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Degenerate integrodifferential equations of parabolic type with Robin boundary conditions: L^2-theory

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Abstract. This paper is devoted to solving a degenerate parabolic integrodifferential equation with the Robin boundary condition. We begin with solving the equation without the integral delay term. For that purpose we introduce some new unknown function following Favini and Yagi [**4**] and construct the fundamental solution to the equation to be satisfied by it by the method of Kato and Tanabe [**5**]. Using this fundamental solution we transform the original problem to an easily solvable integral equation for the time derivative of the new unknown function.

1. Introduction.

The present paper is concerned with the initial value problem for the following degenerate integrodifferential equation of parabolic type

$$\frac{d}{dt}(M(t)u(t)) + L(t)u(t) + \int_0^t B(t,s)u(s)ds = f(t), \quad 0 < t \le T, \tag{1.1}$$

$$M(0)u(0) = M(0)u_0. \tag{1.2}$$

Here $L(t)$ is the realization of a second-order linear elliptic differential operator in $L^2(\Omega)$ with the Robin boundary condition, $M(t)$ is the multiplication operator by some nonnegative function satisfying some smoothness assumptions for each $0 \le t \le T$, and $B(t,s)$ is a linear second-order partial differential operator for each $0 \le s \le t \le T$.

At first we will solve the problem without the integral term

$$\frac{d}{dt}(M(t)u(t)) + L(t)u(t) = f(t), \quad 0 < t \le T, \tag{1.3}$$

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$$M(0)u(0) = M(0)u_0. \tag{1.4}$$

Let us introduce the possibly multivalued linear operator $A(t) = L(t)M(t)^{-1}$. By the method of Favini and Yagi [**4**] we can show that $-A(t)$ generates an infinitely differentiable semigroup for any $t \in [0, T]$. The problem (1.3)–(1.4) is transformed into the following one for the new unknown function $v(t) = M(t)u(t)$:

$$\frac{d}{dt}v(t) + A(t)v(t) \ni f(t), \quad 0 < t \le T, \tag{1.5}$$

$$v(0) = v_0 = M(0)u_0. \tag{1.6}$$

We begin by constructing the fundamental solution to problem (1.5)–(1.6) using the method of Kato and Tanabe [**5**]. Using this fundamental solution, we transform the problem (1.1)–(1.2) into an integral equation to be satisfied by $(Mu)'$, which is easily solvable by successive approximations.

The construction of the fundamental solution to a problem of the type (1.3)–(1.4) is discussed in detail in Chapter IV of Favini and Yagi [**4**], essentially under the assumption that the domains $D(M(t))$ and $D(L(t))$ of $M(t)$ and $L(t)$ be *independent* of time. More exactly, Propositions 4.13, 4.14 and 4.15 in [**4**] are concerned with the case when either of $M(t)$ or $L(t)$ is *constant*.

We stress that the case where M and L are both dependent on t is rather difficult to handle. In this respect we believe that our Theorem 4.1, in which a fundamental solution with satisfactory properties is constructed, is of some independent interest. This fundamental solution enables us to solve not only (1.3)–(1.4) but also the original problem (1.1)–(1.2) by first transforming the equation (1.1) into the one with $(Mu)'$ instead of $B(\cdot, \cdot)u$ in the integral term, using an idea of Crandall and Nohel [**2**] concerning equations containing a multivalued operator.

General results for nondegenerate equations with $M = I$ were obtained by Prüss [**7**] for both hyperbolic equations and parabolic ones. For degenerate equations a result analogous to the one of the present paper was obtained in the case of the Dirichlet boundary condition in the space $L^p(\Omega)$ in Favini, Lorenzi and Tanabe [**3**] for $p \in (1, 3/2)$, and in Lorenzi and Tanabe [**6**] for p satisfying $p \in (1, 2)$ together with some other conditions. If the boundary condition is of Robin type, the operator $L(t)$ has a variable domain unlike the case of the Dirichlet condition, which makes the situation difficult. Therefore we consider the operator $L(t)$ also in the space $H^1(\Omega)^*$, which has a negative norm, so that the corresponding operator has a constant domain $H^1(\Omega)$. This is essentially used in the proofs of Lemmata 4.2 and 4.4. Consequently, we are obliged to consider the problem in the space $L^2(\Omega)$ instead of in $L^p(\Omega)$, $1 < p < \infty$.

As for a degenerate system of ordinary integrodifferential equations we refer to Bulatov [**1**].

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2. Assumptions.

Let Ω be a bounded open set of $\boldsymbol{R}^n$ with a C^2-boundary $\partial\Omega$. Let

$$\mathscr{L}(t) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{i,j}(x,t)\frac{\partial}{\partial x_i}\right) + \sum_{i=1}^{n} a_i(x,t)\frac{\partial}{\partial x_i} + a_0(x,t), \quad 0 \le t \le T, \tag{2.1}$$

be a linear second order differential operator such that $a_{i,j}$, a_i and a_0 are real-valued functions satisfying

$$\begin{aligned} &a_{i,j},\ \frac{\partial a_{i,j}}{\partial x_j},\ a_i,\ \frac{\partial a_i}{\partial x_i},\ a_0,\ \frac{\partial a_{i,j}}{\partial t},\ \frac{\partial^2 a_{i,j}}{\partial x_j \partial t},\ \frac{\partial a_i}{\partial t},\ \frac{\partial^2 a_i}{\partial x_i \partial t},\ \frac{\partial a_0}{\partial t} \\ &\quad \in C(\overline{\Omega} \times [0,T]), \quad i,j = 1,\dots,n, \end{aligned} \tag{2.2}$$

$$\begin{aligned} &\frac{\partial a_{i,j}}{\partial t},\ \frac{\partial^2 a_{i,j}}{\partial x_j \partial t},\ \frac{\partial a_i}{\partial t},\ \frac{\partial^2 a_i}{\partial x_i \partial t},\ \frac{\partial a_0}{\partial t} \text{ are uniformly Hölder continuous} \\ &\text{functions of } t \text{ of order } \gamma \in (0,1),\ i,j = 1,\dots,n; \end{aligned} \tag{2.3}$$

$$\begin{aligned} &\{a_{i,j}(x,t)\} \text{ is a positive definite symmetric matrix for each} \\ &(x,t) \in \overline{\Omega} \times [0,T]; \end{aligned} \tag{2.4}$$

$$a_0(x,t) - \frac{1}{2}\sum_{i=1}^{n} \frac{\partial a_i(x,t)}{\partial x_i} > 0, \quad \text{for } (x,t) \in \overline{\Omega} \times [0,T]. \tag{2.5}$$

Let b be a real-valued continuous function belonging to $C^1(\partial\Omega \times [0,T])$ such that $\partial^2 b/\partial x_i \partial t \in C(\partial\Omega \times [0,T])$ and $\partial b/\partial t$ is a uniformly Hölder continuous function of t of order γ. Suppose

$$\frac{1}{2}\sum_{i=1}^{n} a_i(x,t)\nu_i(x) + b(x,t) \ge 0 \quad \text{for } (x,t) \in \partial\Omega \times [0,T], \tag{2.6}$$

where $\nu = (\nu_1,\dots,\nu_n)$ is the outer normal unit vector to $\partial\Omega$. Let $a(t;u,v), u, v \in H^1(\Omega)$, be the sesquilinear form defined by

$$a(t;u,v) = \int_\Omega \left\{ \sum_{i,j=1}^{n} a_{i,j}(\cdot,t)\frac{\partial u}{\partial x_i}\overline{\frac{\partial v}{\partial x_j}} + \sum_{i=1}^{n} a_i(\cdot,t)\frac{\partial u}{\partial x_i}\overline{v} + a_0(\cdot,t)u\overline{v}\right\}dx$$
$$+ \int_{\partial\Omega} b(x,t)u\overline{v}\,dS.$$

The realization $L(t)$ and $\widetilde{L}(t)$ of $\mathscr{L}(t)$ in $L^2(\Omega)$ and $H^1(\Omega)^*$, respectively, with the Robin boundary condition is defined by

$$D(L(t)) = \left\{u \in H^2(\Omega);\ \sum_{i,j=1}^{n} a_{i,j}(\cdot,t)\nu_j\frac{\partial u}{\partial x_i} - b(\cdot,t)u = 0 \text{ on } \partial\Omega\right\},$$
$$L(t)u = \mathscr{L}u(t) \text{ for } u \in D(L(t)),$$
$$D(\widetilde{L}(t)) = H^1(\Omega), \quad (\widetilde{L}(t)u,v)_{H^1(\Omega)^*\times H^1(\Omega)} = a(t;u,v) \quad \text{for } u,v \in H^1(\Omega),$$

respectively. Observe that $\widetilde{L}(t)u = L(t)u$ whenever $u \in D(L(t))$.

From now on C will denote a positive constant that may vary from line to line. By virtue of the assumptions the following inequalities hold:

$$|a(t;u,v) - a(s;u,v)| \le C|t-s|\|u\|_{H^1}\|v\|_{H^1}. \tag{2.7}$$

$$|\dot{a}(t;u,v) - \dot{a}(s;u,v)| \le C|t-s|^\gamma\|u\|_{H^1}\|v\|_{H^1}, \quad \dot{a}(t;u,v) = \frac{\partial}{\partial t}a(t;u,v). \tag{2.8}$$

It also follows from our assumptions that $\widetilde{L}(t)$ is differentiable in $\mathscr{L}(H^1(\Omega), H^1(\Omega)^*)$, and in view of (2.7), (2.8) one has

$$\big\|\widetilde{L}(t) - \widetilde{L}(s)\big\|_{\mathscr{L}(H^1,(H^1)^*)} \le C|t-s|, \tag{2.9}$$

$$\big\|\dot{\widetilde{L}}(t) - \dot{\widetilde{L}}(s)\big\|_{\mathscr{L}(H^1,(H^1)^*)} \le C|t-s|^\gamma, \qquad \dot{\widetilde{L}}(t) = \frac{\partial}{\partial t}\widetilde{L}(t). \tag{2.10}$$

Assume that

$$0 < \rho < 1, \quad 0 < \alpha < 1, \quad 2\rho + \alpha > 2, \quad \frac{2(1-\rho)}{2-\rho} < \gamma < 1. \tag{2.11}$$

Let m be a nonnegative function in $C^1(\overline{\Omega}\times[0,T])$ such that

$$|\nabla_x m(x,t)| \le Cm(x,t)^\rho, \quad |\dot{m}(x,t)| \le Cm(x,t)^\alpha, \tag{2.12}$$

where $\nabla_x m = (\partial m/\partial x_1, \dots, \partial m/\partial x_n)$ and $\dot{m} = \partial m/\partial t$, and $\dot{m}$ is a uniformly Hölder continuous function of t of order γ:

$$|\dot{m}(x,t) - \dot{m}(x,s)| \le C|t-s|^\gamma, \quad t,s \in [0,T]. \tag{2.13}$$

EXAMPLE. Let $m(x,t) = m_0(x,t)^k$, where $m_0 \in C^1(\overline{\Omega} \times [0,T])$, $m_0 \ge 0$ in $\overline{\Omega} \times [0,T]$ and $k > 3$. Then,

$$\nabla_x m(x,t) = k\, m_0(x,t)^{k-1} \nabla_x m_0(x,t), \quad \dot{m}(x,t) = k\, m_0(x,t)^{k-1} \dot{m}_0(x,t).$$

Hence (2.12) is satisfied with $\rho = \alpha = (k-1)/k$. Furthermore, if $\dot{m}_0$ satisfies the condition

$$|\dot{m}_0(x,t) - \dot{m}_0(x,s)| \le C|t-s|^\gamma, \quad t,s \in [0,T],$$

for some exponent $\gamma \in (2/(k+1), 1)$, then (2.13) also holds.

The notation $M(t)$ denotes the multiplication operator by the function $m(\cdot, t)$.

Let $B(t,s)$, $(t,s) \in \Delta = \{(t,s); 0 \le s \le t \le T\}$, be a second-order linear differential operator such that

$$B(t,s) = \sum_{i,j=1}^{n} b_{i,j}(x,t,s) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t,s) \frac{\partial}{\partial x_i} + b_0(x,t,s),$$

whose coefficients $b_{i,j}, b_i, b_0$ are functions belonging to $C(\overline{\Omega} \times \Delta; \boldsymbol{C})$ and are uniformly Hölder continuous of order ω in t, where

$$\frac{1-\rho}{2-\rho} < \omega \le 1. \tag{2.14}$$

3. Preliminaries.

With the aid of integration by parts and the hypotheses (2.4), (2.5), (2.6) one can show without difficulty that there exist positive constants c_0 and c_1 such that the following inequality holds for $u \in H^1(\Omega)$ and $t \in [0,T]$:

$$\operatorname{Re} a(t;u,u) \ge c_0 \int_\Omega |\nabla u|^2 dx + c_1 \|u\|_{L^2}^2.$$

This inequality implies that $L(t)$ and $\widetilde{L}(t)$ have everywhere defined bounded inverses. Furthermore, with the aid of a well-known argument on analytic semigroups it can be shown that there exists a positive constant c_2 such that the inequality

$$|a_\lambda(t,u,u)| \geq c_2\big\{|\lambda|\|\sqrt{m}u\|_{L^2}^2 + \|u\|_{H^1}^2\big\}$$

holds for $\operatorname{Re}\lambda \geq -c_2|\operatorname{Im}\lambda|$ and $u \in H^1(\Omega)$, where

$$a_\lambda(t;u,v) = \lambda(m(\cdot,t)u,v) + a(t;u,v), \qquad \lambda \in \boldsymbol{C},\ t \in [0,T].$$

Hence following the argument of Favini and Yagi [**4**, p. 76], one can show that the following inequality holds:

$$\|M(t)(\lambda M(t) + L(t))^{-1}\|_{\mathscr{L}(L^2)} \leq C|\lambda|^{-1/(2-\rho)}, \quad \lambda \in \Sigma \quad \text{or} \quad |\lambda| \leq c_3, \tag{3.1}$$

where $\mathscr{L}(L^2) = \mathscr{L}(L^2, L^2)$, c_3 is some positive constant and

$$\Sigma = \{\lambda \in \boldsymbol{C} \setminus \{0\} : |\arg\lambda| \leq \theta_0\},$$

θ_0 being an angle such that $\pi/2 < \theta_0 < \pi$.

Also the argument on p. 75 in [**4**] yields that there exists some positive constant c_4 such that for $\lambda \in \Sigma$ or $|\lambda| \leq c_4$ the following inequalities hold:

$$\big\|M(t)(\lambda M(t) + \widetilde{L}(t))^{-1}\big\|_{\mathscr{L}((H^1)^*)} \leq C|\lambda|^{-1}, \tag{3.2}$$

$$\big\|M(t)(\lambda M(t) + \widetilde{L}(t))^{-1}\big\|_{\mathscr{L}((H^1)^*, L^2)} \leq C|\lambda|^{-1/2}. \tag{3.3}$$

Furthermore

$$\big\|(\lambda M(t) + \widetilde{L}(t))^{-1}\big\|_{\mathscr{L}((H^1)^*, H^1)} \leq C, \tag{3.4}$$

$$\big\|\widetilde{L}(t)(\lambda M(t) + \widetilde{L}(t))^{-1}\big\|_{\mathscr{L}((H^1)^*)} \leq C. \tag{3.5}$$

These inequalities readily imply

$$\left.\begin{array}{l} \big\|(\lambda M(t) + \widetilde{L}(t))^{-1}\big\|_{\mathscr{L}((H^1)^*, L^2)} \\ \|(\lambda M(t) + L(t))^{-1}\|_{\mathscr{L}(L^2)} \\ \|(\lambda M(t) + L(t))^{-1}\|_{\mathscr{L}(L^2, H^1)} \end{array}\right\} \leq C, \tag{3.6}$$

$$\big\|\widetilde{L}(t)(\lambda M(t)+L(t))^{-1}\big\|_{\mathscr{L}(L^2,(H^1)^*)} \le C. \tag{3.7}$$

Since $\dot{\widetilde{L}}(t)\widetilde{L}(t)^{-1} \in \mathscr{L}((H^1)^*)$, it follows from (3.5) and (3.7) that

$$\big\|\dot{\widetilde{L}}(t)(\lambda M(t)+\widetilde{L}(t))^{-1}\big\|_{\mathscr{L}((H^1)^*)} \le C, \tag{3.8}$$

$$\big\|\dot{\widetilde{L}}(t)(\lambda M(t)+L(t))^{-1}\big\|_{\mathscr{L}(L^2,(H^1)^*)} \le C. \tag{3.9}$$

Let

$$A(t)=L(t)M(t)^{-1}, \quad \widetilde{A}(t)=\widetilde{L}(t)M(t)^{-1}.$$

Then $A(t)$ and $\widetilde{A}(t)$ are possibly multi-valued operators. However, for $\lambda \in \Sigma$ or $|\lambda| \le \min\{c_3, c_4\}$ one gets

$$(\lambda + A(t))^{-1} = M(t)(\lambda M(t)+L(t))^{-1}, \quad (\lambda+\widetilde{A}(t))^{-1} = M(t)(\lambda M(t)+\widetilde{L}(t))^{-1},$$

which are single-valued. Hence (3.1), (3.2) and (3.3) can be rewritten as

$$\|(\lambda + A(t))^{-1}\|_{\mathscr{L}(L^2)} \le C|\lambda|^{-1/(2-\rho)}, \tag{3.10}$$

$$\big\|(\lambda+\widetilde{A}(t))^{-1}\big\|_{\mathscr{L}((H^1)^*)} \le C|\lambda|^{-1}, \tag{3.11}$$

$$\big\|(\lambda+\widetilde{A}(t))^{-1}\big\|_{\mathscr{L}((H^1)^*,L^2)} \le C|\lambda|^{-1/2}. \tag{3.12}$$

By virtue of (3.10) $-A(t)$ generates, for all $t \in [0,T]$, an infinitely differentiable semigroup $e^{-\tau A(t)}$:

$$e^{-\tau A(t)} = \frac{1}{2\pi i}\int_\Gamma e^{\lambda\tau}(\lambda + A(t))^{-1} d\lambda, \tag{3.13}$$

where Γ is a smooth contour in Σ connecting $\infty e^{-i\theta_0}$ and $\infty e^{i\theta_0}$, satisfying the following estimates:

$$\big\|e^{-\tau A(t)}\big\|_{\mathscr{L}(L^2)} \le C\tau^{(\rho-1)/(2-\rho)}, \tag{3.14}$$

$$\left\|\frac{\partial}{\partial\tau}e^{-\tau A(t)}\right\|_{\mathscr{L}(L^2)} \le C\tau^{(2\rho-3)/(2-\rho)}. \tag{3.15}$$

Furthermore, the following statements hold:

$$\frac{\partial}{\partial \tau} e^{-\tau A(t)} + A(t) e^{-\tau A(t)} \ni 0 \text{ or equivalently } A(t)^{-1} \frac{\partial}{\partial \tau} e^{-\tau A(t)} = -e^{-\tau A(t)}, \tag{3.16}$$

$$\lim_{\tau \to +0} e^{-\tau A(t)} v_0 = v_0 \text{ in the strong topology of } L^2(\Omega) \text{ for } v_0 \in D(A(t)). \tag{3.17}$$

4. Construction of the fundamental solution.

In this section we construct the fundamental solution U to the initial value problem (1.5)–(1.6) following the method of Kato and Tanabe [**5**]. We look for a U of the form

$$U(t,s) = e^{-(t-s)A(t)} + \int_s^t e^{-(t-\tau)A(t)} \Phi(\tau, s) d\tau, \quad 0 \le s < t \le T, \tag{4.1}$$

where Φ solves the integral equation

$$\Phi(t,s) = \Phi_1(t,s) + \int_s^t \Phi_1(t,\tau)\Phi(\tau,s) d\tau, \quad 0 \le s < t \le T, \tag{4.2}$$

with

$$\Phi_1(t,s) = -\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right) e^{-(t-s)A(t)} = -\frac{1}{2\pi i} \int_\Gamma e^{\lambda(t-s)} \frac{\partial}{\partial t} (\lambda + A(t))^{-1} d\lambda. \tag{4.3}$$

The purpose of this section is to establish the following theorem.

Theorem 4.1. *The operator-valued function $U(t,s)$, $0 \le s < t \le T$, defined by* (4.1), (4.2) *and* (4.3), *is considered as the fundamental solution to problem* (1.5)–(1.6) *in the sense that U is differentiable with respect to t in (s,T), $R(U(t,s)) \subset D(A(t))$ for $0 \le s < t \le T$ and it satisfies*

$$\frac{\partial}{\partial t} U(t,s) + A(t) U(t,s) \ni 0, \tag{4.4}$$

and

$$\lim_{t \to s+0} \|U(t,s)v_0 - v_0\|_{L^2(\Omega)} = 0 \tag{4.5}$$

for $v_0 \in D(A(s))$, $0 \le s < T$. *Furthermore the following inequality holds.*

$$\left\| \frac{\partial}{\partial t} U(t,s) \right\|_{\mathscr{L}(L^2)} \le C(t-s)^{(2\rho-3)/(2-\rho)}. \tag{4.6}$$

The proof is given in 3 steps.

STEP 1. Proof of the differentiability of $U(t,s)$ and inequality (4.6).
Set

$$W(t,s) = \int_s^t e^{-(t-\tau)A(t)} \Phi(\tau,s) d\tau, \tag{4.7}$$

so that

$$U(t,s) = e^{-(t-s)A(t)} + W(t,s). \tag{4.8}$$

LEMMA 4.1. *The following inequalities hold for all* $t \in [0,T]$ *and* $\lambda \in \Sigma$ *or* $|\lambda| \le \min\{c_3, c_4\}$:

$$\big\| \dot{M}(t)(\lambda M(t) + L(t))^{-1} \big\|_{\mathscr{L}(L^2)} \le C|\lambda|^{-\alpha/(2-\rho)}, \tag{4.9}$$

$$\big\| \dot{M}(t)(\lambda M(t) + \widetilde{L}(t))^{-1} \big\|_{\mathscr{L}((H^1)^*, L^2)} \le C|\lambda|^{-\alpha/2}. \tag{4.10}$$

PROOF. The assumption (2.12) and Hölder's inequality yield

$$\|\dot{m}u\|_{L^2} \le C\|m^\alpha u\|_{L^2} \le C\|mu\|_{L^2}^\alpha \|u\|_{L^2}^{1-\alpha}. \tag{4.11}$$

Applying this inequality to $u = (\lambda M(t) + L(t))^{-1} f$ and using (3.1) and (3.6) one observes

$$\begin{aligned} \|\dot{m}u\|_{L^2} &\le C\big\|M(t)(\lambda M(t) + L(t))^{-1} f\big\|_{L^2}^\alpha \big\|(\lambda M(t) + L(t))^{-1} f\big\|_{L^2}^{1-\alpha} \\ &\le C\big(|\lambda|^{-1/(2-\rho)} \|f\|_{L^2}\big)^\alpha \|f\|_{L^2}^{1-\alpha} = C|\lambda|^{-\alpha/(2-\rho)} \|f\|_{L^2}, \end{aligned}$$

which implies (4.9). Analogously (4.11), (3.3) and (3.6) yield

$$\|\dot{m}u\|_{L^2} \le C\big(|\lambda|^{-1/2} \|f\|_{(H^1)^*}\big)^\alpha \|f\|_{(H^1)^*}^{1-\alpha} = C|\lambda|^{-\alpha/2} \|f\|_{(H^1)^*},$$

which is (4.10). □

Lemma 4.2. *The following inequalities hold for all $t \in [0,T]$ and $\lambda \in \Sigma$ with $|\lambda| \geq 1$:*

$$\left\| \frac{\partial}{\partial t} \big(\lambda + A(t)\big)^{-1} \right\|_{\mathscr{L}(L^2)} \leq C\big[|\lambda|^{(1-\rho-\alpha)/(2-\rho)} + |\lambda|^{-1/2}\big], \tag{4.12}$$

$$\left\| \frac{\partial}{\partial t} \big(\lambda + \widetilde{A}(t)\big)^{-1} \right\|_{\mathscr{L}((H^1)^*, L^2)} \leq C|\lambda|^{(1-\rho)/(2-\rho)-\alpha/2}, \tag{4.13}$$

$$\left\| \frac{\partial}{\partial t} \big(\lambda + A(t)\big)^{-1} \right\|_{\mathscr{L}(L^2,(H^1)^*)} \leq C|\lambda|^{-\alpha/(2-\rho)}, \tag{4.14}$$

$$\left\| \frac{\partial}{\partial t} \big(\lambda + \widetilde{A}(t)\big)^{-1} \right\|_{\mathscr{L}((H^1)^*)} \leq C|\lambda|^{-\alpha/2}. \tag{4.15}$$

Proof. First note that

$$\begin{aligned}
\frac{\partial}{\partial t} \big(\lambda + \widetilde{A}(t)\big)^{-1} &= \frac{\partial}{\partial t}\big[M(t)(\lambda M(t) + \widetilde{L}(t))^{-1}\big] \\
&= \dot{M}(t)\big(\lambda M(t) + \widetilde{L}(t)\big)^{-1} \\
&\quad - M(t)\big(\lambda M(t) + \widetilde{L}(t)\big)^{-1}\big(\lambda \dot{M}(t) + \dot{\widetilde{L}}(t)\big)\big(\lambda M(t) + \widetilde{L}(t)\big)^{-1} \\
&= \sum_{i=1}^{3} \tilde{J}_i(t,\lambda),
\end{aligned} \tag{4.16}$$

where

$$\begin{aligned}
\tilde{J}_1(t,\lambda) &= \dot{M}(t)\big(\lambda M(t) + \widetilde{L}(t)\big)^{-1}, \\
\tilde{J}_2(t,\lambda) &= -\lambda M(t)\big(\lambda M(t) + \widetilde{L}(t)\big)^{-1}\dot{M}(t)\big(\lambda M(t) + \widetilde{L}(t)\big)^{-1}, \\
\tilde{J}_3(t,\lambda) &= -M(t)\big(\lambda M(t) + \widetilde{L}(t)\big)^{-1}\dot{\widetilde{L}}(t)\big(\lambda M(t) + \widetilde{L}(t)\big)^{-1}.
\end{aligned}$$

Then by virtue of Lemma 4.1, (3.1), (3.3), (3.8) we have

$$\big\|\tilde{J}_1(t,\lambda)f\big\|_{L^2} \leq C \begin{cases} |\lambda|^{-\alpha/(2-\rho)}\|f\|_{L^2}, \\ |\lambda|^{-\alpha/2}\|f\|_{(H^1)^*}. \end{cases} \tag{4.17}$$

$$\begin{aligned}\big\|\tilde J_2(t,\lambda)f\big\|_{L^2} &\le C|\lambda|^{(1-\rho)/(2-\rho)}\big\|\tilde J_1(t,\lambda)f\big\|_{L^2}\\ &\le C\begin{cases}|\lambda|^{(1-\rho-\alpha)/(2-\rho)}\|f\|_{L^2},\\ |\lambda|^{(1-\rho)/(2-\rho)-\alpha/2}\|f\|_{(H^1)^*}.\end{cases}\end{aligned} \tag{4.18}$$

$$\begin{aligned}\big\|\tilde J_3(t,\lambda)f\big\|_{L^2} &\le C|\lambda|^{-1/2}\big\|\dot{\widetilde L}(t)(\lambda M(t)+\widetilde L(t))^{-1}f\big\|_{(H^1)^*}\\ &\le C|\lambda|^{-1/2}\|f\|_{(H^1)^*}.\end{aligned} \tag{4.19}$$

In view of (4.16), (4.17), (4.18), (4.19), the inequality

$$\frac{1-\rho}{2-\rho}-\frac{\alpha}{2}>-\frac{\alpha}{2}>-\frac{1}{2},$$

which follows from (2.11), and the assumption $|\lambda|\ge 1$, the following inequality holds for $f\in L^2(\Omega)$ or $f\in H^1(\Omega)^*$:

$$\left\|\frac{\partial}{\partial t}\big(\lambda+\widetilde A(t)\big)^{-1}f\right\|_{L^2}\le C\begin{cases}\big[|\lambda|^{(1-\rho-\alpha)/(2-\rho)}+|\lambda|^{-1/2}\big]\|f\|_{L^2},\\ |\lambda|^{(1-\rho)/(2-\rho)-\alpha/2}\|f\|_{(H^1)^*}.\end{cases}$$

This implies (4.12) and (4.13). With the aid of

$$\big\|\tilde J_1(t,\lambda)f\big\|_{(H^1)^*}\le C\big\|\tilde J_1(t,\lambda)f\big\|_{L^2},\quad \big\|\tilde J_2(t,\lambda)f\big\|_{(H^1)^*}\le C\big\|\tilde J_1(t,\lambda)f\big\|_{(H^1)^*},$$

$$\big\|\tilde J_3(t,\lambda)f\big\|_{(H^1)^*}\le C|\lambda|^{-1}\big\|\dot{\widetilde L}(t)(\lambda M(t)+\widetilde L(t))^{-1}f\big\|_{(H^1)^*}\le C|\lambda|^{-1}\|f\|_{(H^1)^*}$$

(cf. (3.2), (3.8)) and (4.17) one obtains for $f\in L^2(\Omega)$ or $f\in H^1(\Omega)^*$

$$\left\|\frac{\partial}{\partial t}\big(\lambda+\widetilde A(t)\big)^{-1}f\right\|_{(H^1)^*}\le C\begin{cases}|\lambda|^{-\alpha/(2-\rho)}\|f\|_{L^2},\\ |\lambda|^{-\alpha/2}\|f\|_{(H^1)^*}.\end{cases}$$

Thus (4.14) and (4.15) are established, and the proof is complete. □

LEMMA 4.3. *The following estimates hold for all $0\le s<t\le T$:*

$$\|\Phi_1(t,s)\|_{\mathscr L(L^2)}\le C\big[(t-s)^{(2\rho+\alpha-3)/(2-\rho)}+(t-s)^{-1/2}\big], \tag{4.20}$$

$$\|\Phi(t,s)\|_{\mathscr L(L^2)}\le C\big[(t-s)^{(2\rho+\alpha-3)/(2-\rho)}+(t-s)^{-1/2}\big]. \tag{4.21}$$

Proof. In (4.3) choose Γ as the contour consisting of two half lines $\{re^{\pm\theta_0}; T(t-s)^{-1} \le r < \infty\}$ and an arc $\{T(t-s)^{-1}e^{i\psi}; -\theta_0 \le \psi \le \theta_0\}$. Then, inequality (4.12) holds on Γ, since $\Gamma \subset \Sigma \cap \{\lambda \in \boldsymbol{C}; |\lambda| \ge 1\}$, and (4.20) is easily shown by this choice of Γ. Inequality (4.21) is an easy consequence of (4.2) and (4.20). $\square$

Proposition 4.1. *The following estimates hold for all* $0 \le s < t \le T$:

$$\|W(t,s)\|_{\mathscr{L}(L^2)} \le C\big[(t-s)^{(2\rho+\alpha-2)/(2-\rho)} + (t-s)^{\rho/[2(2-\rho)]}\big], \tag{4.22}$$

$$\|U(t,s)\|_{\mathscr{L}(L^2)} \le C(t-s)^{(\rho-1)/(2-\rho)}. \tag{4.23}$$

Proof. Since $(\rho-1)/(2-\rho) > -1$ and $(2\rho+\alpha-3)/(2-\rho) > -1$ in view of (2.11), inequality (4.22) follows from (3.14) and (4.21). Inequality (4.23) is a simple consequence of (3.14) and (4.22). $\square$

Lemma 4.4. *The following estimates hold for all* $0 \le t, s \le T$ *and* $\lambda \in \Sigma$ *with* $|\lambda| \ge 1$:

$$\begin{aligned}&\left\|\frac{\partial}{\partial t}(\lambda + A(t))^{-1} - \frac{\partial}{\partial s}(\lambda + A(s))^{-1}\right\|_{\mathscr{L}(L^2)}\\ &\qquad \le C\big[|t-s|^{\gamma}|\lambda|^{(1-\rho)/(2-\rho)} + |t-s||\lambda|^{(3-2\rho-\alpha)/(2-\rho)}\big].\end{aligned} \tag{4.24}$$

Proof. Let $J_i(t,\lambda)$ be the restriction of $\tilde{J}_i(t,\lambda)$ to $L^2(\Omega)$ for $i = 1,2,3$:

$$\begin{aligned}
J_1(t,\lambda) &= \dot{M}(t)(\lambda M(t) + L(t))^{-1},\\
J_2(t,\lambda) &= -\lambda M(t)\big(\lambda M(t) + \widetilde{L}(t)\big)^{-1}\dot{M}(t)(\lambda M(t) + L(t))^{-1},\\
J_3(t,\lambda) &= -M(t)\big(\lambda M(t) + \widetilde{L}(t)\big)^{-1}\dot{\widetilde{L}}(t)(\lambda M(t) + L(t))^{-1}.
\end{aligned}$$

To show (4.24) we begin by estimating the increments of the first term in the last side of (4.16) with $\tilde{J}_i(t,\lambda)$ being replaced by $J_i(t,\lambda)$, $i = 1,2,3$. For this purpose we consider the identity:

$$\begin{aligned}J_1(t,\lambda) - J_1(s,\lambda) &= \big(\dot{M}(t) - \dot{M}(s)\big)(\lambda M(t) + L(t))^{-1}\\ &\quad + \dot{M}(s)\big[(\lambda M(t) + L(t))^{-1} - (\lambda M(s) + L(s))^{-1}\big].\end{aligned} \tag{4.25}$$

Inequalities (2.13) and (3.6) yield

$$\left\|(\dot{M}(t)-\dot{M}(s))(\lambda M(t)+L(t))^{-1}\right\|_{\mathscr{L}(L^2)} \le C|t-s|^{\gamma}. \tag{4.26}$$

Analogously

$$\left\|(M(t)-M(s))(\lambda M(t)+L(t))^{-1}\right\|_{\mathscr{L}(L^2)} \le C|t-s|. \tag{4.27}$$

One has

$$\begin{aligned}
&\dot{M}(s)\big[(\lambda M(t)+L(t))^{-1}-(\lambda M(s)+L(s))^{-1}\big]\\
&\quad=\dot{M}(s)\big[(\lambda M(t)+\widetilde{L}(t))^{-1}-(\lambda M(s)+\widetilde{L}(s))^{-1}\big]\big|_{L^2}\\
&\quad=\dot{M}(s)\big(\lambda M(s)+\widetilde{L}(s)\big)^{-1}\big[\lambda M(s)+\widetilde{L}(s)-\lambda M(t)-\widetilde{L}(t)\big](\lambda M(t)+L(t))^{-1}\\
&\quad=\lambda\dot{M}(s)(\lambda M(s)+L(s))^{-1}(M(s)-M(t))(\lambda M(t)+L(t))^{-1}\\
&\qquad+\dot{M}(s)\big(\lambda M(s)+\widetilde{L}(s)\big)^{-1}\big(\widetilde{L}(s)-\widetilde{L}(t)\big)(\lambda M(t)+L(t))^{-1}.
\end{aligned} \tag{4.28}$$

In view of (4.9) and (4.27) one observes

$$\begin{aligned}
&\left\|\lambda\dot{M}(s)(\lambda M(s)+L(s))^{-1}(M(s)-M(t))(\lambda M(t)+L(t))^{-1}\right\|_{\mathscr{L}(L^2)}\\
&\quad\le|\lambda|\left\|\dot{M}(s)(\lambda M(s)+L(s))^{-1}\right\|_{\mathscr{L}(L^2)}\left\|(M(s)-M(t))(\lambda M(t)+L(t))^{-1}\right\|_{\mathscr{L}(L^2)}\\
&\quad\le C|\lambda|^{1-\alpha/(2-\rho)}|t-s|.
\end{aligned} \tag{4.29}$$

By virtue of (4.10), (2.9) and (3.6)

$$\begin{aligned}
&\left\|\dot{M}(s)(\lambda M(s)+\widetilde{L}(s))^{-1}(\widetilde{L}(s)-\widetilde{L}(t))(\lambda M(t)+L(t))^{-1}\right\|_{\mathscr{L}(L^2)}\\
&\quad\le\left\|\dot{M}(s)(\lambda M(s)+\widetilde{L}(s))^{-1}\right\|_{\mathscr{L}((H^1)^*,L^2)}\\
&\qquad\times\left\|\widetilde{L}(s)-\widetilde{L}(t)\right\|_{\mathscr{L}(H^1,(H^1)^*)}\left\|(\lambda M(t)+L(t))^{-1}\right\|_{\mathscr{L}(L^2,H^1)}\\
&\quad\le C|\lambda|^{-\alpha/2}|t-s|.
\end{aligned} \tag{4.30}$$

It follows from (4.28), (4.29) and (4.30) that

$$\begin{aligned}
&\left\|\dot{M}(s)\big[(\lambda M(t)+L(t))^{-1}-(\lambda M(s)+L(s))^{-1}\big]\right\|_{\mathscr{L}(L^2)}\\
&\quad\le C|\lambda|^{1-\alpha/(2-\rho)}|t-s|+C|\lambda|^{-\alpha/2}|t-s|\le C|\lambda|^{1-\alpha/(2-\rho)}|t-s|.
\end{aligned} \tag{4.31}$$

From (4.25), (4.26) and (4.31) one obtains the following inequality

$$\|J_1(t,\lambda)-J_1(s,\lambda)\|_{\mathscr{L}(L^2)} \le C\big[|t-s|^\gamma + |t-s||\lambda|^{1-\alpha/(2-\rho)}\big]. \tag{4.32}$$

Next we consider the increments of the second term in the last side of (4.16):

$$\begin{aligned} &J_2(t,\lambda)-J_2(s,\lambda)\\ &\quad= -\lambda\big[M(t)(\lambda M(t)+L(t))^{-1} - M(s)(\lambda M(s)+L(s))^{-1}\big]J_1(t,\lambda)\\ &\qquad - \lambda M(s)(\lambda M(s)+L(s))^{-1}\big[J_1(t,\lambda)-J_1(s,\lambda)\big]. \end{aligned} \tag{4.33}$$

From (4.12) one deduces

$$\begin{aligned} &\big\|M(t)(\lambda M(t)+L(t))^{-1} - M(s)(\lambda M(s)+L(s))^{-1}\big\|_{\mathscr{L}(L^2)}\\ &= \Big\|\int_s^t \frac{\partial}{\partial r}(\lambda + A(r))^{-1}\,dr\Big\|_{\mathscr{L}(L^2)} \le C|t-s|\big[|\lambda|^{(1-\rho-\alpha)/(2-\rho)} + |\lambda|^{-1/2}\big]. \end{aligned} \tag{4.34}$$

With the aid of (4.33), (4.34), (4.9) (or (4.17)), (3.1) and (4.32) one obtains

$$\begin{aligned} &\|J_2(t,\lambda)-J_2(s,\lambda)\|_{\mathscr{L}(L^2)}\\ &\quad\le C|t-s||\lambda|^{1-\alpha/(2-\rho)}\big[|\lambda|^{(1-\rho-\alpha)/(2-\rho)} + |\lambda|^{-1/2}\big]\\ &\qquad + C|\lambda|^{1-1/(2-\rho)}\big[|t-s|^\gamma + |t-s||\lambda|^{1-\alpha/(2-\rho)}\big]\\ &\quad= C|t-s|\big[|\lambda|^{(3-2\rho-2\alpha)/(2-\rho)} + |\lambda|^{(2-\rho-2\alpha)/[2(2-\rho)]}\big]\\ &\qquad + C\big[|t-s|^\gamma|\lambda|^{(1-\rho)/(2-\rho)} + |t-s||\lambda|^{(3-2\rho-\alpha)/(2-\rho)}\big]\\ &\quad\le C\big[|t-s|^\gamma|\lambda|^{(1-\rho)/(2-\rho)} + |t-s||\lambda|^{(3-2\rho-\alpha)/(2-\rho)}\big]. \end{aligned} \tag{4.35}$$

Consider now the following identity concerning the last term of (4.16):

$$\begin{aligned} &J_3(t,\lambda)-J_3(s,\lambda)\\ &\quad= -\big[M(t)(\lambda M(t)+\widetilde{L}(t))^{-1} - M(s)(\lambda M(s)+\widetilde{L}(s))^{-1}\big]\dot{\widetilde{L}}(t)(\lambda M(t)+L(t))^{-1}\\ &\qquad - M(s)(\lambda M(s)+\widetilde{L}(s))^{-1}\big[\dot{\widetilde{L}}(t)(\lambda M(t)+L(t))^{-1} - \dot{\widetilde{L}}(s)(\lambda M(s)+L(s))^{-1}\big]. \end{aligned} \tag{4.36}$$

With the aid of (4.13) one gets

$$
\begin{aligned}
&\big\|M(t)(\lambda M(t)+\widetilde{L}(t))^{-1}-M(s)(\lambda M(s)+\widetilde{L}(s))^{-1}\big\|_{\mathscr{L}((H^1)^*,L^2)} \\
&\quad=\bigg\|\int_s^t \frac{\partial}{\partial r}\big(\lambda+\widetilde{A}(r)\big)^{-1}dr\bigg\|_{\mathscr{L}((H^1)^*,L^2)} \le C|t-s||\lambda|^{(1-\rho)/(2-\rho)-\alpha/2}. \qquad (4.37)
\end{aligned}
$$

Next we consider the identities

$$
\begin{aligned}
&\dot{\widetilde{L}}(t)\big(\lambda M(t)+\widetilde{L}(t)\big)^{-1}-\dot{\widetilde{L}}(s)\big(\lambda M(s)+\widetilde{L}(s)\big)^{-1} \\
&\quad=\dot{\widetilde{L}}(t)\widetilde{L}(t)^{-1}\widetilde{L}(t)\big(\lambda M(t)+\widetilde{L}(t)\big)^{-1}-\dot{\widetilde{L}}(s)\widetilde{L}(s)^{-1}\widetilde{L}(s)\big(\lambda M(s)+\widetilde{L}(s)\big)^{-1} \\
&\quad=\big[\dot{\widetilde{L}}(t)\widetilde{L}(t)^{-1}-\dot{\widetilde{L}}(s)\widetilde{L}(s)^{-1}\big]\widetilde{L}(t)\big(\lambda M(t)+\widetilde{L}(t)\big)^{-1} \\
&\qquad+\dot{\widetilde{L}}(s)\widetilde{L}(s)^{-1}\big[\widetilde{L}(t)(\lambda M(t)+\widetilde{L}(t))^{-1}-\widetilde{L}(s)(\lambda M(s)+\widetilde{L}(s))^{-1}\big] \\
&\quad=\big[\dot{\widetilde{L}}(t)\widetilde{L}(t)^{-1}-\dot{\widetilde{L}}(s)\widetilde{L}(s)^{-1}\big]\widetilde{L}(t)\big(\lambda M(t)+\widetilde{L}(t)\big)^{-1} \\
&\qquad+\dot{\widetilde{L}}(s)\widetilde{L}(s)^{-1}\big[I-\lambda M(t)(\lambda M(t)+\widetilde{L}(t))^{-1}-I+\lambda M(s)(\lambda M(s)+\widetilde{L}(s))^{-1}\big] \\
&\quad=\big[\dot{\widetilde{L}}(t)\widetilde{L}(t)^{-1}-\dot{\widetilde{L}}(s)\widetilde{L}(s)^{-1}\big]\widetilde{L}(t)\big(\lambda M(t)+\widetilde{L}(t)\big)^{-1} \\
&\qquad-\lambda\dot{\widetilde{L}}(s)\widetilde{L}(s)^{-1}\big[M(t)(\lambda M(t)+\widetilde{L}(t))^{-1}-M(s)(\lambda M(s)+\widetilde{L}(s))^{-1}\big]. \qquad (4.38)
\end{aligned}
$$

As is easily seen

$$
\big\|\dot{\widetilde{L}}(t)\widetilde{L}(t)^{-1}-\dot{\widetilde{L}}(s)\widetilde{L}(s)^{-1}\big\|_{\mathscr{L}((H^1)^*)}\le C|t-s|^{\gamma}. \qquad (4.39)
$$

With the aid of (4.14)

$$
\begin{aligned}
&\big\|M(t)(\lambda M(t)+L(t))^{-1}-M(s)(\lambda M(s)+L(s))^{-1}\big\|_{\mathscr{L}(L^2,(H^1)^*)} \\
&\quad=\bigg\|\int_s^t \frac{\partial}{\partial r}(\lambda+A(r))^{-1}dr\bigg\|_{\mathscr{L}(L^2,(H^1)^*)}\le C|t-s||\lambda|^{-\alpha/(2-\rho)}. \qquad (4.40)
\end{aligned}
$$

Then equality (4.38) and inequalities (4.39), (3.7), (4.40) yield

$$
\begin{aligned}
&\big\|\dot{\widetilde{L}}(t)(\lambda M(t)+\widetilde{L}(t))^{-1}-\dot{\widetilde{L}}(s)(\lambda M(s)+\widetilde{L}(s))^{-1}\big\|_{\mathscr{L}(L^2,(H^1)^*)}\\
&\quad\le \big\|\dot{\widetilde{L}}(t)\widetilde{L}(t)^{-1}-\dot{\widetilde{L}}(s)\widetilde{L}(s)^{-1}\big\|_{\mathscr{L}((H^1)^*)}\big\|\widetilde{L}(t)(\lambda M(t)+\widetilde{L}(t))^{-1}\big\|_{\mathscr{L}(L^2,(H^1)^*)}\\
&\qquad+|\lambda|\big\|\dot{\widetilde{L}}(s)\widetilde{L}(s)^{-1}\big\|_{\mathscr{L}((H^1)^*)}\\
&\qquad\times\big\|M(t)(\lambda M(t)+\widetilde{L}(t))^{-1}-M(s)(\lambda M(s)+\widetilde{L}(s))^{-1}\big\|_{\mathscr{L}(L^2,(H^1)^*)}\\
&\quad\le C|t-s|^\gamma+C|t-s||\lambda|^{1-\alpha/(2-\rho)}. \qquad (4.41)
\end{aligned}
$$

It follows from (4.36), (4.37), (3.9), (3.3), (4.41) that

$$
\begin{aligned}
&\big\|J_3(t,\lambda)-J_3(s,\lambda)\big\|_{\mathscr{L}(L^2)}\\
&\quad\le \big\|M(t)(\lambda M(t)+\widetilde{L}(t))^{-1}-M(s)(\lambda M(s)+\widetilde{L}(s))^{-1}\big\|_{\mathscr{L}((H^1)^*,L^2)}\\
&\qquad\times\big\|\dot{\widetilde{L}}(t)(\lambda M(t)+L(t))^{-1}\big\|_{\mathscr{L}(L^2,(H^1)^*)}\\
&\qquad+\big\|M(s)(\lambda M(s)+\widetilde{L}(s))^{-1}\big\|_{\mathscr{L}((H^1)^*,L^2)}\\
&\qquad\times\big\|\dot{\widetilde{L}}(t)(\lambda M(t)+L(t))^{-1}-\dot{\widetilde{L}}(s)(\lambda M(s)+L(s))^{-1}\big\|_{\mathscr{L}(L^2,(H^1)^*)}\\
&\quad\le C|t-s||\lambda|^{(1-\rho)/(2-\rho)-\alpha/2}+C|\lambda|^{-1/2}\big[|t-s|^\gamma+|t-s||\lambda|^{1-\alpha/(2-\rho)}\big]\\
&\quad\le C\big[|t-s|^\gamma|\lambda|^{-1/2}+|t-s||\lambda|^{1/2-\alpha/(2-\rho)}\big], \qquad (4.42)
\end{aligned}
$$

since

$$\left(\frac{1}{2}-\frac{\alpha}{2-\rho}\right)-\left(\frac{1-\rho}{2-\rho}-\frac{\alpha}{2}\right)=\frac{\rho(1-\alpha)}{2(2-\rho)}>0.$$

The desired inequality (4.24) follows from (4.16), (4.32), (4.35), (4.42) and

$$\frac{3-2\rho-\alpha}{2-\rho}-\left(\frac{1}{2}-\frac{\alpha}{2-\rho}\right)=\frac{4-3\rho}{2(2-\rho)}>0. \qquad \square$$

Lemma 4.5. *The following inequality holds for all* $0\le s<\tau<t\le T$:

$$
\begin{aligned}
&\|\Phi_1(t,s)-\Phi_1(\tau,s)\|_{\mathscr{L}(L^2)}\\
&\quad\le C\left\{(t-\tau)^\gamma(t-s)^{(2\rho-3)/(2-\rho)}+\frac{t-\tau}{t-s}\big[(\tau-s)^{(2\rho+\alpha-3)/(2-\rho)}+(\tau-s)^{-1/2}\big]\right\}.
\end{aligned}
$$

PROOF. In view of (4.3) one obtains

$$\begin{aligned}
&\Phi_1(t,s)-\Phi_1(\tau,s)\\
&\quad=-\frac{1}{2\pi i}\int_\Gamma e^{\lambda(t-s)}\left[\frac{\partial}{\partial t}(\lambda+A(t))^{-1}-\frac{\partial}{\partial\tau}(\lambda+A(\tau))^{-1}\right]d\lambda\\
&\quad\quad-\frac{1}{2\pi i}\int_\Gamma\big[e^{\lambda(t-s)}-e^{\lambda(\tau-s)}\big]\frac{\partial}{\partial\tau}(\lambda+A(\tau))^{-1}d\lambda=:\sum_{j=1}^{2}I_j(t,\tau,s),
\end{aligned}\tag{4.43}$$

where Γ is a smooth contour in Σ connecting $\infty e^{-i\theta_0}$ and $\infty e^{i\theta_0}$. Using the analiticity of the integrands and deforming the path Γ to the one in the proof of Lemma 4.3 and using Lemma 4.4, we obtain without difficulty

$$\begin{aligned}
&\|I_1(t,\tau,s)\|_{\mathscr{L}(L^2)}\\
&\quad\le C\big[(t-\tau)^\gamma(t-s)^{(2\rho-3)/(2-\rho)}+(t-\tau)(t-s)^{(3\rho+\alpha-5)/(2-\rho)}\big]\\
&\quad\le C\left[(t-\tau)^\gamma(t-s)^{(2\rho-3)/(2-\rho)}+\frac{t-\tau}{t-s}(\tau-s)^{(2\rho+\alpha-3)/(2-\rho)}\right].
\end{aligned}\tag{4.44}$$

A change of the order of integration yields

$$\begin{aligned}
I_2(t,\tau,s)&=-\frac{1}{2\pi i}\int_\Gamma\left(\int_\tau^t\frac{\partial}{\partial\sigma}e^{\lambda(\sigma-s)}d\sigma\right)\frac{\partial}{\partial\tau}(\lambda+A(\tau))^{-1}d\lambda\\
&=-\frac{1}{2\pi i}\int_\tau^t\left(\int_\Gamma\lambda e^{\lambda(\sigma-s)}\frac{\partial}{\partial\tau}(\lambda+A(\tau))^{-1}d\lambda\right)d\sigma.
\end{aligned}\tag{4.45}$$

For a fixed $\sigma\in(\tau,t)$, deforming Γ to the contour consisting of two half lines $\{re^{\pm i\theta_0};\ T(\sigma-s)^{-1}\le r<\infty\}$ and of an arc $\{T(\sigma-s)^{-1}e^{i\psi};\ -\theta_0\le\psi\le\theta_0\}$, we deduce that

$$\begin{aligned}
&\left\|\int_\Gamma\lambda e^{\lambda(\sigma-s)}\frac{\partial}{\partial\tau}(\lambda+A(\tau))^{-1}d\lambda\right\|_{\mathscr{L}(L^2)}\\
&\quad\le C\big[(\sigma-s)^{(3\rho+\alpha-5)/(2-\rho)}+(\sigma-s)^{-3/2}\big].
\end{aligned}\tag{4.46}$$

It follows from (4.45) and (4.46) that

$$\|I_2(t,\tau,s)\|_{\mathscr{L}(L^2)}\le C\int_\tau^t\big[(\sigma-s)^{(3\rho+\alpha-5)/(2-\rho)}+(\sigma-s)^{-3/2}\big]d\sigma.\tag{4.47}$$

Observe now that $0 < (\tau-s)/(t-s) < 1$ and, according to (2.11), one has $\alpha+\rho > 1$, implying $0 < (3-2\rho-\alpha)/(2-\rho) < 1$. Therefore

$$\begin{aligned}
&\int_\tau^t (\sigma-s)^{(3\rho+\alpha-5)/(2-\rho)}d\sigma \\
&\quad = \frac{(\tau-s)^{(2\rho+\alpha-3)/(2-\rho)} - (t-s)^{(2\rho+\alpha-3)/(2-\rho)}}{(3-2\rho-\alpha)/(2-\rho)} \\
&\quad = \frac{(\tau-s)^{(2\rho+\alpha-3)/(2-\rho)}}{(3-2\rho-\alpha)/(2-\rho)}\left[1-\left(\frac{\tau-s}{t-s}\right)^{(3-2\rho-\alpha)/(2-\rho)}\right] \\
&\quad < \frac{(\tau-s)^{(2\rho+\alpha-3)/(2-\rho)}}{(3-2\rho-\alpha)/(2-\rho)}\left(1-\frac{\tau-s}{t-s}\right) = \frac{(\tau-s)^{(2\rho+\alpha-3)/(2-\rho)}}{(3-2\rho-\alpha)/(2-\rho)}\frac{t-\tau}{t-s}. \qquad (4.48)
\end{aligned}$$

Analogously

$$\int_\tau^t (\sigma-s)^{-3/2}d\sigma < 2(\tau-s)^{-1/2}\frac{t-\tau}{t-s}. \tag{4.49}$$

It follows from (4.47), (4.48) and (4.49) that

$$\|I_2(t,\tau,s)\|_{\mathscr{L}(L^2)} \le C\frac{t-\tau}{t-s}\big[(\tau-s)^{(2\rho+\alpha-3)/(2-\rho)} + (\tau-s)^{-1/2}\big]. \tag{4.50}$$

The assertion of the lemma follows from (4.43), (4.44) and (4.50). $\square$

Proposition 4.2. *For $0 \le s < t < T$, $\int_s^t e^{-(t-\tau)A(t)}\Phi_1(\tau,s)d\tau$ is differentiable in t, and*

$$\begin{aligned}
\frac{\partial}{\partial t}\int_s^t e^{-(t-\tau)A(t)}\Phi_1(\tau,s)d\tau &= -\int_s^t \Phi_1(t,\tau)\Phi_1(\tau,s)d\tau \\
&\quad - \int_s^t \frac{\partial}{\partial\tau}e^{-(t-\tau)A(t)}[\Phi_1(\tau,s)-\Phi_1(t,s)]d\tau + e^{-(t-s)A(t)}\Phi_1(t,s), \qquad (4.51)
\end{aligned}$$

$$\begin{aligned}
&\left\|\frac{\partial}{\partial t}\int_s^t e^{-(t-\tau)A(t)}\Phi_1(\tau,s)d\tau\right\|_{\mathscr{L}(L^2)} \\
&\quad \le C\big[(t-s)^{\gamma+(3\rho-4)/(2-\rho)} + (t-s)^{(3\rho+\alpha-4)/(2-\rho)} + (t-s)^{(3\rho-4)/[2(2-\rho)]}\big]. \qquad (4.52)
\end{aligned}$$

PROOF. For $s < t - \varepsilon < t$ one deduces by the usual manner (cf. (4.3))

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_s^{t-\varepsilon} e^{-(t-\tau)A(t)}\Phi_1(\tau,s)d\tau \\
&\quad = e^{-\varepsilon A(t)}\big[\Phi_1(t-\varepsilon,s)-\Phi_1(t,s)\big] - \int_s^{t-\varepsilon}\Phi_1(t,\tau)\Phi_1(\tau,s)d\tau \\
&\qquad - \int_s^{t-\varepsilon}\frac{\partial}{\partial\tau}e^{-(t-\tau)A(t)}\big[\Phi_1(\tau,s)-\Phi_1(t,s)\big]d\tau + e^{-(t-s)A(t)}\Phi_1(t,s). \qquad (4.53)
\end{aligned}$$

In view of (3.14) and Lemma 4.5

$$\begin{aligned}
&\big\|e^{-\varepsilon A(t)}\big[\Phi_1(t-\varepsilon,s)-\Phi_1(t,s)\big]\big\|_{\mathscr{L}(L^2)} \le C\varepsilon^{(\rho-1)/(2-\rho)} \\
&\quad \times\left\{\varepsilon^\gamma(t-s)^{(2\rho-3)/(2-\rho)} + \frac{\varepsilon}{t-s}\big[(t-\varepsilon-s)^{(2\rho+\alpha-3)/(2-\rho)} + (t-\varepsilon-s)^{-1/2}\big]\right\} \\
&\quad \to 0 \qquad (4.54)
\end{aligned}$$

as $\varepsilon \to +0$, due to assumption (2.11). It follows from (4.20) that

$$\begin{aligned}
&\int_s^t \|\Phi_1(t,\tau)\Phi_1(\tau,s)\|_{\mathscr{L}(L^2)}d\tau \\
&\quad \le C\int_s^t \big[(t-\tau)^{(2\rho+\alpha-3)/(2-\rho)} + (t-\tau)^{-1/2}\big] \\
&\qquad \times\big[(\tau-s)^{(2\rho+\alpha-3)/(2-\rho)} + (\tau-s)^{-1/2}\big]d\tau \\
&\quad \le C\big[(t-s)^{(3\rho+2\alpha-4)/(2-\rho)} + (t-s)^{(3\rho+2\alpha-4)/[2(2-\rho)]} + 1\big], \qquad (4.55)
\end{aligned}$$

and from (3.15) and Lemma 4.5 that

$$\begin{aligned}
&\int_s^t \left\|\frac{\partial}{\partial\tau}e^{-(t-\tau)A(t)}\big[\Phi_1(\tau,s)-\Phi_1(t,s)\big]\right\|_{\mathscr{L}(L^2)}d\tau \\
&\le C\int_s^t (t-\tau)^{(2\rho-3)/(2-\rho)} \\
&\quad \times\left\{(t-\tau)^\gamma(t-s)^{(2\rho-3)/(2-\rho)} + \frac{t-\tau}{t-s}\big[(\tau-s)^{(2\rho+\alpha-3)/(2-\rho)} + (\tau-s)^{-1/2}\big]\right\}d\tau \\
&\le C\big[(t-s)^{\gamma+(3\rho-4)/(2-\rho)} + (t-s)^{(3\rho+\alpha-4)/(2-\rho)} + (t-s)^{(3\rho-4)/[2(2-\rho)]}\big]. \qquad (4.56)
\end{aligned}$$

The assertion of the proposition follows from (4.53), (4.54), (4.55), (4.56), (3.14) and (4.20). □

As is easily seen (cf. (4.2) and (4.7)),

$$W(t,s) = \int_s^t e^{-(t-\tau)A(t)}\Phi_1(\tau,s)d\tau + \int_s^t \Big(\int_\sigma^t e^{-(t-\tau)A(t)}\Phi_1(\tau,\sigma)d\tau\Big)\Phi(\sigma,s)d\sigma. \tag{4.57}$$

Analogously to (4.22) one has

$$\Big\| \int_s^t e^{-(t-\tau)A(t)}\Phi_1(\tau,s)d\tau \Big\|_{\mathscr{L}(L^2)} \leq C\big[(t-s)^{(2\rho+\alpha-2)/(2-\rho)} + (t-s)^{\rho/[2(2-\rho)]}\big]. \tag{4.58}$$

Hence using Proposition 4.2, and noting that assumption (2.11) implies

$$\frac{2\rho+\alpha-2}{2-\rho} > 0, \quad \gamma + \frac{3\rho-4}{2-\rho} > -1, \quad \frac{3\rho+\alpha-4}{2-\rho} > -1 \tag{4.59}$$

and that

$$\frac{3\rho-4}{2(2-\rho)} = -1 + \frac{\rho}{2(2-\rho)} > -1, \tag{4.60}$$

we obtain from (4.57) and (4.58) that $W(t,s)$ is differentiable with respect to t in (s,T) and

$$\frac{\partial}{\partial t}W(t,s) = \frac{\partial}{\partial t}\int_s^t e^{-(t-\tau)A(t)}\Phi_1(\tau,s)d\tau + \int_s^t \Big(\frac{\partial}{\partial t}\int_\sigma^t e^{-(t-\tau)A(t)}\Phi_1(\tau,\sigma)d\tau\Big)\Phi(\sigma,s)d\sigma. \tag{4.61}$$

Owing to (4.52) and (4.21) it is easily seen that

$$\left\| \frac{\partial}{\partial t} W(t,s) \right\|_{\mathscr{L}(L^2)} \le C\big[(t-s)^{\gamma+(3\rho-4)/(2-\rho)} \\ + (t-s)^{(3\rho+\alpha-4)/(2-\rho)} + (t-s)^{(3\rho-4)/[2(2-\rho)]}\big]. \quad (4.62)$$

On the other hand, (4.3), (3.15) and (4.20) imply

$$\left\| \frac{\partial}{\partial t} e^{-(t-s)A(t)} \right\|_{\mathscr{L}(L^2)} \le \left\| \frac{\partial}{\partial s} e^{-(t-s)A(t)} \right\|_{\mathscr{L}(L^2)} + \|\Phi_1(t,s)\|_{\mathscr{L}(L^2)} \\ \le C(t-s)^{(2\rho-3)/(2-\rho)}. \quad (4.63)$$

Thus we have proved (cf. (4.1)) that U is differentiable with respect to t in (s,T). Moreover, from (4.62) and (4.63) it follows that the inequality (4.6) holds.

STEP 2. Proof of $R(U(t,s)) \subset D(A(t))$ for $0 \le s < t \le T$ and (4.4). It suffices to show that

$$A(t)^{-1} \frac{\partial}{\partial t} U(t,s) = -U(t,s). \quad (4.64)$$

By virtue of (4.3) and (3.16)

$$A(t)^{-1} \frac{\partial}{\partial t} e^{-(t-s)A(t)} = -A(t)^{-1}\Phi_1(t,s) - A(t)^{-1} \frac{\partial}{\partial s} e^{-(t-s)A(t)} \\ = -A(t)^{-1}\Phi_1(t,s) - e^{-(t-s)A(t)}. \quad (4.65)$$

Let $0 < s < t-\varepsilon < T$. Since

$$\int_s^{t-\varepsilon} e^{-(t-\tau)A(t)} d\tau \\ = \int_s^{t-\varepsilon} \frac{\partial}{\partial \tau} e^{-(t-\tau)A(t)} A(t)^{-1} d\tau \\ = \big[e^{-\varepsilon A(t)} - e^{-(t-s)A(t)}\big] A(t)^{-1} \to \big[I - e^{-(t-s)A(t)}\big] A(t)^{-1} \quad \text{as } \varepsilon \to +0,$$

one has

$$\int_s^t e^{-(t-\tau)A(t)} d\tau = \big[I - e^{-(t-s)A(t)}\big] A(t)^{-1}. \quad (4.66)$$

In view of (4.51), (3.16) and (4.66)

$$
\begin{aligned}
&A(t)^{-1}\frac{\partial}{\partial t}\int_s^t e^{-(t-\tau)A(t)}\Phi_1(\tau,s)d\tau\\
&\quad= -A(t)^{-1}\int_s^t \Phi_1(t,\tau)\Phi_1(\tau,s)d\tau\\
&\qquad - A(t)^{-1}\int_s^t \frac{\partial}{\partial\tau}e^{-(t-\tau)A(t)}\big[\Phi_1(\tau,s)-\Phi_1(t,s)\big]d\tau + A(t)^{-1}e^{-(t-s)A(t)}\Phi_1(t,s)\\
&\quad= -A(t)^{-1}\int_s^t \Phi_1(t,\tau)\Phi_1(\tau,s)d\tau - \int_s^t e^{-(t-\tau)A(t)}\big[\Phi_1(\tau,s)-\Phi_1(t,s)\big]d\tau\\
&\qquad + A(t)^{-1}e^{-(t-s)A(t)}\Phi_1(t,s)\\
&\quad= -A(t)^{-1}\int_s^t \Phi_1(t,\tau)\Phi_1(\tau,s)d\tau - \int_s^t e^{-(t-\tau)A(t)}\Phi_1(\tau,s)d\tau\\
&\qquad + \big[I-e^{-(t-s)A(t)}\big]A(t)^{-1}\Phi_1(t,s) + A(t)^{-1}e^{-(t-s)A(t)}\Phi_1(t,s)\\
&\quad= -A(t)^{-1}\int_s^t \Phi_1(t,\tau)\Phi_1(\tau,s)d\tau - \int_s^t e^{-(t-\tau)A(t)}\Phi_1(\tau,s)d\tau + A(t)^{-1}\Phi_1(t,s).
\end{aligned}
\tag{4.67}
$$

Using (4.67) and changing suitably the order of integration one gets

$$
\begin{aligned}
&A(t)^{-1}\int_s^t \left(\frac{\partial}{\partial t}\int_\sigma^t e^{-(t-\tau)A(t)}\Phi_1(\tau,\sigma)d\tau\right)\Phi(\sigma,s)d\sigma\\
&\quad= \int_s^t \bigg[-A(t)^{-1}\int_\sigma^t \Phi_1(t,\tau)\Phi_1(\tau,\sigma)d\tau\\
&\qquad\qquad - \int_\sigma^t e^{-(t-\tau)A(t)}\Phi_1(\tau,\sigma)d\tau + A(t)^{-1}\Phi_1(t,\sigma)\bigg]\Phi(\sigma,s)d\sigma\\
&\quad= -A(t)^{-1}\int_s^t \Phi_1(t,\tau)d\tau\int_s^\tau \Phi_1(\tau,\sigma)\Phi(\sigma,s)d\sigma\\
&\qquad - \int_s^t e^{-(t-\tau)A(t)}d\tau\int_s^\tau \Phi_1(\tau,\sigma)\Phi(\sigma,s)d\sigma + A(t)^{-1}\int_s^t \Phi_1(t,\tau)\Phi(\tau,s)d\tau.
\end{aligned}
\tag{4.68}
$$

From (4.1), (4.2), (4.65), (4.61), (4.67) and (4.68) one easily derives (4.64).

STEP 3. Proof of (4.5).
We first show the following lemma.

LEMMA 4.6. *The following inequality holds for $0 \le s < t \le T$ and $\tau > 0$:*

$$\big\|e^{-\tau A(t)} - e^{-\tau A(s)}\big\|_{\mathscr{L}(L^2)} \le C(t-s)\big[\tau^{(2\rho+\alpha-3)/(2-\rho)} + \tau^{-1/2}\big].$$

PROOF. Due to (4.12) one easily concludes

$$\begin{aligned}\big\|e^{-\tau A(t)} - e^{-\tau A(s)}\big\|_{\mathscr{L}(L^2)} &= \left\|\frac{1}{2\pi i}\int_{\tau\Gamma} e^{\lambda\tau}\big[(\lambda + A(t))^{-1} - (\lambda + A(s))^{-1}\big]d\lambda\right\|_{\mathscr{L}(L^2)} \\ &= \left\|\frac{1}{2\pi i}\int_{\tau\Gamma} e^{\lambda\tau}\int_s^t \frac{\partial}{\partial r}(\lambda + A(r))^{-1} dr d\lambda\right\|_{\mathscr{L}(L^2)} \\ &\le C(t-s)\int_{\tau\Gamma} e^{\mathrm{Re}\lambda\tau}\big[|\lambda|^{(1-\rho-\alpha)/(2-\rho)} + |\lambda|^{-1/2}\big]|d\lambda| \\ &\le C(t-s)\big[\tau^{(2\rho+\alpha-3)/(2-\rho)} + \tau^{-1/2}\big]. \qquad \square\end{aligned}$$

Let $v_0 \in D(A(s))$, $0 \le s < T$. Then with the aid of Lemma 4.6 and assumption (2.11) one obtains

$$\begin{aligned}&\big\|e^{-(t-s)A(t)}v_0 - e^{-(t-s)A(s)}v_0\big\|_{L^2} \\ &\quad \le C\big[(t-s)^{(\rho+\alpha-1)/(2-\rho)} + (t-s)^{1/2}\big]\|v_0\|_{L^2}.\end{aligned}$$

In view of (3.17)

$$\big\|e^{-(t-s)A(s)}v_0 - v_0\big\|_{L^2} \to 0$$

as $t \to s+0$. Furthermore, by virtue of (4.22)

$$\|W(t,s)v_0\|_{L^2} \le C\big[(t-s)^{(2\rho+\alpha-2)/(2-\rho)} + (t-s)^{\rho/[2(2-\rho)]}\big]\|v_0\|_{L^2}.$$

Hence

$$U(t,s)v_0 = \big(e^{-(t-s)A(t)}v_0 - e^{-(t-s)A(s)}v_0\big) + e^{-(t-s)A(s)}v_0 + W(t,s)v_0 \to v_0$$

as $t \to s+0$. Thus the proof of Theorem 4.1 is complete.

5. Solving problem (1.3)–(1.4).

We begin this section by proving the following theorem.

Theorem 5.1. *Suppose that* $f \in C^{\omega}([0,T];L^2(\Omega))$ *with* ω *satisfying* (2.14) *and* $v_0 \in D(A(0))$. *Then the function defined by*

$$v(t) = U(t,0)v_0 + \int_0^t U(t,s)f(s)ds \tag{5.1}$$

is a solution to (1.5)–(1.6) *in the following sense*:

$$\begin{aligned}&v \in C([0,T];L^2(\Omega)) \cap C^1((0,T];L^2(\Omega)), \quad v(t) \in D(A(t)) \text{ for } 0 < t \leq T,\\ &(1.5) \text{ and } (1.6) \text{ hold.}\end{aligned} \tag{5.2}$$

Proof. Let $\varepsilon > 0$ be such that $0 < t - \varepsilon < t$. Then by the usual method one deduces

$$\begin{aligned}&\frac{d}{dt}\int_0^{t-\varepsilon} e^{-(t-s)A(t)}f(s)ds\\ &\quad = e^{-\varepsilon A(t)}[f(t-\varepsilon) - f(t)] - \int_0^{t-\varepsilon} \Phi_1(t,s)f(s)ds\\ &\quad - \int_0^{t-\varepsilon}\left(\frac{\partial}{\partial s}e^{-(t-s)A(t)}\right)\cdot[f(s)-f(t)]ds + e^{-tA(t)}f(t),\end{aligned} \tag{5.3}$$

Φ_1 being defined by (4.3). In view of (2.14)

$$\left\|e^{-\varepsilon A(t)}[f(t-\varepsilon) - f(t)]\right\|_{L^2} \leq C\varepsilon^{(\rho-1)/(2-\rho)+\omega}|f|_{C^{\omega}([0,T];L^2)} \to 0 \tag{5.4}$$

as $\varepsilon \to +0$. In view of (3.15) and (2.14) one gets

$$\begin{aligned}&\int_0^t \left\|\left(\frac{\partial}{\partial s}e^{-(t-s)A(t)}\right)\cdot[f(s)-f(t)]\right\|_{L^2(\Omega)} ds\\ &\quad \leq C\int_0^t (t-s)^{(2\rho-3)/(2-\rho)+\omega}ds\,|f|_{C^{\omega}([0,T];L^2(\Omega))}\\ &\quad = C\frac{t^{(\rho-1)/(2-\rho)+\omega}}{(\rho-1)/(2-\rho)+\omega}\,|f|_{C^{\omega}([0,T];L^2(\Omega))}.\end{aligned} \tag{5.5}$$

Integrating both sides of (5.3) from t to t', where $0 < t < t' \leq T$, and then letting $\varepsilon \to +0$, we easily see, using (5.4) and (5.5), that $\int_0^t e^{-(t-s)A(t)} f(s)ds$ is differentiable with respect to t in $(0, T)$ and

$$\begin{aligned} &\frac{d}{dt}\int_0^t e^{-(t-s)A(t)} f(s)ds \\ &\quad = -\int_0^t \Phi_1(t,s)f(s)ds - \int_0^t \frac{\partial}{\partial s} e^{-(t-s)A(t)} \cdot [f(s) - f(t)]ds + e^{-tA(t)} f(t). \end{aligned} \tag{5.6}$$

By virtue of (2.11), (4.22) and (4.62) $\int_0^t W(t,s)f(s)ds$ is differentiable in t and

$$\frac{d}{dt}\int_0^t W(t,s)f(s)ds = \int_0^t \frac{\partial}{\partial t} W(t,s)f(s)ds. \tag{5.7}$$

From (5.6) and (5.7) one obtains

$$\begin{aligned} \frac{d}{dt}\int_0^t U(t,s)f(s)ds = &-\int_0^t \left\{ \Phi_1(t,s)f(s) + \frac{\partial}{\partial s} e^{-(t-s)A(t)} \cdot [f(s) - f(t)] \right\} ds \\ &+ e^{-tA(t)} f(t) + \int_0^t \frac{\partial}{\partial t} W(t,s)f(s)ds. \end{aligned} \tag{5.8}$$

From Theorem 4.1, Lemma 4.3 and estimates (3.15), (4.62) we deduce that $v \in C([0,T]; L^2(\Omega)) \cap C^1((0,T]; L^2(\Omega))$. Therefore

$$\begin{aligned} &A(t)^{-1} \frac{d}{dt}\int_0^t U(t,s)f(s)ds \\ &\quad = -\int_0^t \left\{ A(t)^{-1}\Phi_1(t,s)f(s) + A(t)^{-1} \frac{\partial}{\partial s} e^{-(t-s)A(t)} \cdot [f(s) - f(t)] \right\} ds \\ &\qquad + A(t)^{-1} e^{-tA(t)} f(t) + A(t)^{-1} \int_0^t \frac{\partial}{\partial t} W(t,s)f(s)ds. \end{aligned} \tag{5.9}$$

In view of (4.3) and (3.16) the integrand of the first term in the right-hand side of (5.9) equals

$$
\begin{aligned}
&- A(t)^{-1}\Big(\frac{\partial}{\partial t}+\frac{\partial}{\partial s}\Big)e^{-(t-s)A(t)}\cdot f(s)+A(t)^{-1}\frac{\partial}{\partial s}e^{-(t-s)A(t)}\cdot [f(s)-f(t)]\\
&= -A(t)^{-1}\frac{\partial}{\partial t}e^{-(t-s)A(t)}f(s)-A(t)^{-1}\frac{\partial}{\partial s}e^{-(t-s)A(t)}f(t)\\
&= -A(t)^{-1}\frac{\partial}{\partial t}e^{-(t-s)A(t)}f(s)-e^{-(t-s)A(t)}f(t).
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
A(t)^{-1}\frac{d}{dt}\int_0^t U(t,s)f(s)ds &= \int_0^t \Big\{A(t)^{-1}\frac{\partial}{\partial t}e^{-(t-s)A(t)}f(s)+e^{-(t-s)A(t)}f(t)\Big\}ds\\
&\quad + A(t)^{-1}e^{-tA(t)}f(t)+\int_0^t A(t)^{-1}\frac{\partial}{\partial t}W(t,s)f(s)ds.
\end{aligned}
\tag{5.10}
$$

By virtue of (4.8), (4.64) and (4.66) the right-hand side in (5.10) is equal to

$$
\begin{aligned}
&\int_0^t \Big\{A(t)^{-1}\frac{\partial}{\partial t}U(t,s)f(s)+e^{-(t-s)A(t)}f(t)\Big\}ds+A(t)^{-1}e^{-tA(t)}f(t)\\
&\quad = -\int_0^t U(t,s)f(s)ds+\big[I-e^{-tA(t)}\big]A(t)^{-1}f(t)+A(t)^{-1}e^{-tA(t)}f(t)\\
&\quad = -\int_0^t U(t,s)f(s)ds+A(t)^{-1}f(t).
\end{aligned}
$$

Thus it has been shown that

$$
A(t)^{-1}\frac{d}{dt}\int_0^t U(t,s)f(s)ds=-\int_0^t U(t,s)f(s)ds+A(t)^{-1}f(t).
$$

From this equality, (5.1) and (4.64) it follows that

$$
\begin{aligned}
&A(t)^{-1}(f(t)-v'(t))\\
&\quad = A(t)^{-1}f(t)-A(t)^{-1}\frac{d}{dt}U(t,0)v_0-A(t)^{-1}\frac{d}{dt}\int_0^t U(t,s)f(s)ds\\
&\quad = \int_0^t U(t,s)f(s)ds+U(t,0)v_0=v(t).
\end{aligned}
$$

This implies that $v(t) \in D(A(t))$ for all $t \in (0,T]$ and (1.5) holds. The initial condition (1.6) follows from (4.5). $\square$

From formula (5.8) in the proof of Theorem 5.1 it is seen that the following inequality holds:

$$\left\| \frac{d}{dt} \int_0^t U(t,s) f(s) ds \right\|_{L^2} \leq C t^{(\rho-1)/(2-\rho)} \|f\|_{C^\omega([0,T];L^2(\Omega))}. \tag{5.11}$$

Let $f \in C([0,T];L^2(\Omega))$ and $v_0 = M(0)u_0,\ u_0 \in L^2(\Omega)$, and suppose that v is a solution of (1.5)–(1.6) in the sense (5.2). Set

$$u(t) = L(t)^{-1}(f(t) - v'(t)).$$

Since

$$L(t)M(t)^{-1}v(t) = A(t)v(t) \ni f(t) - v'(t),$$

one has

$$M(t)^{-1}v(t) \ni L(t)^{-1}(f(t) - v'(t)) = u(t),$$

which implies

$$v(t) = M(t)u(t).$$

Hence

$$L(t)u(t) = f(t) - v'(t) = f(t) - (M(t)u(t))'.$$

Furthermore

$$M(t)u(t) = v(t) \to v_0 = M(0)u_0$$

as $t \to +0$. Consequently u is a solution to problem (1.3)–(1.4) in the following sense:

$$\begin{aligned} &u \in C((0,T];L^2(\Omega)), \quad Mu \in C([0,T];L^2(\Omega)) \cap C^1((0,T];L^2(\Omega)), \\ &u(t) \in D(L(t)) \text{ for } 0 < t \leq T \text{ and } L(\cdot)u(\cdot) \in C((0,T];L^2(\Omega)), \\ &\text{(1.3) and (1.4) hold.} \end{aligned} \tag{5.12}$$

Conversely, if u is a solution to problem (1.3)–(1.4) in the sense (5.12), it is easy to see that $v = Mu$ is a solution to problem (1.5)–(1.6) in the sense (5.2).

Combining this and Theorem 5.1, one establishes the following theorem.

Theorem 5.2. *Suppose that $f \in C^\omega([0,T]; L^2(\Omega))$ with ω satisfying* (2.14) *and $u_0 \in D(L(0))$. Then there exists a solution to problem* (1.3)–(1.4) *in the sense* (5.12).

Now we turn to the proof of the uniqueness of the solution to problem (1.3)–(1.4). For this purpose we prepare another operator-valued function $V(t,s)$ defined by

$$V(t,s) = e^{-(t-s)A(s)} + \int_s^t \Psi(t,\tau) e^{-(\tau-s)A(s)} d\tau \equiv e^{-(t-s)A(s)} + Z(t,s), \quad (5.13)$$

$$\Psi(t,s) = \Psi_1(t,s) + \int_s^t \Psi(t,\tau)\Psi_1(\tau,s) d\tau, \quad (5.14)$$

$$\Psi_1(t,s) = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right) e^{-(t-s)A(s)}. \quad (5.15)$$

The following lemma is similar to Lemma 4.3 and can be easily shown.

Lemma 5.1. *For $0 \le s < t \le T$ the following inequalities hold*:

$$\|\Psi_1(t,s)\|_{\mathscr{L}(L^2)} \le C\big[(t-s)^{(2\rho+\alpha-3)/(2-\rho)} + (t-s)^{-1/2}\big], \quad (5.16)$$

$$\|\Psi(t,s)\|_{\mathscr{L}(L^2)} \le C\big[(t-s)^{(2\rho+\alpha-3)/(2-\rho)} + (t-s)^{-1/2}\big]. \quad (5.17)$$

With the aid of (3.14) and (5.17) (cf. also Section 4) one observes

$$\|V(t,s)\|_{\mathscr{L}(L^2)} \le C(t-s)^{(\rho-1)/(2-\rho)}, \quad (5.18)$$

$$\|Z(t,s)\|_{\mathscr{L}(L^2)} \le C\big[(t-s)^{(2\rho+\alpha-2)/(2-\rho)} + (t-s)^{\rho/[2(2-\rho)]}\big]. \quad (5.19)$$

First we show the uniqueness of the solution to problem (1.5)–(1.6). Let v be a solution to problem (1.5)–(1.6) in the sense (5.2) for $f \in C([0,T]; L^2(\Omega))$ and $v_0 \in L^2(\Omega)$. One has

$$v(t) = A(t)^{-1}(f(t) - v'(t)), \quad 0 < t \le T. \quad (5.20)$$

By virtue of (5.15), (5.20) and (3.16)

$$\begin{aligned}
&\frac{\partial}{\partial s}\big(e^{-(t-s)A(s)}v(s)\big) = \frac{\partial}{\partial s}e^{-(t-s)A(s)} \cdot v(s) + e^{-(t-s)A(s)}v'(s) \\
&= \Psi_1(t,s)v(s) - \frac{\partial}{\partial t}e^{-(t-s)A(s)}v(s) + e^{-(t-s)A(s)}v'(s) \\
&= \Psi_1(t,s)v(s) - \frac{\partial}{\partial t}e^{-(t-s)A(s)}A(s)^{-1}(f(s) - v'(s)) + e^{-(t-s)A(s)}v'(s) \\
&= \Psi_1(t,s)v(s) + e^{-(t-s)A(s)}(f(s) - v'(s)) + e^{-(t-s)A(s)}v'(s) \\
&= \Psi_1(t,s)v(s) + e^{-(t-s)A(s)}f(s). \qquad (5.21)
\end{aligned}$$

LEMMA 5.2. *For $0 < s < t \leq T$ one has*

$$\begin{aligned}
&\frac{\partial}{\partial s}(Z(t,s)v(s)) \\
&\quad = \int_s^t \Psi(t,\tau)\big[\Psi_1(\tau,s)v(s) + e^{-(\tau-s)A(s)}f(s)\big]d\tau - \Psi(t,s)v(s). \qquad (5.22)
\end{aligned}$$

PROOF. We begin by showing that the right derivative of $Z(t,s)v(s)$ at s is equal to the right-hand side in (5.22). For $s < s+h < t$

$$\begin{aligned}
&\frac{1}{h}\big[Z(t,s+h)v(s+h) - Z(t,s)v(s)\big] \\
&\quad = \frac{1}{h}\int_{s+h}^t \Psi(t,\tau)\big[e^{-(\tau-s-h)A(s+h)}v(s+h) - e^{-(\tau-s)A(s)}v(s)\big]d\tau \\
&\qquad - \frac{1}{h}\int_s^{s+h} \Psi(t,\tau)e^{-(\tau-s)A(s)}v(s)d\tau. \qquad (5.23)
\end{aligned}$$

In view of (5.21) one has

$$\begin{aligned}
&e^{-(\tau-s-h)A(s+h)}v(s+h) - e^{-(\tau-s)A(s)}v(s) \\
&\quad = \int_s^{s+h} \frac{\partial}{\partial r}\big(e^{-(\tau-r)A(r)}v(r)\big)dr \\
&\quad = \int_s^{s+h} \big(\Psi_1(\tau,r)v(r) + e^{-(\tau-r)A(r)}f(r)\big)dr.
\end{aligned}$$

Hence

$$\begin{aligned}
&\Big\|\frac{1}{h}\big[e^{-(\tau-s-h)A(s+h)}v(s+h)-e^{-(\tau-s)A(s)}v(s)\big]\Big\|_{L^2}\\
&\quad\le \frac{C}{h}\int_s^{s+h}\big\{\big[(\tau-r)^{(2\rho+\alpha-3)/(2-\rho)}+(\tau-r)^{-1/2}\big]\|v(r)\|_{L^2}\\
&\qquad\qquad+(\tau-r)^{(\rho-1)/(2-\rho)}\|f(r)\|_{L^2}\big\}dr\\
&\quad\le C\Big\{\big[(\tau-s-h)^{(2\rho+\alpha-3)/(2-\rho)}+(\tau-s-h)^{-1/2}\big]\sup_{s\le r\le s+h}\|v(r)\|_{L^2}\\
&\qquad\qquad+(\tau-s-h)^{(\rho-1)/(2-\rho)}\sup_{s\le r\le s+h}\|f(r)\|_{L^2(\Omega)}\Big\}. \qquad (5.24)
\end{aligned}$$

For $s<s+h<s+h_0<t$ write

$$\begin{aligned}
&\Big\|\frac{1}{h}\int_{s+h}^{t}\Psi(t,\tau)\big[e^{-(\tau-s-h)A(s+h)}v(s+h)-e^{-(\tau-s)A(s)}v(s)\big]d\tau\\
&\qquad-\int_s^t\Psi(t,\tau)\frac{\partial}{\partial s}\big(e^{-(\tau-s)A(s)}v(s)\big)d\tau\Big\|_{L^2}\\
&\quad\le\Big\|\frac{1}{h}\int_{s+h}^{s+h_0}\Psi(t,\tau)\big[e^{-(\tau-s-h)A(s+h)}v(s+h)-e^{-(\tau-s)A(s)}v(s)\big]d\tau\Big\|_{L^2}\\
&\qquad+\Big\|\frac{1}{h}\int_{s+h_0}^{t}\Psi(t,\tau)\big[e^{-(\tau-s-h)A(s+h)}v(s+h)-e^{-(\tau-s)A(s)}v(s)\big]d\tau\\
&\qquad\qquad-\int_{s+h_0}^t\Psi(t,\tau)\frac{\partial}{\partial s}\big(e^{-(\tau-s)A(s)}v(s)\big)d\tau\Big\|_{L^2}\\
&\qquad+\Big\|\int_s^{s+h_0}\Psi(t,\tau)\frac{\partial}{\partial s}\big(e^{-(\tau-s)A(s)}v(s)\big)d\tau\Big\|_{L^2}. \qquad (5.25)
\end{aligned}$$

Let ε be an arbitrary positive number. By virtue of (5.24) the first term in the right-hand side in (5.25) is dominated by

$$\begin{aligned}
&C\sup_{s+h<\tau<s+h_0}\|\Psi(t,\tau)\|_{\mathscr{L}(L^2)}\\
&\quad\times\int_{s+h}^{s+h_0}\big[(\tau-s-h)^{(2\rho+\alpha-3)/(2-\rho)}\\
&\qquad\qquad+(\tau-s-h)^{-1/2}+(\tau-s-h)^{(\rho-1)/(2-\rho)}\big]d\tau
\end{aligned}$$

$$
\begin{aligned}
&= C \sup_{s+h<\tau<s+h_0} \|\Psi(t,\tau)\|_{\mathscr{L}(L^2)} \\
&\quad \times \left[\frac{(h_0-h)^{(\rho+\alpha-1)/(2-\rho)}}{(\rho+\alpha-1)/(2-\rho)} + \frac{(h_0-h)^{1/2}}{1/2} + \frac{(h_0-h)^{1/(2-\rho)}}{1/(2-\rho)} \right] \\
&\le C \sup_{s<\tau<s+h_0} \|\Psi(t,\tau)\|_{\mathscr{L}(L^2)} \big[h_0^{(\rho+\alpha-1)/(2-\rho)} + h_0^{1/2} + h_0^{1/(2-\rho)} \big].
\end{aligned}
$$

Hence if h_0 is sufficiently small, the first term in the right-hand side in (5.25) is smaller than ε for any $h \in (0, h_0)$, and so is the third term. If h is so small that $0 < h < h_0/2$, then in view of Lemma 5.1 and (5.24) one has, for $s + h_0 < \tau < t$,

$$
\begin{aligned}
&\left\| \frac{1}{h} \Psi(t,\tau) \big[e^{-(\tau-s-h)A(s+h)} v(s+h) - e^{-(\tau-s)A(s)} v(s) \big] \right\|_{L^2} \\
&\le C \big[(t-\tau)^{(2\rho+\alpha-3)/(2-\rho)} + (t-\tau)^{-1/2} \big] \\
&\quad \times \big[(\tau-s-h)^{(2\rho+\alpha-3)/(2-\rho)} + (\tau-s-h)^{-1/2} + (\tau-s-h)^{(\rho-1)/(2-\rho)} \big] \\
&\le C \big[(t-\tau)^{(2\rho+\alpha-3)/(2-\rho)} + (t-\tau)^{-1/2} \big] \\
&\quad \times \left[\left(\frac{h_0}{2} \right)^{(2\rho+\alpha-3)/(2-\rho)} + \left(\frac{h_0}{2} \right)^{-1/2} + \left(\frac{h_0}{2} \right)^{(\rho-1)/(2-\rho)} \right].
\end{aligned}
$$

Hence one can apply the dominated convergence theorem to show that the second term in the right-hand side in (5.25) tends to 0 as $h \to +0$. Thus from (5.25) it follows

$$
\begin{aligned}
&\frac{1}{h} \int_{s+h}^{t} \Psi(t,\tau) \big[e^{-(\tau-s-h)A(s+h)} v(s+h) - e^{-(\tau-s)A(s)} v(s) \big] d\tau \\
&\quad \to \int_s^t \Psi(t,\tau) \frac{\partial}{\partial s} \big(e^{-(\tau-s)A(s)} v(s) \big) d\tau
\end{aligned} \tag{5.26}
$$

as $h \to +0$. With the aid of (3.17) one deduces that as $h \to +0$

$$
\begin{aligned}
&\frac{1}{h} \int_s^{s+h} \Psi(t,\tau) e^{-(\tau-s)A(s)} v(s) d\tau - \Psi(t,s) v(s) \\
&\quad = \frac{1}{h} \int_s^{s+h} \big[\Psi(t,\tau) e^{-(\tau-s)A(s)} v(s) - \Psi(t,s) v(s) \big] d\tau
\end{aligned}
$$

$$= \frac{1}{h} \int_s^{s+h} \Psi(t,\tau)\big[e^{-(\tau-s)A(s)}v(s) - v(s)\big]d\tau$$

$$+ \frac{1}{h} \int_s^{s+h} \big[\Psi(t,\tau)v(s) - \Psi(t,s)v(s)\big]d\tau \to 0. \tag{5.27}$$

It follows from (5.23), (5.26) and (5.27) that the right derivative of $Z(t,s)v(s)$ at s equals the right-hand side in (5.22).

The proof of the statement on the left derivative is approximately the same. Indeed, for $0 \le s-h < s$ one has

$$\frac{1}{h}\big[Z(t,s)v(s) - Z(t,s-h)v(s-h)\big]$$

$$= \frac{1}{h}\int_s^t \Psi(t,\tau)\big[e^{-(\tau-s)A(s)}v(s) - e^{-(\tau-s+h)A(s-h)}v(s-h)\big]d\tau$$

$$- \frac{1}{h}\int_{s-h}^s \Psi(t,\tau)e^{-(\tau-s+h)A(s-h)}v(s-h)d\tau. \tag{5.28}$$

Since

$$\Big\|\frac{1}{h}\big[e^{-(\tau-s)A(s)}v(s) - e^{-(\tau-s+h)A(s-h)}v(s-h)\big]\Big\|_{L^2}$$

$$\le \frac{C}{h}\int_{s-h}^s \big\{\big[(\tau-r)^{(2\rho+\alpha-3)/(2-\rho)} + (\tau-r)^{-1/2}\big]\|v(r)\|_{L^2}$$

$$+ (\tau-r)^{(\rho-1)/(2-\rho)}\|f(r)\|_{L^2}\big\}dr$$

$$\le C\Big\{\big[(\tau-s)^{(2\rho+\alpha-3)/(2-\rho)} + (\tau-s)^{-1/2}\big] \sup_{s-h\le r\le s}\|v(r)\|_{L^2}$$

$$+ (\tau-s)^{(\rho-1)/(2-\rho)} \sup_{s-h\le r\le s}\|f(r)\|_{L^2}\Big\},$$

the proof of the fact that the first term in the right-hand side in (5.28) converges to

$$\int_s^t \Psi(t,\tau)\frac{\partial}{\partial s}\big(e^{-(\tau-s)A(s)}v(s)\big)d\tau$$

is simpler than that of (5.26). In view of (3.14) and Lemma 4.6 one has

$$
\begin{aligned}
&\big\|e^{-(\tau-s+h)A(s-h)}v(s-h)-v(s)\big\|_{L^2}\\
&\quad=\big\|e^{-(\tau-s+h)A(s-h)}[v(s-h)-v(s)]\\
&\qquad\quad+\big[e^{-(\tau-s+h)A(s-h)}-e^{-(\tau-s+h)A(s)}\big]v(s)+e^{-(\tau-s+h)A(s)}v(s)-v(s)\big\|_{L^2}\\
&\quad\le C(\tau-s+h)^{(\rho-1)/(2-\rho)}\|v(s-h)-v(s)\|_{L^2}\\
&\qquad+Ch\big[(\tau-s+h)^{(2\rho+\alpha-3)/(2-\rho)}+(\tau-s+h)^{-1/2}\big]\|v(s)\|_{L^2}\\
&\qquad+\big\|e^{-(\tau-s+h)A(s)}v(s)-v(s)\big\|_{L^2}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\left\|\frac{1}{h}\int_{s-h}^{s}\Psi(t,\tau)\big[e^{-(\tau-s+h)A(s-h)}v(s-h)-v(s)\big]d\tau\right\|_{L^2}\\
&\quad\le C\sup_{s-h\le\tau\le s}\|\Psi(t,\tau)\|_{\mathscr{L}(L^2)}\frac{1}{h}\int_{s-h}^{s}\big\{(\tau-s+h)^{(\rho-1)/(2-\rho)}\|v(s-h)-v(s)\|_{L^2}\\
&\qquad+h\big[(\tau-s+h)^{(2\rho+\alpha-3)/(2-\rho)}+(\tau-s+h)^{-1/2}\big]\|v(s)\|_{L^2}\\
&\qquad+\big\|e^{-(\tau-s+h)A(s)}v(s)-v(s)\big\|_{L^2}\big\}d\tau. \qquad (5.29)
\end{aligned}
$$

Since

$$
\begin{aligned}
&\frac{1}{h}\int_{s-h}^{s}(\tau-s+h)^{(\rho-1)/(2-\rho)}\|v(s-h)-v(s)\|_{L^2}d\tau\\
&\quad=(2-\rho)h^{1/(2-\rho)}\left\|\frac{v(s-h)-v(s)}{h}\right\|_{L^2}\to 0,\\
&\int_{s-h}^{s}\big[(\tau-s+h)^{(2\rho+\alpha-3)/(2-\rho)}+(\tau-s+h)^{-1/2}\big]d\tau\\
&\quad=\frac{2-\rho}{\rho+\alpha-1}h^{(\rho+\alpha-1)/(2-\rho)}+2h^{1/2}\to 0,\\
&\frac{1}{h}\int_{s-h}^{s}\big\|e^{-(\tau-s+h)A(s)}v(s)-v(s)\big\|_{L^2}d\tau\le\sup_{0\le\sigma\le h}\big\|e^{-\sigma A(s)}v(s)-v(s)\big\|_{L^2}\to 0,
\end{aligned}
$$

the right-hand side of (5.29) tends to 0 as $h\to+0$. Thus the proof of Lemma 5.2 is complete. □

THEOREM 5.3. *Suppose $f\in C([0,T];L^2(\Omega))$ and $v_0\in L^2(\Omega)$. If a solution* *to problem* (1.5)–(1.6) *in the sense* (5.2) *exists, then it is uniquely determined by* f *and* v_0.

Proof. The desired result follows from the equality:

$$v(t) = V(t,0)v_0 + \int_0^t V(t,s)f(s)ds. \tag{5.30}$$

In view of (5.21), Lemma 5.2, (5.14) and (5.13)

$$\begin{aligned}
&\frac{\partial}{\partial s}(V(t,s)v(s)) \\
&\quad = \frac{\partial}{\partial s}\big[e^{-(t-s)A(s)}v(s) + Z(t,s)v(s)\big] \\
&\quad = \Psi_1(t,s)v(s) + e^{-(t-s)A(s)}f(s) \\
&\qquad + \int_s^t \Psi(t,\tau)\big[\Psi_1(\tau,s)v(s) + e^{-(\tau-s)A(s)}f(s)\big]d\tau - \Psi(t,s)v(s) \\
&\quad = V(t,s)f(s).
\end{aligned} \tag{5.31}$$

Hence, integrating both sides in (5.31), one obtains

$$V(t,t-\varepsilon)v(t-\varepsilon) - V(t,0)v(0) = \int_0^{t-\varepsilon} V(t,s)f(s)ds. \tag{5.32}$$

One has

$$V(t,t-\varepsilon)v(t-\varepsilon) = V(t,t-\varepsilon)(v(t-\varepsilon)-v(t)) + \big[e^{-\varepsilon A(t-\varepsilon)} + Z(t,t-\varepsilon)\big]v(t). \tag{5.33}$$

By virtue of (5.18)

$$\|V(t,t-\varepsilon)(v(t-\varepsilon)-v(t))\|_{L^2} \le C\varepsilon^{1/(2-\rho)}\left\|\frac{v(t-\varepsilon)-v(t)}{\varepsilon}\right\|_{L^2} \to 0. \tag{5.34}$$

Lemma 4.6 and (3.17) yield

$$\begin{aligned}
&\big\|e^{-\varepsilon A(t-\varepsilon)}v(t) - v(t)\big\|_{L^2} \\
&\quad = \big\|\big[e^{-\varepsilon A(t-\varepsilon)} - e^{-\varepsilon A(t)}\big]v(t) + \big[e^{-\varepsilon A(t)}v(t) - v(t)\big]\big\|_{L^2} \\
&\quad \le C\varepsilon\big[\varepsilon^{(2\rho+\alpha-3)/(2-\rho)} + \varepsilon^{-1/2}\big]\|v(t)\|_{L^2} + \big\|e^{-\varepsilon A(t)}v(t) - v(t)\big\|_{L^2} \to 0.
\end{aligned} \tag{5.35}$$

Owing to (5.19) and (2.11) one has

$$\|Z(t,t-\varepsilon)v(t)\|_{L^2} \le C\big[\varepsilon^{(2\rho+\alpha-2)/(2-\rho)} + \varepsilon^{\rho/[2(2-\rho)]}\big]\|v(t)\|_{L^2} \to 0. \tag{5.36}$$

It follows from (5.33), (5.34), (5.35) and (5.36) that

$$V(t,t-\varepsilon)v(t-\varepsilon) \to v(t)$$

as $\varepsilon \to +0$. Letting $\varepsilon \to +0$ in (5.32), we obtain (5.30). □

THEOREM 5.4. *Suppose $f \in C([0,T];L^2(\Omega))$ and $u_0 \in L^2(\Omega)$. If a solution to problem* (1.3)–(1.4) *in the sense* (5.12) *exists, then it is uniquely determined by f and $M(0)u_0$.*

PROOF. Supposing that there exist two solutions to (1.3)–(1.4), let their difference be denoted by u. Then, by what was stated just before Theorem 5.2, $v = Mu$ is a solution to (1.5)–(1.6) in the sense (5.2) with $f = 0$ and $v_0 = 0$. In view of Theorem 5.3 $v(t) \equiv 0$. Substituting this in (1.3) one obtains $L(t)u(t) \equiv 0$, which implies $u(t) \equiv 0$. □

REMARK 5.1. Let $f \in C^\omega([0,T];L^2(\Omega))$ and

$$v(t) = \int_0^t U(t,s)f(s)ds.$$

Then in view of Theorem 5.1 v is a solution to problem (1.5)–(1.6) with $v_0 = 0$ in the sense (5.2). Hence by virtue of the proof of Theorem 5.3 the equality (5.30) holds with $v_0 = 0$:

$$\int_0^t V(t,s)f(s)ds = \int_0^t U(t,s)f(s)ds. \tag{5.37}$$

Let w_0 be an arbitrary element of $L^2(\Omega)$ and $\rho_\varepsilon(t) = \varepsilon^{-1}\rho(t/\varepsilon)$ be a mollifier. Taking $f(t) = \rho_\varepsilon(t-s_0)w_0$ in (5.37), where $0 < s_0 < t$, and letting $\varepsilon \to +0$, one obtains $V(t,s_0)w_0 = U(t,s_0)w_0$. Thus we have proved

$$V(t,s) = U(t,s), \quad 0 \le s < t \le T. \tag{5.38}$$

6. The problem with the integral term.

In this section we establish Theorem 6.1:

Theorem 6.1. *Let $f \in C^\omega([0,T];L^2(\Omega))$ with ω satisfying* (2.14) *and $u_0 \in D(L(0))$. Then, there exists a solution u to problem* (1.1)–(1.2) *in the following sense*:

$$\begin{aligned}
&u \in C((0,T];L^2(\Omega)), \quad Mu \in C([0,T];L^2(\Omega)) \cap C^1((0,T];L^2(\Omega)),\\
&u(t) \in D(L(t)) \text{ for } 0 < t \le T \text{ and } L(\cdot)u(\cdot) \in C((0,T];L^2(\Omega)),\\
&t^{1-\delta}\left\|\frac{d}{dt}(M(t)u(t))\right\|_{L^2} \text{ and } t^{1-\delta}\|L(t)u(t)\|_{L^2} \text{ are bounded in } 0 < t < T,\\
&(1.1) \text{ and } (1.2) \text{ hold,}
\end{aligned} \tag{6.1}$$

where

$$\delta = \min\left\{\gamma + \frac{2(\rho-1)}{2-\rho}, \frac{2\rho+\alpha-2}{2-\rho}, \frac{\rho}{2(2-\rho)}\right\} \in (0,1). \tag{6.2}$$

Furthermore, a solution to (1.1)–(1.2) *in this sense is unique.*

Setting

$$K(t,s) = B(t,s)L(s)^{-1} \tag{6.3}$$

we write (1.1)–(1.2) as

$$\frac{d}{dt}(M(t)u(t)) + L(t)u(t) + \int_0^t K(t,s)L(s)u(s)ds = f(t), \quad 0 < t \le T, \tag{6.4}$$

$$M(0)u(0) = M(0)u_0. \tag{6.5}$$

Clearly $K \in C(\Delta;\mathscr{L}(L^2(\Omega)))$ and

$$\|K(t',s) - K(t,s)\|_{\mathscr{L}(L^2)} \le C|t'-t|^\omega. \tag{6.6}$$

If we substitute $L(\cdot)u(\cdot)$ by $A(\cdot)v(\cdot)$ in (6.4) as in the previous section, then a multivalued operator appears also in the integral term. In order to avoid this difficulty we transform (6.4) to the equation with $(Mu)'$ instead of Lu in the integral term using the idea of Crandall and Nohel [**2**, Proposition 1].

The (generalized) convolution $F * G$ of two operator-valued functions F and G from Δ to $\mathscr{L}(X)$ and that $F * f$ of an operator-valued function F and a vector-valued function f are defined by

$$(F * G)(t,s) = \int_s^t F(t,r)G(r,s)dr,$$

$$(F * f)(t) = \int_0^t F(t,s)f(s)ds,$$

respectively, whenever they make sense.

We note that the associative law holds for both of them. Moreover, the convolution $F * G$ is well-defined when both F and G have weak singularities on the line $t = s$, while the convolution $F * f$ is well-defined when F has a weak singularity on the line $t = s$ and f belongs to $L^\infty(0,T)$.

Then (6.4) is briefly rewritten as

$$(Mu)' + Lu + K * Lu = f. \tag{6.7}$$

Since $K \in C(\Delta; \mathscr{L}(L^2(\Omega)))$, the integral equation

$$R + K + K * R = 0 \tag{6.8}$$

admits a unique solution $R \in C(\Delta; \mathscr{L}(L^2(\Omega)))$ that can be constructed by successive approximations:

$$R = -K + K * K - K * K * K + \cdots .$$

It is an easy task to check that R satisfies the estimate

$$\|R(t',s) - R(t,s)\|_{\mathscr{L}(L^2)} \le C|t'-t|^\omega. \tag{6.9}$$

Furthermore one has $K * R = R * K$. Hence

$$R + K + R * K = 0. \tag{6.10}$$

Assume now $G \in C(\Delta; \mathscr{L}(L^2(\Omega)))$. Then with the aid of (6.8) and (6.10) it is easy to show that the equation $F + K * F = G$ admits a unique solution in $C(\Delta; \mathscr{L}(L^2(\Omega)))$ so that

$$F + K * F = G \quad \text{if and only if} \quad F = G + R * G. \tag{6.11}$$

Consequently, equation (6.7), considered as a convolution equation for Lu is equivalent to

$$(Mu)' + Lu = f + R * f - R * (Mu)'. \tag{6.12}$$

Let $v = Mu$ be the new unknown function. Then, in view of (6.12) one has

$$\begin{aligned} &v' + Av \ni f + R * f - R * v' \\ &v(0) = v_0 = M(0)u_0. \end{aligned} \tag{6.13}$$

This problem is transformed into the (equivalent) integrodifferential equation

$$\begin{aligned} v(t) &= U(t,0)v_0 + \int_0^t U(t,s)\big[f(s) + (R * f)(s) - (R * v')(s)\big]ds \\ &= g(t) - \int_0^t \int_s^t U(t,\tau)R(\tau,s)d\tau\, v'(s)ds = g(t) - (U * R * v')(t), \end{aligned} \tag{6.14}$$

where

$$g(t) = U(t,0)v_0 + \int_0^t U(t,s)\big[f(s) + (R * f)(s)\big]ds. \tag{6.15}$$

Differentiation of (6.14) yields

$$v'(t) = g'(t) - \int_0^t Q(t,s)v'(s)ds, \tag{6.16}$$

where

$$Q(t,s) = \frac{\partial}{\partial t}(U * R)(t,s) = \frac{\partial}{\partial t}\int_s^t U(t,r)R(r,s)dr. \tag{6.17}$$

Equation (6.16) can be considered as the integral equation to be satisfied by v'.

We plan to solve problem (1.1)–(1.2) as follows. First we determine the solution w to the integral equation

$$w(t) = g'(t) - \int_0^t Q(t,s)w(s)ds. \tag{6.18}$$

Then we show that the function v defined by

$$v(t) = v_0 + \int_0^t w(s)ds \tag{6.19}$$

is a solution to problem (6.13). Next, using the argument by which it was shown that $v = Mu$ is a solution to (1.5)–(1.6) in the sense (5.2) if and only if u is a solution to (1.3)–(1.4) in the sense (5.12), we obtain a function u satisfying (6.12). Finally, with the aid of (6.11) we show that u is a desired solution to problem (1.1)–(1.2).

By the usual method one first derives the following representation for Q:

$$\begin{aligned} Q(t,s) = {} & \int_s^t \Phi_1(t,r)R(r,s)dr - \int_s^t \frac{\partial}{\partial r} e^{-(t-r)A(t)}(R(r,s) - R(t,s))dr \\ & + e^{-(t-s)A(t)}R(t,s) + \int_s^t \frac{\partial}{\partial t} W(t,r)R(r,s)dr. \end{aligned}$$

Hence using (2.11), (4.20), (3.15), (6.9), (3.14) and (4.62) one observes that

$$\|Q(t,s)\|_{\mathscr{L}(L^2)} \leq C(t-s)^{(\rho-1)/(2-\rho)}. \tag{6.20}$$

Consequently, the integral kernel $Q(t,s)$ of the equation (6.18) has a weak singularity.

Lemma 6.1. *The following inequality holds for all* $0 \leq s < t \leq T$:

$$\left\| \frac{\partial}{\partial t} e^{-(t-s)A(t)} - \frac{\partial}{\partial t} e^{-(t-s)A(s)} \right\|_{\mathscr{L}(L^2)} \leq C\big[(t-s)^{(2\rho+\alpha-3)/(2-\rho)} + (t-s)^{-1/2}\big].$$

Proof. First we note

$$\begin{aligned} & \frac{\partial}{\partial t} e^{-(t-s)A(t)} - \frac{\partial}{\partial t} e^{-(t-s)A(s)} \\ & \quad = \frac{\partial}{\partial t} \frac{1}{2\pi i} \int_{(t-s)\Gamma} e^{\lambda(t-s)} \big[(\lambda + A(t))^{-1} - (\lambda + A(s))^{-1}\big] d\lambda \\ & \quad = \frac{1}{2\pi i} \int_{(t-s)\Gamma} \lambda e^{\lambda(t-s)} \big[(\lambda + A(t))^{-1} - (\lambda + A(s))^{-1}\big] d\lambda - \Phi_1(t,s). \end{aligned} \tag{6.21}$$

Using (4.12), we observe

$$\begin{aligned}
&\left\| \frac{1}{2\pi i} \int_{(t-s)\Gamma} \lambda e^{\lambda(t-s)} \big[(\lambda + A(t))^{-1} - (\lambda + A(s))^{-1}\big] d\lambda \right\|_{\mathscr{L}(L^2)} \\
&\quad = \left\| \frac{1}{2\pi i} \int_{(t-s)\Gamma} \lambda e^{\lambda(t-s)} \int_s^t \frac{\partial}{\partial r} (\lambda + A(r))^{-1} dr d\lambda \right\|_{\mathscr{L}(L^2)} \\
&\quad \le C(t-s) \int_{(t-s)\Gamma} e^{\mathrm{Re}\lambda(t-s)} \big[|\lambda|^{(3-2\rho-\alpha)/(2-\rho)} + |\lambda|^{1/2}\big] |d\lambda| \\
&\quad \le C\big[(t-s)^{(2\rho+\alpha-3)/(2-\rho)} + (t-s)^{-1/2}\big].
\end{aligned} \tag{6.22}$$

The desired inequality follows from (6.21), (6.22) and (4.20). □

In view of (3.14) and (3.16) one has for $\varphi \in A(0)v_0$

$$\begin{aligned}
\left\| \frac{d}{dt} e^{-tA(0)} v_0 \right\|_{L^2} &= \left\| \frac{d}{dt} e^{-tA(0)} A(0)^{-1} \varphi \right\|_{L^2} = \left\| A(0)^{-1} \frac{d}{dt} e^{-tA(0)} \varphi \right\|_{L^2} \\
&= \left\| e^{-tA(0)} \varphi \right\|_{L^2} \le C t^{(\rho-1)/(2-\rho)} \|\varphi\|_{L^2}.
\end{aligned}$$

Hence

$$\left\| \frac{d}{dt} e^{-tA(0)} v_0 \right\|_{L^2} \le C t^{(\rho-1)/(2-\rho)} \|v_0\|_{D(A(0))}, \tag{6.23}$$

where $\|v_0\|_{D(A(0))} = \inf_{\varphi \in A(0)v_0} \|\varphi\|_{L^2}$. Lemma 6.1 and (6.23) yield

$$\begin{aligned}
\left\| \frac{d}{dt} e^{-tA(t)} v_0 \right\|_{L^2} &\le \left\| \frac{d}{dt} (e^{-tA(t)} - e^{-tA(0)}) v_0 \right\|_{L^2} + \left\| \frac{d}{dt} e^{-tA(0)} v_0 \right\|_{L^2} \\
&\le C\big[t^{(2\rho+\alpha-3)/(2-\rho)} + t^{-1/2}\big] \|v_0\|_{L^2} + C t^{(\rho-1)/(2-\rho)} \|v_0\|_{D(A(0))}.
\end{aligned} \tag{6.24}$$

By virtue of (4.62) one has

$$\left\| \frac{d}{dt} W(t,0) v_0 \right\|_{L^2} \le C\big[t^{\gamma+(3\rho-4)/(2-\rho)} + t^{(3\rho+\alpha-4)/(2-\rho)} + t^{(3\rho-4)/[2(2-\rho)]}\big] \|v_0\|_{L^2}, \tag{6.25}$$

and by (5.11)

$$\begin{aligned}&\left\|\frac{d}{dt}\int_0^t U(t,s)\big[f(s)+(R*f)(s)\big]ds\right\|_{L^2(\Omega)}\\&\quad\le Ct^{(\rho-1)/(2-\rho)}\|f+R*f\|_{C^\omega([0,T];L^2)}.\end{aligned}\tag{6.26}$$

It follows from (6.15), (6.24), (6.25) and (6.26) that the following inequality holds:

$$\|g'(t)\|_{L^2}\le C\big[t^{\gamma+(3\rho-4)/(2-\rho)}+t^{(3\rho+\alpha-4)/(2-\rho)}+t^{(3\rho-4)/[2(2-\rho)]}\big],\tag{6.27}$$

since

$$\frac{3\rho+\alpha-4}{2-\rho}<\frac{2\rho+\alpha-3}{2-\rho},\quad \frac{3\rho-4}{2(2-\rho)}<-\frac{1}{2}<\frac{\rho-1}{2-\rho}.$$

By virtue of the inequalities (6.20) and (6.27) together with (4.59) and (4.60) the integral equation (6.18) can be solved by successive approximation, and the unique solution w satisfies

$$\|w(t)\|_{L^2}\le C\big[t^{\gamma+(3\rho-4)/(2-\rho)}+t^{(3\rho+\alpha-4)/(2-\rho)}+t^{(3\rho-4)/[2(2-\rho)]}\big].\tag{6.28}$$

Let v be the function defined by (6.19). Integrating (6.16) from 0 to t and noting $g(0)=v_0$ we easily obtain (6.14). For $0\le s<t\le T$ one has

$$\begin{aligned}&\|(R*v')(t)-(R*v')(s)\|_{L^2}\\&\quad=\left\|\int_s^t R(t,\sigma)v'(\sigma)d\sigma+\int_0^s(R(t,\sigma)-R(s,\sigma))v'(\sigma)d\sigma\right\|_{L^2}\\&\quad\le C\int_s^t\|w(\sigma)\|_{L^2}d\sigma+C(t-s)^\omega\int_0^s\|w(\sigma)\|_{L^2}d\sigma\\&\quad\le C(t-s)\big[s^{\gamma+(3\rho-4)/(2-\rho)}+s^{(3\rho+\alpha-4)/(2-\rho)}+s^{(3\rho-4)/[2(2-\rho)]}\big]\\&\qquad+C(t-s)^\omega\big[s^{\gamma+2(\rho-1)/(2-\rho)}+s^{(2\rho+\alpha-2)/(2-\rho)}+s^{\rho/[2(2-\rho)]}\big].\end{aligned}$$

Hence following the proof of Theorem 5.1 it can be shown that $\int_0^t U(t,s)(R*v')(s)ds$ is differentiable and

$$A(t)^{-1}\frac{d}{dt}\int_0^t U(t,s)(R*v')(s)ds=-\int_0^t U(t,s)(R*v')(s)ds+A(t)^{-1}(R*v')(t).$$

Using this fact and noting that $f + R * f \in C^\omega([0,T]; L^2(\Omega))$ we see that v is a solution to problem (6.13) in the sense of (5.2). Hence one observes with the aid of the argument which was used to show that $v = Mu$ is a solution to (1.5)–(1.6) in the sense (5.2) if and only if u is a solution to (1.3)–(1.4) in the sense (5.12) that the function u defined by

$$u(t) = L(t)^{-1}[f(t) + (R * f)(t) - (R * v')(t) - v'(t)]$$

satisfies (6.12). In view of (6.28)

$$\begin{aligned}\left\| \frac{d}{dt}(M(t)u(t)) \right\|_{L^2} &= \|w(t)\|_{L^2} \\ &\le C\big[t^{\gamma+(3\rho-4)/(2-\rho)} + t^{(3\rho+\alpha-4)/(2-\rho)} + t^{(3\rho-4)/[2(2-\rho)]}\big],\end{aligned}$$

and hence

$$\|L(t)u(t)\|_{L^2} \le C\big[t^{\gamma+(3\rho-4)/(2-\rho)} + t^{(3\rho+\alpha-4)/(2-\rho)} + t^{(3\rho-4)/[2(2-\rho)]}\big].$$

Since

$$\delta - 1 = \min\left\{\gamma + \frac{3\rho-4}{2-\rho}, \frac{3\rho+\alpha-4}{2-\rho}, \frac{3\rho-4}{2(2-\rho)}\right\},$$

from the last two inequalities it follows that $t^{1-\delta}\|d(M(t)u(t))/dt\|_{L^2}$ and $t^{1-\delta}\|L(t)u(t)\|_{L^2}$ are both bounded.
Therefore estimates in (6.1) hold with δ being defined by (6.2).

Finally, using (6.11) one concludes that u is the desired solution to problem (1.1)–(1.2).

Next, we show the uniqueness of the solution. Let $f \in C([0,T]; L^2(\Omega))$ and $u_0 \in L^2(\Omega)$, and let u be the difference of a couple of solutions to problem (1.1)–(1.2) in the sense (6.1). Then u is a solution to problem (1.1)–(1.2) with $f = 0$, $u_0 = 0$ in the sense (6.1). The argument by which (6.12) was derived from (6.7) yields

$$(Mu)' + Lu = -R * (Mu)'.$$

Therefore $v = Mu$ satisfies

$$
\begin{aligned}
&v' + Av \ni -R * v', \quad v(0) = 0, \\
&\|v'(t)\|_{L^2} \le Ct^{\delta-1} \text{ for some constants } C > 0 \text{ and } \delta \in (0,1).
\end{aligned}
\tag{6.29}
$$

Since $R * v' \in C([0,T]; L^2(\Omega))$, the proof of Theorem 5.3 and (5.38) yield

$$
v(t) = -\int_0^t U(t,s)(R * v')(s)ds = -\int_0^t (U * R)(t,s)v'(s)ds. \tag{6.30}
$$

Differentiating both sides of (6.30) one obtains

$$
v'(t) = -\int_0^t Q(t,s)v'(s)ds. \tag{6.31}
$$

In view of (6.20) and (6.29) it follows from (6.31) that $(M(t)u(t))' = v'(t) \equiv 0$. Substituting this in (6.4) one gets

$$
L(t)u(t) + \int_0^t K(t,s)L(s)u(s)ds = 0,
$$

from which it follows that $L(t)u(t) \equiv 0$, and hence $u(t) \equiv 0$. Thus the proof of Theorem 6.1 is complete.

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