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A generalized Pohozaev identity and its applications

By Nichiro KAWANO[1], Wei-Ming NI[2]
and Shoji YOTSUTANI[3]

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§ 1. Introduction.

In this paper we establish a generalized Pohozaev identity and its variant for the radial solutions of the following quasilinear elliptic equation,

$$\mathrm{div}(A(|Du|)Du)+f(|x|, u)=0 \tag{1.1}$$

in $\boldsymbol{R}^n$, where Du is the gradient of u, $f(|x|, u)$ and $A(p)$ are given functions. The Pohozaev identity is useful to investigate the existence and non-existence of the ground state of (1.1). By a ground state we mean a positive solution u in $\boldsymbol{R}^n$, which tends to zero at ∞.

The Pohozaev identity was used by Pohozaev [**15**] in 1965 to show the non-existence of non-trivial solutions of non-linear eigenvalue problems for semi-linear elliptic equations. Identities of this kind were first discovered by Rellich [**17**] in 1940 in his study of the first eigenvalue of Δ, and by Nehari [**5**] in 1960. The idea was applied to investigate the properties of solutions for non-linear elliptic equations (see, e.g., [**1**], [**2**], [**3**], [**4**], [**6**], [**7**], [**8**], [**9**], [**10**], [**11**], [**12**], [**13**], [**14**], [**16**]). Especially, Ding and Ni [**2**] found that the Pohozaev-type identity is useful to get the non-existence theorems for the ground state in the anomalous case, $f_u(|x|, 0)=0$, by employing suitable change of variables. Recently, Ni and Serrin [**9, 10, 11**] established some generalized Pohozaev identities and used them to investigate the solutions of the quasilinear elliptic equations,

$$\mathrm{div}(A(|Du|)Du)+f(u)=0\,. \tag{1.2}$$

They extend the argument employed by Ding and Ni to the quasilinear case. Their results are sharp, however their arguments are tricky and difficult. Our

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aim is to simplify, unify and generalize the method. We have found that the essence of the argument is clarified by using a new kind of Pohozaev-type identity (see the identity (PII) in Section 2).

The organization of this paper is as follows. In Section 2 we state the main theorems of this paper. In Section 3 applications to the generalized Laplace equations are included, which are useful to investigate the existence and non-existence of the ground state. Sections 4-8 contain some related topics and the proofs of theorems stated in Section 3. In Section 4 we collect some preliminary lemmas which will be used frequently in the subsequent sections. In Section 5 we investigate the asymptotic behavior of solutions, which is fundamental for further analysis. In Section 6 we establish the existence and uniqueness of the solutions of the initial value problem. In Section 7 we give the proof of the generalized Pohozaev identity for the generalized Laplace equation, and prove the existence theorem for the ground state. In Section 8 we prove the non-existence theorems of the positive solutions and the ground state. Section 9 contains some slight modifications of the theorems obtained in previous sections for a special nonlinearity, and Section 10 contains some concluding remarks.

§ 2. Main theorems.

We consider the radial solutions of (1.1). Let $u=u(r)$ be a radial solution of (1.1), then u satisfies the equation

$$r^{1-n}(r^{n-1}A(|u'|)u')'+f(r, u)=0\,, \qquad r>0\,, \tag{2.1}$$

where n is a positive integer, and $u'=u'(r)=du(r)/dr$.

THEOREM 2.1. *Suppose that* $A(p)\in C^1((0, \infty))$, $pA(p)\to 0$ *as* $p\to 0$, *and* $f(r, u)$, $f_r(r, u)\in C((0, \infty)\times(-\infty, \infty))$. *Let* $u(r)\in C([0, \infty))\cap C^2((0, \infty))$ *satisfy* (2.1),

$$\lim_{r\to 0} r^n\int_0^{|u'(r)|}\rho E(\rho)d\rho=0\,, \tag{2.2}$$

$$\lim_{r\to 0} r^{n-1}A(|u'(r)|)u'(r)=0\,, \tag{2.3}$$

and

$$\lim_{r\to 0} r^n F(r, u(r))=0\,. \tag{2.4}$$

Then the following generalized Pohozaev identity holds:

$$\begin{aligned} &R^n\left\{\int_0^{|u'(R)|}\rho E(\rho)d\rho+F(R, u(R))+aR^{-1}A(|u'(R)|)u'(R)u(R)\right\}\\ &=\int_0^R\left\{n\int_0^{|u'|}\rho E(\rho)d\rho+(a+1-n)A(|u'|)|u'|^2\right.\\ &\qquad\left.+nF(r, u)+rF_r(r, u)-auf(r, u)\right\}r^{n-1}dr \end{aligned} \tag{PI}$$

for all $R>0$, *where* a *is an arbitrary constant,*

$$F(r, u)=\int_0^u f(r, \xi)d\xi$$

and

$$E(p)=A(p)+p\frac{dA(p)}{dp}, \qquad p>0.$$

REMARK 2.1. It holds that

$$\int_0^p \rho E(\rho)d\rho=\int_0^p \rho(\rho A(\rho))_\rho d\rho=Ap^2-\int_0^p \rho A(\rho)d\rho, \tag{2.5}$$

since $pA(p)\to 0$ as $p\to 0$. Define

$$\varphi(p)=\begin{cases} pA(p), & p>0, \\ 0, & p=0, \end{cases}$$

and

$$\psi(p)=\begin{cases} p^2A(p)-\int_0^p \rho A(\rho)d\rho, & p>0, \\ 0, & p=0. \end{cases}$$

For the sake of convenience, we simply denote $\varphi(p)$ and $\psi(p)$ by $pA(p)$ and $\int_0^p \rho E(\rho)d\rho$, respectively.

REMARK 2.2. The equation (2.1) is equivalent to

$$E(|u'|)u''+\frac{n-1}{r}A(|u'|)u'+f(r, u)=0 \tag{2.6}$$

for all $r>0$ with $u'\neq 0$.

REMARK 2.3. This identity is a variant of Theorem 2.1 in Ni and Serrin [**9**]. The conditions (2.2)-(2.4) look too technical. However it is not difficult to check them in the concrete examples (see, e.g., the proof of Theorem 3.1 in Section 7).

The following result is our main theorem.

THEOREM 2.2. *Under the assumptions of Theorem* 2.1, *the following identity holds:*

$$\begin{aligned}
&\sigma(\sigma+1)\{(m-1)A(|u'(R)|)-E(|u'(R)|)\}R^{n-2\sigma-2}w(R)^2 \\
&+\{-R^2w''(R)w(R)+2\sigma Rw'(R)w(R)\}\{E(|u'(R)|)-A(|u'(R)|)\}R^{n-2\sigma-2} \\
&+\Big\{-R^2w''(R)w(R)+\Big(\frac{\lambda}{c}\Big(\frac{n+k}{q+1}-2(1-c)\sigma\Big)-(n-1)+2\sigma\Big)Rw'(R)w(R) \\
&\quad+\frac{\lambda}{c}(1-c)R^2w'(R)^2\Big\}R^{n-2\sigma-2}A(|u'(R)|)+\frac{\lambda}{c}R^n\Big\{cA(|u'(R)|)|u'(R)|^2
\end{aligned}$$

$$-\int_0^{|u'(R)|}\rho A(\rho)d\rho\Big\}+R^n\Big\{\frac{\lambda}{c}F(R,\ u(R))-u(R)f(R,\ u(R))\Big\}$$

(PII)

$$=\frac{\lambda}{c}\int_0^R\Big\{nF(r,\ u)+rF_r(r,\ u)-\frac{n+k}{q+1}uf(r,\ u)+\frac{(m+k)(1-\theta)}{m(q+1)}A(|u'|)|u'|^2$$

$$+n\Big(\frac{1}{m}A(|u'|)|u'|^2-\int_0^{|u'|}\rho A(\rho)d\rho\Big)\Big\}r^{n-1}dr$$

for all $R>0$ *with* $u'(R)\neq 0$, *where*

$$w(r)=r^\sigma u(r)\,,\qquad \sigma=\frac{m+k}{q+1-m}\,,$$

$$q=\frac{(m-1)(n+k)(1-\theta)+\{(m-1)n+m+mk\}\theta}{n-m}\,,\qquad \lambda=\frac{c(q+1)\theta}{c(q+1)-(1-\theta)}\,,$$

and m, k, θ, c *are arbitrary constants such that* $1<m<n$, $k>-m$, $0\leqq\theta\leqq 1$, $c>0$, $cm\geqq 1-\theta$.

REMARK 2.4. Note that $q+1-m=(m+k)(m-1+\theta)/(n-m)>0$, $c(q+1)-(1-\theta)\geqq c(q+1-m)>0$. Therefore σ, q and λ are positive constants. In fact, the above theorem holds for any choice of the constants m, k, θ, c as long as the other constants σ, q, λ are well-defined. However, applications to partial differential equations usually occur in the ranges restricted above.

REMARK 2.5. We should note that

$$(m-1)A(p)-E(p)\longrightarrow 0\qquad\text{as } p\to 0$$

in the following important examples by choosing suitable m.

(i) The generalized Laplacian: $A(p)=p^{\mu-2}$.
Take $m=\mu$, then $(m-1)A-E=0$.

(ii) The generalized mean curvature operator: $A(p)=(1+p^2)^{\mu/2-1}$.
Take $m=2$, then $(m-1)A-E=(2-\mu)(1+p^2)^{\mu/2-2}p^2\to 0$.

Therefore the coefficient of $R^{n-2\sigma-2}w(R)^2$ in (PII) vanishes as $u'(R)\to 0$. Actually we shall adjust λ so as to eliminate this coefficient as $p\to 0$, because we can not get the precise information for $w(R)$ in the applications (see, e.g., the proof of Theorem 3.3 in Section 8). The arrangement of the left-hand side of (PII) is closely related to Lemmas 4.1 and 4.2, which will appear later.

REMARK 2.6. The case $\theta=1$ and $c=1/m$ is most important. In this situation $q=((m-1)n+m+mk)/(n-m)$, $\sigma=(n-m)/m$ and $\lambda=1$.

REMARK 2.7. This theorem is very useful to obtain the non-existence theorems of the ground state in the anomalous case. The arguments employed in Ding and Ni [**2**, Theorem 5.13] and their generalizations to quasilinear equations in Ni and Serrin [**10**, Theorems 4.1, 4.2, 5.3, 5.4, 6.5, 6.6] are considerably simplified by using the identity (PII). Furthermore we can naturally understand the meaning and sharpness of the assumptions in those papers. We shall see these facts in the subsequent sections.

PROOF OF THEOREM 2.1. It follows from the assumption $pA(p)\to 0$ as $p\to 0$ that the function $\int_0^{|t|}\rho A(\rho)d\rho$ is continuously differentiable in $t\in(-\infty, \infty)$. Thus we have

$$\int_0^{|u'|}\rho A(\rho)d\rho \in C^1((0, \infty))$$

and

$$\frac{d}{dr}\int_0^{|u'|}\rho A(\rho)d\rho = A(|u'|)u'u''$$

which implies

$$\frac{d}{dr}\int_0^{|u'|}\rho E(\rho)d\rho = \frac{d}{dr}\left\{A(|u'|)|u'|^2-\int_0^{|u'|}\rho A(\rho)d\rho\right\}$$
$$= \frac{d}{dr}\left\{(r^{1-n}u')(r^{n-1}Au')-\int_0^{|u'|}\rho A(\rho)d\rho\right\}$$
$$= (1-n)r^{-n}u'(r^{n-1}Au')+r^{1-n}u''(r^{n-1}Au')+r^{1-n}u'(r^{n-1}Au')'-Au'u''$$
$$= (1-n)r^{-1}A|u'|^2-u'f$$

by using (2.1). Consequently we obtain

$$\frac{d}{dr}\left\{r^n\left(\int_0^{|u'|}\rho E(\rho)d\rho+F(r, u)+ar^{-1}A(|u'|)u'u\right)\right\}$$
$$= \frac{d}{dr}\left\{r^n\int_0^{|u'|}\rho E(\rho)d\rho+r^nF(r, u)+a(r^{n-1}Au')u\right\}$$
$$= nr^{n-1}\int_0^{|u'|}\rho E(\rho)d\rho+r^n\frac{d}{dr}\int_0^{|u'|}\rho E(\rho)d\rho$$
$$+nr^{n-1}F+r^n(F_r+fu')+a(r^{n-1}Au')'u+ar^{n-1}A|u'|^2$$
$$= nr^{n-1}\int_0^{|u'|}\rho E(\rho)d\rho+r^n\{(1-n)r^{-1}A|u'|^2-u'f\}$$
$$+nr^{n-1}F+r^n(F_r+fu')-ar^{n-1}fu+ar^{n-1}A|u'|^2$$
$$= r^{n-1}\left\{n\int_0^{|u'|}\rho E(\rho)d\rho+(a+1-n)A|u'|^2+nF+rF_r-auf\right\}$$

by virtue of (2.1). Integrating the above equality over $[0, R]$, we get (PI) by noting (2.2), (2.3) and (2.4). Q. E. D.

Now we are ready for the proof of the main theorem.

PROOF OF THEOREM 2.2. We note that

$$\int_0^p \rho E(\rho)d\rho = Ap^2-\int_0^p \rho A(\rho)d\rho = (1-c)Ap^2+\left(cAp^2-\int_0^p \rho A(\rho)d\rho\right).$$

Taking $a=(n+k)/(q+1)$ in (PI) and using the above equality, we have

$$P(R) = \int_0^R Q(r)dr\,, \tag{2.7}$$

where

$$P(r) = r^n(1-c)Ap^2+r^nF+\frac{n+k}{q+1}r^{n-1}Auu'+r^n\left(cAp^2-\int_0^p \rho A(\rho)d\rho\right), \tag{2.8}$$

$$\begin{aligned} Q(r) = r^{n-1}\Big\{&n\left(\frac{1}{m}Ap^2-\int_0^p \rho A(\rho)d\rho\right) \\ &+\frac{(m+k)(1-\theta)}{m(q+1)}Ap^2+\left(nF+rF_r-\frac{n+k}{q+1}uf\right)\Big\}, \end{aligned} \tag{2.9}$$

and $p=|u'|$. Here we use the following identity to get the coefficient of Ap^2:

$$\frac{n+k}{q+1}+1-n+\frac{n(m-1)}{m} = \frac{(m+k)(1-\theta)}{m(q+1)}.$$

We shall rearrange $P(r)$. Introduce a change of variables,

$$u = r^{-\sigma}w\,, \qquad \sigma = \frac{m+k}{q+1-m}.$$

By direct calculation, for r with $u'(r)\neq 0$ we obtain

$$u' = -\sigma r^{-\sigma-1}w+r^{-\sigma}w'\,, \qquad u'' = \sigma(\sigma+1)r^{-\sigma-2}w-2\sigma r^{-\sigma-1}w'+r^{-\sigma}w''\,,$$

$$Ew''+\Big\{(n-1)A-2\sigma E\Big\}\frac{w'}{r}+\sigma\{(\sigma+1)E-(n-1)A\}\frac{w}{r^2}+r^\sigma f = 0\,, \tag{2.10}$$

and

$$\begin{aligned} P(r) = &(1-c)(r^{n-2\sigma}(w')^2-2\sigma r^{n-2\sigma-1}ww'+\sigma^2r^{n-2\sigma-2}w^2)A \\ &+\frac{n+k}{q+1}(r^{n-2\sigma-1}ww'-\sigma r^{n-2\sigma-2}w^2)A+r^nF+r^n\left(cAp^2-\int_0^p \rho A(\rho)d\rho\right). \end{aligned} \tag{2.11}$$

Multiplying (2.10) by $-r^{n-2\sigma}w$, and (2.11) by λ/c, adding them up, we get

$$\begin{aligned} \frac{\lambda}{c}P(r) = &\sigma(\sigma+1)\{(m-1)A-E\}r^{n-2\sigma-2}w^2-Er^{n-2\sigma}ww'' \\ &+\Big\{-((n-19A-2\sigma E)+\frac{\lambda}{c}\left(-2\sigma(1-c)+\frac{n+k}{q+1}\right)A\Big\}r^{n-2\sigma-1}ww' \\ &+\frac{\lambda}{c}(1-c)r^{n-2\sigma}(w')^2A+\frac{\lambda}{c}r^n\left(cp^2A-\int_0^p \rho A(\sigma)d\rho\right)+r^n\left(\frac{\lambda}{c}F-uf\right). \end{aligned} \tag{2.12}$$

Here we have used the following identity to obtain the coefficient of $r^{n-2\sigma-2}w^2$:

$$-\sigma(\sigma+1)E+\sigma\Big\{(n-1)+\frac{\lambda}{c}\Big((1-c)\sigma-\frac{n+k}{q+1}\Big)\Big\}A=-\sigma(\sigma+1)E+\sigma(\sigma+1)(m-1)A.$$

Thus we obtain (PII) by combining (2.7), (2.9) and (2.12). Q.E.D.

§ 3. Applications to the generalized Laplace equation.

In this section we state some results obtained by virtue of generalized Pohozaev identities (PI) and (PII).

We shall treat the following generalized Laplace equation

(3.1) $$\mathrm{div}(|Du|^{m-2}Du)+f(|x|,u)=0, \qquad x\in \boldsymbol{R}^n,$$

where m is a constant. This equation corresponds to the case

(3.2) $$A(p)=p^{m-2}$$

in the equation (1.1). We are only interested in the positive radial solutions of (3.1). Thus we consider the ordinary differential equation

(F_α) $$\begin{cases} r^{1-n}(r^{n-1}|u'|^{m-2}u')'+f(r,u^+)=0, & r>0,\\ u(0)=\alpha>0, \end{cases}$$

where $u^+=\max\{u,0\}$. *We shall assume throughout this paper that*

$$1<m<n.$$

We now collect the hypotheses which will be assumed under various circumstances (but not simultaneously) in the subsequent sections. Concerning the equation (F_α), we introduce

(F.1) $\begin{cases} f(r,u)\in C((0,\infty)\times[0,\infty)); \text{ and}\\ \text{for every } M, R>0, \quad \sup\{r^{-\nu}|f(r,u)|: 0<r\leqq R,\ 0\leqq u\leqq M\}<\infty, \end{cases}$

(F.2) $f(r,u)\geqq 0$ on $(0,\infty)\times[0,\infty)$,

(F.3) $\begin{cases} \text{if } m>2, \text{ then for every } L, M, R>0,\\ \inf\{r^{-\nu}f(r,u): 0<r\leqq R,\ L\leqq u\leqq M\}>0, \end{cases}$

(F.4) for every $L, M, R>0$, $\sup\{r^{-\nu}|f_u(r,u)|: 0<r\leqq R,\ L\leqq u\leqq M\}<\infty$,

(F.5) $f_r(r,u)\in C((0,\infty)\times[0,\infty))$,

(F.6) $nF(r,u)+rF_r(r,u)\leqq \dfrac{n-m}{m}uf(r,u)$ for all $u>0$, $r>0$,

(F.7) $f(r,u)\geqq \text{Pos. Const. } r^k u^q$ for all $u>0$

and sufficiently large $r>0$, where k and q are constants satisfying $k\geqq -m$ and $q\neq m-1$, (F.8) $nF(r, u)+rF_r(r, u) \geqq \frac{n-m}{m} uf(r, u)$ for all $u>0$, $r>0$

with the strict inequality holding for some sequence of values u tending to zero and all sufficiently large $r>0$,

(F.9) $uf(r, u) \geqq mF(r, u)$ for all sufficiently small $u>0$ and sufficiently large $r>0$,

(F.2)* $\begin{cases} \text{there exists } \alpha_0>0 \text{ such that } f(r, u)<0 \text{ (resp. } f(r, u)>0) \\ \text{for all } u>\alpha_0 \text{ (resp. } u<\alpha_0) \text{ and sufficiently small } r>0; \text{ and} \\ f(r, \alpha_0)=0 \text{ for all sufficiently small } r>0, \end{cases}$

(F.7)* $f(r, u) \geqq \text{Pos. Const.}\, r^k u^q$ for sufficiently small $u>0$

and sufficiently large $r>0$, where k and q are constants satisfying $k \geqq -m$ and $q \neq m-1$, where ν is a constant satisfying $\nu>-m$, and

$$F(r, u)=\int_0^u f(r, \xi^+)d\xi .$$

We state our results.

THEOREM 3.1. *Suppose that* (F.1)-(F.5) *hold. Then there exists a unique solution* $u(r;\alpha) \in C([0, \infty)) \cap C^2((0, \infty))$ *of* (F_α), *and* $u(r)=u(r;\alpha)$ *satisfies the generalized Pohozaev-type identities,*

$$(3.3)\quad \begin{aligned} &R^n\left\{\frac{m-1}{m}|u'(R)|^m+F(R, u(R))+\frac{n-m}{m}R^{-1}|u'(R)|^{m-2}u'(R)u(R)\right\} \\ &=\int_0^R\left\{nF(r, u(r))+rF_r(r, u(r))-\frac{n-m}{m}u(r)f(r, u^+(r))\right\}r^{n-1}dr \end{aligned}$$

and

$$(3.4)\quad \begin{aligned} &(m-2)\{-R^2w''(R)w(R)+2\sigma Rw'(R)w(R)\}R^{n-2\sigma-2}|u'(R)|^{m-2} \\ &+\{-R^2w''(R)w(R)+(4\sigma-2m\sigma-m+1)Rw'(R)w(R) \\ &+(m-1)R^2w'(R)^2\}R^{n-2\sigma-2}|u'(R)|^{m-2}+R^n\{mF(R, u(R))-u(R)f(R, u^+(R))\} \\ &=m\int_0^R\left\{nF(r, u(r))+rF_r(r, u(r))-\frac{n-m}{m}u(r)f(r, u^+(r))\right\}r^{n-1}dr \end{aligned}$$

where R is an arbitrary positive number, $w(r)=r^\sigma u(r)$ *and* $\sigma=(n-m)/m$.

REMARK 3.1. We note that

$$\begin{aligned} \int_0^R F_r(r, u(r))r^n dr &= \lim_{\delta\to 0}\int_\delta^R F_r(r, u(r))r^n dr \\ &= F(R, u(R))R^n-n\int_0^R F(r, u(r))r^{n-1}dr-\int_0^R f(r, u^+(r))u'(r)r^n dr \end{aligned}$$

and

$$0 \leqq \int_0^R F(r, u(r)) r^{n-1} dr \leqq \int_0^R F(r, \alpha) r^{n-1} dr \leqq \alpha \int_0^R \tilde{f}(r, \alpha) r^{n-1} dr < \infty$$

$$0 \leqq \int_0^R f(r, u^+(r))(-u'(r)) r^n dr \leqq \text{Pos. Const.} \int_0^R \tilde{f}(r, \alpha) r^{n+(\nu+1)/(m-1)} dr < \infty$$

by the assumptions (F.1), (F.2) and the estimate (c) of Proposition 6.1 in Section 6 together with the fact that $u(r) \leqq \alpha$ (see (6.1)), where

$$\tilde{f}(r, \alpha) = \sup\{f(r, u);\ 0 \leqq u \leqq \alpha\}.$$

We shall investigate the properties of solutions of (F_α). The following theorem gives a sufficient condition for the existence of ground states.

THEOREM 3.2. *Suppose that* (F.1)-(F.5) *and* (F.6) *hold. Then, for every* $\alpha>0$, $u(r;\alpha)$ *is positive on* $[0, \infty)$.

Moreover if (F.7) *with* $q>m-1$ *holds, then* $u(r;\alpha)\to 0$ *as* $r\to\infty$.

We shall also give some sufficient conditions for the solutions of the equation (F_α) having a zero.

THEOREM 3.3. *Suppose that* (F.1)-(F.5), (F.7) *with* $q \leqq ((m-1)n+m+mk)/(n-m)$, (F.8) *and* (F.9) *hold. Then, for every* $\alpha>0$, $u(r;\alpha)$ *has a finite zero on* $[0, \infty)$.

REMARK 3.2. Theorems 3.2 and 3.3 are closely related to Theorems 3.2 and 4.1 in Ni and Serrin [**10**].

We now explain the meaning of the above theorems. Consider the equation,

$$r^{1-n}(r^{n-1}|u'|^{m-2}u')' + r^k (u^+)^q = 0\,, \qquad u(0)=\alpha>0\,, \tag{3.5}$$

where $1<m<n$, $k>-m$, and $q>m-1$. For every $\alpha>0$, (3.5) has a unique solution $u=u(r;\alpha)\in C([0,\infty))\cap C^2((0,\infty))$ by Theorem 3.1. In view of Theorems 3.2 and 3.3, the structure of solutions is as follows;

(i) If $q \geqq ((m-1)n+m+mk)/(n-m)$, then $u(r;\alpha)$ is positive on $[0,\infty)$ and tends to zero as $r\to\infty$ for every $\alpha>0$.

(ii) If $q<((m-1)n+m+mk)/(n-m)$, then $u(r;\alpha)$ has a finite zero on $[0,\infty)$ for every $\alpha>0$.

The case $f(r,u)=r^k u^{((m-1)n+m+mk)/(n-m)}$ lies on the borderline of the existence and non-existence. Here small perturbations can seriously affect the situation. If

$$f(r, u) = K(r) u^{((m-1)n+m+mk)/(n-m)}\,,$$

where $K(r)=Q(r)r^k$, $Q\in C^1([0,\infty))$, $Q(r)>0$ and $Q'(r)\leqq 0$, then $u(r;\alpha)$ is positive on $[0,\infty)$ and tends to zero as $r\to\infty$ for every $\alpha>0$. On the other hand, if

$$f(r, u) = \tilde{K}(r) u^{((m-1)n+m+mk)/(n-m)}\,,$$

where $\tilde{K}(r)=\tilde{Q}(r)r^k$, $\tilde{Q}\in C^1([0, \infty))$, $\tilde{Q}(r)>0$, $\tilde{Q}'(r)\geqq 0$ and $\tilde{Q}'(r)\not\equiv 0$, then $u(r;\alpha)$ has a zero on $[0, \infty)$ for any $\alpha>0$. These are obtained by applying Theorems 3.2 and 3.3.

The conditions for $K(r)$ and $\tilde{K}(r)$ in the above examples can be weakened by Theorems 9.2 and 9.3 in Section 9.

We consider a different kind of perturbations to the nonlinearity $f(r, u)$ $=r^k u^{((m-1)n+m+mk)/(n-m)}$. If

$$f(r, u)=r^k u^{((m-1)n+m+mk)/(n-m)}+\varepsilon r^k u^{q'},$$

where $\varepsilon>0$, $k>-m$ and $q'>((m-1)n+m+mk)/(n-m)$, then $u(r;\alpha)$ is positive on $[0, \infty)$ and tends to zero as $r\to\infty$ for every $\alpha>0$ by Theorem 3.2 (or Theorem 9.2 in Section 9). On the other hand, if

$$f(r, u)=r^k u^{((m-1)n+m+mk)/(n-m)}-\varepsilon r^k u^{q'},$$

where $\varepsilon>0$, $k>-m$ and $q'>((m-1)n+m+mk)/(n-m)$, then (F_α) has no ground state in the class $C([0, \infty))\cap C^2(0, \infty))$ by the following result.

THEOREM 3.4. *Suppose that* (F.1), (F.2)*, (F.5), (F.7)* *with* $q\leqq ((m-1)n+m+mk)/(n-m)$, (F.8) *and* (F.9) *hold. Then* (F_α) *does not admit any positive solution in* $C([0, \infty))\cap C^2((0, \infty))$ *which tends to zero as* $r\to\infty$.

§ 4. Preliminaries.

In this section, we collect some fundamental facts which will be frequently used in the proofs of the theorems stated in Section 3.

LEMMA 4.1. *Suppose that the function* $w(r)\in C^2((0, \infty))$ *is bounded and monotone for sufficiently large* $r>0$, *then there exists a sequence* $r_j\to\infty$ *such that*

$$w(r_j)\longrightarrow C \tag{4.1}$$

$$r_j w'(r_j)\longrightarrow 0 \tag{4.2}$$

$$r_j^2 w''(r_j)\longrightarrow 0 \tag{4.3}$$

as $j\to\infty$, *where* C *is a constant.*

PROOF. We have only two possibilities:

(i) $w'(r)\geqq 0$ for sufficiently large r,
(ii) $w'(r)\leqq 0$ for sufficiently large r.

In both cases, we may assume that

$$w(r)\longrightarrow C \tag{4.4}$$

as $r\to\infty$, since w is bounded.

Set $w_1(r)=rw'(r)$, $r>0$. (4.4) implies that

$$\liminf_{r\to\infty}|w_1(r)| = 0 .$$

Thus we have

$$(4.5)\qquad \begin{cases} \text{in case (i) } w_1(r)\geqq\liminf_{r\to\infty} w_1(r)=0 & \text{for large } r, \\ \text{in case (ii) } w_1(r)\leqq\limsup_{r\to\infty} w_1(r)=0 & \text{for large } r. \end{cases}$$

From (4.5) we assert that there exists a sequence $r_j\to\infty$ such that $w_1(r_j)\to 0$ and $r_j w_1'(r_j)\to 0$ by Lemma 5.25 of Ding and Ni [**2**]. Thus

$$r_j w'(r_j) = w_1(r_j) \longrightarrow 0$$

$$r_j^2 w''(r_j) = r_j w_1'(r_j) - w_1(r_j) \longrightarrow 0$$

as $j\to\infty$. Q. E. D.

LEMMA 4.2. *Under the assumptions of Lemma* 4.1, *if* $r_j\to\infty$ *is a sequence which satisfies* (4.1) *and* (4.2), *then*

$$(4.6)\qquad |u'(r)|^{m-2} r^{n-2\sigma-2} w(r)$$

and

$$(4.7)\qquad |u'(r)|^{m-2} r^{n-2\sigma-2} |rw'(r)|$$

are bounded for $r=r_j$, *where*

$$u(r) = r^{-\sigma} w(r) , \qquad \sigma = \frac{m+k}{q+1-m} ,$$

$$q = \frac{(m-1)(n+k)(1-\theta)+\{(m-1)n+m+mk\}\theta}{n-m} ,$$

and m, k, θ *are arbitrary constants such that* $1<m<n$, $k>-m$, $0\leqq\theta\leqq 1$.

PROOF. We note that $q>m-1$, $\sigma>0$, $-(1-\theta)\sigma\leqq 0$, and

$$\begin{aligned} |u'|^{m-2} r^{n-2\sigma-2} &= |r^{-\sigma} w' - \sigma r^{-\sigma-1} w|^{m-2} r^{n-2\sigma-2} \\ &= |rw'-\sigma w|^{m-2} r^{-(\sigma+1)(m-2)+n-2\sigma-2} \\ &= |rw'-\sigma w|^{m-2} r^{-(1-\theta)\sigma} \leqq |rw'-\sigma w|^{m-2} \end{aligned}$$

for sufficiently large r. We divide the proof into the following two cases.

Case 1: $m\geqq 2$ or $C\neq 0$. We have

$$|u'|^{m-2} r^{n-2\sigma-2} w \leqq |rw'-\sigma w|^{m-2} w ,$$

and

$$|u'|^{m-2} r^{n-2\sigma-2} |rw'| \leqq |rw'-\sigma w|^{m-2} |rw'| ,$$

for sufficiently large $r>0$. Hence the boundedness of (4.6) and (4.7) holds by virtue of (4.1) and (4.2).

Case 2: $1<m<2$ and $C=0$. In this case, it must hold that $w'(r)\leqq 0$ for sufficiently large $r>0$ or $w'(r)\geqq 0$ for sufficiently large $r>0$. Since $w\to 0$ monotonically as $r\to\infty$, w' and w have opposite signs. Thus we obtain

$$|rw'-\sigma w| \geqq |\sigma||w|, \qquad |rw'-\sigma w| \geqq |rw'|,$$

which implies

$$|u'|^{m-2}r^{n-2\sigma-2}|w| \leqq \sigma^{m-2}|w|^{m-1}, \qquad |u'|^{m-2}r^{n-2\sigma-2}|rw'| \leqq |rw'|^{m-1},$$

and the conclusion holds. Q. E. D.

§ 5. Asymptotic behavior.

In this section, we consider the asymptotic behaviors at ∞ of radial solutions of (3.1). Although these are essentially in Ni and Serrin [**10**], we need to extend them slightly for our purpose.

LEMMA 5.1. *Suppose that* $f(r, u)\in C((0, \infty)\times[0, \infty))$ *and* $f(r, u)\geqq 0$ *for* u *near* 0 *and sufficiently large* r. *Let* u *be a positive radial solution of* (3.1) *defined for* $r\geqq r_0$ *and tending to zero as* $r\to\infty$, *where* $r_0>0$. *Then*

(5.1) $$u \geqq \text{Pos. Const.}\, r^{-(n-m)/(m-1)} \qquad u' \leqq \text{Neg. Const.}\, r^{-(n-1)/(m-1)}$$

for all sufficiently large r.

PROOF. For a radial solution $u=u(r)$, we can write (3.1) in the following form

(5.2) $$(r^{n-1}|u'|^{m-2}u')'+r^{n-1}f(r, u) = 0\,.$$

Since $u\to 0$ as $r\to\infty$, one can suppose without loss of generality that $f(r, u)\geqq 0$ for $u=u(r)$, $r\geqq r_0$. Hence

$$(r^{n-1}|u'|^{m-2}u')' \leqq 0 \qquad \text{for } r\geqq r_0$$

and $r^{n-1}|u'|^{m-2}u'$ is a decreasing function. Since u' obviously cannot be everywhere non-negative, it follows that

(5.3) $$r^{n-1}|u'|^{m-2}u' \longrightarrow \text{negative limit} \quad (\text{possibly } -\infty)$$

as $r\to\infty$, hence $u'<0$ for all suitably large r. In particular for all large r, we have

(5.4) $$r^{n-1}|u'|^{m-2}u' \leqq -C$$

where C is some positive constant. Since $u'<0$, we get

$$u' \leqq \text{Neg. Const.}\, r^{-(n-1)/(m-1)}$$

for sufficiently large r. Integrating this inequality from a large fixed value r to ∞ yields

$$u \geqq \text{Pos. Const.}\, r^{-(n-m)/(m-1)}$$

for sufficiently large r. Q. E. D.

The following theorems concerning the asymptotic behavior of solutions at ∞ are generalizations of Theorems 2.2 and 6.2 of Ni and Serrin [**10**].

THEOREM 5.1. *Suppose that* (F.1) *and* (F.7) *hold. Let u be a positive radial solution of* (3.1) *defined for $r \geqq r_0$. Then u has the asymptotic behavior*

$$u = \begin{cases} O(r^{-(m+k)/(q-m+1)}) & \text{at } r=\infty, \text{ if } m>-k, \\ O((\log r)^{-(m-1)/(q-m+1)}) & \text{at } r=\infty, \text{ if } m=-k. \end{cases} \tag{5.5}$$

Here and in what follows $u(r)=O(\xi(r))$ at $r=\infty$ implies $\limsup_{r\to\infty} u(r)/\xi(r) < \infty$.

THEOREM 5.2. *Suppose that* (F.1), (F.7) *with $q \leqq (m-1)(n+k)/(n-m)$ hold. Then, for every $r_0>0$, there can exist no positive radial solution of* (3.1) *defined on $[r_0, \infty)$.*

COROLLARY 5.1. *Suppose that* (F.1) *and* (F.7)* *hold. Let u be a positive radial solution of* (3.1) *defined for $r \geqq r_0$ and tending to zero as $r \to \infty$. Then u has the asymptotic behavior*

$$u = O(r^{-(m+k)/(q-m+1)}) \quad \text{at } r=\infty.$$

PROOF OF THEOREMS 5.1 AND 5.2. We have by (F.7)

$$f(r, u) \geqq \text{Pos. Const.}\, r^k u^q \quad \text{for } r \geqq r_1, \tag{5.6}$$

where r_1 $(>r_0)$ is sufficiently large. It follows from (5.2) that

$$(r^{n-1}|u'|^{m-2}u')' = -r^{n-1}f(r, u) \leqq 0 \quad \text{for } r \geqq r_1, \tag{5.7}$$

which implies that $r^{n-1}|u'|^{m-2}u'$ is a decreasing function. Thus

$$r^{n-1}|u'|^{m-2}u' \longrightarrow C\ (\geqq -\infty) \quad \text{as } r \to \infty, \tag{5.8}$$

where C is a constant. Suppose that $C \geqq 0$. Then we have

$$u'(r) \geqq 0 \quad \text{for } r \geqq r_1, \tag{5.9}$$

so $u(r)$ is non-decreasing. We have $u(r) \geqq u(r_1) > 0$. Integrating (5.7) over $[r_1, r]$, we obtain

$$\begin{aligned} -r^{n-1}|u'(r)|^{m-2}u'(r) &= -r_1^{n-1}|u'(r_1)|^{m-2}u'(r_1) + \int_{r_1}^{r} s^{n-1} f(s, u(s))\,ds \\ &\geqq -r_1^{n-1}|u'(r_1)|^{m-2}u'(r_1) + \text{Pos. Const.}\int_{r_1}^{r} s^{n-1+k} u(r_1)^q\,ds \\ &\geqq \text{Const.} + \text{Pos. Const.}\, r^{n+k} > 0 \end{aligned}$$

for sufficiently large r. Thus

$$u'(r) < 0 \quad \text{for sufficiently large } r, \tag{5.10}$$

which contradicts (5.9). Therefore we see that $C<0$ and (5.10) holds.

Integrating (5.7) over $[r_1, r]$ gives

$$-r^{n-1}|u'(r)|^{m-2}u'(r)+r_1^{n-1}|u'(r_1)|^{m-2}u'(r_1)=\int_{r_1}^{r}s^{n-1}f(s, u(s))ds.$$

Hence by (F.7)

$$\int_{r_1}^{r}s^{n-1}f(s, u(s))ds \geqq \text{Pos. Const.}\int_{r_1}^{r}s^{n-1+k}u(s)^q ds$$

$$\geqq \text{Pos. Const. } u(r)^q\int_{r_1}^{r}s^{n-1+k}ds=\text{Pos. Const.}(r^{n+k}-r_1^{n+k})u(r)^q,$$

where we have used the fact that $u(s)>u(r)$ for $s<r$. Choosing $r\geqq 2r_1$ and noting again that $u'(r_1)<0$, we have

$$-r^{n-1}|u'(r)|^{m-2}u'(r)\geqq \text{Pos. Const.}(1-2^{-(n+k)})r^{n+k}u(r)^q,$$

that is

$$-u^{-q/(m-1)}u'\geqq cr^{(1+k)/(m-1)}, \tag{5.11}$$

where

$$c=\{\text{Pos. Const.}(1-2^{-(n+k)})\}^{1/(m-1)}.$$

Integrating (5.11) from $s_0=2r_1$ to r $(>s_0)$, we obtain

$$\frac{m-1}{q+1-m}\{u(r)^{(m-1-q)/(m-1)}-u(s_0)^{(m-1-q)/(m-1)}\}$$

$$\geqq\begin{cases}\dfrac{c(m-1)}{m+k}(r^{(m+k)/(m-1)}-s_0^{(m+k)/(m-1)}), & \text{if } k>-m,\\ c\log(r/s_0), & \text{if } k=-m.\end{cases}$$

When $q<m-1$ an immediate contradiction results by noting $u>0$ and letting $r\to\infty$. If $q>m-1$, then, for all sufficiently large r, we have

$$u\leqq\begin{cases}\text{Pos. Const. } r^{-(m+k)/(q+1-m)}, & \text{if } k>-m,\\ \text{Pos. Const.}(\log r)^{-(m-1)/(q+1-m)}, & \text{if } k=-m.\end{cases} \tag{5.12}$$

This is the required estimate (5.5).

Next suppose that $m-1<q<(m-1)(n+k)/(n-m)$. In this case (5.12) contradicts the conclusion of Lemma 5.1, namely that

$$u\geqq \text{Pos. Const. } r^{-(n-m)/(m-1)}.$$

Hence no solution of the type under consideration can exist.

It remains to show the same result when $q=(m-1)(n+k)/(n-m)$. In this case, using Lemma 5.1 and the hypothesis $f(r, u)\geqq$Pos. Const. $r^k u^{(m-1)(n+k)/(n-m)}$, we see that

$$f(r, u)\geqq \text{Pos. Const. } r^{-n}$$

for all sufficiently large r. Hence by (5.7)

$$(r^{n-1}|u'|^{m-2}u')' = -r^{n-1}f(r, u) \leqq -C_0 r^{-1},$$

where C_0 is a positive constant. By integration from s to r, where s is suitably large and $r>s$, we obtain

$$r^{n-1}|u'(r)|^{m-2}u'(r)-s^{n-1}|u'(s)|^{m-2}u'(s) \leqq -C_0\log\left(\frac{r}{s}\right).$$

It follows that $r^{n-1}|u'|^{m-2}u'$ tends to $-\infty$ as $r\to\infty$. By the argument used at the end of the proof of Lemma 5.1 this in turn implies that $r^{n-m}u^{m-1}\to\infty$ as $r\to\infty$. On the other hand, since $q=(m-1)(n+k)/(n-m)$ in the present case, the inequality (5.12) implies that $u=O(r^{-(n-m)/(m-1)})$. The resulting contradiction establishes that there can be no solutions of the type under consideration even when $q=(m-1)(n+k)/(n-m)$, and thus completes the proof of the theorems. Q. E. D.

PROOF OF COROLLARY 5.1. It is obvious from the proof of Theorem 5.1. Q.E.D.

§6. Initial value problems.

Most of the results in this section are extensions of some well-known theorems in semilinear elliptic equations. We include a brief proof for each one of them for the reader's convenience.

PROPOSITION 6.1. *Suppose that* (F.1) *and* (F.2) *hold, and* $\alpha>0$. *Then the following two conditions are equivalent:*

(i) $u \in C([0, \infty))\cap C^2((0, \infty))$ *satisfies* (F_α),

(ii) $u\in C([0, \infty))$ *satisfies*

$$u(r) = \alpha-\int_0^r\left\{\int_0^t\left(\frac{s}{t}\right)^{n-1}f(s, u^+(s))ds\right\}^{1/(m-1)}dt. \tag{6.1}$$

Moreover in both cases, the following properties hold:

(a) $\lim_{r\to 0} r^{n-1}|u'(r)|^{m-2}u'(r) = 0$,

(b) $u'(r) = -\left\{\int_0^r\left(\frac{s}{r}\right)^{n-1}f(s, u^+(s))ds\right\}^{1/(m-1)} \leqq 0$

for all $r>0$,

(c) $u'(r) = O(r^{(\nu+1)/(m-1)})$ *at* $r=0$,

(d) u *is non-increasing on* $[0, \infty)$, *and furthermore if* $f(r, u)>0$ *on* $(0, \infty)\times(0, \infty)$, *then* $u'(r)<0$ *for all* $r>0$,

(e) *if* $u(R)=0$ *with* $R>0$, *then* $u'(R)<0$,

(f) *if* $\sup\{f(r, \xi): 0\leqq\xi\leqq\alpha\}\in L^1_{\text{loc}}([0, \infty))$, *then* $u\in C^1([0, \infty))\cap C^2((0, \infty))$ *and*

$u'(0)=0$,

(g) *if* $f(r, u) \in C([0, \infty)\times[0, \infty))$ *and* $m \leqq 2$, *then* $u \in C^2([0, \infty))$ *and* $u'(0)=0$.

REMARK 6.1. This result is a generalization of Ni [7], and Ni and Yotsutani [**13**, Proposition 4.1].

PROOF. We first show that (i) implies (ii). Put $U(s)=u(r)$, $s=r^{-(n-m)/(m-1)}$, $U_s=dU(s)/ds$. Then

$$(|U_s|^{m-2}U_s)_s = -(m-1)^m(n-m)^{-m}s^{-m(n-1)/(n-m)}f(s^{-(m-1)/(n-m)}, U^+) \leqq 0,$$

for all $s \in (0, \infty)$, and

$$\lim_{s\to\infty} U(s) = \alpha > 0.$$

Thus $|U_s|^{m-2}U_s$ is non-increasing in $s \in (0, \infty)$, $|U_s|^{m-2}U_s \geqq 0$ for all $s \in (0, \infty)$, and $|U_s|^{m-2}U_s \downarrow 0$ as $s \to \infty$. Consequently we have

$$0 = \lim_{s\to\infty} |U_s(s)|^{m-1} = \lim_{r\to 0}\left|\left(\frac{m-1}{n-m}\right)^{m-1} r^{n-1}|u'|^{m-2}u'\right| = 0, \tag{6.2}$$

which implies (a). Multiplying (F_α) by r^{n-1}, we have

$$(r^{n-1}|u'|^{m-2}u')' = -r^{n-1}f(r, u). \tag{6.3}$$

Integrating (6.3) over $[\varepsilon, r]$ and letting $\varepsilon \to 0$, we obtain (b) from (a). (c) is obvious. Integrating (b) over $[0, r]$ and using $u(0)=\alpha$, we have (6.1). (d), (f) and (g) follow from (b) and (F_α). We shall now show (e). Suppose that $u(R)=0$ and $u'(R)=0$. It follows from (b) that $f(r, u(r)) \equiv 0$ on $(0, R]$. Thus $u'(r) \equiv 0$ on $(0, R]$ by (b), which implies that $u(r) \equiv \alpha > 0$ on $[0, R]$. This is a contradiction. It is readily seen that (ii) implies (i). Q.E.D.

For the existence and uniqueness of the solutions of the problem (F_α), we have

PROPOSITION 6.2. *Suppose that* (F.1)-(F.4) *hold. Then there exists a unique solution of* (F_α), *which has the properties* (a)-(g) *of Proposition* 6.1.

PROOF. By Proposition 6.1, we only have to prove the uniqueness and existence of the solutions of the integral equation (6.1). We first show the uniqueness. Let $u_1(r)$ and $u_2(r)$ be solutions of (6.1). We note that $u_i(r)$ $(i=1, 2)$ are non-increasing and $u_i(r) \leqq \alpha$ $(i=1, 2)$ for $r>0$ by (b) of Proposition 6.1. Suppose that $u_i(r)>0$ on $[0, R_i]$ $(i=1, 2)$. Put $R=\min\{R_1, R_2\}$. It holds from (6.1) that

$$
\begin{aligned}
&|u_1(r)-u_2(r)| \\
&= \left|-\int_0^r \left\{\left(\int_0^t \left(\frac{s}{t}\right)^{n-1} f(s, u_1)ds\right)^{1/(m-1)} - \left(\int_0^t \left(\frac{s}{t}\right)^{n-1} f(s, u_2)ds\right)^{1/(m-1)}\right\} dt\right| \\
&\leqq \int_0^r \left|\int_0^1 \frac{d}{d\theta}\left\{\int_0^t \left(\frac{s}{t}\right)^{n-1}(\theta f(s, u_1)+(1-\theta)f(s, u_2))ds\right\}^{1/(m-1)} d\theta\right| dt \\
&= \int_0^r \frac{1}{m-1}\left|\int_0^1 \left\{\int_0^t \left(\frac{s}{t}\right)^{n-1}(\theta f(s, u_1)+(1-\theta)f(s, u_2))ds\right\}^{1/(m-1)-1}\right. \\
&\qquad\qquad \left.\cdot\left\{\int_0^t \left(\frac{s}{t}\right)^{n-1}(f(s, u_1)-f(s, u_2))ds\right\} d\theta\right| dt \\
&\leqq \int_0^r \frac{1}{m-1}\int_0^1 \left\{\int_0^t \left(\frac{s}{t}\right)^{n-1} s^{\nu} s^{-\nu}(\theta f(s, u_1)+(1-\theta)f(s, u_2))ds\right\}^{1/(m-1)-1} \\
&\qquad\qquad \cdot\left\{\int_0^t \left(\frac{s}{t}\right)^{n-1}|f(s, u_1)-f(s, u_2)|ds\right\} d\theta dt \\
&\leqq \int_0^r (m-1)^{-1}\left\{C_1\int_0^t \left(\frac{s}{t}\right)^{n-1} s^{\nu} ds\right\}^{1/(m-1)-1} t^{-n+1}\left\{\int_0^t s^{n-1}|f(s, u_1)-f(s, u_2)|ds\right\} dt \\
&= \int_0^r C_2 t^{-(\nu+1)(m-2)/(m-1)-n+1}\left\{\int_0^t s^{n-1}|f(s, u_1)-f(s, u_2)|ds\right\} dt \\
&= C_2\int_0^r s^{n-1}|f(s, u_1)-f(s, u_2)|\left\{\int_s^r t^{-(\nu+1)(m-2)/(m-1)-n+1} dt\right\} ds \\
&= C_2\int_0^r s^{-\nu}|f(s, u_1)-f(s, u_2)|\, s^{n+\nu-1}\left\{\int_s^r t^{-(\nu+1)(m-2)/(m-1)-n+1} dt\right\} ds \\
&\leqq C_3\int_0^r G(s)|u_1(s)-u_2(s)|ds\,,
\end{aligned}
$$

for any $r\in(0, R]$, where

$$
\begin{aligned}
&C_1 = \begin{cases} \sup\{s^{-\nu}f(s, u): 0<s\leqq R,\ \varepsilon\leqq u\leqq\alpha\} & \text{if } 1<m\leqq 2\,, \\ \inf\{s^{-\nu}f(s, u): 0<s\leqq R,\ \varepsilon\leqq u\leqq\alpha\} & \text{if } m>2\,, \end{cases} \\
&C_2 = (m-1)^{-1}(n+\nu)^{-1}C_1^{1/(m-1)-1}\,, \\
&C_3 = C_2\sup\{s^{-\nu}|f_u(s, u)| : 0<s\leqq R,\ \varepsilon\leqq u\leqq\alpha\}, \qquad \text{and} \\
&G(s) = s^{n+\nu-1}\int_s^R t^{-(\nu+1)(m-2)/(m-1)-n+1} dt\,,
\end{aligned}
$$

with $\varepsilon=\min\{u_1(R), u_2(R)\}$. Since

$$\int_0^R G(s)ds = \frac{m-1}{(n+\nu)(m+\nu)} R^{(m+\nu)/(m-1)} < \infty\,,$$

by $n>m>1$ and $\nu>-m$, we have

$$|u_1(r)-u_2(r)| = 0 \qquad \text{on } [0, R]$$

by using Gronwall's inequality.

Define $R_*(u_i)$ by $\inf\{r>0: u_i(r)=0\}$ $(i=1, 2)$. This is well-defined and $0<R_*(u_i)\leqq\infty$ $(i=1, 2)$. It follows from the above argument that $0<R_*=R_*(u_1)=R_*(u_2)\leqq\infty$, and $u_1=u_2$ on $[0, R_*)$. If $R_*<\infty$, then we obtain $u_1=u_2$ on $[R_*, \infty)$ as well, since u_1 and u_2 are decreasing and satisfy

$$(r^{n-1}|u_1'|^{m-2}u_1')'=(r^{n-1}|u_2'|^{m-2}u_2')'=-r^{n-1}f(r, 0) \qquad \text{in } (R_*, \infty)$$
$$u_1(R_*)=u_2(R_*)=0\,, \qquad u_1'(R_*)=u_2'(R_*)\,.$$

Thus the uniqueness holds.

It is easily seen from the above estimates that the existence follows from the standard arguments. Q. E. D.

§7. Special case of the generalized Pohozaev identity.

In this section we shall give the proofs of Theorems 3.1 and 3.2 stated in Section 3.

PROOF OF THEOREM 3.1. The existence and uniqueness of the solution of (F_α) follow from Proposition 6.2. We shall show (3.3) and (3.4). Take $A(p)=p^{m-2}$, $a=(n-m)/m$, $c=1/m$, $\theta=1$ and $f(r, u^+)$ instead of $f(r, u)$ in (PI) and (PII). From (c) of Proposition 6.1, we see that $|u'|=O(r^{(\nu+1)/(m-1)})$ at $r=0$. Thus we have

$$r^n\int_0^{|u'|}\rho E(\rho)d\rho=\frac{m-1}{m}r^n|u'|^m=O(r^{n+m(\nu+1)/(m-1)}) \qquad \text{at } r=0\,, \tag{7.1}$$

and

$$r^{n-1}A(|u'|)|u'|=r^{n-1}|u'|^{m-1}=O(r^{n+\nu}) \qquad \text{at } r=0\,. \tag{7.2}$$

Since $1<m<n$ and $-m<\nu$, the assumptions (2.2), (2.3) and (2.4) in Theorem 2.1 are satisfied. Thus we obtain (3.3) from (PI). Similarly we obtain (3.4) from (PII). Q. E. D.

PROOF OF THEOREM 3.2. Let $\alpha>0$ be arbitrary but fixed. Suppose that there exists $R>0$ such that

$$u(r;\alpha)>0 \qquad \text{for } r\in[0, R)\,, \qquad u(R;\alpha)=0\,.$$

It follows from (3.3), (F.6) and (e) of Proposition 6.1 that

$$0<\frac{m-1}{m}R^n|u'(R)|^m=\int_0^R\Big\{nF(r, u)+rF_r(r, u)-\frac{n-m}{m}uf(r, u)\Big\}r^{n-1}dr\leqq 0\,,$$

which is a contradiction. Thus $u(r;\alpha)$ is positive on $[0, \infty)$.

Moreover if (F.7) holds, then $u(r;\alpha)\to 0$ as $r\to\infty$ by Theorem 5.1. Q. E. D.

§ 8. Non-existence theorems for the ground state.

In this section we shall give the proofs of Theorems 3.3 and 3.4 stated in Section 3.

PROOF OF THEOREM 3.3. Suppose that $u(r)=u(r;\alpha)$ is a solution of (F_α) which is positive on $[0, \infty)$. Setting

$$w(r)=r^{\sigma}u(r)\,, \qquad \sigma=(n-m)/m\,, \tag{8.1}$$

we have (3.4) by (F.1)-(F.5) and Theorem 3.1, and also

$$0<w(r)\leqq \text{Pos. Const.} \qquad \text{for } r\geqq 1 \tag{8.2}$$

by (F.7) with $q\leqq((m-1)n+m+mk)/(n-m)$ and Theorem 5.1. It follows from (F.8) that there exist positive constants C_1 and R_1 such that

$$\text{the right-hand side of (3.4)} \geqq C_1>0 \qquad \text{for all } R>R_1\,. \tag{8.3}$$

Next, we claim that $w(r)$ has no local minimum for large $r>0$. To see this, suppose that $w(r)$ has a local minimum at $r=\rho$ which is so large that (F.9) holds. Then we have

$$w(\rho)>0\,, \qquad w'(\rho)=0\,, \qquad w''(\rho)\geqq 0\,. \tag{8.4}$$

It follows from (8.4) and (F.9) that

$$\begin{aligned}\text{the left-hand side of (3.4)} &= -(m-1)\rho^2 w''(\rho)w(\rho)\rho^{n-2\sigma-2}|u'(\rho)|^{m-2}\\ &\quad +\rho^n\{mF(\rho, u(\rho))-u(\rho)f(\rho, u(\rho))\}\leqq 0\,,\end{aligned} \tag{8.5}$$

which contradicts (8.3), and our assertion is established.

It is now clear that we have only two possibilities:

(i) $w'(r)\geqq 0$ for sufficiently large r,
(ii) $w'(r)\leqq 0$ for sufficiently large r.

Thus it follows from (8.2) and Lemmas 4.1 and 4.2 that there exists a sequence $r_j\to\infty$ such that

$$w(r_j)\longrightarrow C\,, \tag{8.6}$$

$$r_j w'(r_j)\longrightarrow 0\,, \tag{8.7}$$

$$r_j^2 w''(r_j)\longrightarrow 0\,, \tag{8.8}$$

$$|u'(r_j)|^{m-2}r_j^{n-2\sigma-2}w(r_j) \quad \text{are bounded,} \tag{8.9}$$

$$|u'(r_j)|^{m-2}r_j^{n-2\sigma-2}|r_j w'(r_j)| \quad \text{are bounded,} \tag{8.10}$$

as $j\to\infty$. Taking $R=r_j$ in (3.4), and letting $j\to\infty$, we see from (F.9) and (8.3) that $0\geqq C_1>0$, which is a contradiction. Q.E.D.

PROOF OF THEOREM 3.4. Suppose that $u(r)=u(r;\alpha)$ is a solution of (F_α)

which is positive on $[0, \infty)$ and tends to zero as $r\to\infty$. We divide the proof into the following two cases.

Case 1: $\alpha\neq\alpha_0$. It follows from the assumption (F.2)* and the proof of Proposition 6.1 that $r^{n-1}|u'(r)|^{m-2}u'(r)\to 0$ as $r\to 0$. Thus we obtain (3.4) by virtue of the proof of Theorem 3.1. It follows from Corollary 5.1 and the proof of Theorem 3.3 that $u(r;\alpha)$ has a zero, which is a contradiction.

Case 2: $\alpha=\alpha_0$. It follows from (F.2)* that

$$(r^{n-1}|u'|^{m-2}u')' = -r^{n-1}f(r, u) > 0 \qquad (\text{resp. } <0)$$

for sufficiently small $r>0$ and $u(r)>\alpha$ (resp. $u(r)<\alpha$). Thus $u(r)$ can not cross the line $u=\alpha$, which implies that $f(r, u(r))$ does not change sign near $r=0$. Therefore we have $r^{n-1}|u'(r)|^{m-2}u'(r)\to 0$ as $r\to 0$ by the proof of Proposition 6.1, and we complete the proof by the same arguments as in Case 1. Q. E. D.

§ 9. Special nonlinearity.

As a special case of (F_α), we consider the following problem:

$$(\mathrm{K}_\alpha) \qquad \begin{cases} r^{1-n}(r^{n-1}|u'|^{m-2}u')'+\sum_{i=1}^{I} K_i(r)(u^+)^{q_i} = 0, & r>0, \\ u(0)=\alpha>0, \end{cases}$$

where $u^+=\max\{u, 0\}$. In this special case, we can improve Theorems 3.1-3.3 by using the idea introduced by Kusano and Naito [3] (see the proof of Theorem 9.1). The following hypotheses will be assumed under various circumstances (but not simultaneously).

(K.1) $\quad K_i\in C^1((0,\infty))$ and $K_i(r)=O(r^{\nu_i})$ at $r=0$,

(K.2) $\quad \begin{cases} K_i\geqq 0 \quad \text{on } (0,\infty); \text{ and} \\ \text{if } m>2, \text{ then, for every } R>0,\ \inf\{r^{-\nu_i}K_i(r);\ 0<r\leqq R\}>0, \end{cases}$

(K.3) $\quad \int_0^R (r^{-\mu_i}K_i(r))' r^{n+\mu_i} dr \leqq 0 \quad$ for all $R>0$,

(K.4) $\quad \sum_{i=1}^{I} (r^{-\mu_i}K_i(r))' \not\equiv 0$,

(K.5) $\quad \int_0^R (r^{-\mu_i}K_i(r))' r^{n+\mu_i} dr \geqq 0 \quad$ for all $R>0$,

for all $1\leqq i\leqq I$, where q_i, ν_i, μ_i are constants satisfying

$$q_i>m-1, \qquad \nu_i>-m,$$
$$\mu_i=\{(n-m)q_i-((m-1)n+m)\}/m.$$

Now we state our results.

THEOREM 9.1. *Suppose that* (K.1) *and* (K.2) *hold. Then there exists a unique solution* $u(r;\alpha)\in C([0,\infty))\cap C^2((0,\infty))$ *of* (K_α). *Moreover,* $u=u(r;\alpha)$ *satisfies the following generalized Pohozaev identity*

$$R^n\left\{\frac{m-1}{m}|u'(R)|^m+\sum_{i=1}^{I}\frac{1}{q_i+1}K_i(R)u^+(R)^{q_i+1}+\frac{n-m}{m}R^{-1}|u'(R)|^{m-2}u'(R)u(R)\right\}$$

$$(9.1)\quad =\sum_{i=1}^{I}\frac{1}{q_i+1}u^+(R)^{q_i+1}\int_0^R(r^{-\mu_i}K_i(r))'r^{n+\mu_i}dr$$
$$+\sum_{i=1}^{I}\int_0^R\left\{\int_0^r(s^{-\mu_i}K_i(s))'s^{n+\mu_i}ds\right\}u^+(r)^{q_i}(-u'(r))dr$$

and its variant

$$(m-2)\{-R^2w''(R)w(R)+2\sigma Rw'(R)w(R)\}R^{n-2\sigma-2}|u'(R)|^{m-2}$$
$$+\{-R^2w''(R)w(R)+(4\sigma-2m\sigma-m+1)Rw'(R)w(R)+(m-1)R^2w'(R)^2\}$$
$$(9.2)\quad \cdot R^{n-2\sigma-2}|u'(R)|^{m-2}+R^n\sum_{i=1}^{I}\left(\frac{m}{q_i+1}-1\right)K_i(R)u^+(R)^{q_i+1}$$
$$=\sum_{i=1}^{I}\frac{m}{q_i+1}u^+(R)^{q_i+1}\int_0^R(r^{-\mu_i}K_i(r))'r^{n+\mu_i}dr$$
$$+m\sum_{i=1}^{I}\int_0^R\left\{\int_0^r(s^{-\mu_i}K_i(s))'s^{n+\mu_i}ds\right\}u^+(r)^{q_i}(-u'(r))dr,$$

where R *is an arbitrary positive number,* $w(r)=r^\sigma u(r)$ *and* $\sigma=(n-m)/m$.

REMARK 9.1. By the definition of μ_i, we have

$$(9.3)\quad q_i=((m-1)n+m+m\mu_i)/(n-m).$$

As a consequence of the above theorem, we obtain the following theorems.

THEOREM 9.2. *Suppose that* (K.1), (K.2) *and* (K.3) *hold. Then, for every* $\alpha>0$, $u(r;\alpha)$ *is positive on* $[0,\infty)$.

Moreover if there exist constants $C>0$, $k>-m$ *and* i *with* $1\leqq i\leqq I$ *such that* $K_i(r)\geqq Cr^k$ *for sufficiently large* r, *then* $u(r;\alpha)\to 0$ *as* $r\to\infty$.

THEOREM 9.3. *Suppose that* (K.1), (K.2), (K.4) *and* (K.5) *hold. Then, for every* $\alpha>0$, $u(r;\alpha)$ *has a finite zero on* $[0,\infty)$.

We now give the proofs of Theorems 9.1, 9.2 and 9.3.

PROOF OF THEOREM 9.1. We obtain the existence and uniqueness of the solutions of (K_α) by (K.1), (K.2) and Theorem 3.1. We shall show (9.1). It follows from (K.1), (K.2) and Theorem 3.1 that (3.3) and (3.4) hold. We note that

$$(9.4)\qquad mF(r,\ u)-uf(r,\ u^{+})=\sum_{i=1}^{I}\Big(\frac{m}{q_i+1}-1\Big)K_i(r)u^{+}(r)^{q_i+1}.$$

Rearranging the right-hand sides of (3.3) and (3.4), we have

$$\Big\{nF(r,\ u)+rF_r(r,\ u)-\frac{n-m}{m}uf(r,\ u^{+})\Big\}r^{n-1}$$
$$=\sum_{i=1}^{I}\frac{1}{q_i+1}\{(-\mu_i)K_i+K_i'r\}r^{n-1}u^{+}(r)^{q_i+1}$$
$$=\sum_{i=1}^{I}\frac{1}{q_i+1}(r^{-\mu_i}K_i)'r^{n+\mu_i}u^{+}(r)^{q_i+1}$$
$$=\sum_{i=1}^{I}\frac{1}{q_i+1}\Big(\int_0^r(s^{-\mu_i}K_i(s))'s^{n+\mu_i}ds\Big)'u^{+}(r)^{q_i+1}.$$

Thus the right-hand sides of (3.3) and (3.4) become that of (9.1) and (9.2) in view of integration by parts, and the proof is complete. (The rearrangement of the right-hand side was employed in Lemma 1 of Kusano and Naito [**3**].) Q.E.D.

PROOF OF THEOREM 9.2. The assertion follows obviously from (9.1) and the proof of Theorem 3.2. Q. E. D.

PROOF OF THEOREM 9.3. It follows from (K.5) that

$$\int_0^r(s^{-\mu_i}K_i(s))'s^{n+\mu_i}ds\geqq 0\qquad\text{for all } r>0,$$

which implies, by integration by parts,

$$r^nK_i(r)\geqq(n+\mu_i)\int_0^r s^{n-1}K_i(s)ds.$$

Solving this integral inequality, we have

$$K_i(r)\geqq \text{Pos. Const. } r^{\mu_i}\qquad\text{for all } r\geqq 1.$$

Therefore $w(r)=r^{(n-m)/m}u(r)$ is bounded on $[1,\infty)$ by virtue of Theorem 5.1 and Remark 9.1. Thus we obtain the conclusion by (9.2), (9.4) and the argument similar to the proof of Theorem 3.3. Q. E. D.

§ 10. Concluding remarks.

In this paper we have established the generalized Pohozaev identities (PI) and (PII). Although we have only applied them to the generalized Laplace equation, the identities are useful for other kind of equations (e. g., prescribed mean curvature equations). In fact, we can simplify the proofs of Theorems 4.1, 4.2, 5.3, 5.4, 6.5 and 6.6 in Ni and Serrin [**10**]. Moreover, it is not difficult to generalize those results to the case $f(r,\ u)$ with the r-dependence instead of $f(u)$.

We should note that in that situation we shall need to assume further that $|u'(r)|\to 0$ as $r\to\infty$.

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Nichiro KAWANO
Department of Mathematics
Faculty of Education
Miyazaki University
Miyazaki 889-21
Japan

Wei-Ming NI
School of Mathematics
University of Minnesota
Minneapolis
Minnesota 55455
U.S.A.

Shoji YOTSUTANI
Department of Applied Mathematics and Informatics
Faculty of Science and Technology
Ryukoku University
Ohtsu 520-21
Japan