J. Math. Soc. Japan
Vol. 34, No. 1, 1982

On the Brun-Titchmarsh theorem

By Henryk IWANIEC

(Received May 23, 1980)

1. Introduction.

Let a and q be relatively prime positive integers and let $\pi(x, q, a)$ stand for the number of primes $p\leqq x$ congruent to a mod q. The prime number theorem of Siegel and Walfisz (see Prachar's book [31]) states that

$$\pi(x\,;q,\,a)=\frac{Lix}{\varphi(q)}+O(x\exp(-A(\log x)^{1/2}))$$

uniformly for $q\leqq(\log x)^{B}$ where B is any positive constant and $A=A(B)>0$. This theorem has the defect that it holds only for relatively small values of q. The Extended Riemann Hypothesis yields (see Titchmarsh [34])

$$\pi(x\,;q,\,a)=\frac{Lix}{\varphi(q)}+O(x^{1/2}\log x)$$

uniformly for $q\leqq x^{1/2}(\log x)^{-2}$, but even this is not always sufficient. H. L. Montgomery conjectures that

$$\pi(x\,;q,\,a)=\frac{Lix}{\varphi(q)}+O\Big(\Big(\frac{x}{q}\Big)^{1/2+\varepsilon}\Big)$$

uniformly for all $q<x^{1-\varepsilon}$ (actually Montgomery formulated the conjecture for $\Psi(x\,;q,\,a)$, see [24], the above version follows easily by partial summation). Here and in what follows ε stands for any positive constant to be regarded as being small and not necessarily the same at each occurrence; the constants implied in the symbols 0 and $\ll$ depend at most on ε.

In 1930 E. C. Titchmarsh [34] used Brun's sieve to prove that if $q<x^{1-\varepsilon}$ then

$$\pi(x\,;q,\,a)\ll\frac{x}{\varphi(q)\log x}.$$

This bound represents the true order of magnitude of $\pi(x\,;q,\,a)$ in the whole range $q<x^{1-\varepsilon}$. Although Titchmarsh' result is much less precise than the hypothetical asymptotic formula of Montgomery it has been recognized to be equally fruitful in various problems. Titchmarsh himself applied his result for

evaluating the sum

$$\sum_{p\leq x} d(p-1).$$

For this particular purpose a numerical value of the constant implied in $\ll$ is immaterial while in other circumstances its exact value would be of great importance.

A careful application of Selberg's sieve has led van Lint and Richert [**22**] to

$$\pi(x\,;q,\,a)<\frac{2x}{\varphi(q)\log x/q}\Big(1+O\Big(\frac{1}{\log x/q}\Big)\Big)$$

uniformly for $q<x$. We remark that the same bound can be extracted from the combinatorial sieve of Rosser through the results of [**16**].

Other proofs have been given by Bombieri and Davenport in [**2**] and by Montgomery and Vaughan in [**25**] via different large sieve inequalities. Montgomery and Vaughan obtained very neat bound without error term $O((\log x/q)^{-1})$ at all! A reduction in the value of the factor 2 would have important consequences for the location of Siegel's zero (see [**35**] [**33**] and [**29**]), but this seems not to be attainable by sieve methods alone.

Most applications require estimates for $\pi(x\,;q,\,a)$ on average over q. In 1972 C. Hooley [**12**] started studying such estimates introducing several beautiful ideas to the subject. He was the first who succeeded in treating the remainder terms in sieve estimates non-trivially. Although the details of Hooley's ideas will be given in the proofs of our theorems, it may be helpful to make a few introductory remarks now.

Given $q<x$ we consider the sequence

$$\mathcal{A}^{(q)}=\{l\leqq x\,;\ l\equiv a(\mathrm{mod}\ q)\}$$

and for $(d,\,q)=1$ we denote

$$r(\mathcal{A}^{(q)},\,d)=|\{l\in\mathcal{A}^{(q)}\,;\ l\equiv 0(\mathrm{mod}\ d)\}|-\frac{x}{qd}.$$

A direct application of either Selberg's Λ^2-method or Rosser's combinatorial sieve leads to

$$\pi(x\,;q,\,a)<\frac{(2+\varepsilon)x}{\varphi(q)\log D}+R(\mathcal{A}^{(q)},\,D)$$

where $\varepsilon>0$, $x>x_0(\varepsilon)$, $2<D<x$ and

$$R(\mathcal{A}^{(2)},\,D)=\sum_{\substack{d<D\\(d,q)=1}}\rho_d r(\mathcal{A}^{(q)},\,d)$$

with some coefficients $\rho_d=\rho(d,\,D)$ bounded by $3^{\nu(d)}$ in absolute value. The remainder term $R(\mathcal{A}^{(q)}, D)$ is required to be $\ll x^{1-\varepsilon}/\varphi(q)$. The larger D we can admit the better our result will be. Traditionally, one uses a trivial estimate $|r(\mathcal{A}^{(q)}, d)|\leqq 1$ which allows us to take $D=x^{1-2\varepsilon}/q$ and consequently giving

$$\pi(x\,;\,q,\,a)<\frac{(2+\varepsilon)x}{\varphi(q)\log x/q}. \tag{1.1}$$

This is rather crude method of treating $R(\mathcal{A}^{(q)}, D)$ because on summing the error terms $r(\mathcal{A}^{(q)}, d)$ over absolute values one gives up a possibility of cancellations when summing over weights ρ_d instead. At first look it seems to be unrealistic to get a great cancellation because the weights ρ_d involve Möbius function $\mu(d)$ about which we know very little. In Hooley's methods the cancellation of the terms $\rho_d r(\mathcal{A}^{(q)}, d)$ is due to the extra averaging over q rather than to a particular shape of the sieve weights ρ_d. To be clear, Hooley considered (implicitely) bilinear forms of the type

$$\sum_{\substack{Q<q\leqq 2Q\\(a,q)=1}}\ \sum_{\substack{d<D\\(d,q)=1}}\alpha_q\rho_d r(\mathcal{A}^{(q)}, d)\,.$$

He developed various methods to dealing with such sums. In his first paper [12] Hooley expressed each remainder term $R(\mathcal{A}^{(q)}, D)$ by exponential sums, used Cauchy's inequality to change the coefficients α_q into more suitable $\alpha_q=1$, and a reciprocity relation

$$\frac{\bar{p}}{q}+\frac{\bar{q}}{p}\equiv\frac{1}{pq}\qquad(\mathrm{mod}\ 1)$$

where $\bar{p}p\equiv 1\ (\mathrm{mod}\ q)$, $\bar{q}q\equiv 1\ (\mathrm{mod}\ p)$ for $(p, q)=1$, to arrive finally at a number of incomplete Kloosterman-Ramanujan sums being estimated through the use of the celebrated result of Weil. In the second paper on the subject [13] Hooley improved his first results by means of the large sieve inequality. And in the third paper [14] he applied a simple variant of the Linnik dispersion method getting in a surprisingly elementary manner still stronger estimates for almost all q in vicinity of x. Summerizing the above three papers of Hooley one may reformutate his results as follows.

THEOREM 1 (Hooley). *Let ε, ε_1 and A be arbitrary positive numbers and $x>x_0\,(\varepsilon, \varepsilon_1, A)$. We then have*

$$\pi(x\,;\,q,\,a)<\frac{(4+\varepsilon)x}{\varphi(q)\log x} \tag{1.2}$$

save for at most $Q(\log Q)^{-A}$ exceptional values of q in $(Q, 2Q]$ with $x^{1/2}<Q\leqq x^{3/4}$ and

$$\pi(x\,;\,q,\,a)<\frac{(4+\varepsilon)x}{\varphi(q)\log q} \tag{1.3}$$

save for at most $Q(\log Q)^{-A}$ *exceptional values of* q *in* $(Q, 2Q]$ *with* $x^{3/4}<Q\leqq x^{1-\varepsilon_1}$.

In 1974 Y. Motohashi demonstrated remarkable refinements of Selberg's sieve to improve (1.1) for all single q in $(x^{\varepsilon}, x^{1/2-\varepsilon})$. Let me quote here a few of his estimates (see [27] and [28]).

THEOREM 2 (Motohashi). *Let* $\varepsilon>0$ *and* $x>x_0(\varepsilon)$. *Then*

$$\pi(x\,;q,a)<\frac{(2+\varepsilon)x}{\varphi(q)\log D(x,q)} \tag{1.4}$$

where

$$D(x,q)=xq^{-3/8} \quad \textit{if} \quad q<x^{1/3-\varepsilon} \tag{1.5}$$

$$D(x,q)=xq^{-1/2} \quad \textit{if} \quad q\leqq x^{2/5} \tag{1.6}$$

$$D(x,q)=x^2q^{-3} \quad \textit{if} \quad x^{2/5}<q\leqq x^{1/2}. \tag{1.7}$$

Motohashi's works were very pioneering for the sieve theory because this was the first instance when a particular property of sieve weights had been made use of in a such effective manner. The relevant property is displayed in the following binary form for the remainder term of Selberg's sieve

$$\Lambda(\mathcal{A},D)=\sum_{d_1<\sqrt{D}}\sum_{d_2<\sqrt{D}}\lambda_{d_1}\lambda_{d_2}r(\mathcal{A}^{(q)},[d_1,d_2]).$$

In estimating the above remainder term Motohashi used the analytic technique familiar from the theory of L-functions. The bilinear form $\Lambda(\mathcal{A}, D)$ is utilized in an application of the mean-square theorem for Dirichlet's polynomials (see Lemma 4) and the fundamental structure of the sequence $\mathcal{A}^{(q)}$ is utilized in an application of various estimates for L-functions, just to mention the Burgess 3/16-theorem and the fourth moment theorem for $L(s,\chi)$ on the half-line (see Lemma 2). On the Extended Lindelöf Hypothesis, namely

$$L\Big(\frac{1}{2}+it,\chi\Big)\ll(|t|+1)q^{\varepsilon}$$

Motohashi was also able to show that (1.4) holds with

$$D(x,q)=x \quad \text{if} \quad q<x^{1/3-\varepsilon}. \tag{1.8}$$

Later some of these results were improved slightly by D. Goldfeld [8] and D. Wolke [36].

Very recently, having received an inspiration from Motohashi's works, we found a bilinear form for the remainder term of Rosser's sieve which in more than one respect has an advantage over that of Selberg's sieve. In the situation considered here Rosser's sieve gives (see [18])

PROPOSITION. *Let* $\varepsilon>0$, $A=\exp(8\varepsilon^{-3})$, $x>x_0(\varepsilon)$, $M\geqq1$, $N\geqq1$ *and* $D=MN<x$.

We then have

$$\pi(x\,;q,a)<\frac{(2+\varepsilon c)x}{\varphi(q)\log D}+\sum_{\alpha\le A}R_{\alpha}(\mathcal{A}^{(q)},M,N) \tag{1.9}$$

where c is an absolute constant and

$$R_{\alpha}(\mathcal{A}^{(q)},M,N)=\sum_{\substack{m<M\ n<N\\(mn,q)=1}}a_m b_n r(\mathcal{A}^{(q)},mn) \tag{1.10}$$

with some coefficients a_m, b_n depending at most on ε, α, M, N and bounded by 1 *in absolute value. In addition,* (1.9) *holds with remainder terms of the type* (1.10) *having the variables of the summation m and n coprime and squarefree.*

A direct injection of the analytic arguments into the above version of Rosser's sieve yields Motohashi's bounds for $\pi(x\,;q,a)$ but in wider ranges; namely (1.5) for $q<^{8/19-\varepsilon}$, and (1.6) for $q^{1/2-\varepsilon}$. Somewhat stronger estimates will be obtained by more elaborated analytic technique in Section 2. No analytic method works in the range $q>x^{1/2}$ for much the some reason as the large sieve is not applicable for Bombieri-Vinogradov's type theorems. In this respect Hooley's arguments reveal to have a great advantage. His ideas related to the Fourier analysis, incomplete Kloosterman-Ramanujan sums and the dispersion method all together are adopted to treat the bilinear forms $R_{\alpha}(\mathcal{A}^{(q)},M,N)$ for single $q<x^{1-\varepsilon}$ as well as for almost all $q<x^{1-\varepsilon}$ in Sections 3 and 4 respectively. In Section 5 we briefly sketch two applications. The first problem deals with the greatest prime factor of shifted primes and the second problem concerns of the least almost prime P_2 in arithmetic progressions.

Some results on the Brun-Titchmarsh theorem for short intervals are stated without proofs in the last Section 6.

ACKNOWLEDGEMENT. The author expresses his gratitude to Professor Christopher Hooley for several stimulating discussions and fruitful suggestions.

2. A character sums approach.

In this section we shall appeal to estimates for character sums of various kinds.

DEFINITION. Let θ be a non-negative constant with the property that for any $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)>0$ such that

$$\sum_{l\le L}\chi(l)\ll Lq^{-\delta} \tag{2.1}$$

for all non-principal characters $\chi\ (\mathrm{mod}\ q)$ and all $L\geqq q^{\theta+\varepsilon}$.

LEMMA 1 (Burgess). *For any q we have $\theta=3/8$ and for cube-free q we even have $\theta=1/4$.*

Actually Burgess proved more precise results, see [3] and [4]. For our purpose any positive δ in (2.1) will be sufficient. However, it follows from the Extended Lindelöf Hypothesis that the true bound for (2.1) should be $c(\varepsilon)L^{1/2}q^{\varepsilon}$ for all $L\geqq 1$. In particular this implies the commonly known

HYPOTHESIS. *For any q we have $\theta=0$.*

Even this seems to be very deep and difficult to prove in general. For special q having fixed prime factors the hypothesis can be proved by Postnikov-Gallagher method (see [1], [6], [17] and [5]).

LEMMA 2. *For $T\geqq 2$ we have*

$$\sum_{\chi(\bmod q)}\int_{-T}^{T}\left|L\left(\frac{1}{2}+it,\chi\right)\right|^{4}dt\ll qT(\log qT)^{4}.$$

This lemma is a simple consequence of the fourth moment estimate for $L\left(\frac{1}{2}+it,\chi\right)$ with primitive characters, see [23]. Hence one can easily derive.

LEMMA 3. *For partial sums of $L\left(\frac{1}{2}+it,\chi\right)$ we have*

$$\sum_{\chi\neq\chi_0}\Big|\sum_{l\leqq L}\chi(l)l^{-1/2-it}\Big|^{4}\ll q(|t|+1)\log^{6}qL(|t|+1).$$

We shall also need results about frequency of large values of general Dirichlet's polynomials.

LEMMA 4 (the mean-square theorem). *For any complex numbers a_n we have*

$$\sum_{\chi(\bmod q)}\Big|\sum_{N<n\leqq 2N}a_n\chi(n)\Big|^{2}\ll(N+q)\sum_{N<n\leqq 2N}|a_n|^{2}.$$

This lemma is almost trivial (for a proof see [23]). The next lemma is much deeper.

LEMMA 5 (the large values theorem of Huxley). *For any complex numbers a_n and for a positive V we have*

$$\#\{\chi(\bmod q);\Big|\sum_{N<n\leqq 2N}a_n\chi(n)\Big|>V\}\ll GNV^{-2}+q^{1+\varepsilon}G^{3}NV^{-6}$$

where $G=\sum|a_n|^{2}$.

Two important arguments are involved in the proof of Lemma 5, namely the Halász-Montgomery inequality and the Huxley reflexion method. For a simple and elegant proof see Jutila [19].

On the basis of (2.1), Lemmas 3, 4 and 5 we shall demonstrate the proof of

THEOREM 3. *For any $\varepsilon>0$, $x>x_0(\varepsilon)$ and $q\leqq x^{9/20-\varepsilon}$ we have*

$$\pi(x;q,a)\leqq\frac{(2+\varepsilon)x}{\varphi(q)\log D}\tag{2.2}$$

with $D=D(x,q)=\min(xq^{-\theta},x^{2}q^{-12/5})$.

In virtue to Burgess' results one gets (2.2) unconditionally with $D=xq^{-3/8}$ for any q and with $D=xq^{-1/4}$ for cube-free q. Moreover, assuming the hypothesis $\theta=0$ one may take

$$D=\begin{cases} x & \text{if } q\leqq x^{5/12} \\ x^2q^{-12/5} & \text{if } q>x^{5/12}. \end{cases} \tag{2.3}$$

By the linear sieve results (1.9)-(1.10) the proof of Theorems 3 reduces to showing that

$$\sum_{M<m\leqq 2M}\sum_{N<n\leqq 2N} a_m b_n r(\mathcal{A}^{(q)}, mn)\ll x^{1-\delta}/\varphi(q) \tag{2.4}$$

for $M, N\leqq D^{1/2}x^{-\varepsilon}$ where ε is any positive constant, $\delta=\delta(\varepsilon)>0$ and the coefficients a_m, b_n are bounded by 1 in absolute value. For simplicity of analytic arguments it is convenient to work with Riesz' means

$$A_k(x, d)=\frac{1}{k!}\sum_{\substack{l\leqq x \\ l\equiv a \ (\mathrm{mod}\ q) \\ l\equiv 0 \ (\mathrm{mod}\ d)}}\left(\log\frac{x}{l}\right)^k$$

and

$$r_k(x, d)=A_k(x, d)-\frac{x}{qd}.$$

We have $A_k(x, d)=\int_1^x A_{k-1}(y, d)\frac{dy}{y}$ hence $A_k(x, d)$ is nondecreasing function of x and consequently for any $\lambda>0$ we can write

$$\frac{1}{\lambda}\int_{e^{-\lambda}x}^{x} A_{k-1}(y, d)\frac{dy}{y}\leqq A_{k-1}(x, d)\leqq\frac{1}{\lambda}\int_x^{e^{\lambda}x} A_{k-1}(y, d)\frac{dy}{y}.$$

The integrals are equal to

$$A_k(x, d)-A_k(e^{-\lambda}x, d) \quad\text{and}\quad A_k(e^{\lambda}x, d)-A_k(x, d).$$

Therefore, extracting the main term x/qd we obtain

$$\begin{aligned} r_{k-1}(x, d)&\leqq\left(\frac{e^{\lambda}-1}{\lambda}-1\right)\frac{x}{qd}+\frac{1}{\lambda}[r_k(e^{\lambda}x, d)-r_k(x, d)] \\ r_{k-1}(x, d)&\geqq\left(\frac{1-e^{-\lambda}}{\lambda}-1\right)\frac{x}{qd}+\frac{1}{\lambda}[r_k(x, d)-r_k(e^{-\lambda}x, d)] \end{aligned} \tag{2.5}$$

for $k=1, 2, \cdots$. Letting

$$R_k(x\,; M, N)=\sum_{\substack{M<m\leqq 2M \\ (mn,q)=1}}\sum_{N<n\leqq 2N} a_m b_n r_k(x\,; mn)$$

we deduce from (2.5) the following implication:

$$\text{if}\quad R_k(x\,;\,M,\,N)\ll x^{1-\delta}/\varphi(q)\quad\text{then}\quad R_{k-1}(x\,;\,M,\,N)\ll x^{1-\delta/2}/\varphi(q)\,.$$

Therefore the proof of (2.3) reduces to showing that

$$R_4(x\,;\,M,\,N)\ll x^{1-\delta}/\varphi(q)$$

subject to $M, N<D^{1/2}x^{-\varepsilon}$ with any $\varepsilon>0$ and some $\delta=\delta(\varepsilon)>0$. By the orthogonality of characters we have for $(d, q)=1$

$$\begin{aligned} r_4(x,\,d)&=\frac{1}{24}\,\frac{1}{\varphi(q)}\sum_{\chi\ (\mathrm{mod}\ q)}\bar{\chi}(a)\chi(d)\sum_{b\leq x/d}\chi(b)\Big(\log\frac{x}{bd}\Big)^4-\frac{x}{qd}\\ &=\frac{1}{24}\,\frac{1}{\varphi(q)}\sum_{\substack{\chi\ (\mathrm{mod}\ q)\\ \chi\neq\chi_0}}\bar{\chi}(a)\chi(d)\sum_{b\leq x/d}\chi(b)\Big(\log\frac{x}{bd}\Big)^4+O\Big(\frac{x^{\varepsilon}}{q}\Big)\,. \end{aligned}$$

Hence, letting $L=x/MN$,

$$B(s,\,\chi)=\sum_{l\leq L}\chi(l)l^{-s}$$

$$M(s\,;\,\chi)=\sum_{M<m\leq 2M}a_m\chi(m)m^{-s}$$

$$N(s\,;\,\chi)=\sum_{N<n\leq 2N}b_n\chi(n)n^{-s}$$

we obtain

$$R_4(x\,;\,M,\,N)=\frac{1}{2\pi i}\int_{(1/2)}\frac{x^s}{s^5}\,\frac{1}{\varphi(q)}\sum_{\chi\neq\chi_0}\bar{\chi}(a)B(s,\,\chi)M(s,\,\chi)N(s,\,\chi)ds+O\Big(\frac{x^{\varepsilon}}{q}MN\Big)\,.$$

Now, it is sufficient to show that

$$\sum_{\chi\neq\chi_0}|B(s,\,\chi)M(s,\,\chi)N(s,\,\chi)|\ll|s|^3x^{1/2-\delta}\,.$$

We have trivial estimates

$$|B(s,\,\chi)|\leqq 2L^{1/2},\quad |M(s,\,\chi)|\leqq M^{1/2},\quad |N(s,\,\chi)|\leqq N^{1/2}$$

thus the characters $\chi\neq\chi_0$ for which one of the above three bounds is less than $(\varphi(q)x^{\delta})^{-1}$ can be neglected. The set of remaining characters $\chi\neq\chi_0$ can be classified into $\ll(\log x)^3$ subsets $S(U, V, W)$ of characters satisfying simultaneous conditions

$$U<|B(x,\,\chi)|\leqq 2U,\quad V<|M(s,\,\chi)|\leqq 2V\quad\text{and}\quad W<|N(s,\,\chi)|\leqq 2W$$

where $U=2^{1-u}L^{1/2}$, $V=2^{-v}M^{1/2}$, $W=2^{-w}N^{1/2}$, $u, v, w=1, 2, \cdots, [2\log x]$. It is, therefore, sufficient to show that for every U, V, W in question

$$UVW|S(U,\,V,\,W)|\ll|s|^3x^{1/2-2\delta}\,. \tag{2.6}$$

Here $|S(U, V, W)|$ stands for the cardinality of $S(U, V, W)$. A sufficient information about $|S(U, V, W)|$ can be derived from Lemmas 3, 4 and 5. By the mean-square theorem we deduce that

$$|S(U, V, W)| \ll MV^{-2}+qV^{-2}$$

$$|S(U, V, W)| \ll NW^{-2}+qW^{-2}$$

and by Lemma 3 we deduce that

$$|S(U, V, W)| \ll qU^{-4}|s| \ (\log qL|s|)^{6}.$$

By Huxley's large values theorem we deduce that

$$|S(U, V, W)| \ll MV^{-2}+q^{1+\varepsilon}MV^{-6}$$

$$|S(U, V, W)| \ll NW^{-2}+q^{1+\varepsilon}NW^{-6}$$

$$|S(U, V, W)| \ll (L^{2}U^{-4}+q^{1+\varepsilon}L^{2}U^{-12}) \ (\log L)^{6}.$$

In addition to the above we need an upper bound for U. To this end we utilize (2.1). By partial summation we deduce that, unless $S(U, V, W)$ is emply,

$$U \ll |s| L^{1/2}x^{-3\delta}, \quad \delta=\delta(\varepsilon)>0 \tag{2.7}$$

subject to $L \geqq q^{\theta+\varepsilon}$. This restriction is satisfied because $L=x/MN$, $M \leqq D^{1/2}x^{-\varepsilon}$, $N \leqq D^{1/2}x^{-\varepsilon}$ and D is chosen in Theorem 3 just for this statement to hold.

Now we are ready to prove (2.6). We apply Heath-Brown's arguments which had been used in [**11**] to estimate the number of primes in short intervals. Burgess' estimates for $\sum_{l \leq L} \chi(l)$ play here the same rôle as van der Corput estimates for partial sums of the Riemann zeta-function does in [**11**]. Our result (2.2) is an analogue of Lemma 2 from [**11**]. For simplicity we denote

$$F=\min\Big(\frac{M+q}{V^{2}}, \frac{N+q}{W^{2}}, \frac{q}{U^{4}}, \frac{M}{V^{2}}+\frac{qM}{V^{6}}, \frac{N}{W^{2}}+\frac{qN}{W^{6}}, \frac{L^{2}}{U^{4}}+\frac{qL^{2}}{U^{12}}\Big)$$

with the aim of showing that

$$UVWF \ll |s| x^{1/2-3\delta}.$$

We consider four cases

Case 1. $F \leqq 2V^{-2}M$, $2W^{-2}N$. In this case, by (2.12) we get

$$UVWF \leqq 2UVW \min(V^{-2}M, W^{2}N) \leqq 2U(MN)^{1/2} \ll |s| x^{1/2-3\delta}.$$

Case 2. $F>2V^{-2}M$, $2W^{-2}N$. In this case we have

$$\begin{aligned} F \leqq\ & 2\min\{qV^{-2}, qW^{-2}, qMV^{-6}, qNW^{-6}, qU^{-4}, L^{2}U^{-4}\} \\ & +2\min\{qV^{-2}, qW^{-2}, qMV^{-6}, qNW^{-6}, qU^{-4}, qL^{2}U^{-12}\} \end{aligned}$$

$$\leqq 2(qV^{-2})^{5/16}(qW^{-2})^{5/16}(qMV^{-6})^{1/16}(qNW^{-6})^{1/16}(\min(qU^{-4},\ L^2U^{-4}))^{1/4}$$

$$+2\min\{(qV^{-2})^{5/16}(qW^{-2})^{5/16}(qMV^{-6})^{1/16}(qNW^{-6})^{1/16}(qU^{-4})^{1/4},$$

$$(qV^{-2})^{7/16}(qW^{-2})^{7/16}(qMV^{-6})^{1/48}(qNW^{-6})^{1/48}(qL^2U^{-12})^{1/12}\}$$

$$=2(UVW)^{-1}q(MN)^{1/16}\{\min(1,\ q^{-1/4}L^{1/2})+\min(1,\ L^{1/6}(MN)^{-1/24})\}$$

$$\ll(UVW)^{-1}(x^{1/16}q^{31/32}+x^{1/20}q)\ll(UVW)^{-1}x^{1/2-\varepsilon}.$$

Case 3. $F>2V^{-2}M$, $F\leqq 2W^{-2}N$. In this case we have

$$F\leqq 2\min\{qV^{-2},\ NW^{-2},\ qMV^{-6},\ qU^{-4},\ L^2U^{-4}\}$$

$$+2\min\{qV^{-2},\ NW^{-2},\ qMV^{-6},\ qU^{-4},\ qL^2U^{-12}\}$$

$$\leqq 2(qV^{-2})^{1/8}(NW^{-2})^{1/2}(qMV^{-6})^{1/8}(\min(qU^{-4},\ L^2U^{-4}))^{1/4}$$

$$+2\min\{(qV^{-2})^{1/8}(NW^{-2})^{1/2}(qMV^{-6})^{1/8}(qU^{-4})^{1/4},$$

$$(qV^{-2})^{3/8}(NW^{-2})^{1/2}(qMV^{-6})^{1/24}(qL^2U^{-12})^{1/12}\}$$

$$=2(UVW)^{-1}(qN)^{1/2}M^{1/8}\{\min(1,\ q^{-1/4}L^{1/2})+\min(1,\ L^{1/6}M^{-1/12})\}$$

$$\ll(UVW)^{-1}(x^{1/8}q^{7/16}N^{3/8}+x^{1/12}q^{1/2}N^{5/12})\ll(UVW)^{-1}x^{1/2-\varepsilon/2}.$$

Case 4. $F>2W^{-2}N$, $F\leqq 2V^{-2}M$. We may reduce this case to the previous one by interchanging M with N and V with W.

The proof of Theorem 3 is complete.

3. A Kloosterman sums approach.

In this section we present another treatment of $R_\alpha(\mathcal{A}^{(q)}, M, N)$ which depends on several ideas of Hooley. We first consider

$$D(x\,;\,M,\,N)=\sum_{\substack{M<m\leqq 2M\\(m,q)=1}}\Big|\sum_{\substack{N<n\leqq 2N\\(n,q)=1}}b_n r(\mathcal{A}^{(q)},\,mn)\Big|^2$$

with the aim of proving the following basic theorem.

THEOREM 4. *For any complex numbers* b_n *we have*

$$D(x\,;\,M,\,N)\ll\mathcal{D}\sum_{N<n\leqq 2N}|b_n|^2 \tag{3.1}$$

where

$$\mathcal{D}=\Big(\frac{x}{qN}+\frac{x}{q^{3/2}}+q^{1/2}N+\frac{x}{q^{3/4}M^{1/2}}+\frac{x^{3/2}}{q^{5/4}MN^{1/2}}+\frac{x^{3/2}}{q^2M^{1/2}N^{1/2}}+\frac{x^2}{q^{7/4}M^{3/2}N}\Big)x^{\varepsilon}.$$

On using the Cauchy-Schwarz inequality one can easily derive from Theorem 4 the following result.

THEOREM 5. *Let* $\varepsilon>0$, $x^{2/5}<q\leqq x^{2/3-6\varepsilon}$, $M=x^{1-3\varepsilon}/q$, $N=x^{1/2-4\varepsilon}/q^{3/4}$, $|a_m|\leqq 1$, $|b_n|\leqq 1$. *We then have*

$$\sum_{\substack{m\leqq M \\ (mn,q)=1}}\sum_{n\leqq N} a_m b_n r(\mathcal{A}^{(q)}, mn) \ll x^{1-\varepsilon}/\varphi(q). \tag{3.2}$$

This theorem admits us to take in (1.9) $D=x^{3/2-7\varepsilon}/q^{7/4}$ thus getting

THEOREM 6. *For every* $\varepsilon>0$ *and* $x>x_0(\varepsilon)$ *we have*

$$\pi(x\,;q,a)\leqq\frac{(8+\varepsilon)x}{\varphi(q)\log(x^6q^{-7})} \tag{3.3}$$

provided $(a,q)=1$ *and* $x^{2/5}<q\leqq x^{2/3}$.

Proceeding to the proof of Theorem 4 we first observe that it suffices to consider b_n real. We then write

$$D(x\,;M,N)=\sum_{\substack{M<m\leqq 2M \\ (m,q)=1}}\Bigl(\sum_{\substack{N<n\leqq 2N \\ (n,q)=1}} b_n \sum_{\substack{r\leqq x \\ r\equiv a\,(q) \\ r\equiv 0\,(mn)}} 1-\frac{x}{qm}\sum_{\substack{N<n\leqq 2N \\ (n,q)=1}}\frac{b_n}{n}\Bigr)^2 \tag{3.4}$$

$$=W(x\,;M,N)-2\frac{x}{q}V(x\,;M,N)+\Bigl(\frac{x}{q}\Bigr)^2U(M,N),$$

say. Each term will be evaluated separately.

i) *Evalution of* $V(x\,;M,N)$.

By the definition

$$V(x\,;M,N)=\Bigl(\sum_{\substack{N<n_1\leqq 2N \\ (n_1,q)=1}}\frac{b_{n_1}}{n_1}\Bigr)\sum_{\substack{N<n_2\leqq 2N \\ (n_2,q)=1}} b_{n_2}\sum_{\substack{M<m\leqq 2M \\ (m,q)=1}}\frac{1}{m}\sum_{\substack{r\leqq x \\ r\equiv a\,(q) \\ r\equiv 0\,(mn_2)}} 1.$$

We reinterpret the congruences $r\equiv a(\mathrm{mod}\ q)$, $r\equiv 0(\mathrm{mod}\ mn_2)$ by writing $r=mn_2l$ where $l\equiv a\overline{mn_2}\ (\mathrm{mod}\ q)$ and $l\leqq x/mn_2$. Therefore

$$\sum_{\substack{r\leqq x \\ r\equiv a\,(q) \\ r\equiv 0\,(mn_2)}} 1=\sum_{\substack{l\leqq x/mn_2 \\ l\equiv a\overline{mn_2}\,(q)}} 1=\Bigl[\frac{x}{mn_2q}-a\frac{\overline{mn_2}}{q}\Bigr]-\Bigl[a\frac{\overline{mn_2}}{q}\Bigr]$$

$$=\frac{x}{mn_2q}+\Psi\Bigl(\frac{x}{mn_2q}-a\frac{\overline{mn_2}}{q}\Bigr)-\Psi\Bigl(-a\frac{\overline{mn_2}}{q}\Bigr)$$

where $\Psi(\xi)=[\xi]-\xi+\frac{1}{2}$. The main term x/mn_2q contributes to $V(x\,;M,N)$ exactly

$$V_1(x\,;M,N)=\frac{x}{q}U(M,N). \tag{3.5}$$

To estimate the contribution of the terms $\Psi\Bigl(\frac{\chi}{mn_2q}-a\frac{\overline{mn_2}}{q}\Bigr)$ where $\chi=x$ or 0 we appeal to the following.

LEMMA 6. *For any non-negative numbers a_n and real x_n we have*

$$\Big|\sum_n a_n \Psi(x_n)\Big| \ll \frac{1}{H}\sum_n a_n + \sum_{h=1}^{H}\frac{1}{h}\Big|\sum_n a_n l(hx_n)\Big|$$

the constant implied in the symbol $\ll$ being absolute.

This is an analogue of the Erdös-Turan theorem. The proof is in [**20**]. By lemma 6 we get

$$\sum_{\substack{M<m\leqq 2M\\ (m,q)=1}} \frac{1}{m}\Psi\Big(\frac{\chi}{mn_2q}-a\frac{\overline{mn_2}}{q}\Big) \ll \frac{1}{H}+\sum_{h=1}^{H}\frac{1}{h}|S_h(\chi, q, M, n_2)|$$

where

$$S_h(\chi, q, M, n_2)=\sum_{\substack{M<m\leqq 2M\\ (m,q)=1}}\frac{1}{m}e\Big(\frac{h\chi}{mn_2q}-ah\frac{\overline{mn_2}}{q}\Big).$$

To estimate $S_n(\chi, q, M, n_2)$ we need the following Lemma which may be deduced from Lemma 3 of [**12**].

LEMMA 7 (Hooley). *If $\nu_2>\nu_1$ then*

$$\sum_{\substack{\nu_1<\nu\leqq\nu_2\\ (\nu,q)=1}} e\Big(b\frac{\bar{\nu}}{q}\Big)\ll (b, q)^{1/2}q^{1/2+\varepsilon}+(b, q)\frac{\nu_2-\nu_1}{q}.$$

Moreover, for any real number y we have

$$\sum_{\substack{\nu_1<\nu\leqq\nu_2\\ (\nu,q)=1}} e\Big(y\nu+b\frac{\bar{\nu}}{q}\Big)\ll (b, q)^{1/2}q^{1/2+\varepsilon}\Big(1+\frac{\nu_2-\nu_1}{q}\Big).$$

By partial summation we obtain

$$S_h(\chi, q, M, n_2)\ll M^{-1}\Big(1+\frac{hx}{qMN}\Big)\Big\{(h, q)^{1/2}q^{1/2}+(h, q)\frac{M}{q}\Big\}q^{\varepsilon}$$

and hence

$$\frac{1}{H}+\sum_{h=1}^{H}\frac{1}{h}|S_h(\chi, q, M, n_2)|\ll\frac{1}{H}+\frac{1}{M}\Big(1+\frac{Hx}{qMN}\Big)\Big(q^{1/2}+\frac{M}{q}\Big)q^{\varepsilon}\log 2H$$

because

$$\sum_{h\leqq H}\frac{(h, q)}{h}\ll d(q)\log 2H.$$

On taking H in an optimal manner we deduce from the above result that the total contribution of terms $\Psi\Big(\frac{\chi}{mn_2q}-a\frac{\overline{mn_2}}{q}\Big)$ to $V(x\,;\,M, N)$ is

$$V_2(x\,;M,N)\ll x^{\varepsilon}(q^{1/2}M^{-1}+q^{-1}+x^{1/2}q^{-1/4}M^{-1}N^{-1/2}+x^{1/2}q^{-1}M^{-1/2}N^{-1/2})\sum|b_n|^2. \tag{3.6}$$

ii) *Preliminary to the evaluation of* $W(x; M, N)$.

By the definition

$$W(x;M,N)=\sum_{\substack{N<n_1,n_2\leq 2N\\(n_1n_2,q)=1}} b_{n_1}b_{n_2}\sum_{\substack{M<m\leq 2M\\(m,q)=1}}\ \sum_{\substack{r_1,r_2\leq x\\r_1\equiv r_2\equiv a\,(q)\\r_1\equiv 0\,(mn_1)\\r_2\equiv 0\,(mn_2)}} 1.$$

We reinterpret the congruences $r_1\equiv r_2\equiv a(\mathrm{mod}\,q)$, $r_1\equiv 0(\mathrm{mod}\,mn_1)$ and $r_2\equiv 0$ $(\mathrm{mod}\,mn_2)$ by writing

$$r_1=mn_1l_1,\quad r_2=mn_2l_2$$

$$r_1l_1\equiv n_2l_2\ (\mathrm{mod}\,q),\quad (l_1l_2,q)=1$$

$$m\equiv a\overline{n_1l_1}\ (\mathrm{mod}\,q).$$

Letting $M_1=M_1(l_1,l_2)=\min\left(2M,\dfrac{x}{n_1l_1},\dfrac{x}{n_2l_2}\right)$ on changing the order of summation we get

$$W(x;M,N)=\sum_{\substack{N<n_1,n_2\leq 2N\\n_1l_1,n_2l_2\leq x/M\\n_1l_1\equiv n_2l_2\,(q)\\(n_1l_1n_2l_2,q)=1}}\ \sum_{l_1,l_2} b_{n_1}b_{n_2}\sum_{\substack{M<m\leq M_1\\m\equiv a\overline{n_1l_1}\,(q)}} 1.$$

We express

$$\sum_{\substack{M<m\leq M_1\\m\equiv a\overline{n_1l_1}\,(q)}} 1=\frac{M_1-M}{q}+\Psi\left(\frac{M_1-a\overline{n_1l_1}}{q}\right)-\Psi\left(\frac{M-a\overline{n_1l_1}}{q}\right)$$

and accordingly we denote by $W_1(x;M,N)$ and $W_2(x;M,N)$ the total contributions to $W(x;M,N)$ of terms $(M_1-M)/q$ and $\Psi(M_1-a\overline{n_1l_1})/q)-\Psi((M-a\overline{n_1l_1})/q)$ respectively. Unlikely to the situation before there is a difficulty in evaluating of $W_1(x;M,N)$.

iii) *Evaluation of* $W_1(x; M, N)$.

To avoid a heavy partial summation it is convenient to approximate $(M_1-M)/q$ by $\dfrac{1}{q}\sum_{M<m\leq M_1} 1$ with the error term $O(q^{-1})$ thus getting

$$W_1(x;M,N)=\frac{1}{q}\sum_{M<m\leq 2M}\ \sum_{\substack{N<n_1,n_2\leq 2N\\n_1l_1,n_2l_2\leq x/m\\n_1l_1\equiv n_2l_2\,(q)\\(n_1l_1n_2l_2,q)=1}} b_{n_1}b_{n_2}+O\left(\left(1+\frac{x}{qM}\right)\frac{x\log x}{qMN}\sum|b_n|^2\right).$$

The ranges of variable l_1, l_2 are too short to carry out the summation simply. To this end we use characters $\chi(\mathrm{mod}\,q)$ giving

$$\sum_{n_1,n_2,l_1,l_2} b_{n_1}b_{n_2}=\frac{1}{\varphi(q)}\sum_{\chi\,(\mathrm{mod}\,q)}\Big|\sum_{\substack{ln\leq x/m\\N<n\leq 2N}}\chi(ln)b_n\Big|^2.$$

The contribution of the principal character $\chi=\chi_0$ is equal to

$$\frac{1}{\varphi(q)}\Big(\sum_{\substack{N<n\leq 2N\\(n,q)=1}} b_n\Big(\frac{\varphi(q)}{q}\frac{x}{mn}+O(d(q))\Big)\Big)^2$$

$$=\frac{\varphi(q)x^2}{q^2m^2}\Big(\sum_{\substack{N<n\leq 2N\\(n,q)=1}}\frac{b_n}{n}\Big)^2+O\Big(\frac{d^2(q)}{q}\Big(\frac{x}{m}+N\Big)\sum|b_n|^2\Big)$$

and the contribution of the non-principal characters is estimated by

$$\frac{1}{\varphi(q)}\sum_{\chi\neq\chi_0}\Big|\sum_{\substack{N<n\leq 2N\\ ln\leq x/m}}\chi(ln)b_n\Big|^2\ll(q+Nq^{1/2})\frac{x^{1+\varepsilon}}{qmN}\sum|b_n|^2 .$$

The latter easily follows from the fourth moment estimate for $L(s,\chi)$ (see Lemma 2) and the mean-square theorem for Dirichlet's polynomials (see Lemma 4). Gathering together the above estimates we arrive at

$$W_1(x\,;M,N)=\Big(\frac{x}{q}\Big)^2U(M,N)+O\Big(x^\varepsilon\Big(\frac{x}{qN}+\frac{x}{q^{3/2}}+\frac{MN}{q^2}+\frac{x^2}{q^2M^2N}\Big)\sum|b_n|^2\Big). \tag{3.7}$$

iv) *Estimation of* $W_2(x\,;M,N)$.

By Lemma 7 and the definition of $W_2(x\,;M,N)$ we get

$$W_2(x\,;M,N)\ll\frac{1}{H}\sum_{\substack{N<n_1,n_2\leq 2N\\ n_1l_1,n_2l_2\leq x/M\\ n_1l_1\equiv n_2l_2\,(q)\\ (n_1l_1n_2l_2,q)=1}}$$

$$+\sum_{h=1}^{H}\frac{1}{h}\sum_{\substack{N<n_1,n_2\leq 2N\\(n_1n_2,q)=1}}|b_{n_1}b_{n_2}|(|T_h(M_1,n_1,n_2)|+|T_h(M,n_1,n_2)|$$

where for $g=g(l_1,l_2)=M_1$ or M we denoted

$$T_h(g,n_1,n_2)=\sum_{\substack{l_1\leq x/(n_1M)\\ n_1l_1\equiv n_2l_2\,(q)\\(l_1l_2,q)=1}}\sum_{l_2\leq x/(n_2M)}e\Big(h\frac{g(l_1,l_2)-a\overline{n_1l_1}}{q}\Big).$$

To avoid a heavy partial summation in the case of $g=M_1(l_1,l_2)$ we first write

$$e\Big(\frac{h}{q}M_1\Big)=e\Big(\frac{h}{q}M\Big)+2\pi i\frac{h}{q}\int_M^{M_1}e\Big(\frac{h}{q}\xi\Big)d\xi$$

getting

$$T_h(M_1,n_1,n_2)=T_h(M,n_1,n_2)+2\pi i\frac{h}{q}\int_M^{2M}e\Big(\frac{h}{q}\xi\Big)U_h\Big(\frac{x}{n_1\xi},\frac{x}{n_2\xi},n_1,n_2\Big)d\xi$$

with

$$U_h(L_1, L_2, n_1, n_2)=\sum_{\substack{l_1\leq L_1 \\ n_1l_1\equiv n_2l_2(q) \\ (l_1l_2,q)=1}}\sum_{l_2\leq L_2} e\Big(-ah\frac{n_1l_1}{q}\Big).$$

By Lemma 7 we deduce

$$U_h(L_1, L_2, n_1, n_2)=\frac{1}{q}\sum_{u(\bmod q)}\sum_{\substack{l_1\leq L_1 \\ (l_1,q)=1}} e\Big(\frac{-un_1l_1-ah\overline{n_1l_1}}{q}\Big)\sum_{l_2\leq L_2} e\Big(\frac{un_2l_2}{q}\Big)$$

$$\ll\frac{1}{q}\sum_{u(\bmod q)}(h, q)^{1/2}q^{1/2+\varepsilon}\Big(1+\frac{L_1}{q}\Big)\min\Big(L_2\frac{1}{\|un_2/q\|}\Big)$$

$$\ll(h, q)^{1/2}q^{1/2+\varepsilon}\Big(1+\frac{x}{qMN}\Big)^2.$$

This yields

$$|T_n(g, n_1, n_2)|\ll\Big(1+\frac{h}{q}M\Big)\Big(1+\frac{x}{qMN}\Big)^2(h, q)^{1/2}q^{1/2+\varepsilon}$$

and hence

$$W_2(x\,;M, N)\ll\frac{1}{H}\Big(1+\frac{x}{qM}\Big)\frac{x\log x}{MN}\sum|b_n|^2$$

$$+\Big(1+\frac{HM}{q}\Big)\Big(1+\frac{x}{qMN}\Big)^2q^{1/2+\varepsilon}N(\log 2H)\sum|b_n|^2.$$

On taking H in an optimal manner we conclude that

$$W_2(x\,;M, N)\ll x^{\varepsilon}\Big[\Big(1+\frac{x}{qMN}\Big)^2q^{1/2}N+\Big(1+\frac{x}{qM}\Big)^{1/2}\Big(1+\frac{x}{qMN}\Big)\frac{x^{1/2}}{q^{1/4}}\Big]\sum|b_n|^2. \tag{3.8}$$

v) *Completion of the proof.*

If we introduce the results (3.5), (3.6), (3.7) and (3.8) into (3.4) we find out that the main terms disappear throughout and we are left with the error terms only giving (3.1) with

$$\mathcal{D}\ll\Big(\frac{x}{q^{1/2}M}+\frac{x^{3/2}}{q^{5/4}MN^{1/2}}+\frac{x^{3/2}}{q^2M^{1/2}N^{1/2}}+\frac{x}{qN}+\frac{x}{q^{3/2}}+\frac{MN}{q^2}+q^{1/2}N$$
$$+\frac{x^{1/2}}{q^{1/4}}+\frac{x}{q^{3/4}M^{1/2}}+\frac{x^2}{q^{7/4}M^{3/2}N}\Big)x^{\varepsilon}. \tag{3.9}$$

We also have trivial bounds $\mathcal{D}\ll MN$ and for $MN>x$ we even have $\mathcal{D}\ll\Big(\frac{x}{q}\Big)^2(MN)^{-1}$. Combining these three estimates we see that some of terms in (3.9) are redundant as stated in Theorem 4.

The limitation for modulus q in Theorem 4 comes out from various places, the most responsible being the estimates in Lemma 7 for the incomplete

Kloosterman-Ramanujan sums. These estimates become worse than trivial one if the summation is over an interval (ν_1, ν_2) shorter than $q^{1/2}$. Such situation just occurs in our applications when $q \geqq x^{2/3}$. In 1972 Hooley [12] in connection with the Brun-Titchmarsh theorem stated a hypothesis that the true bound for the incomplete Kloosterman-Ramanujan sum

$$K(\nu_2, \nu_1; q, b) = \sum_{\substack{\nu_1 < \nu \leqq \nu_2 \\ (\nu, q)=1}} e\left(b\frac{\bar{\nu}}{q}\right)$$

should be $c(\varepsilon)(b, q)^{1/2}(\nu_2-\nu_1)^{1/2}q^{\varepsilon}$ provided $q^{1/4} \leqq \nu_2-\nu_1 < q$. Later [15] he extended the hypothesis into

HYPOTHESIS R* (Hooley). *The estimate*

$$K(\nu_1, \nu_2; q, b) \ll (b, q)^{1/2}(\nu_2-\nu_1)^{1/2}q^{\varepsilon}$$

holds if $1 \leqq \nu_2-\nu_1 \leqq q$.

A simple examination of our arguments shows

THEOREM 7. *On Hypothesis* R* *we have*

$$\sum_{m \leqq N} \sum_{n \leqq N} a_m b_n r(\mathcal{A}^{(q)}, mn) \ll \frac{x^{1-\varepsilon}}{\varphi(q)} \tag{3.10}$$

subjects to $x^{4/9} < q \leqq x^{2/3-6\varepsilon}$, $M = x^{1-3\varepsilon}/q$, $N = x^{2/3-4\varepsilon}/q$ *and* $|a_m|, |b_n| \leqq 1$. Hence, by Proposition we deduce

THEOREM 8. *On Hypothesis* R* *we have*

$$\pi(x; q, a) \leqq \frac{(6+\varepsilon)x}{\varphi(q)\log(x^5q^{-6})} \tag{3.11}$$

for all $(a, q)=1$, $x^{4/9} < q < x^{2/3}$ *provided* $x > x_0(\varepsilon)$.

It turns out that having the Hooley Hypothesis R* the dispersion method is not always the best tool for the problem under the consideration. We state, without proof, what can be obtained by the Fourier series method.

THEOREM 9. *On Hypothesis* R* *we have*

$$\pi(x; q, a) \leqq \frac{(5/3+\varepsilon)x}{\varphi(q)\log x/q} \tag{3.12}$$

for all $(a, q)=1$, $x^{4/9} < q \leqq x^{1-\varepsilon}$ *provided* $x > x_0(\varepsilon)$.

Notice that (3.12) is sharper than (3.11) for $q > x^{7/12}$.

4. Statistical results.

We shall be concerned with bounds for $\pi(x; q, a)$ for almost all q in intervals of the type $(Q, 2Q]$. The extra variable q offers us another arrangement of the dispersion which turns out to yield sharper results in certain ranges of

Q. Our aim is to prove

THEOREM 10. *Let a be a fixed non-zero integer and let ε, ε_1 and A be any positive constants. Then, provided $x > x_0(\varepsilon, \varepsilon_1, A)$, we have, for $(a, q)=1$ and $Q < q \leqq 2Q$, that*

$$\pi(x\,;q,a) \leqq \frac{(12+\varepsilon)x}{\varphi(q)\log(x^5 q^{-3})} \qquad \text{if} \quad x^{1/2} < Q \leqq x^{2/3} \tag{4.1}$$

and

$$\pi(x\,;q,a) \leqq \frac{(4+\varepsilon)x}{\varphi(q)\log x} \qquad \text{if} \quad x^{2/3} < Q \leqq x^{1-\varepsilon_1} \tag{4.2}$$

save for at most $Q(\log Q)^{-A}$ exceptional values of q.

THEOREM 11. *Let the notation and the assumptions of Theorem 10 be adopted. Then on Hypothesis* R* *we have*

$$\pi(x\,;q,a) \leqq \frac{(5+\varepsilon)x}{\varphi(q)\log(x^2 q^{-1})} \qquad \textit{if} \quad x^{1/2} < Q \leqq x^{1-\varepsilon_1} \tag{4.3}$$

save for at most $Q(\log Q)^{-A}$ exceptional values of q in $(Q, 2Q]$.

The proofs will be reduced to estimating the dispersion

$$T(x\,;Q,M,N) = \sum_{\substack{Q<q\leqq 2Q\\ (a,q)=1}} \sum_{\substack{M<m\leqq 2M\\ (m,q)=1}} \Big| \sum_{\substack{N<n\leqq 2N\\ (n,mq)=1}} b_n r(\mathcal{A}^{(q)}, mn) \Big|^2$$

where $|b_n| \leqq 1$ and $b_n = 0$ for non-squarefree n. Suppose we have

$$T(x\,;Q,M,N) \ll x^{2-\varepsilon}/QM \tag{4.4}$$

for any $M \leqq M_0$, $N \leqq N_0$. This yields

$$R_a(\mathcal{A}^{(q)}, M_0, N_0) \ll x^{1-\varepsilon/4}/q$$

save for at most $Q(\log Q)^{-A}$ exceptional values of q, $(a, q)=1$, in $(Q, 2Q]$. Hence by (1.9) we conclude that

$$\pi(x\,;q,a) \leqq \frac{(2+\varepsilon)x}{\varphi(q)\log M_0 N_0}$$

for the same q's, provided $x > x_0(\varepsilon, \varepsilon_1, A)$. We shall show that (4.4) holds for any M and N subject to either

$$M < x^{1-\varepsilon}Q^{-1},\ N < Q^{1/2}x^{-1/6} \qquad \text{if} \quad x^{1/3} < Q \leqq x^{1-2\varepsilon} \tag{4.5}$$

or

$$M < x^{1-2\varepsilon}Q^{-1},\ N < Qx^{-1/2} \qquad \text{if} \quad x^{1/2} < Q \leqq x^{1-2\varepsilon}. \tag{4.6}$$

By the above discussion the first result will complete the proof of (4.1) and the latter that of (4.2). Assuming Hypothesis R* we shall briefly prove that (4.4) holds for any M and N subject to

$$M<x^{1-4\varepsilon}Q^{-1},\ N<Q^{3/5}x^{-1/5},\ x^{1/3}<Q\leqq x^{1-4\varepsilon}. \tag{4.7}$$

This will complete the proof of (4.3).

To make the exposition clear we shall deal with $a=\pm 1$ only, the general ease being similar but a minor complication occurs when a is not prime to the variable n in $T(x\,;Q,M,N)$.

i) *An elementary treatment of* $T(x\,;Q,M,N)$.

We start with proving that the restrictions (4.5) are sufficient for (4.4) to hold. Our arguments will be entirely elementary. We write

$$T(x\,;Q,M,N)=\sum_{\substack{Q<q\leqq 2Q\\(a,q)=1}}\ \sum_{\substack{M<m\leqq 2M\\(m,q)=1}}\Bigg(\sum_{\substack{N<n\leqq 2N\\(n,mq)=1}} b_n \sum_{\substack{r\leqq x\\ r\equiv 0\,(q)\\ r\equiv 0\,(mn)}} 1-\frac{x}{qm}\sum_{\substack{N<n\leqq 2N\\(n,mq)=1}}\frac{b_n}{n}\Bigg)^2 \tag{4.8}$$

$$=C(x\,;Q,M,N)-2xB(x\,;Q,M,N)+x^2A(Q,M,N)$$

where

$$A(Q,M,N)=\sum_{\substack{Q<q\leqq 2Q\\(a,q)=1}}\frac{1}{q^2}\sum_{\substack{M<m\leqq 2M\\(m,q)=1}}\frac{1}{m^2}\Bigg(\sum_{\substack{N<n\leqq 2N\\(n,mq)=1}}\frac{b_n}{n}\Bigg)^2$$

and $B(x\,;Q,M,N)$, $C(x\,;Q,M,N)$ are defined analogously. We shall refer to $A(Q,M,N)$ as a main term. Since $A(Q,M,N)$ will occur in formulas for $B(x\,;Q,M,N)$ and $C(x\,;Q,M,N)$ we do not need to evaluate $A(Q,M,N)$ at all.

By the definition we have

$$B(x\,;Q,M,N)=\sum_q\frac{1}{q}\sum_m\frac{1}{m}\Bigg(\sum_{n_2}\frac{b_{n_2}}{n_2}\Bigg)\Bigg(\sum_{n_1}b_{n_1}\sum_{\substack{r\leqq x\\ r\equiv a\,(q)\\ r\equiv 0\,(mn_1)}}1\Bigg).$$

We reinterpret the congruence $r\equiv a(\mathrm{mod}\,q)$ by writing $r=a+lq$. We then have $l<\dfrac{x-a}{Q}$, $(l,mn_1)=1$, $Q<q\leqq\min\Big(2Q,\dfrac{x-a}{l}\Big)$ and $q\equiv -a\bar{l}\ (\mathrm{mod}\,mn_1)$. On changing the order of summation we get

$$B(x\,;Q,M,N)=\sum_m\frac{1}{m}\sum_{n_1,n_2}b_{n_1}\frac{b_{n_2}}{n_2}\sum_l\sum_{\substack{Q<q\leqq\min(2Q,(x-a)/l)\\ q\equiv -a\bar{l}\,(mn_1)\\ (q,n_2/(n_1,n_2))=1}}\frac{1}{q}. \tag{4.9}$$

We shall relax the congruence conditions in the inner sum over q by means of the following elementary lemma.

LEMMA 8. *For* $(a,b)=1$ *we have*

$$\sum_{\substack{l\leqq\xi\\ l\equiv a\,(\mathrm{mod}\,b)\\ (l,c)=1}}1=\frac{\varphi(c)(b,c)}{\varphi((b,c))bc}\xi+O(d(c)).$$

On applying Lemma 8 twice by partial summation we deduce that

(4.10)
$$\sum_{\substack{Q<q\leq\min(2Q,(x-a)/l)\\ q\equiv-a\bar{l}(mn_1)\\ (q,n_2/(n_1,n_2))=1}}\frac{1}{q}=\frac{1}{\varphi(mn_1)}\sum_{\substack{Q<q\leq\min(2Q,(x-a)/l)\\ (q,mn_1n_2)=1}}\frac{1}{q}+O(d(n_2)q^{-1}).$$

Now insert this into (4.9) and change the order of summation again getting

$$B(x,Q,M,N)=\sum_q\frac{1}{q}\sum_m\frac{1}{m}\sum_{n_1,n_2}b_{n_1}\frac{b_{n_2}}{n_2}\frac{1}{\varphi(mn_1)}\sum_{\substack{l\leq(x-a)/q\\ (l,mn_1)=1}}1+O(xQ^{-2}N\log N).$$

By lemma 8 one gets

$$\sum_{\substack{l\leq(x-a)/q\\ (l,mn_1)=1}}1=\frac{\varphi(mn_1)}{mn_1}\frac{x}{q}+O(d(mn_1))$$

thus

(4.11)
$$B(x;Q,M,N)=xA(Q,M,N)+O(\log^2MN+xQ^{-2}N\log N).$$

We deal with $C(x;Q,M,N)$ in a similar way. By the definition we have

$$C(x;Q,M,N)=\sum_q\sum_m\sum_{n_1,n_2}b_{n_1}b_{n_2}\sum_{\substack{r_1,r_2\leq x\\ r_1\equiv r_2\equiv a(q)\\ r_1\equiv0(mn_1)\\ r_2\equiv0(mn_2)}}1.$$

We reinterpret the congruences $r_1\equiv r_2\equiv a(\bmod q)$ by writing

$$r_1=a+l_1q,\quad r_2=a+l_2q.$$

We then have

$$l_1,\ l_2\leqq\frac{x-a}{Q},\ l_1\equiv l_2\bmod(m(n_1,n_2)),\ (l_1,mn_1)=1,\ (l_2,mn_2)=1.$$

Let $Q_1=Q_1(l_1,l_2)=\min\left(2Q,\frac{x-a}{l_1},\frac{x-a}{l_2}\right)$ and α be the common solution of

$$\alpha\equiv-a\bar{l}_1\ (\bmod mn_1),\quad \alpha\equiv-a\bar{l}_2\ (\bmod mn_2).$$

On changing the order of summation we get

(4.12)
$$C(x;Q,M,N)=\sum_m\sum_{n_1,n_2}b_{n_1}b_{n_2}\sum_{l_1,l_2}\sum_{\substack{Q<q\leq Q_1\\ q\equiv\alpha(m[n_1,n_2])}}1.$$

By Lemma 8 we deduce that

(4.13)
$$\sum_{\substack{Q<q\leq Q_1\\ q\equiv\alpha(m[n_1,n_2])}}1=\frac{(n_1,n_2)}{\varphi(mn_1n_2)}\sum_{\substack{Q<q\leq Q_1\\ (q,mn_1n_2)=1}}1+O(d(mn_1n_2)).$$

Now insert this into (4.11) and change the order of summation again getting

$$C(x;Q,M,N)=\sum_q\sum_m\sum_{n_1,n_2}b_{n_1}b_{n_2}\frac{(n_1,n_2)}{\varphi(mn_1n_2)}\sum_{\substack{l_1,l_2\leq(x-a)/q\\ l_1\equiv l_2(m(n_1,n_2))\\ (l_1,mn_1)=1\\ (l_2,mn_2)=1}}1$$
$$+O((xQ^{-1}MN^2+x^2Q^{-2}N^2)\log^8MN).$$

By Lemma 8 one gets

$$\sum_{\substack{l_1, l_2 \leq (x-\alpha)/q \\ l_1 \equiv l_2 (m(n_1, n_2)) \\ (l_1, mn_1)=1 \\ (l_2, mn_2)=1}} 1 = \frac{\varphi(mn_1n_2)}{(n_1, n_2)n_1n_2m^2q^2} x + O\Big(\frac{x}{Q} d(n_1n_2)\Big)$$

thus

$$C(x\,;\,Q,\,M,\,N) = x^2A(Q,\,M,\,N) + O((x + xQ^{-1}MN^2 + x^2Q^{-2}N^2)\log^8 MN)\,. \tag{4.14}$$

If we introduce (4.11) and (4.14) into (4.8) we find out that the main terms disappear throughout and we are left with the error terms only giving

$$T(x\,;\,Q,\,M,\,N) \ll (x + xQ^{-1}MN^2 + x^2Q^{-2}N^2)\log^8 MN.$$

Hence we deduce that (4.4) holds for any M and N subject to (4.6).

ii) *Kloosterman's sums approach*.

The only essential distinction in proving that (4.5) is sufficient for (4.4) pertains to sums (4.10) and (4.13) which will be evaluated in greater precision by means of Lemmas 6 and 7. We deal with $C(x\,;\,Q,\,M,\,N)$ only, the case of $B(x\,;\,Q,\,M,\,N)$ being analogous and simpler.

We start with (4.12). For the inner sum over q we first write

$$\sum_{\substack{Q < q \leq Q_1 \\ q \equiv \alpha (m[n_1, n_2])}} 1 = \frac{Q_1 - Q}{m[n_1,\,n_2]} + \Psi\Big(\frac{Q_1 - \alpha}{m[n_1,\,n_2]}\Big) - \Psi\Big(\frac{Q - \alpha}{m[n_1,\,n_2]}\Big)\,.$$

The main term $(Q_1 - Q)/m[n_1,\,n_2]$ can be written as before

$$\frac{Q_1 - Q}{m[n_1,\,n_2]} = \frac{(n_1,\,n_2)}{\varphi(mn_1n_2)} \sum_{\substack{Q < q \leq Q_1 \\ (q, mn_1n_2)=1}} 1 + O\Big(\frac{(n_1,\,n_2)}{\varphi(mn_1n_2)} d(mn_1n_2)\Big)$$

by Lemma 8. Therefore we gained the factor $(n_1,\,n_2)/\varphi(mn_1n_1)$ in the error term compared with that of (4.13). As a result we arrive at (4.14) with error term

$$O((x + xQ^{-1} + x^2Q^{-2}M^{-1})\log^8 MN) + \Psi(x\,;\,Q,\,M,\,N) \tag{4.15}$$

where $\Psi(x\,;\,Q,\,M,\,N)$ stands for the total contribution of terms $\Psi((Q_1 - \alpha)/m[n_1n_2])$ and $\Psi((Q - \alpha)/m[n_1n_2])$. The first part of (4.15) is admissible.

It remains to estimate $\Psi(x\,;\,Q,\,M,\,N)$. In the former treatment we estimated each term $\Psi((Q_1 - \alpha/m[n_1n_2])$ and $\Psi((Q - \alpha)/m[n_1n_2])$ trivially by 1 in absolute value.

Now, by means of Lemmas 6 and 7, we shall get a great cancellation by summing these terms over l_1 and l_2 in question. By the definition we have

$$\Psi(x\,;\,Q,\,M,\,N) = \sum_m \sum_{n_1, n_2} b_{n_1} b_{n_2} \sum_{l_1, l_2} \Big\{\Psi\Big(\frac{Q_1 - \alpha}{m[n_1,\,n_2]}\Big) - \Psi\Big(\frac{Q - \alpha}{m[n_1,\,n_2]}\Big)\Big\}$$

and hence by an application of Lemma 6 we get

(4.16) $\quad \Psi(x;Q,M,N)\ll H^{-1}N^{2}(x/Q)^{2}\log x$

$$+\sum_{h=1}^{H}\frac{1}{h}\sum_{m}\sum_{n_1,n_2}\left\{\left|\sum_{l_1,l_2}e\left(h\frac{Q_1-\alpha}{m[n_1,n_2]}\right)\right|+\left|\sum_{l_1,l_2}e\left(h\frac{Q-\alpha}{m[n_1,n_2]}\right)\right|\right\}.$$

To avoid a messy partial summation in respect to two variables l_1, l_2 in the sum involving $Q_1=Q_1(l_1,l_2)$ we arrange each term as follows

$$e\left(h\frac{Q_1-\alpha}{m[n_1,n_2]}\right)=e\left(h\frac{Q-\alpha}{m[n_1,n_2]}\right)+\frac{2\pi ih}{m[n_1,n_2]}\int_{Q}^{Q_1}e\left(h\frac{\xi-\alpha}{m[n_1,n_2]}\right)d\xi.$$

Hence, on changing the order of integration over ξ with the summation over l_1, l_2 we obtain

(4.17) $$\sum_{l_1,l_2\leqq(x-a)/Q}e\left(h\frac{Q_1-\alpha}{m[n_1,n_2]}\right)=\sum_{l_1,l_2\leqq(x-a)/Q}e\left(h\frac{Q-\alpha}{m[n_1,n_2]}\right)$$
$$+\frac{2\pi ih}{m[n_1,n_2]}\int_{Q}^{2Q}\sum_{l_1,l_2\leqq(x-a)/\xi}e\left(h\frac{\xi-a}{m[n_1,n_2]}\right)d\xi.$$

In this way we arrived at the problem of estimating sums

$$L_h(y,m,n_1,n_2)=\sum_{\substack{l_1,l_2\leqq y\\ l_1\equiv l_2(m(n_1,n_2))\\ (l_1,mn_1)=1\\ (l_2,mn_2)=1}}e\left(\frac{-h\alpha}{m[n_1,n_2]}\right)$$

where $\dfrac{x-a}{2Q}<y\leqq\dfrac{x-a}{Q}$ and $\alpha=\alpha(l_1,l_2)$ is the simultaneous solution of the congruences

$$\alpha\equiv-a\bar{l}_1 \pmod{mn_1},\quad \alpha\equiv-a\bar{l}_2 \pmod{mn_2}.$$

Letting $n_2^*=n_2/(n_1,n_2)$ we see that

$$\frac{-\alpha}{m[n_1,n_2]}\equiv a\frac{\overline{l_1n_2^*}}{mn_1}+a\frac{\overline{l_2mn_1}}{n_2^*}\pmod 1$$

which by using additive characters mod $m(n_1,n_2)$ yields

$$L_h(y,m,n_1,n_2)=\frac{1}{m(n_1,n_2)}\sum_{u\,(\mathrm{mod}\,m(n_1,n_2))}$$
$$\cdot\left(\sum_{\substack{l_1\leqq y\\(l_1,mn_1)=1}}e\left(\frac{ul_1}{m(n_1,n_2)}+ah\frac{\overline{l_1n_2^*}}{mn_1}\right)\right)\left(\sum_{\substack{l_2\leqq y\\(l_2,n_2^*)=1}}e\left(\frac{-ul_2}{m(n_1,n_2)}+ah\frac{\overline{l_2mn_1}}{n_2^*}\right)\right).$$

A straightforward application of lemma 7 leads to

$$(4.18)\qquad \sum_{\substack{l_1\leq y\\(l_1, mn_1)=1}} e\Big(\frac{ul_1}{m[n_1, n_2]}+ah\frac{\overline{l_1n_2^*}}{mn_1}\Big)\ll\Big(1+\frac{x}{QMN}\Big)(h, mn_1)^{1/2}(MN)^{1/2+\varepsilon}.$$

One may do much the same with $\sum_{l_2}$ but since the length of summation is rather long relatively to the modulus n_2^* the bound so obtained could be very weak. We shall gain a lot by receiving a cancellation not only from $e(ah\,\overline{l_2mn_1}/n_2^*)$ but also from the exponentials $e(-ul_2/m(n_1, n_2))$. To cross both aspects of the summation we write

$$l_2=\nu+n_2^*s \quad\text{with}\quad (\nu, n_2^*)=1,\ 0<\nu\leqq n_2^*,\ 0\leqq s<\Big[\frac{y}{n_2^*}\Big]$$

we then obtain

$$\sum_{\substack{l_2\leq y\\(l_2, n_2^*)=1}} e\Big(\frac{ul_2}{m(n_1, n_2)}+ah\frac{\overline{l_2mn_1}}{n_2^*}\Big)=\sum_{0\leq s<[y/n_2^*]} e\Big(-\frac{un_2^*}{m(n_1, n_2)}s\Big)$$

$$\cdot\sum_{\substack{0<\nu\leq n_2^*\\(\nu, n_2^*)=1}} e\Big(\frac{-u\nu}{m(n_1, n_2)}+ah\frac{\overline{\nu mn_1}}{n_2^*}\Big)+\sum_{\substack{n_2^*[y/n_2^*]<l_2\leq y\\(l_2, n_2^*)=1}} e\Big(\frac{-ul_2}{m(n_1, n_2)}+ah\frac{\overline{l_2mn_1}}{n_2^*}\Big)$$

$$\leqq(h, n_2^*)^{1/2}(n_2^*)^{1/2+\varepsilon}\Big\{1+\min\Big(\frac{y}{n_2^*},\ \frac{1}{\|un_2^*/m(n_1n_2)\|}\Big)\Big\}$$

by Lemma 7. On summing over $u(\operatorname{mod} m(n_1, n_2))$ we get

$$(4.19)\qquad L_h(y, m, n_1, n_2)\ll\Big(1+\frac{x}{QMN}\Big)^2(h, mn_1n_2)^{1/2}M^{1/2}Nx^{\varepsilon}.$$

Hence, by (4.17) we infer from (4.16) that

$$\Psi(x\,;Q, M, N)$$

$$\ll H^{-1}N^2\Big(\frac{x}{Q}\Big)^2x^{\varepsilon}+\sum_{h=1}^{H}\frac{d(h)}{h}MN^2\Big(1+\frac{hQ}{MN^2}\Big)\Big(1+\frac{x}{QMN}\Big)^2M^{1/2}Nx^{2\varepsilon}$$

$$\ll\Big(H^{-1}N^2x^2Q^{-2}+\Big(1+\frac{HQ}{MN^2}\Big)\Big(1+\frac{x}{QMN}\Big)^2M^{3/2}N^3\Big)x^{3\varepsilon}$$

for any positive H. On taking $H=MN^2Q^{-1}x^{4\varepsilon}$ we finally obtain

$$(4.20)\qquad \Psi(x\,;Q, M, N)\ll(QM)^{-1}x^{2-\varepsilon}+\Big(1+\frac{x}{QMN}\Big)^2M^{3/2}N^3x^{7\varepsilon}.$$

This together with (4.15) yields (4.14) with the error term

$$(4.21)\qquad O(x^{1+\varepsilon}+x^{2+\varepsilon}Q^{-2}M^{-1}+x^{2-\varepsilon}Q^{-1}M^{-1}+x^{7\varepsilon}M^{3/2}N^3+x^{2+7\varepsilon}Q^{-2}M^{-1/2}N).$$

We can evaluate $xB(x\,;Q, M, N)$ with the same precision in much similar manner. The arguments are slightly simpler because the summation over l_2 is absent. Since the main terms $x^2A(Q, M, N)$, $xA(Q, M, N)$ in the formulas for $C(x; Q, M, N)$ and $B(x; Q, M, N)$ respectively will disappear after introducing them into (4.8) the quantity (4.21) represents an upper bound for the dispersion $T(x; Q, M, N)$. From this we immediately deduce that (4.4) holds for any M and N subject to (4.5). The proof of Theorem 10 is complete.

iii) *Proof of Theorem* 11.

There is nothing essentially new in the proof of (4.3) in comparison to that of (4.1). Assuming Hypothesis R* one gets bound

$$\sum_{\substack{l_1\leq y\\ (l, mn_1)=1}} e\left(\frac{ul_1}{m(n_1, n_2)}+ah\frac{\overline{l_1n_2^*}}{mn_1}\right)\ll\frac{x}{QMN}(h, mn_1)^{1/2}(MN)^{1/2+\varepsilon}$$
$$+(h, mn_1)^{1/2}\left(\frac{x}{Q}\right)^{1/2}(MN)^{\varepsilon}$$

in place of (4.18). This leads to

$$L_h(y, m, n_1, n_2)\ll\left(1+\frac{x}{QMN}\right)^{3/2}(h, mn_1n_2)^{1/2}(xN/Q)^{1/2}x^{\varepsilon}$$

in place of (4.19) and finally we have

$$T(x; Q, M, N)\ll x^{1+\varepsilon}+x^{2+\varepsilon}Q^{-2}M^{-1}+x^{2-\varepsilon}Q^{-1}M^{-1}$$
$$+(MN^{5/2}x^{1/2}Q^{1/2}+x^2Q^{-2}M^{-1/2}N)x^{7\varepsilon}$$

in place of (4.21). From this we deduce that (4.4) holds for any M and N subject to (4.7). The proof of Theorem 11 is complete.

5. Two applications.

i) *The greatest prime factor of $p+a$.*

There are several applications of statistical estimates for $\pi(x; q, a)$ to the theory of numbers. As an example we consider the problem of the greatest prime factor of $p+a$ which had been previously investigated by Goldfeld [7], Motohashi [26] and Hooley [12], [13]. We shall prove the following result.

THEOREM 12. *If*

$$\theta<\theta_0=\frac{5}{3}-(2^{-4}3^{-5}5^{-3}17^5)^{1/7}e^{-1/8}=.6381089\cdots$$

then infinity often the greatest prime factor of $p+a$ exceeds p^{θ}.

The value of θ_0 given above should be compared with the value

$$\theta_0=1-\frac{1}{2}e^{-1/4}=.611059\cdots$$

that was obtained by Motohashi [26] and with the values

$$\theta_0=2-\frac{3}{2}e^{-1/12}=.6197\cdots$$

and $\theta_0=5/8=.625$ that were obtained by Hooley in [12] and [13] respectively.

Proof. By Bombieri-Vinogradov's theorem one easily deduces that

$$\sum_{x^{1/2}<q\leqq P(x)}\pi(x\,;q,-a)\log q\sim\frac{1}{2}x$$

as $x\rightarrow\infty$ where q indicates a prime number and $P(x)$ is the greatest prime factor of $\prod_{-a<p\leqq x}(p+a)$ (see [7]).

We consider for $1/2<\theta\leqq 2/3$ the sum

$$T(x,\theta)=\sum_{x^{1/2}<q\leqq x^{\theta}}\pi(x\,;q,-a)\log q$$

and use Theorems 6 and 9 to find an upper bound for it. For all q in $(x^{1/2}, x^{8/15})$ we apply (3.3) and for almost all q in $(x^{8/15}, x^{\theta})$ we apply (4.1). For the exceptional values of q in the latter interval we use trivial bounds

$$\pi(x,q,-a)\leqq\frac{x}{q}.$$

These exceptional modulus q contribute to $T(x,\theta)$ very little because in each interval of the type $[Q, 2Q]$ there are $O(Q(\log Q)^{-2})$ of them. We therefore obtain

$$T(x,\theta)\leqq\sum_{x^{1/2}<q\leqq x^{8/15}}\frac{8x\log q}{\varphi(q)\log(x^6q^{-7})}+\sum_{x^{8/15}<q\leqq x^{\theta}}\frac{12x\log q}{\varphi(q)\log(x^5q^{-3})}+o(x)$$

$$=\tau(\theta)x+o(x),$$

as $x\rightarrow\infty$ with

$$\tau(\theta)=\int_{1/2}^{8/15}\frac{8}{6-7u}du+\int_{8/15}^{\theta}\frac{12}{5-3u}du$$

by prime number theorem. A computation shows that $\tau(\theta)<1/2$ for $\theta<\theta_0$ which completes the proof.

ii) *The least P_2 in arithmetic progressions.*

Theorem 5 can be easily injected into weighted sieve theory to give substantial improvements for the least almost prime in arithmetic progressions. We demonstrate the power of (3.2) by proving

Theorem 13. *Let $(a, q)=1$, $q\geqq 2$. Then the least $P_2\equiv a(\mathrm{mod}\,q)$ is $\ll q^{1.845}$.*

The best estimate known till now was $P_2\ll q^{1.965}$ which is due to R. Heath-Brown [10] (first result with $P_2\ll q^2$).

PROOF. Let $\mathcal{A}=\{n\,;\ n\leqq x,\ n\equiv a(\bmod q)\}$, $(a, q)=1$, $x^{1/2}<q\leqq x^{3/5}$, $P=\{p\,;\ p\nmid q\}$. Put $M=x^{1-3\varepsilon}q^{-1}$, $N=x^{1/2-4\varepsilon}q^{-3/4}$ and $D=MN$. We consider the simplest weighted sum with Richert's weights of logarithmic type

$$W(\mathcal{A}, z, M)=\sum_{\substack{n\in\mathcal{A}\\(n,P(z))=1}}\left\{1-\frac{1}{\lambda}\sum_{\substack{p\mid n\\ z\leqq p<M}}\left(1-\frac{\log p}{\log M}\right)\right\}$$

with $z=D^{1/4}$, $\lambda=3-\dfrac{\log x}{\log M}$. For $(n, P(z))=1$ we have

$$W(n):=1-\frac{1}{\lambda}\sum_{\substack{p\mid n\\ z\leqq p<M}}\left(1-\frac{\log p}{\log M}\right)\leqq 1-\frac{1}{\lambda}\left(\nu(n)-\frac{\log x}{\log M}\right)$$

where $\nu(n)$ stands for the number of distinct prime factors of n. Hence, if $W(n)>0$ then $\nu(n)\leqq 2$. Therefore

$$|\{n\in\mathcal{A}\,;\ n=P_2\}|\geqq W(\mathcal{A}, z, M)+O\left(\frac{\varepsilon x}{\varphi(q)\log x}\right),$$

the error term being taken to care non-squarefree numbers. To estimate $W(\mathcal{A}, z, M)$ from below we first write it, in the usual sieve notation,

$$W(\mathcal{A}, z, M)=S(\mathcal{A}, P, z)-\frac{1}{\lambda}\sum_{z\leqq p<M,\, p\nmid q}\left(1-\frac{\log p}{\log M}\right)S(\mathcal{A}_p, P, z)$$

and then appeal to [18] for linear sieve results with bilinear forms for the remainder term. In case of $S(\mathcal{A}, P, z)$ it simply gives

$$S(\mathcal{A}, P, z)\geqq\frac{x}{\varphi(q)\log D}\{2\log 3+O(\varepsilon)\}.$$

The remaining sum over $p\in[z, M)$ can be split up into $\ll\log x$ sums of the type

$$\sum_{\substack{P\leqq p<P_1\\ p\nmid q}}\left(1-\frac{\log p}{\log M}\right)S(\mathcal{A}_p, P, z)\leqq\left(1-\frac{\log P}{\log M}\right)\sum_{\substack{P\leqq p<P_1\\ p\nmid q}}S(\mathcal{A}_p, P, z).$$

For each $S(\mathcal{A}_p, P, z)$ above, by Theorem 1 of [18] we have

$$S(\mathcal{A}_p, P, z)\leqq\frac{\{(2+O(\varepsilon)\}x}{\varphi(q)p\log D/p}+\sum_{l<\exp(8\varepsilon^{-3})}\sum_{m\leqq M/P}\sum_{\substack{n\leqq N\\(mn,q)=1}}a_m(l)b_n(l)r(\mathcal{A}, pmn)$$

where $|a_m(l)|$, $|b_n(l)|\leqq 1$. Summing over $p\in[P, P_1)$, $p\nmid q$ with an interpretation pm as one variable of the summation and n as the other, by Theorem 5, the total remainder term arising is $\ll x^{1-\varepsilon}/\varphi(q)$. Hence we conclude that

$$\sum_{\substack{z\leqq p<M\\ p\nmid q}}\left(1-\frac{\log p}{\log M}\right)S(\mathcal{A}_p, P, z)\leqq\frac{\{2+O(\varepsilon)\}x}{\varphi(q)\log M}\sum_{z\leqq p<M}\frac{1}{p}\frac{\log M/p}{\log D/p}.$$

Letting $x=q^{\theta}$, $\delta=(6\theta-7)/4(\theta-1)$ by partial summation and by prime number theorem an easy computation shows

$$\frac{\log D}{\log M}\sum_{z\leq p<M}\frac{1}{p}\,\frac{\log M/p}{\log D/p}=\log\frac{3}{\delta-1}-\delta\log\frac{3\delta}{4(\delta-1)}+O(\varepsilon).$$

Finally,

$$W(\mathcal{A},z,M)\geqq\frac{2x}{\varphi(q)\log D}\left\{\log 3-\frac{\theta-1}{2\theta-3}\left(\log\frac{3}{\delta-1}-\delta\log\frac{3\delta}{4(\delta-1)}\right)+O(\varepsilon)\right\}.$$

For $\theta=1.845$ the number in the brackets { } is $>10^{-4}$. This completes the proof of Theorem 13.

REMARKS. There are several possibilities for further improvements. First of all one may try to use more efficient weights as for example these of M. Laborde [**21**] or even better these of G. Greaves [**9**]. Unfortunately the improvement for $\theta=1.845$ is very small and not proportional to efforts required in applying them. Another possibility rests on extending the range of the summation over p in the weighted sum $W(\mathcal{A},z,M)$ beyond $M=x^{1-\varepsilon}/q$, that is to say, to a range where Theorem 5 is not applicable directly (notice that M is nearly as large as the number $|\mathcal{A}|$ of elements in $\mathcal{A}$). For example, on applying two dimensional sieve of Selberg one may show that

$$\sum_{\substack{M\leq p<y\\ p\nmid q}}\left(1-\frac{\log p}{\log y}\right)S(\mathcal{A}_p,P,z)\leqq\frac{\{1+O(\varepsilon)\}x}{\varphi(q)\log y}\left(\frac{\log y/M}{\log N}\right)^2$$

for any $M\leqq y<D$, $N\leqq z<D^{1/4}$. The proof is quite long and an improvement which it yields for θ is again little.

6. Brun-Titchmarsh theorems for short intervals.

Here we shall state, without proofs, several results about $\pi(x+x^{\theta})-\pi(x)$ with $0<\theta<1$, $x\geqq 3$. Most of them had been deduced jointly with R. C. Vaughan in December 1977 during the meeting at the Institut Mittag Leffler in Djursholm.

When estimating the remainder terms in sieve bounds for $\pi(x+x^{\theta})-\pi(x)$ one arrives at exponential sums of the type

$$\sum_{N<n\leq N_1}e\left(\frac{x}{n}\right)$$

with $N<N_1\leqq 2N$, $N=x^{\alpha}$, $0<\alpha<1$. By van der Corput's method or using general theory of exponent pairs (for the definition and the theory the reader is referred to [**30**]) one is able to prove non-trivial estimates

$$\sum_{N<n\leq N_1}e\left(\frac{x}{n}\right)\ll Nx^{-\delta} \tag{6.1}$$

with some $\delta=\delta(\alpha)>0$ for all $0<\alpha<1$. Hence it follows

THEOREM 14. *For any* $0<\theta<1$ *there exists* $\eta=\eta(\theta)>\theta$ *such that*

$$\pi(x+x^{\theta})-\pi(x)<\frac{(2+\varepsilon)x^{\theta}}{\eta(\theta)\log x}, \quad x>x_0(\varepsilon, \theta).$$

A precise value for $\eta(\theta)$ depends on the exponent pair being used for (6.1). Letting (κ, λ) be an exponent pair we have

$$\sum_{N<n\leq N_1} e\Big(\frac{x}{n}\Big)\ll(xN^{-2})^{\kappa}N^{\lambda} \qquad \text{if} \quad N<x^{1/2}.$$

This gives

$$\eta(\theta)=\Bigg(1+\frac{1-\lambda+2\kappa}{3-\lambda-\frac{1}{2}\kappa}\Bigg)\theta-\frac{\kappa}{3-\lambda-\frac{1}{2}\kappa}.$$

On taking $(\kappa, \lambda)=(1/2, 1/2)$ we find that

$$\eta(\theta)=\frac{5}{3}\theta-\frac{2}{9}$$

which is $>\theta$ ($\eta(\theta)=\theta$ is trivial) for $\theta>1/3$.

To the analogy with Hooley's Hypothesis R* is the following

CONJECTURE (exponent pairs conjecture). *For any* $\varepsilon>0$, $(\varepsilon, 1/2+\varepsilon)$ *is an exponent pair.*

On this conjecture we find that $\eta(\theta)=\frac{6}{5}\theta$.

Finally, we remark that for very small θ Vinogradov's method yields sharper bounds than van der Corput's exponent pairs do. For $\theta>1/2$ it is better to use the theory of the Riemann zeta-function giving $\eta(\theta)=(1+\theta)/2$.

References

[1] M.B. Barban, Yu. V. Linnik and N.G. Tshudakov, On prime numbers in an arithmetic progression with a prime-power difference, Acta Arith., **9** (1964), 375-390.

[2] E. Bombieri and H. Davenport, On the large sieve method, Abh. aus Zahlentheorie und Analysis zur Erinnerung an Edmund Landau, Deutscher Verlag Wiss., Berlin 1968, 11-22.

[3] D. Burgess, On character sums and L-series, Proc. London Math. Soc., (3) **12** (1962), 193-206.

[4] D. Burgess, On character sums and L-series, II, Proc. London Math. Soc., (3) **13** (1963), 524-536.

[5] A. Fujii, P.X. Gallagher and H.L. Montgomery, Some hybrid bounds for character sums and Dirichlet's L-series, Topics in number theory (Proc. Colloq. Debrecen (1974), 41-47), North-Holland, Amsterdam, 1976.

[6] P.X. Gallagher, Primes in progressions to prime power modulus, Invent. Math., **16** (1972), 191-201.

[7] D. Goldfeld, On the number of primes p for which $p+a$ has a large prime factor, Mathematatika, **16** (1969), 23-27.
[8] D. Goldfeld, A further improvement of the Brun-Titchmarsh theorem, J. London Math. Soc., (2) **11** (1977), 434-444.
[9] G. Greaves, A weighted sieve of Brun's type, Acta Atith., (to appears).
[10] R. Heath-Brown, Almost-primes in arithmetic progressions and short intervals, Math. Proc. Cambridge Philos. Soc., **83** (1978), 357-375.
[11] R. Heath-Brown and H. Iwaniec, On the difference between consecutive primes, Invent. Math., **55** (1979), 49-69.
[12] C. Hooley, On the Brun-Titchmarsh theorem, J. Reine Angew. Math., **255** (1972), 60-79.
[13] C. Hooley, On the largest prime factor of $p+a$, Mathematika, **20** (1973), 135-143.
[14] C. Hooley, On the Brun Titchmarsh theorem, II, Proc. London Math. Soc., (3) **30** (1975), 114-128.
[15] C. Hooley, On the greatest prime factor of a cubic polynomial, J. Reine Angew. Math., **303/304** (1978), 21-50.
[16] H. Iwaniec, On the error term in the linear sieve, Acta Arith., **19** (1971), 1-30.
[17] H. Iwaniec, On zeros of Dirichlet's L-series, Invent. Math., **23** (1974), 97-104.
[18] H. Iwaniec, A new form of the error term in the linear sieve, Acta Arith., **37** (1980), 307-320.
[19] M. Jutila, Zero density estimates for L-functions, Acta Arith., **32** (1977), 55-62.
[20] M. Laborde, Nombres presque-premiers dans de petits intervalles, Séminaire de Théorie des Nombres, Bordeaux, 1977-1978.
[21] M. Laborde, Buchstab's sifting weights, Mathematika, **26** (1979), 250-257.
[22] J.H. van Lint and H.E. Richert, On primes in arithmetic progressions, Acta Arith., **11** (1965), 209-216.
[23] H.L. Montgomery, Topics in Multiplicative Number Theory, Lecture Notes in Math., **227**, Berlin-New York, 1971.
[24] H.L. Montgomery, Problems concerning prime numbers, Proc. Symp. Pure Math., **28** (1976), 307-310.
[25] H.L. Montgomery and R.C. Vaughan, On the large sieve, Mathematika, **20** (1973), 119-134.
[26] Y. Motohashi, A note on the least prime in an arithmetic progression with a prime difference, Acta Arith., **17** (1970), 283-285.
[27] Y. Motohashi, On some improvements of the Brun-Titchmarsh theorem, J. Math. Soc. Japan, **26** (1974), 306-323.
[28] Y. Motohashi, On some improvements of the Brun-Titchmarsh theorem III, J. Math. Soc. Japan, **27** (1975), 444-453.
[29] Y. Motohashi, A note on Siegel's zero, Proc. Japan Acad., **55** (1979), 190-192.
[30] E. Philips, The zeta-function of Riemann, Quart. J. of Math. Oxford Ser., **4** (1933), 209-225.
[31] K. Prachar, Primzahlverteilung, Berlin-Göttingen-Heidelberg, 1957.
[32] H.E. Richert, Selberg's sieve with weights, Mathematika, **16** (1969), 1-22.
[33] H. Siebert, On a question of P. Turan, Acta Arith., **26** (1975), 303-305.
[34] E.C. Titchmarsh, A divisor problem, Rend. Circ. Mat. Palermo, **54** (1930), 414-429.
[35] P. Turan, Uber die Siegel-Nullestelle der Dirichletschen Funktionen, Acta Arith., **24** (1973), 135-141.
[36] D. Wolke, Eine weitere Möglichkeit zur Verbesserung des Satzes von Brun-Titchmarsh (unpublished manuscript).

H. IWANIEC

Université de Bordeaux I
U.E.R. de Mathématiques et d'Informatique
351, Cours de la Libération
33405 Talence
France

and

Instytut Matematyczny
Polskiej Akademii Nauk
ul. Śniadeckich 8
00-950 Warszawa
Poland