n-1} a_{i+1,j}(x)\partial_{j+1}, \quad i = 0, 1, \dots, n-2,$$

Similarly to Section 2 there is defined a square matrix of order $(n-1)$ associated with vector fields $V^0, V^1, \dots, V^{n-2}$:

$$M = \|a_{ij}(x)\|_{0 \le i \le n-1,\ 2 \le j \le n}.$$

The following conditions on the system $V = \langle V^0, V^1, \dots, V^{n-2}\rangle$ are similar to Condition 1 on triples of vector fields:

Condition 2.
(i) $a_{1i}(x) = a_{i1}(x) = (i+1)x_{i+1}, \quad i = 0, \dots, n-2;$
(ii) $[V^0, V^j] = V^j, \quad j = 1, 2, \dots, n-2;$
(iii) the system V is involutive, that is, R-module generated by V^j, $j = 0, \dots, n-2$, is a Lie algebra over R.

In particular, the first requirement (i) of Condition 2 implies that $V^0 = E$.

Lemma 2 (cf. [**15**]). *Assume that $V^0, V^1, \dots, V^{n-2}$ satisfy Condition 2. Then one has the following relations*

$$V^jF(x) = c_j(x)F(x), \quad j = 0, 1, \dots, n-2,$$

where $c_j(x)$, $j = 0, 1, \dots, n-2$, are polynomials.

A typical example of polynomials in question is the discriminant polynomial of $P_n(t)$. To be more precise, if $F(x)$ is the discriminant polynomial of $P_n(t)$, then there are vector fields $V^0, V^1, \dots, V^{n-2}$ satisfying Condition 2 such that $F(x) = \det(M)$ (cf. [**15**], [**21**], [**22**]).

Example 1 (see [**23**]). Let us consider the following system of vector fields:

$$
\begin{aligned}
V^0 &= 2x_2\partial_2 + 3x_3\partial_3 + 4x_4\partial_4 + 5x_5\partial_5 + 6x_6\partial_6,\\
V^1 &= 3x_3\partial_2 + (4x_4 - \tfrac{4}{3}x_2^2)\partial_3 + (5x_5 - x_2x_3)\partial_4 + (6x_6 - \tfrac{2}{3}x_2x_4)\partial_5 - \tfrac{1}{3}x_2x_5\partial_6,\\
V^2 &= 4x_4\partial_2 + (\tfrac{40}{3}x_5 - \tfrac{2}{3}x_2x_3)\partial_3 + (36x_6 - \tfrac{4}{3}x_2x_4)\partial_4 - 2x_2x_5\partial_5\\
&\quad + (\tfrac{8}{3}x_2x_6 - \tfrac{4}{3}x_3x_5 + \tfrac{4}{9}x_4^2)\partial_6,\\
V^3 &= 5x_5\partial_2 + (16x_6 - \tfrac{16}{9}x_2x_4 + \tfrac{1}{2}x_3^2)\partial_3 - 2x_2x_5\partial_4 - \tfrac{1}{2}x_3x_5\partial_5 - \tfrac{1}{9}x_4x_5\partial_6,\\
V^4 &= 6x_6\partial_2 + (-\tfrac{8}{9}x_2x_5 + \tfrac{1}{9}x_3x_4)\partial_3 + (\tfrac{8}{3}x_2x_6 - \tfrac{4}{3}x_3x_5 + \tfrac{4}{9}x_4^2)\partial_4\\
&\quad - \tfrac{1}{9}x_4x_5\partial_5 + (-\tfrac{10}{27}x_5^2 + \tfrac{8}{9}x_4x_6 + \tfrac{8}{27}x_2^2x_6 + \tfrac{2}{81}x_2x_4^2 - \tfrac{2}{27}x_2x_3x_5)\partial_6.
\end{aligned}
$$

Then R-module generated by the system of vector fields $V = \langle V^0, V^1, V^2, V^3, V^4\rangle$ is endowed with structure of a Lie algebra over R. Moreover these vector fields satisfy Condition 2 and $\det(M)$ is a polynomial which can be considered as an analogue of the discriminant of the polynomial $P_6(t)$ defining a versal unfolding of an A_5-singularity.

It should be remarked that Lie algebra $\mathscr{L}(V)$ generated by V^j, $j = 0, 1, \ldots, 4$, was found in the study of the prehomogeneous vector space $(SL(5)\times GL(4), \begin{smallmatrix}\square\\ \square\end{smallmatrix}\otimes\square)$ by T. Yano and J. Sekiguchi under the guidance of M. Sato. Based on this example M. Sato posed the following question.

Problem 2 (M. Sato). How to describe the set of all polynomials having the form $F(x) = \det(M)$ up to weighted changes of variables?

This problem can be directly generalized to the case of Weyl groups, or more generally, to the case of Coxeter groups (cf. [**15**], [**21**]).

We are going to mention some results related with Problem 2. The simplest example is the following: $\det(M) = x_n^{n-1}$. Here are other well-known examples:

(1) If $n = 2$, then $F(x) = x_2$, the discriminant of $P_2(t)$.

(2) If $n = 3$, then $F(x) = 4x_2^3 - 27x_3^2$, the discriminant of $P_3(t)$.

(3) If $n = 4$, then there are only two polynomials $F_{A,1}$ and $F_{A,2}$ given by Theorem 1.

(4) If $n = 5$, then there are at least two polynomials satisfying Condition 2. The first one is equal to the discriminant of $P_5(t)$, while the second is equal to the determinant of the following matrix

$$
M = \begin{pmatrix}
2x_2 & 3x_3 & 4x_4 & 5x_5\\
3x_3 & x_2^2 + 4x_4 & \frac{5}{4}x_5 & \frac{3}{10}x_2x_4\\
4x_4 & 5x_5 & -\frac{5}{8}x_2x_4 & -\frac{5}{8}x_2x_5\\
5x_5 & 2x_2x_4 & \frac{15}{16}x_3x_4 & \frac{3}{5}x_4^2 + \frac{15}{16}x_3x_5
\end{pmatrix}.
$$

More exactly, $\det(M) = -\frac{1}{20}(3125x_5^4 - 1500x_2x_4^2x_5^2 + 1200x_3x_4^3x_5 - 60x_2^2x_4^4 - 768x_4^5)$.

(5) If $n = 6$, then there are at least two polynomials satisfying Condition 2. One is the discriminant of $P_6(t)$, another is the polynomial defined as the determinant of the matrix associated with fields $V^0, V^1, \dots, V^4$ of Example 1.

(6) In the case $n > 6$ the author knows only examples which are the discriminants of polynomials $P_{k+1}(t)$, $k \geq 6$, versal unfoldings of A_k-singularities.

Finally we mention about an interesting relationship between Example 1 and deformations of complete intersections. Let us consider the total space $V \subset \boldsymbol{C}^3_{(x,y,z)} \times \boldsymbol{C}^6_{(u,v,a,b,c,d)}$ of the minimal versal deformation of a simple space curve singularity of type S_5. It is defined by the following two equations

$$\begin{aligned} x^2 + y^2 + z^3 + by + cz + dz^2 &= u, \\ yz + ax &= v. \end{aligned}$$

Set $S = \boldsymbol{C}^6_{(u,v,a,b,c,d)}$ and consider a natural projection $\varphi : V \to S$. Let D be the discriminant set of the map φ. A system of free generators $\langle \delta^{(0)}, \delta^{(1)}, \dots, \delta^{(5)} \rangle$ of $\boldsymbol{C}[u, v, a, b, c, d]$-module $\mathrm{Der}_S(\log D)$ has been computed by A. G. Aleksandrov (see [**2**], p. 237). Set $Y_j = \delta^{(j)}|_{a=0}$, $j = 0, 1, 2, 4$, and $Y_5 = \frac{1}{v}(\delta^{(5)} - \frac{u}{20}\delta^{(0)})|_{a=0}$. Then by direct computations one gets that Y_j, $j = 0, 1, 2, 4, 5$, are holomorphic vector fields on $\boldsymbol{C}^5_{(u,v,b,c,d)}$. Moreover, Lie algebra $\mathscr{L}(Y)$ is isomorphic to $\mathscr{L}(V)$ for the system $V = \langle V^0, V^1, \dots, V^4 \rangle$ of Example 1 under the following change of variables

$$d = -\tfrac{1}{3}x_2,\ b = \tfrac{1}{8}x_3,\ c = -\tfrac{1}{12}(x_4 - \tfrac{1}{3}x_2^2),\ v = \tfrac{1}{24}x_5,\ u = \tfrac{1}{8}(x_6 - \tfrac{1}{9}x_2x_4).$$

8. Appendix.

Here we write out matrices M associated with polynomials given by Theorem 1 and also write out $[V^1, V^2]$ for the vector fields V^1, V^2.

$$\text{(Ai)} \quad \begin{pmatrix} 2x & 3y & 4z \\ 3y & -x^2 + 4z & -\frac{1}{2}xy \\ 4z & -\frac{1}{2}xy & \frac{1}{4}(8xz - 3y^2) \end{pmatrix}, \quad [V^1, V^2] = \tfrac{1}{2}yV^0 - \tfrac{1}{2}xV^1$$

$$\text{(Aii)} \quad \begin{pmatrix} 2x & 3y & 4z \\ 3y & \frac{1}{2}(z - x^2) & 6xy \\ 4z & -2xy & 16x^3 + 24y^2 - 8xz \end{pmatrix}, \quad [V^1, V^2] = -6yV^0 + 14xV^1$$

(Bi) $\begin{pmatrix} x & 2y & 3z \\ 2y & xy+3z & 2xz \\ 3z & 2xz & yz \end{pmatrix}$, $[V^1, V^2] = -zV^0 + xV^2$

(Bii) $\begin{pmatrix} x & 2y & 3z \\ 2y & -\frac{2}{3}(2xy-9z) & -4xz \\ 3z & -\frac{2}{3}(y^2+3xz) & -2yz \end{pmatrix}$, $[V^1, V^2] = \frac{2}{3}yV^1 - \frac{8}{3}xV^2$

(Biii) $\begin{pmatrix} x & 2y & 3z \\ 2y & -\frac{3}{5}(xy-5z) & -\frac{6}{5}xz \\ 3z & -\frac{3}{5}y^2 & -\frac{6}{5}yz \end{pmatrix}$, $[V^1, V^2] = \frac{3}{5}yV^1 - \frac{6}{5}xV^2$

(Biv) $\begin{pmatrix} x & 2y & 3z \\ 2y & 3(3xy+z) & 6xz \\ 3z & 0 & -3yz \end{pmatrix}$, $[V^1, V^2] = -9zV^0 + 9xV^2$

(Bv) $\begin{pmatrix} x & 2y & 3z \\ 2y & -24xy+2z & -2y^2-32xz \\ 3z & -9y^2 & -12yz \end{pmatrix}$, $[V^1, V^2] = 24zV^0 + 6yV^1 - 40xV^2$

(Bvi) $\begin{pmatrix} x & 2y & 3z \\ 2y & 3xy+\frac{5}{2}z & \frac{9}{2}y^2+\frac{15}{2}xz \\ 3z & \frac{3}{4}(15y^2+xz) & 18yz \end{pmatrix}$, $[V^1, V^2] = \frac{15}{2}zV^0 - \frac{9}{2}yV^1 + \frac{9}{2}xV^2$

(Bvii) $\begin{pmatrix} x & 2y & 3z \\ 2y & \frac{1}{3}(-4xy+7z) & y^2-\frac{14}{3}xz \\ 3z & \frac{3}{2}(7y^2-6xz) & 12yz \end{pmatrix}$, $[V^1, V^2] = 14zV^0 - 9yV^1 - \frac{10}{3}xV^2$

(Hi) $\begin{pmatrix} x & 3y & 5z \\ 3y & 2z+2x^2y & 7xy^2+2x^4y \\ 5z & 7xy^2+2x^4y & \frac{1}{2}(15y^3+4x^4z+18x^3y^2) \end{pmatrix}$,

$[V^1, V^2] = (4x^3y+2y^2)V^0 + 4xyV^1$

(Hii) $\begin{pmatrix} x & 3y & 5z \\ 3y & 36x^2y+6z & 90xy^2+90x^2z \\ 5z & -\frac{10}{3}(12x^3-55y)xy & -\frac{50}{3}(6x^3y^2-y^3+6x^4z-18xyz) \end{pmatrix}$,

$[V^1, V^2] = (-60x^3y+150y^2+180xz)V^0 + (60x^4-\frac{250}{3}xy)V^1 + 54x^2V^2$

$$\text{(Hiii)}\quad \begin{pmatrix} x & 3y & 5z \\ 3y & \frac{1}{10}(x^2y+2z) & \frac{23}{10}xy^2+\frac{3}{20}x^2z \\ 5z & 5xy^2 & \frac{15}{2}y(2y^2+xz) \end{pmatrix},\quad [V^1,V^2]=4y^2V^0-\frac{5}{2}xyV^1+\frac{3}{20}x^2V^2$$

$$\text{(Hiv)}\quad \begin{pmatrix} x & 3y & 5z \\ 3y & \frac{1}{5}(-4x^2y+6z) & \frac{2}{5}xy^2-2x^2z \\ 5z & -\frac{20}{3}xy^2 & \frac{10}{3}y(y^2-5xz) \end{pmatrix},$$

$$[V^1,V^2]=-8y^2V^0+10xyV^1-2x^2V^2$$

$$\text{(Hv)}\quad \begin{pmatrix} x & 3y & 5z \\ 3y & -\frac{9}{5}(4x^2y-z) & -\frac{3}{5}x(9y^2+16xz) \\ 5z & -15xy^2 & -5y(y^2+4xz) \end{pmatrix},$$

$$[V^1,V^2]=(-12y^2+12xz)V^0+10xyV^1-12x^2V^2$$

$$\text{(Hvi)}\quad \begin{pmatrix} x & 3y & 5z \\ 3y & -\frac{3}{5}(3x^2y-4z) & -\frac{18}{5}x(-y^2+2xz) \\ 5z & -\frac{5}{3}x(-8y^2+5xz) & \frac{10}{3}y(2y^2+xz) \end{pmatrix},$$

$$[V^1,V^2]=(8y^2+16xz)V^0-10xyV^1-\frac{27}{5}x^2V^2$$

$$\text{(Hvii)}\quad \begin{pmatrix} x & 3y & 5z \\ 3y & -\frac{3}{5}(2x^2y+z) & -\frac{3}{5}x(-y^2+3xz) \\ 5z & \frac{10}{3}xy^2 & -\frac{5}{3}y(y^2-3xz) \end{pmatrix},$$

$$[V^1,V^2]=(3y^2+3xz)V^0-\frac{10}{3}xyV^1-\frac{12}{5}x^2V^2$$

$$\text{(Hviii)}\quad \begin{pmatrix} x & 3y & 5z \\ 3y & -\frac{3}{5}(24x^2y-7z) & -\frac{9}{5}x(-3y^2+28xz) \\ 5z & -\frac{5}{3}x(7y^2+20xz) & \frac{5}{3}y(7y^2-52xz) \end{pmatrix},$$

$$[V^1,V^2]=(-28y^2+28xz)V^0+30xyV^1-36x^2V^2.$$

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