J. Math. Soc. Japan
Vol. 43, No. 3, 1991

Compact Liouville surfaces

Dedicated to Professor Noboru Tanaka on his 60th birthday

By Kazuyoshi KIYOHARA

(Received Feb. 13, 1990)
(Revised Oct. 1, 1990)

Introduction.

A (local) Liouville surface is by definition a surface which is equipped with a riemannian metric of the following form:

$$g=(f_1(x_1)+f_2(x_2))(dx_1^2+dx_2^2),$$

where $x=(x_1, x_2)$ is a coordinate system, and f_i is a function of the single variable x_i $(i=1, 2)$. This type of metric is called a Liouville metric. A remarkable property of a Liouville surface is that the geodesic flow has the following first integral F. Let (x, ξ) be the canonical coordinate system on the cotangent bundle, and let

$$E=\frac{1}{2}\frac{1}{f_1(x_1)+f_2(x_2)}(\xi_1^2+\xi_2^2)$$

be the energy function associated with the riemannian metric g. If we put

$$F=\frac{1}{f_1(x_1)+f_2(x_2)}(f_2(x_2)\xi_1^2-f_1(x_1)\xi_2^2),$$

then it is easy to see that the Poisson bracket $\{E, F\}$ vanishes. The ellipsoid is a classical example of the Liouville surface, which is originally due to Jacobi (see Darboux [3] and Klingenberg [7] on this example and historical remarks).

The main purpose of this paper is to give a proper definition of compact Liouville surfaces, and to classify them. It is classically known that a Liouville surface is locally characterized as a 2-dimensional riemannian manifold whose geodesic flow has a first integral which is a homogeneous polynomial of degree 2 on each fibre (cf. Darboux [3] Livre VI, Chapitre II). In §1 we first review this fact. This leads us to the following definition: A compact Liouville surface (S, g, F) is a compact 2-dimensional riemannian manifold (S, g) wnose geodesic flow has a first integral F which is fibrewise a homogeneous polynomial of degree 2, and which is not a constant multiple of the energy function E. We also assume that F does not come from a local Killing vector field, which means that F is *not* of the form $\pm H+rE$, where $r\in \boldsymbol{R}$ and $H\in C^{\infty}(T^*S)$ is fibrewise the square of a linear form.

We say that two compact Liouville surfaces (S, g, F) and (S', g', F') are equivalent if there is an isometry $\phi: (S, g)\to(S', g')$ and $r, s\in \boldsymbol{R}$ such that $F'\circ(\phi^*)^{-1}=rF+sE$. In case $r=1$ and $s=0$, then these two Liouville surfaces are said to be isomorphic. We will classify these equivalence classes. In §2 we study the geometric properties of the points where the first integral F is proportional to the energy function E. Let $\mathfrak{N}$ be the set of all such points. Then it turns out that $\#\mathfrak{N}$, the number of the points in $\mathfrak{N}$, is zero if S is diffeomorphic to the torus or the Klein bottle, $\#\mathfrak{N}$ is 2 if S is diffeomorphic to the real projective plane $\boldsymbol{R}P^2$, $\#\mathfrak{N}$ is 4 if S is diffeomorphic to the sphere S^2, and no such metrics exist if the genus of S is greater than 1 (Theorem 2.1).

In §3 the continuation of the natural local coordinate system of the Liouville surface (S, g, F) is studied. As a result, in case S is diffeomorphic to the sphere, we obtain a conformal mapping Φ from a flat torus $\boldsymbol{R}^2/\Gamma$ to (S, g) such that Φ^*g is of the form described before. Here Γ is a lattice generated by an orthogonal basis. The classification in this case is done by using this mapping. A similar result is shown for the case where S is diffeomorphic to the torus.

In §4 we concentrate our attention on a certain type of Liouville surfaces (S^2, g, F) which includes both ellipsoids and the standard sphere. There we find families of riemannian manifolds which are not isometric, but whose geodesic flows are mutually symplectically isomorphic. (A similar result has been obtained in Kiyohara [**6**] in the case of surfaces of revolution.) In particular we find concrete examples of C_l-metrics on S^2 which seem to be new. A C_l-metric on a compact manifold is by definition a riemannian metric such that every geodesic is closed and has the same length l. Although Guillemin [**4**] proved that there are many C_l-metrics around the standard metric on S^2, their explicit forms were not known except for Zoll's examples. (See Besse [**2**] for the generality of the C_l-metric.)

In §5 we prove that if $(S, g, F_i)(i=1, 2)$ are two Liouville surfaces, then $F_2\in \boldsymbol{R}F_1+\boldsymbol{R}E$, unless (S, g) has positive constant gaussian curvature. This means that our equivalence classes are nothing but the isometry classes. As a corollary we see that any Liouville surface does not admit a non-zero Killing vector field under the same assumption.

In the last section, §6, we briefly mention about the laplacian acting on functions on a Liouville surface (S, g, F). We show that a second order linear differential operator whose principal symbol is $-F$ is naturally defined, and that it commutes with the laplacian. This enables us to transform the defining equation of eigenfunctions of the laplacian into a pair of single ordinary differential equations of second order. If the underlying surface is the sphere, and if everything is analytic, then this pair of equations are replaced by a single ordinary differential equation on a complex domain.

Preliminary remark.

We assume the differentiability of class C^∞ unless otherwise stated. Let (S, g) be a riemannian manifold. The metric tensor g induces the bundle isomorphism from the tangent bundle TS to the cotangent bundle T^*S as usual. If H is a function on the cotangent bundle T^*S, then $H^\flat$ denotes the function on the tangent bundle which is the pull back of H by this bundle isomorphism. We denote by $H_p \in C^\infty(T_p^*S)$ (resp. $H_p^\flat \in C^\infty(T_pS)$) the restriction of H (resp. $H^\flat$) to the cotangent space T_p^*S (resp. the tangent space T_pS) at $p \in S$. For a tangent vector v to S, $\gamma_v(t)$ denotes the geodesic of (S, g) such that the initial vector $\dot{\gamma}_v(0)$ is v. The space of unit tangent vectors at $p \in S$ is denoted by S_pS.

§1. The local characterization and the global definition.

Let g be a riemannian metric on a neighborhood U of a point $p \in \boldsymbol{R}^2$, and let $E \in C^\infty(T^*U)$ be the corresponding energy function. The following proposition is classical.

PROPOSITION 1.1. *Assume that $F \in C^\infty(T^*U)$ satisfies the following conditions:*
(1) $\{E, F\}=0$,
(2) *F_q is a homogeneous polynomial of degree 2 for every $q \in U$,*
(3) $F_p \notin \boldsymbol{R}E_p$.

Then there is a coordinate system (x_1, x_2) on a (possibly smaller) neighborhood of p, and there are functions $f_i(x_i)$ $(i=1, 2)$ such that

$$g=(f_1(x_1)+f_2(x_2))(dx_1^2+dx_2^2)$$

and

$$F=\frac{1}{f_1(x_1)+f_2(x_2)}(f_2(x_2)\xi_1^2-f_1(x_1)\xi_2^2),$$

where (x, ξ) is the associated canonical coordinates on the cotangent bundle. Furthermore, such coordinate system (x_1, x_2) and functions f_1, f_2 are essentially unique.

PROOF. In view of the condition (3) we may assume that $F_q \notin \boldsymbol{R}E_q$ for every $q \in U$. By diagonalizing the quadratic forms E_q and F_q simultaneously, we see that there exist functions $V_i \in C^\infty(T^*U)$ which are fibrewise linear and $f_i \in C^\infty(U)$ $(i=1, 2)$ such that $f_2 > -f_1$ on U and

$$E=\frac{1}{2}(V_1^2+V_2^2)$$

$$F = f_2 V_1^2 - f_1 V_2^2.$$

The condition (1) then implies that $\{V_1, f_2\} = \{V_2, f_1\} = 0$ and

$$-\{V_1, f_1\}V_2 + \{V_2, f_2\}V_1 - 2(f_1+f_2)\{V_1, V_2\} = 0.$$

Then we have $\{\sqrt{f_1+f_2}\,V_1, \sqrt{f_1+f_2}\,V_2\} = 0$. Let X_i $(i=1, 2)$ be the vector field on U defined by

$$\lambda(X_i) = \sqrt{f_1+f_2}\,V_i(\lambda), \qquad \lambda \in T^*U.$$

Since the above equality implies $[X_1, X_2]=0$, it follows that there is a coordinate system (x_1, x_2) on U such that $\partial/\partial x_i = X_i$ $(i=1, 2)$. Let (x, ξ) be the associated canonical coordinates on T^*U. Then we have $\xi_i = \sqrt{f_1+f_2}\,V_i$ $(i=1, 2)$ and $\{\xi_1, f_2\} = \{\xi_2, f_1\} = 0$. This implies that f_i $(i=1, 2)$ is the function of the single variable x_i. We note that f_i are uniquely determined as the functions on U, and that X_i are also unique except for the sign. ■

In view of this proposition we are led to the global definition of the compact Liouville surface. A triple (S, g, F) is called a compact Liouville surface if (S, g) is a compact 2-dimensional riemannian manifold and F is a C^∞ function on T^*S satisfying the following conditions:

(L.1) $\{E, F\} = 0$,

(L.2) $F_p = F|_{T_p^*S}$ is a homogeneous polynomial of degree 2 for any $p \in S$,

(L.3) F is not of the form $rH + sE$, where $H \in C^\infty(T^*S)$ is fibrewise the square of a linear form, and $r, s \in \boldsymbol{R}$.

Here E denotes the corresponding energy function. It should be noted that the condition (L.3) contains the condition that $F \notin \boldsymbol{R}E$. The meaning of this condition is just as stated in Introduction.

Let (S, g, F) be a compact Liouville surface. We put

$$\mathfrak{N} = \{p \in S \mid F_p = rE_p \text{ for some } r \in \boldsymbol{R}\}.$$

LEMMA 1.2. *The ratio $r = F_p/E_p$ does not depend on the point $p \in \mathfrak{N}$.*

PROOF. Since F is the first integral, the function $t \to F^\flat(\dot{\gamma}(t))$ is constant for any geodesic $\gamma(t)$, where the dot denotes the derivative in t. Therefore the lemma follows from the fact that any two points on S can be joined by a geodesic. ■

Two compact Liouville surfaces (S, g, F) and (S', g', F') are said to be equivalent if there is an isometry $\phi: (S, g) \to (S', g')$ and $r, s \in \boldsymbol{R}$ such that $F' \circ (\phi^*)^{-1} = rF + sE$. It should be noted that if (S, g, F) is a compact Liouville surface, so is $(S, g, F - rE)$ for any $r \in \boldsymbol{R}$, and they belong to the same equivalence class. Moreover the set $\mathfrak{N}$ does not change. Hence, in view of Lemma 1.2 we can assume the following condition without loss of generality:

(L.4) $F_p = 0$ if $p \in \mathfrak{N}$.

In the rest of this paper we shall always assume the condition (L.4) on the Liouville surface unless otherwise stated.

§2. The set $\mathfrak{N}$ and the geodesics with $F=0$

Let (S, g, F) be a compact Liouville surface, and let $\mathfrak{N}$ be the subset of S defined in the previous section. In this section we shall prove the following theorem.

THEOREM 2.1. *$\#\mathfrak{N}$, the number of the points in $\mathfrak{N}$, must be 0 or 2 or 4. In case $\#\mathfrak{N}=4$, then S is diffeomorphic to the sphere S^2. In this case there is a simply closed geodesic L on (S, g) on which every point in the set $\mathfrak{N}$ lies. Let p_1, p_2, p_3, p_4 be the points in $\mathfrak{N}$ ordered along with the geodesic L. Then we see that $Cut(p_i)$, the cut locus of the point p_i, consists of the single point $\{p_{i+2}\}$, where $i=1, 2, 3, 4$ and $i+2$ should be considered modulo 4. Moreover, F_q is indefinite if $q \notin L$, and F_q is degenerate and semi-definite if $q \in L$.*

In case $\#\mathfrak{N}=2$, then S is dffeomorphic to the real projective plane $\boldsymbol{R}P^2$. In this case the double covering of (S, g, F) becomes of the above type. In case $\#\mathfrak{N}=0$, then S is diffeomorphic to the torus or the Klein bottle.

PROOF. First we shall consider the case where $\#\mathfrak{N}=0$. Let V_1 and V_2 be as in the proof of Proposition 1.1. As was noted there V_i^2 $(i=1, 2)$ are uniquely defined, and in this case they are globally defined functions on T^*S. Hence they give the decomposition of T^*S into the sum of two line bundles, which implies that S is diffeomorphic to the torus or the Klein bottle.

In order to consider the case $\#\mathfrak{N}>0$, we need some lemmas.

LEMMA 2.2. *If $F_p=0$ for every point p in an open subset U of S, then F is identically zero.*

PROOF. Fix a point $q \in S$, and let $\gamma(t)$ be a geodesic such that $\gamma(0)=q$ and $\gamma(t_0) \in U$. Then we have $F^\flat(\dot{\gamma}(0))=0$. If we take a vector $v \in T_q S$ near $\dot{\gamma}(0)$, then we also have $\gamma_v(t_0) \in U$, and this implies that $F^\flat(v)=0$. Since $F_q^\flat$ is a quadratic form, it then follows that $F_q^\flat=0$. ∎

LEMMA 2.3. *Suppose that $\#\mathfrak{N} \geqq 3$, and let p_i $(i=1, 2, 3)$ be three points in $\mathfrak{N}$. Assume that $p_2, p_3 \notin Cut(p_1)$, and let $\gamma_i(t)=\gamma_{v_i}(t)$ $(0 \leqq t \leqq t_i$, $v_i \in S_{p_1}S)$ be the minimizing geodesic from p_1 to p_i $(i=2, 3)$. Then we have $v_2+v_3=0$. Furthermore, if we put $w=-\dot{\gamma}_2(t_2) \in S_{p_2}S$, then p_3 is the first conjugate point of p_2 along the* *geodesic* $\gamma_w(t)$ $(0\leqq t\leqq t_2+t_3)$. *In particular we have* $\#\mathfrak{N}<\infty$.

PROOF. Take a point $q\in S$ near p_1 so that $q\notin\gamma_i([-\varepsilon, t_i])$ $(i=2, 3)$ for a fixed small $\varepsilon>0$, and let $\gamma_{u_i}(t)$ $(0\leqq t\leqq t_i')$ $(i=1, 2, 3)$ be the minimizing geodesic from q to p_i $(u_i\in S_qS)$. We first claim that u_2 and u_3 are linearly independent of u_1 if q is sufficiently close to p_1. In fact, assume that $u_2=\pm u_1$ for an infinite sequence of points $\{q_j\}$ converging to p_1. Since $q_j\notin\gamma_2([-\varepsilon, t_2])$, the case where $u_2=u_1$ does not occur for large j. Hence $\gamma_{-u_1}(t)$ $(-t_1'\leqq t\leqq 0)$ and $\gamma_{u_2}(t)$ are joined into a single geodesic from p_1 to p_2 with length $t_1'+t_2'$. Since the length $t_1'+t_2'$ converges to t_2 as $j\to\infty$, it follows that $p_2=\gamma_2(t_2)$ is the conjugate point of p_1 along the geodesic $\gamma_2(t)$. But this contradicts the facts that $\gamma_2(t)$ $(0\leqq t\leqq t_2)$ is minimizing and $p_2\notin Cut(p_1)$. The case for u_3 is similar.

Next we assume that u_2 and u_3 are linearly independent. Then $F^\flat(u_i)=0$ $(i=1, 2, 3)$ and u_i are mutually linearly independent. Since $F_q^\flat$ is a quadratic form on T_qS, it follows that $F_q^\flat=0$. In this case if we take a point q' near q, then the corresponding vectors $u_i'\in T_{q'}S$ are also mutually linearly independent, and we have $F_{q'}^\flat=0$. Then by the previous lemma we see that $F=0$, which contradicts the definition of the Liouville surface. Hence u_2 and u_3 must be linearly dependent.

The case where $u_2=u_3$ clearly does not occur. Hence $u_2+u_3=0$, and the same argument as above shows that $v_2+v_3=0$ and p_3 is the conjugate point of p_2 along the geodesic $\gamma_w(t)$ $(0\leqq t\leqq t_2+t_3)$. If it is not the first conjugate point, then it contradicts the minimizing property of the geodesic segments $\gamma_w(t)$ $(0\leqq t\leqq t_2)$ and $\gamma_w(t)$ $(t_2\leqq t\leqq t_2+t_3)$. Finally if $\#\mathfrak{N}=\infty$, then there are three points in $\mathfrak{N}$ which are sufficiently close to one another. But this contradicts the fact just proved. ∎

Let $\mathcal{T}Conj(p)$ be the tangential first conjugate locus of $p\in S$, and let $\mathcal{T}Cut(p)$ be the tangential cut locus.

LEMMA 2.4. *If* $p\in\mathfrak{N}$, *then* $\mathcal{T}Conj(p)$ *is not empty.*

PROOF. Assume that $p\in\mathfrak{N}$ and $\mathcal{T}Conj(p)$ is empty. Then the exponential mapping $Exp_p\colon T_pS\to S$ is the local diffeomorphism, and hence is the universal covering of S. In this case there are at least two minimizing geodesics joining p and each $q\in Cut(p)$. Assume that the number of such geodesics is just two for every point $q\in Cut(p)$. Let B_p be the closure of the bounded component of $T_pS-\mathcal{T}Cut(p)$ in T_pS. Let σ be the free involution on the boundary $\mathcal{T}Cut(p)$ of B_p such that $v\in\mathcal{T}Cut(p)$ and $\sigma(v)$ are mapped to the same point by Exp_p. Then S becomes homeomorphic to the quotient space of B_p obtained by identifying each $v\in\mathcal{T}Cut(p)$ and $\sigma(v)$, which is clearly homeomorphic to the real projective plane. This contradicts that Exp_p is the universal covering. Hence there is a point $q\in Cut(p)$ such that at least three minimizing geodesics from p to q exist.

Since Exp_p is the local diffeomorphism, there is a neighborhood U of q in S and C^∞ functions $v_i=v_i(q')\in T_pS$ ($q'\in U$, $i=1, 2, 3$) such that $\gamma_{v_i}(1)=q'$ and $v_i\neq v_j$ if $i\neq j$, and $\gamma_{v_i(q)}(t)$ $(0\leqq t\leqq 1)$ are minimizing. If $\gamma_{v_i(q')}(t)$ are mutually transversal at $t=1$, then $F_{q'}^b=0$ and so is true for q'' near q'. Then by Lemma 2.1 we get a contradiction. Hence we see that, for example, $\dot\gamma_{v_1(q')}(1)$ and $\dot\gamma_{v_2(q')}(1)$ are linearly dependent for all q' near q. But this implies, as in the proof of the previous lemma, the point p is the conjugate point of p itself along the geodesic $\gamma_{v_1(q)}(t)$ $(0\leqq t\leqq 2)$. This contradicts the assumption that $\mathcal{T}Conj(p)$ is empty. ∎

Now we shall observe the local behavior of geodesics around a conjugate point. Let $p\in\mathcal{N}$. We introduce the positive functions $t_0(w)$ and $t_1(w)$ of $w\in S_pS$ by $t_0(w)w\in\mathcal{T}Cut(p)$ and $t_1(w)w\in\mathcal{T}Conj(p)$ respectively. Here $t_1(w)$ may be ∞ at some w. We note that the function t_0 is continuous, and that the function t_1 is smooth wherever its value is finite. We next define the positive function $\bar t(v, w)$ of $(v, w)\in S_pS\times S_pS$ on a neighborhood of the set

$$D=\{(v, v)\in S_pS\times S_pS \mid t_1(v)<\infty\}$$

as follows: Let $v\in S_pS$ such that $t_1(v)<\infty$. We first put $\bar t(v, v)=t_1(v)$. If we take $\delta>0$ sufficiently small, then there is a neighborhood I_0 of v in S_pS such that the two geodesic segments $\gamma_v(t)$ $(|t-t_1(v)|<\delta)$ and $\gamma_w(t)$ $(|t-t_1(v)|<\delta)$ intersect exactly at one point for any $w\in I_0-\{v\}$. We then define $\bar t(v, w)\in (t_1(v)-\delta, t_1(v)+\delta)$ so that $\gamma_v(\bar t(v, w))$ is this intersection point. It is easy to see that $\bar t$ is well-defined and continuous on a neighborhood of D.

LEMMA 2.5. *Let $p\in\mathcal{N}$, and fix $v\in S_pS$ such that $t_1(v)<\infty$. Put $p'=\gamma_v(t_1(v))$. Then there is a neighborhood I of v in S_pS such that one of the following two cases occurs:*

(a) *$w\mapsto\bar t(v, w)$ is constant for $w\in I$, and $F_{p'}=0$;*

(b) *$w\mapsto\bar t(v, w)$ is injective for $w\in I$, and $F_{p'}$ is non-zero and semi-definite.*

Furthermore if the case (b) *occurs, then there are $u\in S_{p'}S$ which is orthogonal to $\dot\gamma_v(t_1(v))$, $\varepsilon>0$, and $\delta>0$ such that the set*

$$\{\gamma_w(t) \mid |t-t_1(v)|<\delta,\ w\in I-\{v\}\}$$

does not meet the 'half-slice' $Z=\{Exp_{p'}(su) \mid 0\leqq s\leqq\varepsilon\}$.

PROOF. Assume that $w\mapsto\bar t(v, w)$ is not injective on any neighborhood of v. Then there are $w_j^1, w_j^2\in S_pS$ $(j=1, 2, \cdots)$ such that $w_j^1\neq w_j^2$, $\bar t(v, w_j^1)=\bar t(v, w_j^2)$, and $w_j^1, w_j^2\to v$ $(j\to\infty)$. It is easy to see that the three geodesics γ_v, $\gamma_{w_j^1}$, and $\gamma_{w_j^2}$ intersect mutually transversally at the point $\gamma_v(\bar t(v, w_j^1))$ if w_j^1 and w_j^2 are sufficiently close to v. Hence we have $F_{\gamma_v(\bar{t}(v,w_j^1))}=0$ for large j. In view of Lemma 2.3 this implies that $\bar{t}(v, w_j^1)=t_1(v)$ for large j.

Now let I be a small connected neighborhood of v in S_pS. Fix $e\in S_pS$ which is orthogonal to v, and put

$$I_+ = \{w\in I \mid g(w, e)>0\}, \qquad I_- = \{w\in I \mid g(w, e)<0\}.$$

Assume that there is a vector $u_1\in I_+$ such that $\bar{t}(v, u_1)=t_1(v)$, and also assume that there is a vector $u_2\in I_+$ on the arc between v and u_1 such that $\bar{t}(v, u_2)\neq t_1(v)$. Then the image of the arc between u_1 and u_2 by the mapping $w\mapsto\bar{t}(v, w)$ contains the interval between $t_1(v)$ and $\bar{t}(v, u_2)$, and so does the image of the arc between v and u_2. Hence, if I_+ is sufficiently small, by the same reason as above we have $F_{\gamma_v(t)}=0$ for every t between $t_1(v)$ and $\bar{t}(v, u_2)$. But this contradicts Lemma 2.3. Therefore this case does not occur.

Above two facts show that for a sufficiently small I_+ (resp. I_-) only two cases can occur: (1) $w\mapsto\bar{t}(v, w)$ is injective on $I_+\cup\{v\}$ (resp. $I_-\cup\{v\}$), or (2) $\bar{t}(v, w)=t_1(v)$ for all $w\in I_+$ (resp. $w\in I_-$). Furthermore, by the same reason as above we see that if the case (1) occurs for both I_+ and I_-, then the mapping $w\mapsto\bar{t}(v, w)$ is injective on I. Note that the case (2) occurs for I_+ (resp. I_-) if and only if $t_1(v)=t_1(w)$ for all $w\in I_+$ (resp. $w\in I_-$), and $F_{p'}=0$ in this case. Hence if the derivative of the function t_1 does not vanish at v, then the mapping $w\mapsto\bar{t}(v, w)$ is injective on I.

Now we shall show that if the mapping $w\mapsto\bar{t}(v, w)$ is injective on I, then $F_{p'}^\flat$ is non-zero and semi-definite. Assume that $F_{p'}^\flat$ is zero or indefinite. Then there is a vector $u_0\in S_{p'}S$ such that $u_0\neq\pm\dot{\gamma}_v(t_1(v))$ and $F_{p'}^\flat(u_0)=0$. The injectiveness of the mapping $w\mapsto\bar{t}(v, w)$ implies that if u_0 is suitably chosen, then there are $\varepsilon>0$ and $\delta>0$ such that $\gamma_w(t)$ ($|t-t_1(v)|<\delta$) intersects

$$X_+ = \{Exp_{p'}(su_0) \mid 0\leqq s<\varepsilon\}$$

and does not intersect

$$X_- = \{Exp_{p'}(su_0) \mid -\varepsilon<s<0\}$$

for any $w\in I$. Furthermore we see that the intersection of the set

$$\{\gamma_w(t) \mid |t-t_1(v)| < \delta,\ w\in I_+\cup\{v\}\}$$

and X_+ forms a neighborhood of p' in X_+, and so does for I_-. Hence by the same reason as before we have $F_q=0$ for all $q\in X_+$ near p', which contradicts Lemma 2.3. Therefore if follows that $F_{p'}$ is non-zero and semi-definite under the assumption that the mapping $w\mapsto\bar{t}(v, w)$ is injective on I. The existence of the half-slice Z in this case is also obvious.

Finally we assume that the case (1) occurs for I_+ and the case (2) occurs for I_-. Then as was mentioned above we can take $w\in I_+$ arbitrary near v such that the derivative of the function t_1 at w does not vanish. Fix such $w\in I_+$. By the fact just shown above we see that $F_{\gamma_w(t_1(w))}$ is non-zero and semi-definite. Let γ be the minimizing geodesic joining p' and $\gamma_w(t_1(w))$. Then the geodesics γ and $\gamma_w(t)$ intersect transversally at the point $\gamma_w(t_1(w))$, because $\gamma_w(t)$ does not pass the point p' near the time $t=t_1(v)$. Since $F_{p'}=0$, we obtain a contradiction. This completes the proof of Lemma 2.5. ∎

PROPOSITION 2.6. *If $p\in\mathcal{N}$, then $\mathcal{T}Conj(p)$ is a circle of constant radius. Hence $Conj(p)$ consists of one point $\{q\}$, and $q\in\mathcal{N}$.*

PROOF. Set

$$U=\{v\in S_pS \mid t_1(v)<\infty,\ F_{\gamma_v(t_1(v))}=0\}.$$

We first show that U is open and closed in S_pS. If $v\in U$, then the case (*a*) in Lemma 2.5 occurs for v. As was mentioned in the proof of the lemma, we have $\bar{t}(v, w)=t_1(v)=t_1(w)=\bar{t}(w, v)$ for any w in a neighborhood I of v. This implies $I\subset U$, and hence U is open. Closedness of U is now clear, because t_1 is finite on $\bar{U}$, the closure of U. Therefore if it is shown that U is not empty, then U coincides with S_pS, and t_1 becomes a constant function.

Now assume that U is empty. They $F_{\gamma_v(t_1(v))}$ is non-zero and semi-definite for any $v\in S_pS$ with $t_1(v)<\infty$. First we shall consider the case where $\mathcal{T}Cut(p)=\mathcal{T}Conj(p)$. Let $v\in S_pS$ be any vector, and suppose that there is another minimizing geodesic joining p and $\gamma_v(t_1(v))$ besides $\gamma_v(t)$ $(0\leqq t\leqq t_1(v))$. Since $F_{\gamma_v(t_1(v))}$ is non-zero and semi-definite, these two geodesics form a simple geodesic loop $\gamma_v(t)$ $(0\leqq t\leqq 2t_1(v))$ from p to p. If $\gamma_w(t)$ $(0\leqq t\leqq 2t_1(w))$ is also a geodesic loop for a vector $w\in S_pS$ sufficiently close to v, then these two loops intersects at one point except for p, and as is easily seen it contradicts the fact that four geodesic segments $\gamma_v(t)$ $(0\leqq t\leqq t_1(v),\ t_1(v)\leqq t\leqq 2t_1(v))$ and $\gamma_w(t)$ $(0\leqq t\leqq t_1(w),\ t_1(w)\leqq t\leqq 2t_1(w))$ are minimizing, and that $F_{\gamma_v(t_1(v))}$ and $F_{\gamma_w(t_1(w))}$ are non-zero and semi-definite. Therefore in any way there is a vector $v\in S_pS$ such that $\gamma_v(t)$ $(0\leqq t\leqq t_1(v))$ is the unique minimizing geodesic from p to $\gamma_v(t_1(v))$.

Now let Z be the half-slice for v which is given in Lemma 2.5, and let q_j $(j=1, 2, \cdots)$ be points on Z such that $q_j\to\gamma_v(t_1(v))$ as $j\to\infty$. Let $w_j\in S_pS$ and $t_j>0$ such that $\gamma_{w_j}(t)$ $(0\leqq t\leqq t_j)$ be a minimizing geodesic from p to q_j. By taking a subsequence if necessary, we may assume that $t_j\to t_1(v)$ and w_j converges as $j\to\infty$. In view of the way of the choice of v, w_j must converge to v. But this contradicts the property of Z.

Next we consider the case where $\mathcal{T}Cut(p)\neq\mathcal{T}Conj(p)$. Let $v\in S_pS$ such that $t_0(v)<t_1(v)<\infty$. We note that $\gamma_v(t_1(v))\neq p$, because $\gamma_v(t_1(v))\notin\mathcal{N}$. Since $\gamma_v(t)$ $(0\leqq t\leqq t_1(v))$ is not minimizing, and since $F_{\gamma_v(t_1(v))}$ is semi-definite, it follows that there is a unique minimizing geodesic joining p and $\gamma_v(t_1(v))$. Note that it is also true for $w \in S_pS$ near v. This implies that the minimizing geodesic from p to $\gamma_v(t_1(v))$ is of the form $\gamma_{v'}(t)$ $(0 \leqq t \leqq t_1(v'))$ for some $v' \in S_pS$, and that $t_0(v') = t_1(v')$. Hence the same argument as above is applicable for v', and we get a contradiction. This completes the proof of the proposition. ■

COROLLARY 2.7. *Let $p \in \mathfrak{N}$. If $\mathfrak{T}Cut(p) = \mathfrak{T}Conj(p)$, then S is diffeomorphic to the sphere S^2, and $\#\mathfrak{N} = 2$ or 4. If $\mathfrak{T}Cut(p) \neq \mathfrak{T}Conj(p)$, then S is diffeomorphic to $\boldsymbol{RP}^2$.*

PROOF. We have already seen that the function t_1 is constant on S_pS, Set

$$B(t_1) = \{w \in T_pS \mid g(w, w) \leqq t_1^2\}.$$

Let $\partial B(t_1)$ be its boundary. The quotient space $B(t_1)/\partial B(t_1)$ is then homeomorphic to S^2. The exponential mapping Exp_p induces the continuous mapping $\rho: B(t_1)/\partial B(t_1) \to S$. As is easily seen, this mapping is locally homeomorphic, and hence is the universal covering. Therefore the manifold S is homeomorphic (and hence diffeomorphic) to S^2 or $\boldsymbol{RP}^2$. If $\mathfrak{T}Cut(p) = \mathfrak{T}Conj(p)$, then ρ is bijective, and S is diffeomorphic to S^2. If $\mathfrak{T}Cut(p) \neq \mathfrak{T}Conj(p)$, then ρ is not injective, and S must be diffeomorphic to $\boldsymbol{RP}^2$.

In case S is diffeomorphic to S^2, then $Cut(p)(=Conj(p))$ consists of one point $\{q\}$, and $q \in \mathfrak{N}$. In this case it is clear that $Cut(q) = Conj(q) = \{p\}$. Hence the set $\mathfrak{N}$ becomes the disjoint union of such pairs of points, which implies that $\#\mathfrak{N}$ is even. Assume that $\#\mathfrak{N} \geqq 6$. Then there are three points p_1, p_2, p_3 in $\mathfrak{N}$ such that $p_i \notin Cut(p_j)$ for any i, j. In view of Lemma 2.3 this implies $p_3 \in Conj(p_2)$, etc.. But since $Conj(p_2) = Cut(p_2)$, this is a contradiction. Hence we have $\#\mathfrak{N} = 2$ or 4. ■

LEMMA 2.8. *Suppose that S is diffeomorphic to S^2, and that $\#\mathfrak{N} = 2$. Then F_q is semi-difinite for all $q \in S$.*

PROOF. Let $\mathfrak{N} = \{p, p'\}$. We take a normal polar coordinate system (r, θ) $(0 \leqq r \leqq R,\ \theta \in \boldsymbol{R}/2\pi\boldsymbol{Z})$ with p being the center, i.e., (r, θ) represents the point $Exp_p r(v_1 \cos\theta + v_2 \sin\theta)$, where $\{v_1, v_2\}$ is an orthonormal basis of T_pS, and $t_1(v_1) = R$. Let $\mathcal{J}$ be the set of points $q \in S$ such that F_q is indefinite, and put

$$\mathcal{J}(\theta) = \{(r, \theta) \in \mathcal{J} \mid r \in (0, R)\}.$$

Now assume that $\mathcal{J}$ is not empty, and let $(r_0, \theta_0) \in \mathcal{J}$. Let $\gamma(t)$ be the geodesic such that $\gamma(0) = (r_0, \theta_0)$, $F^\flat(\dot{\gamma}(0)) = 0$, and that $\dot{\gamma}(0)$ is linearly independent of $\partial/\partial r$. Since $\gamma(t)$ is transversal to each radial geodesic $r \mapsto (r, \theta)$, it easily follows that $\gamma(t)$ again intersects the geodesic $\theta = \theta_0$ at a finite time $t > 0$. In particular $\mathcal{J}(\theta)$ is not empty for any θ. Let $r^s(\theta) \in (0, R]$ be the supremum of r such that $(r, \theta) \in \mathcal{J}(\theta)$. If $r^s(\theta) < R$ for some θ, then as is easily seen, the set $\{(r^s(\theta), \theta) \mid \theta \in \boldsymbol{R}/2\pi\boldsymbol{Z}\}$ coincides with the trajectory of a closed geodesic. This implies that $F^\flat(\partial/\partial\theta(r^s(\theta), \theta))$ vanishes, and hence that $F_{(r^s(\theta),\theta)}$ is indefinite. Since $\mathcal{J}$ is open, it contradicts the definition of $r^s(\theta)$. Hence we have $r^s(\theta)=R$ for all θ.

Now let $\gamma_j(t)$ $(0 \leqq t \leqq t_j)$ $(j=1, 2, \cdots)$ be a sequence of geodesics such that $\gamma_j(0) \in \mathcal{J}(0)$, $\gamma_j(0) \to p'$ as $j \to \infty$, $F^\flat(\dot{\gamma}_j(0))=0$, $\gamma_j(t_j) \in \mathcal{J}(0)$, $\gamma_j(t) \notin \mathcal{J}(0)$ if $0<t<t_j$, and $r(\gamma_j(t_j)) \geqq r(\gamma_j(0))$. Then we can easily see that this sequence of geodesic segments converges to the point p'. But this clearly contradicts the existence of a convex neighborhood around p'. ■

We now give the rest of the proof of Theorem 2.1. Suppose that S is diffeomorphic to S^2. First assume that $\#\mathcal{N}=2$. Let (r, θ) be the normal polar coordinates as in the proof of the previous lemma. Let $(r, \theta, \eta_1, \eta_2)$ be the corresponding canonical coordinates on $T^*(S-\mathcal{N})$. Since F is semi-definite everywhere, and since $F^\flat(\partial/\partial r)=0$, F must be of the form $\pm h\eta_2^2$. Here h is a positive function on $S-\mathcal{N}$. Put $H=\sqrt{h}\,\eta_2$ on $T^*(S-\mathcal{N})$ and $H_q=0$ if $q \in \mathcal{N}$. Then H is continuous on $T*S$, smooth on $T^*(S-\mathcal{N})$, and fibrewise linear. Moreover we have $\{E, H\}=0$ on $T^*(S-\mathcal{N})$. Then the differentiability of H at points $\lambda \in T_q^*S-\{0\}(q \in \mathcal{N})$ follows from the differentiability of $\xi_t^*H=H$ at points $\xi_{-t}(\lambda)$, where $\{\xi_t\}$ denotes the geodesic flow. Hence this case is just that excluded by the condition (L.3) in the definition of Liouville surfaces.

Thus we have $\#\mathcal{N}=4$. Let p_1, p_2, p_3, p_4 be points in $\mathcal{N}$ such that $Cut(p_1)=\{p_3\}$, $Cut(p_2)=\{p_4\}$, and vice versa. Let L be the infinite extension of the minimizing geodesic from p_1 to p_2. As is easily seen, L must be a simply closed geodesic, and $d(p_1, p_3)=d(p_2, p_4)=$one half of the length of L, where d denotes the distance function.

Now let q be a point on L between p_1 and p_2, and let q' be a point near q which is not on L. Since $F_{q'} \neq 0$, by considering the minimizing geodesics from q' to p_1, p_2, and q respectively, we have $F^\flat(v) \neq 0$, where v is the tangent vector to the minimizing geodesic from q' to q at q. Since q' is arbitrary, it follows that F_q is semi-definite.

Next let $q \notin L$. Then the two minimizing geodesics from q to p_1 and p_2 intersects transversally at q. Hence F_q is indefinite. This completes the proof of Theorem 2.1. ■

§ 3. The natural coordinate system and the classification

First we shall classify Liouville surfaces whose underlying manifolds are diffeomorphic to the sphere S^2. For this purpose we introduce quadruples $(\alpha_1, \alpha_2, f_1, f_2)$ such that α_1 and α_2 are positive real numbers, and that f_i is a C^∞ function on the circle $\boldsymbol{R}/\alpha_i\boldsymbol{Z}(i=1, 2)$ satisfying the following conditions:

$$f_i(-t)=f_i(t),\quad f_i''(0)>0,\quad f_i(0)=f_i\left(\frac{\alpha_i}{2}\right)=0,$$

$$f_i(t)>0 \text{ if } t\not\equiv 0,\ \frac{\alpha_i}{2}\quad (\mathrm{mod}\,\alpha_i\boldsymbol{Z}),\quad (i=1,2);$$

$$\text{if } f_1(t)\sim\sum_{k\geq 1}a_k t^{2k}\quad (t\to 0),\ \text{then}$$

$$f_2(t)\sim\sum_{k\geq 1}(-1)^{k-1}a_k t^{2k}\quad (t\to 0), \tag{3.1}$$

$$f_1(t)\sim\sum_{k\geq 1}a_k\left(t-\frac{\alpha_1}{2}\right)^{2k}\quad \left(t\to\frac{\alpha_1}{2}\right),$$

$$f_2(t)\sim\sum_{k\geq 1}(-1)^{k-1}\alpha_k\left(t-\frac{\alpha_2}{2}\right)^{2k}\quad \left(t\to\frac{\alpha_2}{2}\right).$$

Here f_i is identified with the periodic function on $\boldsymbol{R}$ with period α_i $(i=1, 2)$, and the symbol $\sim$ stands for the formal Taylor expansion. Let Q be the set of all such quadruples. We say that two quadruples $(\alpha_1, \alpha_2, f_1, f_2)$ and $(\beta_1, \beta_2, h_1, h_2)$ in Q are equivalent if there is a constant $c>0$ and

$$\nu\in\left\{0, \frac{\alpha_1}{2}\right\},\qquad \mu\in\left\{0, \frac{\alpha_2}{2}\right\}$$

such that one of the following conditions is satisfied:

(1) $\beta_i=c\alpha_i$ $(i=1, 2)$, and $c^2h_1(ct)=f_1(t+\nu)$, $c^2h_2(ct)=f_2(t+\mu)$;

(2) $\beta_1=c\alpha_2$, $\beta_2=c\alpha_1$, and $c^2h_1(ct)=f_2(t+\mu)$, $c^2h_2(ct)=f_1(t+\nu)$.

If the case (1) occurs with $c=1$, then these two quadruples are said to be isomorphic.

We can construct a Liouville surface whose underlying manifold is S^2 from each quadruple $(\alpha_1, \alpha_2, f_1, f_2)\in Q$ in the following way. Let $\Gamma=\Gamma(\alpha_1, \alpha_2)$ be the lattice in $\boldsymbol{R}^2$ generated by $(\alpha_1, 0)$ and $(0, \alpha_2)$, and put $T(\alpha_1, \alpha_2)=\boldsymbol{R}^2/\Gamma$. We shall identify $\boldsymbol{R}^2$ with $\boldsymbol{C}$ by taking the complex coordinate $z=x_1+\sqrt{-1}x_2$. Let τ be the involution on $T(\alpha_1, \alpha_2)$ defined by $z\mapsto -z$. We now consider the quotient space $R=T(\alpha_1, \alpha_2)/\{1, \tau\}$. Note that τ has four fixed points in $T(\alpha_1, \alpha_2)$ represented by $z=0$, $\alpha_1/2$, $\sqrt{-1}\alpha_2/2$, $(\alpha_1+\sqrt{-1}\alpha_2)/2$. Let z_0 be one of these four values. By taking $(z-z_0)^2$ as a local coordinate around the point $z=z_0$, the quotient space R can be regarded as the Riemann sphere. Clearly the quotient mapping $\Phi: T(\alpha_1, \alpha_2)\to R$ is holomorphic. By the conditions (3.1) we have a unique riemannian metric g on R such that

$$\Phi^*g=(f_1(x_1)+f_2(x_2))(dx_1^2+dx_2^2),$$

and we also have a unique C^∞ function F on T^*R such that

$$\tilde{F}\circ\Phi^*=F,\qquad \tilde{F}=\frac{1}{f_1(x_1)+f_2(x_2)}(f_2(x_2)\xi_1^2-f_1(x_1)\xi_2^2),$$

where (x, ξ) is the canonical coordinate system of $T^*(T(\alpha_1, \alpha_2))$. Then (R, g, F) is a Liouville surface which satifies the condition (L.4). It is clear that equivalent (resp. isomorphic) quadruples yield equivalent (resp. isomorphic) Liouville surfaces.

THEOREM 3.1. *The above construction gives the one-to-one correspondence between the equivalence classes (resp. isomorphism classes) of quadruples in Q and the equivalence classes (resp. isomorphism classes) of Liouville surfaces whose underlying manifolds are the sphere S^2, and which satisfy the condition* (L.4).

PROOF. We shall give the inverse correspondence. Let (S^2, g, F) be a Liouville surface which satisfies the condition (L.4). As in the proof of Proposition 1.1, the energy function E and the first integral F are described as

$$E = \frac{1}{2}(V_1^2+V_2^2)$$
$$F = f_2V_1^2-f_1V_2^2$$

on a fibred neighborhood of each point in $T^*(S^2-\mathfrak{N})$, where V_1 and V_2 are fibrewise linear. In this case f_1 and f_2 are well-defined C^∞ functions on $S^2-\mathfrak{N}$, and V_1^2 and V_2^2 are well-defined C^∞ functions on $T^*(S^2-\mathfrak{N})$.

Let L be the simply closed geodesic given in Theorem 2.1. Since F is indefinite at each point on S^2-L, and since $f_2>-f_1$ there, it follows that f_1 and f_2 are positive on S^2-L. Let $p_i\in\mathfrak{N}$ $(1\leqq i\leqq 4)$ be as in Theorem 2.1, and let L_i $(1\leqq i\leqq 4)$ be the connected component of $L-\mathfrak{N}$ bounded by p_i and p_{i+1} $(p_5=p_1)$. We have already seen that F is semi-definite at each point on L_i. In fact it is easy to see that the sign of F on L_i and that of L_{i+1} are opposite. Therefore we may assume that $F\geqq 0$ on L_1 and L_3, $F\leqq 0$ on L_2 and L_4. Then we have $f_1=0$, $f_2>0$ on L_1 and L_3, and $f_1>0$, $f_2=0$ on L_2 and L_4. Since $F=0$ on $\mathfrak{N}$, it follows that f_1 and f_2 are continuously extended to the whole S^2 by putting $f_1=f_2=0$ on $\mathfrak{N}$.

Let D_1 and D_2 be the closure of the two connected components of S^2-L respectively. Clearly D_1 and D_2 are diffeomorphic to a closed disc. Hence on $D_1-\mathfrak{N}$ we can take a square root V_i of V_i^2 as a C^∞ function and the corresponding vector field X_i $(i=1, 2)$ as in the proof of Proposition 1.1. Note that the vector fields X_1 and X_2 are also continuously extended to D_1 by putting $X_1=X_2=0$ on $\mathfrak{N}$. We choose V_1(resp. V_2) so that the vector field X_1(resp. X_2) on L_1 (resp. on L_2) tends to the inside of D_1.

LEMMA 3.2. *Consider the ordinary differential equation*

$$\frac{dc_i(t)}{dt} = (X_i)_{c_i(t)} \qquad (i=1, 2)$$

with the initial condition $c_i(0)=q\in D_1-L$. Then there are positive constants a_i and b_i such that the unique solution $c_i(t)$ is defined on the closed interval $[-a_i, b_i]$, *and satisfies*

$$c_i((-a_i, b_i)) \subset (D_1, -L), \; c_i(-a_i) \in L_i, \; c_i(b_i) \in L_{i+2} \; (i=1, 2).$$

Furthermore the value a_i+b_i $(i=1, 2)$ *does not depend on the initial point* q.

PROOF OF LEMMA 3.2. Let (r, θ) be the normal polar coordinate system centered at p_2 such that $L_1 \cup L_4$ is represented by $\theta=0$, $0<r<r_0$, $L_2 \cup L_3$ by $\theta=\pi$, $0<r<r_0$, and D_1-L by $0<\theta<\pi$, $0<r<r_0$, where r_0 is the distance between p_2 and p_4. Since the vectors X_1, X_2 are linearly independent of, and not orthogonal to the vectors defined by $F^\flat=0$ on D_1-L, it easily follows that $\theta(c_1(t))$ (resp. $\theta(c_2(t))$) are the increasing (resp. decreasing) function, and $r(c_i(t))$ $(i=1, 2)$ are the increasing functions. Since $X_1 f_2=0$, and since $f_2=0$ on $L_2 \cup L_4$, it follows that $c_1(t)$ does not pass a neighborhood of the closure of $L_2 \cup L_4$. These facts show the existence of the finte times $-a_1$ and b_1 so that the former conditions are satisfied. Similarly we have the times $-a_2$ and b_2 for $c_2(t)$.

Now let ω_1 and ω_2 be 1-forms on $D_1-\mathfrak{N}$ such that $\omega_i(X_j)=\delta_{ij}$. Clearly they are closed. Let $\gamma(t)$ $(\alpha \leqq t \leqq \beta)$ be a curve in $D_1-\mathfrak{N}$ such that $\gamma(\alpha) \in L_1$ and $\gamma(\beta) \in L_3$. Since the vector field X_2 is tangent to L_1 and L_3, it follows that the integral

$$\int_\gamma \omega_1$$

does not depend on the choice of γ. This shows that the value a_1+b_1 does not depend on q. The constancy of a_2+b_2 is also shown in the same way. ■

We put $\alpha_i=2(a_i+b_i)$ $(i=1, 2)$. Let ω_1 and ω_2 be 1-forms defined in the proof of Lemma 3.2. We define functions x_1 and x_2 on $D_1-\mathfrak{N}$ by

$$x_i(q) = \int_{\gamma_i} \omega_i \qquad (i=1, 2)$$

where γ_i is a curve in $D_1-\mathfrak{N}$ from a point on L_i to q. Moreover we put $x_1(p_1)=x_1(p_2)=x_2(p_2)=x_2(p_3)=0$, $x_1(p_3)=x_1(p_4)=\alpha_1/2$, $x_2(p_1)=x_2(p_4)=\alpha_2/2$. Clearly we have $X_i=\partial/\partial x_i$ on $D_1-\mathfrak{N}$.

LEMMA 3.3. *The functions* (x_1, x_2) *give the homeomorphism from* D_1 *to the rectangle* $[0, \alpha_1/2] \times [0, \alpha_2/2]$ *which is the diffeomorphism from* $D_1-\mathfrak{N}$ *to the rectangle*$-\{4$ *vertices*$\}$.

PROOF OF LEMMA 3.3. In view of Lemma 3.2 it is enough to show the continuity of the functions x_1 and x_2 at four points in $\mathfrak{N}$. Assume that the function x_1 is not continuous at p_1. Then there is a sequence of points $q_k \in D_1-\mathfrak{N}$ $(k=1, 2, \cdots)$ such that $q_k \to p_1$ and $x_1(q_k) \searrow c>0$ as $k \to \infty$. By Lemma 3.2 we have points $q'_k \in L_4$ $(k=1, 2, \cdots)$ such that q_k and q'_k are on the same integral curve of X_2. Since $x_1(q_k)=x_1(q'_k)$, it follows that $x_1(q'_k) \searrow c$ as $k \to \infty$. Since the function x_1 is monotonous on L_4, it then follows that q'_k converges to a point $q'_\infty \in L_4 \cup \{p_1\}$. On the other hand, since $X_2 f_1 = 0$, we have $f_1(q_k) = f_1(q'_k)$. This implies that $f_1(q'_\infty) = 0$, and hence that $q'_\infty = p_1$. Therefore it has been shown that $x_1 > c$ on L_4. But since there is a point in $D_1 - L$ with $x_1 < c$, this clearly contradicts the previous lemma. Other cases are shown in the same way. ■

We now continue the proof of Theorem 3.1. In the similar way to the case of D_1 we can define the vector fields X_i and the functions x_i $(i=1, 2)$ on D_2 so that the functions x_1 and x_2 coincide with those on D_1 on the common boundary L. Therefore the curves x_i=constant are joined into a simply closed curve.

As before, let $\Gamma = \Gamma(\alpha_1, \alpha_2)$ be the lattice in $\boldsymbol{R}^2$ generated by $(\alpha_1, 0)$ and $(0, \alpha_2)$. We denote by $[y_1, y_2] \in \boldsymbol{R}^2/\Gamma$ the image of $(y_1, y_2) \in \boldsymbol{R}^2$ by the quotient mapping $\boldsymbol{R}^2 \to \boldsymbol{R}^2/\Gamma$. Then it is easy to see that the above coordinate systems on D_1 and D_2 induce a continuous mapping $\Phi: \boldsymbol{R}^2/\Gamma \to S^2$ satisfying the following conditions.

(1) Φ maps the set $\{[y_1, y_2] \mid 0 \leqq y_1 \leqq \alpha_1/2,\ 0 \leqq y_2 \leqq \alpha_2/2\}$ homeomorphically onto D_1, and

$$x_i \circ \Phi([y_1, y_2]) \equiv y_i \bmod \alpha_i \boldsymbol{Z} \qquad (i=1, 2).$$

(2) Φ maps the set $\{[y_1, y_2] \mid 0 \leqq y_1 \leqq \alpha_1/2,\ \alpha_2/2 \leqq y_2 \leqq \alpha_2\}$ homeomorphically onto D_2, and

$$x_1 \circ \Phi([y_1, y_2]) \equiv y_1 \bmod \alpha_1 \boldsymbol{Z},$$
$$x_2 \circ \Phi([y_1, y_2]) \equiv -y_2 \bmod \alpha_2 \boldsymbol{Z}.$$

(3) $\Phi([-y_1, -y_2]) = \Phi([y_1, y_2])$.

In view of Lemma 3.3 it is easy to see that Φ is C^∞ outside the four points $[0, 0]$, $[\alpha_1/2, 0]$, $[0, \alpha_2/2]$, $[\alpha_1/2, \alpha_2/2]$, and there Φ is the double covering onto $S^2 - \mathfrak{N}$. Let $\tilde{\mathfrak{N}}$ be the set of these four points. By the definition of the coordinate system (x_1, x_2) we have

$$\Phi^* g = (\Phi^* f_1 + \Phi^* f_2)(dy_1^2 + dy_2^2)$$

on $\boldsymbol{R}^2/\Gamma - \tilde{\mathfrak{N}}$. Here the functions $\Phi^* f_i$ are continuous, and are C^∞ outside $\tilde{\mathfrak{N}}$. It is clear that $\Phi^* f_i$ depends only on the single variable y_i $(i=1, 2)$. We now consider the conformal structure on $\boldsymbol{R}^2/\Gamma$ induced from the flat metric $dy_1^2 + dy_2^2$. Since Φ is continuous, and is the C^∞ conformal mapping outside $\tilde{\mathfrak{N}}$, it follows from an elementary function theory that Φ is C^∞ on the whole $\boldsymbol{R}^2/\Gamma$.

Next we shall show that the functions $\Phi^* f_i$ $(i=1, 2)$ satisfy the condition (3.1), regarding them as the periodic functions on $\boldsymbol{R}$ with periods α_i. Since $\Phi([-y_1, -y_2]) = \Phi([y_1, y_2])$, we first have $\Phi^* f_i(-y_i) = \Phi^* f_i(y_i)$ $(i=1, 2)$. Now let us take the complex coordinate $z = y_1 + \sqrt{-1} y_2$ around the point $[0, 0]$ on $\boldsymbol{R}^2/\Gamma$. Then it is easy to see that the function z^2 induces the local coordinate around $\Phi([0, 0])=p_2$ on S^2. This implies that the symmetric 2-form Φ^*g is described as

$$h(\mathrm{Re}z^2, \mathrm{Im}z^2)dz^2d\bar{z}^2$$

on a neighborhood of $[0, 0]$, where h is a positive C^∞ function in two variables defined on a neighborhood of $(0, 0)$. Hence we have

$$\Phi^*f_1(x_1)+\Phi^*f_2(x_2)=4h(x_1^2-x_2^2, 2x_1x_2)(x_1^2+x_2^2)$$

on a neighborhood of $(x_1, x_2)=(0, 0)$. From this it easily follows that if $\Phi^*f_1(x_1)$ has the Taylor expansion $\sum_{k\geqq 1}a_kx_1^{2k}$ at $x_1=0$, then $a_1>0$, and the Taylor expansion of $\Phi^*f_2(x_2)$ at $x_2=0$ is given by $\sum_{k\geqq 1}(-1)^{k-1}a_kx_2^{2k}$. Other conditions can be obtained in the similar way. Therefore we have obtained the quadruple $(\alpha_1, \alpha_2, \Phi^*f_1, \Phi^*f_2)\in Q$. It is easy to verify that this assignment gives the converse correspondence. This completes the proof of Theorem 3.1. ■

COROLLARY 3.4. *Let (S^2, g, F) be a Louville surface, and let L be the closed geodesic stated in Theorem 2.1. Then the reflection with respect to L is a well-defined isometry of (S^2, g), and it preserves F.*

PROOF. Let $(\alpha_1, \alpha_2, f_1(x_1), f_2(x_2))$ be a quadruple belonging to the corresponding isomorphism class. Then the involution on $\boldsymbol{R}^2/\Gamma(\alpha_1, \alpha_2)$ defined by $(x_1, x_2)\mapsto(x_1, -x_2)$ induces the mapping σ on S^2 whose fixed point set is L. Since $f_2(-x_2)=f_2(x_2)$, it is clear that σ preserves g and F, and that σ is the reflection with respect to L. ■

PROPOSITION 3.5. *A Liouville surface (S^2, g, F) is the double covering of a Liouville surface $(\boldsymbol{RP}^2, g', F')$ if and only if the corresponding quadruples $(\alpha_1, \alpha_2, f_1, f_2)$ satisfy*

$$f_i\left(x_i+\frac{\alpha_i}{2}\right)=f_i(x_i) \qquad (i=1, 2) \tag{3.2}$$

for any $x_i\in\boldsymbol{R}$. In this case the covering transformation is induced from

$$(x_1, x_2)\longmapsto\left(x_1+\frac{\alpha_1}{2}, -x_2+\frac{\alpha_2}{2}\right). \tag{3.3}$$

PROOF. 'if' part is clear. Now assume that (S^2, g, F) is the double covering of $(\boldsymbol{RP}^2, g', F')$, and let τ be the covering transformation on S^2. As was noted in the proof of Theorem 3.1, f_i and $\xi_i^2=(f_1+f_2)V_i^2$ $(i=1, 2)$ are regarded as functions on $S^2-\mathfrak{N}$ and $T^*(S^2-\mathfrak{N})$ respectively. Since E and F are preserved by τ, we have

$$f_i\circ\tau=f_i \tag{3.4}$$

$$\xi_i^2\circ\tau^*=\xi_i^2. \tag{3.5}$$

From the latter formula it follows that τ is induced from the mapping of the form

$$(x_1, x_2) \longmapsto (x_1+a, \pm x_2+b).$$

Since τ is the fixed point free involution, we have (3.3). Then the formula (3.2) follows from (3.4). ■

In the analytic case we have the following

COROLLARY 3.6. *Let (S^2, g, F) be a Liouville surface such that (S^2, g) is the analytic riemannian manifold. Let $(\alpha_1, \alpha_2, f_1(x_1), f_2(x_2))$ be one of the corresponding quadruples. Then f_1, f_2, and F are also analytic, and satisfies*

$$f_i\left(t+\frac{\alpha_i}{2}\right) = f_i(t) \qquad (i=1, 2)$$

for any $t \in \boldsymbol{R}$. Furthermore, extending f_1 as a holomorphic function on a neighborhood of the real axis, we have

$$f_2(t) = -f_1(\sqrt{-1}t)$$

for $t \in \boldsymbol{R}$ near 0. In particular the extended f_1 is a holomorphic function on a neighborhood of the set

$$\left\{z \in \boldsymbol{C} \mid \operatorname{Re} z = \frac{n}{2}\alpha_1 \text{ or } \operatorname{Im} z = \frac{n}{2}\alpha_2,\ n \in \boldsymbol{Z}\right\}$$

and has the double periods $\alpha_1/2$ and $\sqrt{-1}\alpha_2/2$.

PROOF. As is well-known, in case (S^2, g) is analytic, any isothermal coordinates are analytic with respect to the given analytic charts. Therefore it follows that the conformal mapping $\Phi: T(\alpha_1, \alpha_2) \to (S^2, g)$ constructed in the proof of Theorem 3.1 is analytic, and that so is Φ^*g. This implies the analyticity of the functions f_1 and f_2, and hence of the first integral F. The rest part is immediately obtained from the conditions (3.1) for f_1 and f_2. ■

The following corollary is an immediate consequence of Proposition 3.5 and Corollary 3.6.

COROLLARY 3.7. *If (S^2, g, F) is an analytic Liouville surface, then it is the double covering of a Liouville surface whose underlying manifold is the real projective plane $\boldsymbol{R}P^2$.*

We next consider Liouville surfaces whose underlying manifolds are the torus. The simplest example in this case is given as follows: Let Γ be a lattice in $\boldsymbol{R}^2 = \{(x_1, x_2)\}$ generated by an orthogonal basis. After an orthonormal change of the coordinates, we may assume that they are $(\alpha_1, 0)$ and $(0, \alpha_2)$ $(\alpha_1, \alpha_2 > 0)$. Let $f_i \in C^\infty(\boldsymbol{R}/\alpha_i\boldsymbol{Z})$ $(i=1, 2)$ be any non-constant functions such that

$f_1(x_1)+f_2(x_2)>0$ for any x_1, $x_2 \in \boldsymbol{R}$. Put

$$g=(f_1(x_1)+f_2(x_2))(dx_1^2+dx_2^2)$$

$$F=\frac{1}{f_1(x_1)+f_2(x_2)}(f_2(x_2)\xi_1^2-f_1(x_1)\xi_2^2),$$

where (x, ξ) is the associated canonical coordinate system on the cotangent bundle. Then $(\boldsymbol{R}^2/\Gamma, g, F)$ is clearly a Liouville surface. We shall say that this Liouville surface is *of elementary type.*

PROPOSITION 3.8. *Let (S, g, F) be a Liouville surface such that S is diffeomorphic to the torus. Then there is a Liouville surface $(\boldsymbol{R}^2/\Gamma, g', F')$ of elementary type, and a finite covering ϕ: $S \to \boldsymbol{R}^2/\Gamma$ such that $\phi^*g'=g$ and $F\circ\phi^*=F'$.*

PROOF. As in the proof of Proposition 1.1 we can write

$$E=\frac{1}{2}(V_1^2+V_2^2)$$

$$F=f_2V_1^2-f_1V_2^2$$

on a fibred neighborhood of each point in S, where V_i $(i=1, 2)$ are fibrewise linear, and $f_2>-f_1$. In this case Theorem 2.1 shows that E and F are never proportional at any point in S. It thus follows that f_i and V_i^2 are well-defiend C^∞ functions on S and T^*S respectively. Then we have

LEMMA 3.9. *The functions f_1 and f_2 are not constant.*

PROOF. If f_1 is constant, then

$$(f_1+f_2)V_1^2=F+2f_1E$$

becomes also a first integral. But this contradicts the condition (L.3) in the definition of the Liouville surface. ■

As in the proof of Proposition 1.1, we can take the square root V_i of V_i^2 and the corresponding vector X_i at each point in S. Let p be a point in S such that $(df_1)_p\neq 0$, and let $c_0(x_2)$ $(x_2\in\boldsymbol{R})$ be the C^∞ curve in S such that

$$c_0(0)=p, \qquad c_0'(x_2)=\pm(X_2)_{c_0(x_2)}.$$

Since $X_2f_1=0$, it follows that $df_1\neq 0$ at each point $c_0(x_2)$. This implies that the connected component of the closed set $f_1^{-1}(f_1(p))$ containing p coincides with the orbit of $c_0(x_2)(x_2\in\boldsymbol{R})$. Hence the orbit of $c_0(x_2)$ must be a circle, and the curve $x_2\mapsto c_0(x_2)$ has a period. Let $\alpha_2>0$ be the least period.

Now let $x_1\mapsto c(x_1, x_2)$ be the C^∞ curve in S such that

$$c(0, x_2)=c_0(x_2), \qquad \frac{\partial c}{\partial x_1}(x_1, x_2)=\pm(X_1)_{c(x_1, x_2)}.$$

We may assume that $(\partial c/\partial x_1)(0, x_2)$ is C^∞ in the variable x_2. Since the mapping $(x_1, x_2)\mapsto c(x_1, x_2)$ is the local diffeomorphism, it gives a C^∞ covering of S by $\boldsymbol{R}\times\boldsymbol{R}/\alpha_2\boldsymbol{Z}$. Hence there is a point on the circle $\{c(0, x_2)\}$ where df_2 does not vanish. By shifting the parameter x_2, we may assume that this point is $p=c(0, 0)$. Then it follows from the same reason as the case of $c_0(x_2)$ that the curve $x_1\mapsto c(x_1, 0)$ is periodic. Let $\alpha_1>0$ be the least period. Note that the vector fields $(\partial c/\partial x_1)(0, x_2)$ and $(\partial c/\partial x_2)(x_1, 0)$ are also periodic with periods α_2 and α_1 respectively, because S is orientable. Since $[X_1, X_2]=0$ if X_1 and X_2 are locally chosen as C^∞ vector fields, it follows that the mapping $(x_1, x_2)\mapsto c(x_1, x_2)$ induces a C^∞ covering of S by $(\boldsymbol{R}/\alpha_1\boldsymbol{Z})\times(\boldsymbol{R}/\alpha_2\boldsymbol{Z})$. Clearly c^*f_i is a function on the single variable x_i $(i=1, 2)$, and

$$c^*g=(c^*f_1+c^*f_2)(dx_1^2+dx_2^2)$$

$$F=\tilde{F}\circ c^*, \qquad \tilde{F}=\frac{1}{c^*f_1+c^*f_2}(c^*f_2\xi_1^2-c^*f_1\xi_2^2).$$

Let Γ' be the lattice in $\boldsymbol{R}^2$ generated by $(\alpha_1, 0)$ and $(0, \alpha_2)$. By using the covering c, we have a lattice Γ in $\boldsymbol{R}^2$ containing Γ' such that c and the natural covering $\boldsymbol{R}^2/\Gamma'\to\boldsymbol{R}^2/\Gamma$ induces a conformal diffeomorphism between (S, g) and $\boldsymbol{R}^2/\Gamma$ with the flat metric $dx_1^2+dx_2^2$. Therefore we identify S with $\boldsymbol{R}^2/\Gamma$, and c with the natural covering $\boldsymbol{R}^2/\Gamma'\to\boldsymbol{R}^2/\Gamma$. By regarding c^*f_i as a periodic function on $\boldsymbol{R}$, c^*g and $\tilde{F}$ are regarded as a riemannian metric on $\boldsymbol{R}^2$ and a function on $T^*\boldsymbol{R}^2$ which are invariant under the additive action of Γ. Let π_1(resp. π_2): $\boldsymbol{R}^2\to\boldsymbol{R}$ be the projection to the first (resp. the second) variable, and put $\Gamma_i=\pi_i(\Gamma)\subset\boldsymbol{R}$ $(i=1, 2)$. Since Γ contains the sublattice Γ' generated by $(\alpha_1, 0)$ and $(0, \alpha_2)$, it follows that the additive group Γ_i has a generator $\beta_i>0$ $(i=1, 2)$. Let Γ'' be the lattice in $\boldsymbol{R}^2$ generated by $(\beta_1, 0)$ and $(0, \beta_2)$. Then it is clear that the functions c^*f_i are periodic with the period β_i $(i=1, 2)$, and hence that the metric c^*g and the function $\tilde{F}$ are invariant under the action of Γ''. Hence we have a Liouville surface of elementary type whose underlying manifold is $\boldsymbol{R}^2/\Gamma''$, and which is covered by the Liouville surface (S, g, F) by the natural covering $S=\boldsymbol{R}^2/\Gamma\to\boldsymbol{R}^2/\Gamma''$. This completes the proof of Proposition 3.8. ∎

We shall now classify Liouville surfaces whose underlying riemannian manifolds are conformally isomorphic to a prescribed flat torus $(\boldsymbol{R}^2/\Gamma, dx_1^2+dx_2^2)$. As before, we identify $\boldsymbol{R}^2$ with $\boldsymbol{C}$ by taking the complex coordinate $z=x_1+\sqrt{-1}x_2$. As usual, we may assume that the lattice Γ in $\boldsymbol{C}$ is generated by 1 and $w\in U$, where

$$U = \left\{ z \in \boldsymbol{C} \mid |z| \geqq 1, \quad 0 \leqq \operatorname{Re} z \leqq \frac{1}{2}, \operatorname{Im} z > 0 \right\}.$$

Let l_1 and l_2 be real lines through the origin in $\boldsymbol{C}$ which are mutually orthogonal, and such that both $l_1 \cap \Gamma$ and $l_2 \cap \Gamma$ are not $\{0\}$. Let $\mathcal{A} = \mathcal{A}(\Gamma)$ be the set of all such ordered pairs (l_1, l_2) of lines. For each $l = (l_1, l_2) \in \mathcal{A}$ we assign a lattice $\Gamma'(l)$ in the following manner. Let π_i be the orthogonal projection $\boldsymbol{C} \to l_i$ $(i=1, 2)$, and put $\pi_i(\Gamma) = \Gamma'_i$. Note that Γ'_i $(i=1, 2)$ have generators, because $m\Gamma \subset l_1 \cap \Gamma + l_2 \cap \Gamma$ for some $m \in \boldsymbol{Z}$. Then we put $\Gamma'(l) = \Gamma'_1 + \Gamma'_2$. Clearly we have $\Gamma \subset \Gamma'(l)$.

LEMMA 3.10. *$\mathcal{A} = \mathcal{A}(\Gamma)$ is not empty if and only if* $\operatorname{Re} w \in \boldsymbol{Q}$, *or there are $p, q \in \boldsymbol{Q}$ such that $|w|^2 + (p+q)\operatorname{Re} w + pq = 0$, where $\boldsymbol{Q}$ denotes the field of rational numbers. When this condition is satisfied, we have the following two cases.*

(i) *If* $\operatorname{Re} w \notin \boldsymbol{Q}$ *or* $|w|^2 \notin \boldsymbol{Q}$, *then* $\#\mathcal{A} = 2$.

(ii) *If* $\operatorname{Re} w \in \boldsymbol{Q}$ *and* $|w|^2 \in \boldsymbol{Q}$, *then*

$$\mathcal{A} = \{(\boldsymbol{R}(w+p_r), \boldsymbol{R}(w+q_r)) \mid r \in \boldsymbol{Q}^\times\} \cup \{(\boldsymbol{R}, \boldsymbol{R}\sqrt{-1}), (\boldsymbol{R}\sqrt{-1}, \boldsymbol{R})\},$$

where

$$p_r = -(\operatorname{Im} w)^2 r - \operatorname{Re} w, \qquad q_r = r^{-1} - \operatorname{Re} w.$$

PROOF. As is easily seen, a pair of lines

$$(\boldsymbol{R}(w+p), \boldsymbol{R}(w+q)), \qquad p, q \in \boldsymbol{R}$$

is an element of $\mathcal{A}$ if and only if

$$p, q \in \boldsymbol{Q}, \qquad \operatorname{Re}\left(\frac{w+p}{w+q}\right) = 0.$$

This condition is equivalent to

$$p, q \in \boldsymbol{Q}, \qquad (\operatorname{Re} w + p)(\operatorname{Re} w + q) + (\operatorname{Im} w)^2 = 0. \tag{3.6}$$

The condition for $(\boldsymbol{R}, \boldsymbol{R}\sqrt{-1}) \in \mathcal{A}$ is clearly given by $\operatorname{Re} w \in \boldsymbol{Q}$. Hence the first assertion follows.

Now assume that $\operatorname{Re} w \in \boldsymbol{Q}$, or the formula (3.6) is satisfied. If $\operatorname{Re} w \notin \boldsymbol{Q}$, then in the formula (3.6) $p+q$ and pq must be unique. Hence $\#\mathcal{A} = 2$ in this case. If $\operatorname{Re} w \in \boldsymbol{Q}$ and $|w|^2 \notin \boldsymbol{Q}$, then there are no $p, q \in \boldsymbol{Q}$ satisfying (3.6). Hence $\mathcal{A} = \{(\boldsymbol{R}, \boldsymbol{R}\sqrt{-1}), (\boldsymbol{R}\sqrt{-1}, \boldsymbol{R})\}$ in this case. If $\operatorname{Re} w \in \boldsymbol{Q}$ and $|w|^2 \in \boldsymbol{Q}$, then $(\operatorname{Im} w)^2$ is a positive rational number. Hence (3.6) is solved by using the parameter $r \in \boldsymbol{Q}^\times$:

$$\operatorname{Re} w + p = -(\operatorname{Im} w)^2 r, \qquad \operatorname{Re} w + q = r^{-1}.$$

This proves (ii). ■

Let $l=(l_1, l_2)\in\mathcal{A}$. Let $\mathcal{F}(l)$ be the set of pairs of functions $f=(f_1, f_2)$ such that f_i is a function on l_i which is invariant under the action of Γ_i' $(i=1, 2)$, and $f_1+f_2>0$. Here the functions f_i are identified with their pulled back images by the orthogonal projections $\pi_i: \boldsymbol{C}\to l_i$. We put

$$\tilde{\mathcal{B}}(\Gamma)=\{(l, f) \mid l\in\mathcal{A},\ f\in\mathcal{F}(l)\}.$$

For each $(l, f)\in\tilde{\mathcal{B}}(\Gamma)$ we assign a Liouville surface whose underlying riemannian manifold is conformally isomorphic to

$$(\boldsymbol{C}/\Gamma,\ dx_1^2+dx_2^2)$$

as follows. Let (y_1, y_2) be an orthonormal coordinate system on $\boldsymbol{C}=\boldsymbol{R}^2$ so that $\partial/\partial y_i$ is tangent to l_i $(i=1, 2)$. Put

$$g=(f_1(y_1)+f_2(y_2))(dy_1^2+dy_2^2)$$
$$F=\frac{1}{f_1(y_1)+f_2(y_2)}(f_2(y_2)\eta_1^2-f_1(y_1)\eta_2^2),$$

where (y, η) is the associated canonical coordinate system on $T^*\boldsymbol{C}$. Clearly g and F are invariant under the action of Γ. By regarding them as a riemannian metric on $\boldsymbol{C}/\Gamma$ and a function on $T^*(\boldsymbol{C}/\Gamma)$ respectively, we obtain the Liouville surface $(\boldsymbol{C}/\Gamma, g, F)$.

Let G_0 denote the group of all conformal transformations of $\boldsymbol{C}$ which fix the origin 0 and leave Γ invariant. If $\mathrm{Re}\,w\neq 0, 1/2$ and $|w|\neq 1$, then G_0 consists of the multiplication of ± 1. If $\mathrm{Re}\,w=0, 1/2$, $w\neq\sqrt{-1}$, $e^{\pi i/3}$, then G_0 is generated by the mappings $z\mapsto\bar{z}$ and $z\mapsto -z$. If $|w|=1$, $w\neq\sqrt{-1}$, $e^{\pi i/3}$, then G_0 is generated by $z\mapsto w\bar{z}$ and $z\mapsto -z$. If $w=\sqrt{-1}$ or $e^{\pi i/3}$, then G_0 is generated by $z\mapsto\bar{z}$ and $z\mapsto wz$. In each case G_0 is clearly a finite group which preserves the set $\mathcal{A}$ and the metric $dx_1^2+dx_2^2$. Let $G(\Gamma)$ be the group of conformal transformations of $\boldsymbol{C}$ generated by G_0 and all of the translations.

We say that two elements (l, f) and (m, h) in $\tilde{\mathcal{B}}(\Gamma)$ are equivalent if there are $\Phi\in G(\Gamma)$ and $a\in\boldsymbol{R}$ such that one of the following two cases occurs:

(1) $\Phi_0(l_i)=m_i,\ \Phi^*h_i=f_i+(-1)^i a \qquad (i=1, 2);$

(2) $\Phi_0(l_1)=m_2,\ \Phi_0(l_2)=m_1,\ \Phi^*h_2=f_1-a,\ \Phi^*h_1=f_2+a,$

where $\Phi_0\in G_0$ denotes the linear part of Φ. It is easy to see that mutually equivalent elements in $\tilde{\mathcal{B}}(\Gamma)$ give mutually equivalent Liouville surfaces. Let $\mathcal{B}(\Gamma)$ denotes the set of all equivalence classes.

THEOREM 3.11. *The above assignment gives the one-to-one correspondence between the set $\mathcal{B}(\Gamma)$ and the set of equivalence classes of Liouville surfaces whose underlying riemannian manifolds are conformally isomorphic to the flat torus $(\boldsymbol{C}/\Gamma, dx_1^2+dx_2^2)$.*

PROOF. In view of Proposition 3.8 it is clear that the assignment described above is surjective. Now assume that (l, f) and (m, h) in $\tilde{\mathscr{B}}(\Gamma)$ give the same equivalence class of Liouville surfaces. Put

$$g=(f_1+f_2)(dx_1^2+dx_2^2), \qquad g'=(h_1+h_2)(dx_1^2+dx_2^2).$$

Then by the assumption we have an isometry $\phi: (\boldsymbol{C}/\Gamma, g)\to(\boldsymbol{C}/\Gamma, g')$. This gives an isometry $\tilde{\phi}: (\boldsymbol{C}, g)\to(\boldsymbol{C}, g')$ and an automorphism ψ of Γ as an abelian group such that

$$\tilde{\phi}(z+\gamma)=\tilde{\phi}(z)+\psi(\gamma), \qquad z\in\boldsymbol{C},\ \gamma\in\Gamma,$$

where the metrics g and g' on $\boldsymbol{C}/\Gamma$ are identified with their pulled back images on $\boldsymbol{C}$. Hence we can extend ψ on $\boldsymbol{C}$ by putting $\psi(z)=\tilde{\phi}(z)-\tilde{\phi}(0)$. Since $\tilde{\phi}$ is a conformal mapping, it follows that $\psi\in G_0$. Hence $\tilde{\phi}\in G(\Gamma)$, and we have $\tilde{\phi}^*(h_1+h_2)=f_1+f_2$. Since f_i and $h_i (i=1, 2)$ are non-constant functions, it easily follows that $\{\psi(l_1), \psi(l_2)\}=\{m_1, m_2\}$. Therefore we see that (l, f) and (m, h) are equivalent. ∎

Finally we consider the case where the underlying manifold is diffeomorphic to the Klein bottle. Let $w\in\boldsymbol{R}\sqrt{-1}$ such that $\operatorname{Im}w>0$, and let Γ be the lattice in $\boldsymbol{C}$ generated by 1 and w. Put

$$\tilde{c}(z)=\bar{z}+\frac{1}{2}, \qquad z\in\boldsymbol{C}.$$

Clearly $\tilde{c}$ induces a conformal transformation c of $\boldsymbol{C}/\Gamma$ satisfying $c^2=1$. We put

$$K_w=(\boldsymbol{C}/\Gamma)/\{1, c\}.$$

Then K_w is diffeomorphic to the Klein bottle, which possesses the riemannian metric g_w induced from the metric $dx_1^2+dx_2^2$. As is easily seen, any riemannian manifold whose underlying manifold is diffeomorphic to the Klein bottle is conformally isomorphic to one of the flat Klein bottles (K_w, g_w).

Now we fix $w\in\boldsymbol{R}\sqrt{-1}$, $\operatorname{Im}w>0$. Let $f_1(x_1)$ and $f_2(x_2)$ be functions such that $f_1(x_1)+f_2(x_2)>0$ for any $x_1, x_2\in\boldsymbol{R}$ and

$$f_1\left(x_1+\frac{1}{2}\right)=f_1(x_1)$$

$$f_2(-x_2)=f_2(x_2), \qquad f_2(x_2+\operatorname{Im}w)=f_2(x_2).$$

We say that two such pairs of functions (f_1, f_2) and (h_1, h_2) are equivalent if there are $a, b\in\boldsymbol{R}$, $n\in\boldsymbol{Z}$, and $\varepsilon\in\{1, -1\}$ such that

$$h_1(x_1)=f_1(\varepsilon x_1+b)-a$$

$$h_2(x_2)=f_2\left(x_2+\frac{n}{2}\operatorname{Im}w\right)+a.$$

Let $C(w)$ be the set of the equivalence classes of such pairs. For each element of $C(w)$ we assign an equivalence class of Liouville surfaces as follows. Let (f_1, f_2) be a representative. Put

$$g=(f_1(x_1)+f_2(x_2))(dx_1^2+dx_2^2)$$

$$F=\frac{1}{f_1(x_1)+f_2(x_2)}(f_2(x_2)\xi_1^2-f_1(x_1)\xi_2^2).$$

Then clearly g and F are invariant under the action of Γ and the mapping $\tilde{c}$. Hence they induce a Liouville surface whose underlying manifold is K_w, and whose riemannian metric is conformal to g_w. It is easy to see that another representative gives an equivalent Liouville surface.

Then we get

PROPOSITION 3.12. *The above assigment gives the one-to-one correspondence between the set $C(w)$ and the equivalence classes of Liouville surfaces whose underlying riemannian manifolds are conformally isomorphic to (K_w, g_w).*

The proof is easily obtained by using Theorem 3.11.

§ 4. Symplectically isomorphic geodesic flows

Let us consider a Liouville surface (S^2, g, F). Let $(\alpha_1, \alpha_2, f_1, f_2)$ be one of the corresponding quadruples. In this section we shall assume that the following condition is satisfied.

(4.1) There is only one $c_i\in\left(0, \frac{\alpha_i}{2}\right)$ such that $f_i'(c_i)=0$. Moreover the second derivative $f_i''(c_i)$ does not vanish. $(i=1, 2.)$

Put $M=f_1(c_1)+f_2(c_2)>0$. Then there is a number $\beta\in(0, 1)$ such that

$$f_1(c_1)=M(1-\beta), \qquad f_2(c_2)=M\beta.$$

We then introduce new coordinates θ_i $(i=1, 2)$ by

$$\begin{aligned} f_1(x_1)&=M(1-\beta)\sin^2\theta_1 \\ f_2(x_2)&=M\beta\sin^2\theta_2 \end{aligned} \tag{4.2}$$

and the conditions: $dx_i/d\theta_i>0$, and $x_i=0$ corresponds to $\theta_i=0$ $(i=1, 2)$. It is easy to see that this coordinate change is well-defined, and that the mapping $x_i\to\theta_i$ gives a diffeomorphism $\boldsymbol{R}/\alpha_i\boldsymbol{Z}\to\boldsymbol{R}/2\pi\boldsymbol{Z}$.

Now we define positive functions $A_1(t)$ $(|t|\leqq\sqrt{1-\beta})$ and $A_2(t)$ $(|t|\leqq\sqrt{\beta})$ by

$$A_1(\sqrt{1-\beta}\cos\theta_1)=\frac{dx_1}{d\theta_1}\sqrt{M}\sqrt{\beta\cos^2\theta_1+\sin^2\theta_1}$$

$$A_2(\sqrt{\beta}\cos\theta_2)=\frac{dx_2}{d\theta_2}\sqrt{M}\sqrt{(1-\beta)\cos^2\theta_2+\sin^2\theta_2}\,.$$

It is easy to see that $A_1(t)$ and $A_2(t)$ are well-defined C^∞ functions on the closed intervals $[-\sqrt{1-\beta}, \sqrt{1-\beta}]$ and $[-\sqrt{\beta}, \sqrt{\beta}]$ respectively. The riemannian metric g is then described as

$$\begin{aligned} g &= ((1-\beta)\sin^2\theta_1+\beta\sin^2\theta_2) \\ &\quad\times\left(\frac{A_1(\sqrt{1-\beta}\cos\theta_1)^2}{\beta\cos^2\theta_1+\sin^2\theta_1}d\theta_1^2+\frac{A_2(\sqrt{\beta}\cos\theta_2)^2}{(1-\beta)\cos^2\theta_2+\sin^2\theta_2}d\theta_2^2\right) \end{aligned} \tag{4.3}$$

In this case the conditions (3.1) are translated to the following conditions.

$$\begin{aligned} &A_1(t)\sim A_1(-t)\ \ (t\to\sqrt{1-\beta}\,) \\ &A_2(t)\sim A_2(-t)\ \ (t\to\sqrt{\beta}\,) \\ &A_1(t)\sim A_2(\sqrt{1-t^2}\,)\ \ (t\to\sqrt{1-\beta}\,) \end{aligned} \tag{4.4}$$

Here the symbol $\sim$ expresses that both sides have the same Taylor expansion at the given points. Conversely, for any positive number M, and for any positive C^∞ functions $A_1(t)$ and $A_2(t)$ defined on the closed intervals $[-\sqrt{1-\beta}, \sqrt{1-\beta}]$ and $[-\sqrt{\beta}, \sqrt{\beta}]$ which satisfy the conditions (4.4), we get a quadruple and hence a Liouville surface (S^2, g, F) by reversing the above procedure. Clearly another choice of M gives an equivalent Liouville surface. Therefore *we shall assume that* $M=1$ *in the rest of the paper.*

We shall describe some examples. First let us consider the ellipsoid

$$\frac{y_1^2}{b_1}+\frac{y_2^2}{b_2}+\frac{y_3^2}{b_3}=1 \qquad (b_1>b_3>b_2>0)$$

in $\boldsymbol{R}^3=\{(y_1, y_2, y_3)\}$. This ellipsoid is parametrized by $\boldsymbol{R}/2\pi\boldsymbol{Z}\times\boldsymbol{R}/2\pi\boldsymbol{Z}=\{(\theta_1, \theta_2)\}$ as follows:

$$\begin{aligned} y_1 &= \sqrt{b_1}\cos\theta_1\sqrt{(1-\beta)\cos^2\theta_2+\sin^2\theta_2} \\ y_2 &= \sqrt{b_2}\cos\theta_2\sqrt{\beta\cos^2\theta_1+\sin^2\theta_1} \\ y_3 &= \sqrt{b_3}\sin\theta_1\sin\theta_2 \end{aligned} \tag{4.5}$$

where $\beta=(b_3-b_2)/(b_1-b_2)\in(0, 1)$. Then the riemannian metric g induced from the canonical metric on $\boldsymbol{R}^3$ is given by

$$\begin{aligned} g &= ((1-\beta)\sin^2\theta_1+\beta\sin^2\theta_2) \\ &\quad\times\left(\frac{b_3\cos^2\theta_1+b_1\sin^2\theta_1}{\beta\cos^2\theta_1+\sin^2\theta_1}d\theta_1^2+\frac{b_3\cos^2\theta_2+b_2\sin^2\theta_2}{(1-\beta)\cos^2\theta_2+\sin^2\theta_2}d\theta_2^2\right) \end{aligned}$$

(cf. Klingenberg [7] Section 3.5). Next we consider the sphere of radius 1. In this case, put $b_1=b_2=b_3=1$ and let $\beta\in(0, 1)$ be arbitrary in (4.5). Then the induced metric g_0 is given by

$$g_0=((1-\beta)\sin^2\theta_1+\beta\sin^2\theta_2)\times\Big(\frac{d\theta_1^2}{\beta\cos^2\theta_1+\sin^2\theta_1}+\frac{d\theta_2^2}{(1-\beta)\cos^2\theta_2+\sin^2\theta_2}\Big). \tag{4.6}$$

Now we shall show the main theorem in this section. We put

$$A_j^{\pm}(t)=\frac{1}{2}(A_j(t)\pm A_j(-t)) \qquad (j=1, 2).$$

THEOREM 4.1. *Fix* $\beta\in(0, 1)$. *Let* $A_{i1}(t)$ *and* $A_{i2}(t)$ $(i=1, 2)$ *be positive* C^∞ *functions on* $[-\sqrt{1-\beta}, \sqrt{1-\beta}]$ *and* $[-\sqrt{\beta}, \sqrt{\beta}])$ *respectively which satisfy the conditions* (4.4), *and let* (S^2, g_i, F_i) *be the corresponding Liouville surfaces. Assume that* $A_{1j}^+(t)=A_{2j}^+(t)$ $(j=1, 2)$ *Then the geodesic flows of the riemannian manifolds* (S^2, g_i) $(i=1, 2)$ *defined on* $\mathring{T}^*S^2=T^*S^2-\{0\text{-section}\}$ *are mutually sympletically isomorphic.*

PROOF. We put $A_{rj}(t)=(2-r)A_{1j}(t)+(r-1)A_{2j}(t)$ $(1\leqq r\leqq 2)$. Then we have $A_{rj}^+(t)=A_{1j}^+(t)=A_{2j}^+(t)$ $(j=1, 2)$. The corresponding riemannian metric g_r is given by

$$g_r=((1-\beta)\sin^2\theta_1+\beta\sin^2\theta_2)\times\Big(\frac{A_{r1}(\sqrt{1-\beta}\cos\theta_1)^2}{\beta\cos^2\theta_1+\sin^2\theta_1}d\theta_1^2+\frac{A_{r2}(\sqrt{\beta}\cos\theta_2)^2}{(1-\beta)\cos^2\theta_2+\sin^2\theta_2}d\theta_2^2\Big).$$

Here the underlying manifold S^2 is identified with $\boldsymbol{R}/2\pi\boldsymbol{Z}\times\boldsymbol{R}/2\pi\boldsymbol{Z}$ devided by the involution $(\theta_1, \theta_2)\mapsto(-\theta_1, -\theta_2)$. Let E_r be the corresponding energy function, and let $\dot{E}_r$ be its derivative in r. In general for a function u on an open subset of T^*S^2, let X_u denote the symplectic vector field defined by the hamiltonian u.

We shall solve the equation

$$X_{E_r}G_r=\dot{E}_r \qquad (1\leqq r\leqq 2) \tag{4.7}$$

so that G_r is a homogeneous C^∞ function on $\mathring{T}^*S^2$ which is also C^∞ in the parameter r. If such G_r is found, then the one-parameter family of homogeneous symplectic diffeomorphisms Φ_r $(1\leqq r\leqq 2)$ of $\mathring{T}^*S^2$ defined by

$$\frac{d}{dr}\Phi_r(\eta)=(X_{G_r})_{\Phi_r(\eta)}, \qquad \Phi_0(\eta)=\eta, \quad \eta\in\mathring{T}^*S^2$$

satisfies $\Phi_r^*E_r=E_0$, which will prove the theorem.

The function G_r is given as follows. Let (θ, η) be the canonical coordinates associated with $\theta=(\theta_1, \theta_2)$. As is easily seen, the energy function E_r and the first integral F_r are given by

$$E_r=\frac{1}{2}\frac{1}{(1-\beta)\sin^2\theta_1+\beta\sin^2\theta_2}$$
$$\times\Big(\frac{\beta\cos^2\theta_1+\sin^2\theta_1}{A_{r1}(\sqrt{1-\beta}\cos\theta_1)^2}\eta_1^2+\frac{(1-\beta)\cos^2\theta_2+\sin^2\theta_2}{A_{r2}(\sqrt{\beta}\cos\theta_2)^2}\eta_2^2\Big)$$
$$F_r=\frac{1}{(1-\beta)\sin^2\theta_1+\beta\sin^2\theta_2}$$
$$\times\Big(\frac{\beta\cos^2\theta_1+\sin^2\theta_1}{A_{r1}(\sqrt{1-\beta}\cos\theta_1)^2}\beta\sin^2\theta_2\eta_1^2$$
$$-\frac{(1-\beta)\cos^2\theta_2+\sin^2\theta_2}{A_{r2}(\sqrt{\beta}\cos\theta_2)^2}(1-\beta)\sin^2\theta_1\eta_2^2\Big).$$

When $F_r(\theta,\eta)\leqq 0$, then we put

$$G_r(\theta,\eta)=-\frac{\sqrt{2E_r}}{2}\varepsilon_1\int_{(1-\beta)\cos^2\theta_1}^{1-\beta+(F_r/2E_r)}\frac{B_1(y)}{\sqrt{1-y}}\sqrt{\frac{1-\beta+(F_r/2E_r)-y}{1-\beta-y}}dy$$
$$-\frac{\sqrt{2E_r}}{2}\varepsilon_2\int_{\beta\cos^2\theta_2}^{\beta}\frac{B_2(y)}{\sqrt{1-y}}\sqrt{\frac{\beta-(F_r/2E_r)-y}{\beta-y}}dy,$$

and when $F_r(\theta,\eta)>0$,

$$G_r(\theta,\eta)=-\frac{\sqrt{2E_r}}{2}\varepsilon_1\int_{(1-\beta)\cos^2\theta_1}^{1-\beta}\frac{B_1(y)}{\sqrt{1-y}}\sqrt{\frac{1-\beta+(F_r/2E_r)-y}{1-\beta-y}}dy$$
$$-\frac{\sqrt{2E_r}}{2}\varepsilon_2\int_{\beta\cos^2\theta_2}^{\beta-(F_r/2E_r)}\frac{B_2(y)}{\sqrt{1-y}}\sqrt{\frac{\beta-(F_r/2E_r)-y}{\beta-y}}dy,$$

where ε_j denotes the sign of $\eta_j\sin\theta_j$ $(j=1, 2)$. The functions $B_1(y)$ and $B_2(y)$ are defined as follows. By the assumption on $A_{ij}(t)$, we see that $\dot{A}_{rj}(t)=A_{2j}(t)-A_{1j}(t)$ is the odd function in t. Hence there is a unique C^∞ function $B_j(y)$ so that

$$\dot{A}_{rj}(t)=tB_j(t^2) \qquad (j=1, 2).$$

From the above definition it is easy to see that $\{G_r\}$ is a well-defined continuous function on $[1, 2]\times\mathring{T}^*S^2$ which is C^∞ in the region where $F_r\neq 0$, and that the formula (4.7) is satisfied. The smoothness at points where $F_r=0$ is also easily verified in view of the fact that $B_1(y)$ and $B_2(y)$ vanishes up to infinite order at $y=1-\beta$ and $y=\beta$ respectively, which is a consequence of the conditions (4.4) and the assumption that $A_{1j}^+=A_{2j}^+$. ■

A riemannian metric g on S^2 is called a $C_{2\pi}$-metric if every geodesic on (S^2, g) is closed and has the length 2π. We can determine Liouville surfaces whose riemannian metrics are $C_{2\pi}$-metrics.

PROPOSITION 4.2. *Let (S^2, g, F) be a Liouville surface. Then g is $C_{2\pi}$-*

metric if and only if the corresponding quadruples satisfy the condition (4.1), *and the corresponding functions* $A_1(t)(|t|\leqq\sqrt{1-\beta})$ *and* $A_2(t)(|t|\leqq\sqrt{\beta})$ *satisfy*

$$A_1^{\dagger}(t)=A_2^{\dagger}(t)=1.$$

PROOF. If $A_1^{\dagger}=A_2^{\dagger}=1$, then Theorem 4.1 shows that the geodesic flow of (S^2, g) is symplectically isomorphic to that of (S^2, g_0), the sphere of constant curvature 1. Hence g is a $C_{2\pi}$-metric.

Next let (S^2, g, F) be a Liouville surface such that g is a $C_{2\pi}$-metric. Let $(\alpha_1, \alpha_2, f_1(x_1), f_2(x_2))$ be one of the corresponding quadruples. Suppose that $f_1'(x_1)$ vanishes at $x_1=a\in(0, \alpha_1/2)$. Then the closed curve $x_1=a$ is the orbit of a geodesic, and its length is given by

$$\int_0^{\alpha_2}\sqrt{f_1(a)+f_2(x_2)}dx_2.$$

Since this value shoud be 2π, it follows that the value $f_1(a)$ does not depend on a. This implies that $f_1''(a)\leqq 0$. Now assume that $f_1''(a)=0$. Then along the closed geodesic $x_1=a$, the following vector field becomes a Jacobi field:

$$\int_0^t\frac{ds}{f_1(a)+f_2(x_2(s))}\frac{\partial}{\partial x_1},$$

where t and s are the length-parameter of the geodesic $x_1=a$. Clearly this Jacobi field is not periodic, which contradicts the assumption. Hence we have $f_1''(a)<0$, and a must be unique. Since the same is true for $f_2(x_2)$, the condition (4.1) is satisfied.

Now let $A_1(t)$ $(|t|\leqq\sqrt{1-\beta})$ and $A_2(t)$ $(|t|\leqq\sqrt{\beta})$ be the corresponding functions. Let (S^2, g^+, F^+) be the Liouville surface defined from the functions $A_1^{\dagger}$ and $A_2^{\dagger}$. Then it follows from Theorem 4.1 that g^+ is a $C_{2\pi}$-metric. Let E^+ be the corresponding energy function. Then on a solution curve of X_{E^+} with $2E^+=1$ and $F^+=c$, we get

$$\begin{aligned}\frac{dt}{(1-\beta)\sin^2\theta_1+\beta\sin^2\theta_2}&=\frac{\eta_1}{|\eta_1|}\frac{A_1^{\dagger}(\sqrt{1-\beta}\cos\theta_1)d\theta_1}{\sqrt{\beta\cos^2\theta_1+\sin^2\theta_1}\sqrt{(1-\beta)\sin^2\theta_1+c}}\\ &=\frac{\eta_2}{|\eta_2|}\frac{A_2^{\dagger}(\sqrt{\beta}\cos\theta_2)d\theta_2}{\sqrt{(1-\beta)\cos^2\theta_2+\sin^2\theta_2}\sqrt{\beta\sin^2\theta_2-c}},\end{aligned}\tag{4.8}$$

$$\begin{aligned}dt=&\frac{\eta_1}{|\eta_1|}\frac{A_1^{\dagger}(\sqrt{1-\beta}\cos\theta_1)\sqrt{(1-\beta)\sin^2\theta_1+c}}{\sqrt{\beta\cos^2\theta_1+\sin^2\theta_1}}d\theta_1\\ &+\frac{\eta_2}{|\eta_2|}\frac{A_2^{\dagger}(\sqrt{\beta}\cos\theta_2)\sqrt{\beta\sin^2\theta_2-c}}{\sqrt{(1-\beta)\cos^2\theta_2+\sin^2\theta_2}}d\theta_2,\end{aligned}\tag{4.9}$$

where t is the parameter of the solution curve.

First assume that $c<0$. Since the geodesic is closed with length 2π, it follows that there are positive integers p and q which are mutually prime such that

$$2\pi = 2q\int_i^{\pi-i} \frac{A_1^\dagger(\sqrt{1-\beta}\cos\theta_1)\sqrt{(1-\beta)\sin^2\theta_1+c}}{\sqrt{\beta\cos^2\theta_1+\sin^2\theta_1}}d\theta_1$$
$$+2p\int_0^\pi \frac{A_2^\dagger(\sqrt{\beta}\cos\theta_2)\sqrt{\beta\sin^2\theta_2-c}}{\sqrt{(1-\beta)\cos^2\theta_2+\sin^2\theta_2}}d\theta_2,$$

where $i=\arcsin\sqrt{-c/(1-\beta)}\in(0, \pi/2)$. Clearly p and q do not depend on $c<0$. Hence by taking the limit $c\to 0$, we get $p=q=1$ in view of Theorem 2.1. By differentiating the above formula in the variable c, we get

$$\int_i^{\pi-i} \frac{A_1^\dagger(\sqrt{1-\beta}\cos\theta_1)d\theta_1}{\sqrt{\beta\cos^2\theta_1+\sin^2\theta_1}\sqrt{(1-\beta)\sin^2\theta_1+c}} \tag{4.10}$$
$$=\int_0^\pi \frac{A_2^\dagger(\sqrt{\beta}\cos\theta_2)d\theta_2}{\sqrt{(1-\beta)\cos^2\theta_2+\sin^2\theta_2}\sqrt{\beta\sin^2\theta_2-c}}$$

for $c<0$. Similarly we also get

$$\int_0^\pi \frac{A_1^\dagger(\sqrt{1-\beta}\cos\theta_1)d\theta_1}{\sqrt{\beta\cos^2\theta_1+\sin^2\theta_1}\sqrt{(1-\beta)\sin^2\theta_1+c}} \tag{4.11}$$
$$=\int_j^{\pi-j} \frac{A_2^\dagger(\sqrt{\beta}\cos\theta_2)d\theta_2}{\sqrt{(1-\beta)\cos^2\theta_2+\sin^2\theta_2}\sqrt{\beta\sin^2\theta_2-c}}$$

for $c>0$, where $j=\arcsin\sqrt{c/\beta}\in(0, \pi/2)$.

Put

$$A_1^\dagger(\sqrt{1-\beta}) = A_2^\dagger(\sqrt{\beta}) = a\,.$$

Then

$$h_1 = \frac{A_1^\dagger(\sqrt{1-\beta}\cos\theta_1)-a}{(1-\beta)\sin^2\theta_1+\beta\sin^2\theta_2}, \qquad h_2 = \frac{A_2^\dagger(\sqrt{\beta}\cos\theta_2)-a}{(1-\beta)\sin^2\theta_1+\beta\sin^2\theta_2}$$

are well-defined L^2 functions on S^2. Clearly they are even with respect to the antipodal mapping $(\theta_1, \theta_2)\mapsto(\theta_1+\pi, -\theta_2+\pi)$. Let R be the Radon transform on S^2 with the standard $C_{2\pi}$-metric g_0 (cf. Besse [2] p. 123). The operator R transforms the functions f on S^2 to the functions on the manifold of geodesics; the value of $R(f)$ at the geodesic γ is given by

$$R(f)(\gamma) = \int_\gamma f\,.$$

Then the formula (4.8) for the standard metric g_0 (the case where $A_1^\dagger=A_2^\dagger=1$) implies that the value of the function $R(h_1)$ at the geodesic with $2E_0=1$, $F_0=c$ is given by

$$2\int_i^{\pi-i} \frac{A_1^\dagger(\sqrt{1-\beta}\cos\theta_1)-a}{\sqrt{\beta\cos^2\theta_1+\sin^2\theta_1}\sqrt{(1-\beta)\sin^2\theta_1+c}}d\theta_1$$

when $c<0$, and

$$2\int_0^{\pi}\frac{A_1^{+}(\sqrt{1-\beta}\cos\theta_1)-a}{\sqrt{\beta\cos^2\theta_1+\sin^2\theta_1}\sqrt{(1-\beta)\sin^2\theta_1+c}}d\theta_1$$

when $c>0$, where E_0 and F_0 are the corresponding energy function and the first integral. Similarly we can compute the values of $R(h_2)$, and we consequently get $R(h_1)=R(h_2)$ in view of the formulae (4.10) and (4.11). Since the operator R is injective on the space of even functions, it follows that $h_1=h_2$. This implies that

$$A_1^{+}=A_2^{+}=a\,.$$

Then it is obvious that $a=1$. ■

In contrast with Zoll's examples (cf. Besse [**2**] Chapter 4), we have the following.

COROLLARY 4.3. *Let (S^2, g, F) be a Liouville surface such that g is a $C_{2\pi}$-metric. Assume that the riemannian manifold (S^2, g) is analytic. Then (S^2, g) is isometric to (S^2, g_0) of constant curvature* 1.

PROOF. In view of Corollary 3.6 the functions $A_1(t)$ and $A_2(t)$ are even in this case. Hence they are identically equal to 1. ■

§ 5. The isometry class and the equivalence class

In this section we shall prove the following

THEOREM 5.1. *Let $(S, g, F_i)\,(i=1, 2)$ be two compact Liouville surfaces. Assume that the gaussian curvature of (S, g) is not positive constant. Then $F_2\in \boldsymbol{R}F_1+\boldsymbol{R}E$.*

PROOF. By taking a double covering we may assume that S is diffeomorphic to the sphere or the torus. First we consider the case where S is diffeomorphic to S^2. Let L be the closed geodesic given in Theorem 2.1 with respect to the Liouville surface (S, g, F_1), and let σ be the reflection with respect to L. We normalize the metric g so that the length of L is equal to 2π. In view of Corollary 3.4, σ is a well-defined isometry of (S, g) which preserves F_1. Put

$$F_2^{\pm}=\frac{1}{2}(F_2\pm F_2\circ\sigma^*).$$

Then it is clear that F_2^{+} and F_2^{-} are also first integrals.

First we shall show that $F_2^{-}=0$. Assume that this is not the case. Then we have $F_2^{-}\notin \boldsymbol{R}F_1+\boldsymbol{R}E$, because $F_2^{-}\circ\sigma^*=-F_2^{-}$. If $(F_2^{-})_p=rE_p$ at some point $p\in S$, then $(F_2^{-})_{\sigma(p)}=-rE_{\sigma(p)}$. Hence the ratio r must be 0 in view of the proof of Lemma 1.2. Furthermore, the eigenvalues of $(F_2^{-})_q^{\flat}\,(q\in L)$ with respect to the metric g have the same absolute value with opposite sign. Since $(F_2^{-})_q^{\flat}\neq 0$

for some $q\in L$ in view of Lemma 2.3, these facts imply that (S, g, F_2^-) becomes a Liouville surface satisfying the condition (L.4). (Note that Lemmas 1.2 and 2.3 do not need the condition (L.3).)

It is easily seen that $(F_2^-)^\flat(u_p)=0$ at each point $p\in L$, where $u_p\in T_pS$ is tangent to or perpendicular to L. Then by applying Theorem 2.1 to the Liouville surface (S, g, F_2^-), we get a point $o\in S-L$ and $o'=\sigma(o)$ such that $(F_2^-)_o=(F_2^-)_{o'}=0$, and that any minimizing geodesic from o to o' intersects with L orthogonally at the midpoint. Let $\gamma(t)$ be any geodesic with $\dot\gamma(0)\in S_oS$. Then geodesic segment $\gamma(t)$ $(|t|\leqq\pi/2)$ and its image by σ are joined into a simply closed geodesic with length 2π. Note that the tangent vectors to these geodesics give all possible values of F_1. Since F_1 is a first integral, it follows from the well-known theorem on the invariant tori (cf. Arnold [1] §49) that g is a $C_{2\pi}$-metric.

We now apply the arguments in the previous section to the Liouville surface (S, g, F_1). Then metric g is described as

$$
\begin{aligned}
g=&((1-\beta)\sin^2\theta_1+\beta\sin^2\theta_2)\\
&\times\Big(\frac{(1+h_1(\sqrt{1-\beta}\cos\theta_1))^2}{\beta\cos^2\theta_1+\sin^2\theta_1}d\theta_1^2+\frac{(1+h_2(\sqrt{\beta}\cos\theta_2))^2}{(1-\beta)\cos^2\theta_2+\sin^2\theta_2}d\theta_2^2\Big),
\end{aligned}
\tag{5.1}
$$

where $\beta\in(0, 1)$, and $h_1(t)$ $(|t|\leqq\sqrt{1-\beta})$ and $h_2(t)$ $(|t|\leqq\sqrt{\beta})$ are C^∞ odd functions vanishing up to infinite order at the end points. F_1 is given by

$$
\begin{aligned}
F_1=&\frac{1}{(1-\beta)\sin^2\theta_1+\beta\sin^2\theta_2}\\
&\times\Big(\frac{\beta\cos^2\theta_1+\sin^2\theta_1}{(1+h_1(\sqrt{1-\beta}\cos\theta_1))^2}\beta\sin^2\theta_2\eta_1^2\\
&-\frac{(1-\beta)\cos^2\theta_2+\sin^2\theta_2}{(1+h_2(\sqrt{\beta}\cos\theta_2))^2}(1-\beta)\sin^2\theta_1\eta_2^2\Big),
\end{aligned}
$$

where (θ, η) is the associated canonical coordinates. Then on a solution curve of X_E with $2E=1$ and $F_1=c$ we have

$$
\begin{aligned}
\frac{dt}{(1-\beta)\sin^2\theta_1+\beta\sin^2\theta_2}&=\frac{\eta_1}{|\eta_1|}\frac{(1+h_1(\sqrt{1-\beta}\cos\theta_1))d\theta_1}{\sqrt{\beta\cos^2\theta_1+\sin^2\theta_1}\sqrt{(1-\beta)\sin^2\theta_1+c}}\\
&=\frac{\eta_2}{|\eta_2|}\frac{(1+h_2(\sqrt{\beta}\cos\theta_2))d\theta_2}{\sqrt{(1-\beta)\cos^2\theta_2+\sin^2\theta_2}\sqrt{\beta\sin^2\theta_2-c}},
\end{aligned}
\tag{5.2}
$$

where t is the parameter of the solution curve. By considering the cases where $c=\beta$ and $c=-(1-\beta)$, we see that the points o and o' are represented by $(\theta_1, \theta_2)=(\pi/2, \pi/2)$ and $(\pi/2, -\pi/2)$ respectively.

Now assume $c<0$, and consider the two solution curves which pass the points $(\theta_1, \theta_2)=(i, 0)$ and $(\pi-i, 0)$ respectively at $t=0$, and $\eta_2>0$, where $i=$

$\arcsin\sqrt{-c(1-\beta)^{-1}}\in(0, \pi/2)$. Then these solution curves pass the point o at $t=\pi/2$, and $d\theta_1/dt$ is positive on the first solution curve and negative on the second one while $t\in(0, \pi/2]$. Hence we have

$$\begin{aligned}&\int_i^{\pi/2}\frac{1+h_1(\sqrt{1-\beta}\cos\theta_1)}{\sqrt{\beta\cos^2\theta_1+\sin^2\theta_1}\sqrt{(1-\beta)\sin^2\theta_1+c}}d\theta_1\\ &\quad=\int_0^{\pi/2}\frac{1+h_2(\sqrt{\beta}\cos\theta_2)}{\sqrt{(1-\beta)\cos^2\theta_2+\sin^2\theta_2}\sqrt{\beta\sin^2\theta_2-c}}d\theta_2\\ &\quad=\int_{\pi/2}^{\pi-i}\frac{1+h_1(\sqrt{1-\beta}\cos\theta_1)}{\sqrt{\beta\cos^2\theta_1+\sin^2\theta_1}\sqrt{(1-\beta)\sin^2\theta_1+c}}d\theta_1\end{aligned}$$

Since h_1 is odd, it follows that

$$\int_i^{\pi/2}\frac{h_1(\sqrt{1-\beta}\cos\theta_1)}{\sqrt{\beta\cos^2\theta_1+\sin^2\theta_1}\sqrt{(1-\beta)\sin^2\theta_1+c}}d\theta_1=0.$$

By considering the same formula for the standard $C_{2\pi}$-metric g_0, we get

$$\begin{aligned}&\int_i^{\pi/2}\frac{d\theta_1}{\sqrt{\beta\cos^2\theta_1+\sin^2\theta_1}\sqrt{(1-\beta)\sin^2\theta_1+c}}\\ &\quad=\int_0^{\pi/2}\frac{d\theta_2}{\sqrt{(1-\beta)\cos^2\theta_2+\sin^2\theta_2}\sqrt{\beta\sin^2\theta_2-c}}.\end{aligned}$$

Hence we also have

$$\int_0^{\pi/2}\frac{h_2(\sqrt{\beta}\cos\theta_2)d\theta_2}{\sqrt{(1-\beta)\cos^2\theta_2+\sin^2\theta_2}\sqrt{\beta\sin^2\theta_2-c}}=0.$$

In the case where $c>0$ we also get

$$\begin{aligned}&\int_0^{\pi/2}\frac{h_1(\sqrt{1-\beta}\cos\theta_1)d\theta_1}{\sqrt{\beta\cos^2\theta_1+\sin^2\theta_1}\sqrt{(1-\beta)\sin^2\theta_1+c}}\\ &\quad=\int_j^{\pi/2}\frac{h_2(\sqrt{\beta}\cos\theta_2)d\theta_2}{\sqrt{(1-\beta)\cos^2\theta_2+\sin^2\theta_2}\sqrt{\beta\sin^2\theta_2-c}}=0\end{aligned}$$

in the same way, where $j=\arcsin\sqrt{c\beta^{-1}}\in(0, \pi/2)$.

Let $\tilde{h}_1(t)$ ($|t|\leqq\sqrt{1-\beta}$) be the even function such that $\tilde{h}_1(t)=h_1(t)$ for $t\geqq 0$. Then

$$\frac{\tilde{h}_1(\sqrt{1-\beta}\cos\theta_1)}{(1-\beta)\sin^2\theta_1+\beta\sin^2\theta_2}$$

expresses a continuous function on S^2. Let R be the Radon transform on S^2 with the standard $C_{2\pi}$-metric g_0. The formula (5.2) for the standard metric g_0 implies that the function

$$R\left(\frac{\tilde{h}_1(\sqrt{1-\beta}\cos\theta_1)}{(1-\beta)\sin^2\theta_1+\beta\sin^2\theta_2}\right)$$

takes value

$$4\int_i^{\pi/2}\frac{h_1(\sqrt{1-\beta}\cos\theta_1)d\theta_1}{\sqrt{\beta\cos^2\theta_1+\sin^2\theta_1}\sqrt{(1-\beta)\sin^2\theta_1+c}}=0$$

at geodesic with $F_1=c<0$, and

$$4\int_0^{\pi/2}\frac{h_1(\sqrt{1-\beta}\cos\theta_1)d\theta_1}{\sqrt{\beta\cos^2\theta_1+\sin^2\theta_1}\sqrt{(1-\beta)\sin^2\theta_1+c}}=0$$

at geodesic with $F_1=c>0$. Since the operator R is injective on the space of even functions, it follows that $\tilde{h}_1=0$, and hence $h_1=0$. In the same way we also have $h_2=0$. But this indicates $g=g_0$, which contradicts the assumption. Therefore we get $F_2^-=0$ and $F_2=F_2^+$.

Now assume that $F_2\notin\boldsymbol{R}F_1+\boldsymbol{R}E$. In this case it is easy to see that at each point on L the eigenvectors of $F_2^\flat$ with respect to the metric g are proportional to or perpendicular to the tangent vector of L. Therefore for any point $p\in L$ with $(F_2)_p\notin\boldsymbol{R}E_p$, we can find $t\in\boldsymbol{R}^\times$ and $s\in\boldsymbol{R}$ such that $F=F_1+tF_2+sE$ vanishes at p. Then it is clear that $F^\flat$ is semi-definite at each point on L whose value for the tangent vector of L is 0. Let $p'\in L$ be the point whose distance from p along the geodesic L is π. Then by Theorem 2.1 we see that $F_{p'}=0$, and every geodesic starting at p passes the point p' at time π. Let $\gamma(t)$ be a geodesic with $\dot\gamma(0)\in S_pS$. Then $\gamma(-\pi)=\gamma(\pi)=p'$. If $\dot\gamma(0)$ is not perpendicular to L, then $\gamma([-\pi,\pi])$ is not invariant under the action of σ. Furthermore we have $F_1^\flat(\dot\gamma(-\pi))=F_1^\flat(\dot\gamma(\pi))$. These facts imply that $\dot\gamma(-\pi)=\dot\gamma(\pi)$, i. e., $\gamma(t)$ $(|t|\leqq\pi)$ is a closed geodesic with length 2π. Clearly this is also true when $\dot\gamma(0)$ is perpendicular to L. Therefore it has shown that g is a $C_{2\pi}$-metric.

Now again the metric g is described as (5.1), and we get the formula (5.2) on an integral curve of X_E. In this case we consider a geodesic $\gamma(t)$ with $\gamma(0)=p\in L$ and $\gamma(\pi)=p'$. First assume that $F_1^\flat(\dot\gamma(t))=c<0$. Then the point p (resp. p') is represented by $\theta_2=0$ and $\theta_1=\theta_1(p)\in(0,\pi)$ (resp. $\theta_2=\pi$ and $\theta_1=\theta_1(p')\in(0,\pi)$. Clearly we have $\theta_1(p)+\theta_1(p')=\pi$. We take the geodesic $\gamma(t)$ so that $d\theta_2/dt>0$ and $(d\theta_1/dt)|_{t=0}<0$. Then we have

$$\begin{aligned}&\int_0^\pi\frac{dt}{(1-\beta)\sin^2\theta_1+\beta\sin^2\theta_2}\\ &=\int_i^{\theta_1(p)}\frac{(1+h_1(\sqrt{1-\beta}\cos\theta_1))d\theta_1}{\sqrt{\beta\cos^2\theta_1+\sin^2\theta_1}\sqrt{(1-\beta)\sin^2\theta_1+c}}\\ &\quad+\int_i^{\theta_1(p')}\frac{(1+h_1(\sqrt{1-\beta}\cos\theta_1))d\theta_1}{\sqrt{\beta\cos^2\theta_1+\sin^2\theta_1}\sqrt{(1-\beta)\sin^2\theta_1+c}}\end{aligned}$$

$$=\int_0^{\pi}\frac{(1+h_2(\sqrt{\beta}\cos\theta_2))d\theta_2}{\sqrt{(1-\beta)\cos^2\theta_2+\sin^2\theta_2}\sqrt{\beta\sin^2\theta_2-c}},$$

where $i=\arcsin\sqrt{-c(1-\beta)^{-1}}\in(0, \pi/2)$.

Since h_2 is odd, the last formula is equal to

$$\int_0^{\pi}\frac{d\theta_2}{\sqrt{(1-\beta)\cos^2\theta_2+\sin^2\theta_2}\sqrt{\beta\sin^2\theta_2-c}}.$$

By considering the case of the standard metric g_0, this formula is equal to

$$\int_i^{\pi-i}\frac{d\theta_1}{\sqrt{\beta\cos^2\theta_1+\sin^2\theta_1}\sqrt{(1-\beta)\sin^2\theta_1+c}}.$$

Since $\theta_1(p')=\pi-\theta_1(p)$, we then obtain

$$\int_i^{\theta_1(p)}\frac{h_1(\sqrt{1-\beta}\cos\theta_1)}{\sqrt{\beta\cos^2\theta_1+\sin^2\theta_1}\sqrt{(1-\beta)\sin^2\theta_1+c}}d\theta_1=0.$$

Since $\theta_1(p)\in[i, \pi-i]$ is arbitrary for a fixed $c<0$, it follows that $h_1=0$. By considering the case where $c>0$, we also have $h_2=0$. Hence we again get a contradiction. This completes the proof of the case where S is diffeomorphic to the sphere.

Next we shall consider the case where S is diffeomorphic to the torus. In view of the proof of Theorem 3.11, it is easily seen that if two elements (l, f) and (m, h) in $\tilde{\mathscr{B}}(\Gamma)$ induce the mutually isometric Liouville surfaces $(\boldsymbol{C}/\Gamma, g_i, F_i)$ $(i=1, 2)$, then *any* isometry $\phi: (\boldsymbol{C}/\Gamma, g_1)\to(\boldsymbol{C}/\Gamma, g_2)$ is covered by a conformal transformation $\tilde{\phi}\in G(\Gamma)$, which combined with some $a\in\boldsymbol{R}$ gives the equivalence of (l, f) and (m, h). Hence we have $F_2\circ(\phi^*)^{-1}\in\boldsymbol{R}F_1+\boldsymbol{R}E_1$. This completes the proof of Theorem 5.1. ■

COROLLARY 5.2. *Let (S, g, F) be a compact Liouville surface such that the gaussian curvature of (S, g) is not positive constant. Then (S, g) admits no non-zero Killing vector field.*

PROOF. Let Z be a Killing vector field on (S, g), and define $\zeta\in C^\infty(T^*S)$ by

$$\zeta(\lambda)=\lambda(Z),\qquad \lambda\in T^*S.$$

Then it is obvious that the function $F+t\zeta^2$ satisfies the conditions (L.1), (L.2), and (L.3) for small $t\in\boldsymbol{R}$. Hence by Theorem 5.1 there are $a_t, b_t\in\boldsymbol{R}$ such that

$$F+t\zeta^2=a_tF+b_tE.$$

By restricting this formula to a fibre where F and E are linearly independent, we see that a_t and b_t are unique and C^∞ in t. Then we get

$$\zeta^2=\dot{a}_0F+\dot{b}_0E.$$

If $\dot{a}_0 \neq 0$, then it contradicts the condition (L.3). Hence $\dot{a}_0=0$, and clearly $\dot{b}_0$ is also zero. ■

§6. Laplacian and the operator □

Let (S, g, F) be a compact Liouville surface, and let $\triangle$ be the laplacian acting on functions.

PROPOSITION 6.1. *There is a naturally defined differential operator* $\square$ *of second order on* S *such that its principal symbol is* $-F$, *and that* $[\triangle, \square]=0$. *Furthermore the operator* $\square$ *is self-adjoint with respect to the canonical density on* (S, g).

PROOF. As was seen in the proof of Proposition 1.1, we get functions f_1 and f_2 on $S-\mathfrak{N}$ in a unique way. Although the coordinate functions x_1 and x_2 are not uniquely obtained there, it is easy to see that the differential operators $\partial^2/\partial x_1^2$ and $\partial^2/\partial x_2^2$ are unique. Hence we can define a C^∞ operator $\square$ on $S-\mathfrak{N}$ by

$$\square = \frac{1}{f_1+f_2}\Big(f_2\frac{\partial^2}{\partial x_1^2}-f_1\frac{\partial^2}{\partial x_2^2}\Big). \tag{6.1}$$

We shall show that this is a well-defined C^∞ operator on S. Assume that S is diffeomorphic to S^2. Let $U(y_1, y_2)$ be a C^∞ function around $(0, 0)$, and put $u(x_1, x_2)=U(x_1^2-x_2^2, 2x_1x_2)$. Then

$$\begin{aligned}\square u = {}& 4\frac{f_2(x_2)x_1^2-f_1(x_1)x_2^2}{f_1(x_1)+f_2(x_2)}\frac{\partial^2 U}{\partial y_1^2}(x_1^2-x_2^2, 2x_1x_2)\\ &+4\frac{f_2(x_2)x_2^2-f_1(x_1)x_1^2}{f_1(x_1)+f_2(x_2)}\frac{\partial^2 U}{\partial y_2^2}(x_1^2-x_2^2, 2x_1x_2)\\ &+2\frac{\partial U}{\partial y_1}(x_1^2-x_2^2, 2x_1x_2)+8x_1x_2\frac{\partial^2 U}{\partial y_1\partial y_2}(x_1^2-x_2^2, 2x_1x_2).\end{aligned}$$

In view of the condition (3.1) it is easy to see that each line of the right hand side is the C^∞ function of $(x_1^2-x_2^2, 2x_1x_2)$. Since the situation is the same for other three points in $\mathfrak{N}$, it follows that $\square$ is the well-defined C^∞ operator on S. If S is diffeomorphic to $\boldsymbol{R}P^2$, then by taking the double covering we obtain the same result.

The laplacian $\triangle$ is described as

$$\triangle = \frac{-1}{f_1+f_2}\Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big).$$

Hence we have $[\triangle, \square]=0$ by an easy calculation. Other properties are also easily verified. ■

Let (S, g, F) be a compact Liouville surface, and let V_λ be the eigenspace of $\triangle$ corresponding to the eigenvalue λ. Since the operator $\Box$ preserves V_λ, and is self-adjoint, it follows that V_λ is the direct sum of the vector space of simultaneous eigenfunctions $V_{\lambda,\mu}$:

$$V_{\lambda,\mu} = \{u \in C^\infty(S) \mid \triangle u = \lambda u,\ \Box u = \mu u\}.$$

As is easily seen, the equations which define $u \in V_{\lambda,\mu}$ are transformed into a pair of ordinary differential equations of second order in each case. In the rest of this section we shall see the details of this fact in the case where S is diffeomorphic to the sphere S^2.

Let $(\alpha_1, \alpha_2, f_1(x_1), f_2(x_2))$ be one of the corresponding quadruples. Then $u \in V_{\lambda,\mu}$ if and only if

$$\begin{aligned} \frac{\partial^2 u}{\partial x_1^2} &= (-\lambda f_1(x_1)+\mu)u \\ \frac{\partial^2 u}{\partial x_2^2} &= (-\lambda f_2(x_2)-\mu)u\,, \end{aligned} \tag{6.2}$$

where u is regarded as the function on the torus $T(\alpha_1, \alpha_2)$. We now define four functions $v_{i,\pm}(x_i)$ $(i=1, 2)$ on $\boldsymbol{R}$ as the solutions of the ordinary differential equations

$$\begin{aligned} &\frac{d^2 v_{i,\pm}}{dx_i^2} = (-\lambda f_i(x_i)+(-1)^{i-1}\mu)v_{i,\pm} \\ &v_{i,+}(0) = \frac{dv_{i,-}}{dx_i}(0) = 1, \qquad v_{i,-}(0) = \frac{dv_{i,+}}{dx_i}(0) = 0\,. \end{aligned} \tag{6.3}$$

Clearly we have $v_{i,\pm}(-x_i) = \pm v_{i,\pm}(x_i)$.

PROPOSITION 6.2. *$V_{\lambda,\mu} \neq 0$ if and only if both $v_{1,+}$ and $v_{2,+}$ are periodic with periods α_1 and α_2 respectively, or both $v_{1,-}$ and $v_{2,-}$ are periodic with periods α_1 and α_2 respectively. In this case $V_{\lambda,\mu}$ is spanned by $v_{1,+}(x_1)v_{2,+}(x_2)$ or $v_{1,-}(x_1)v_{2,-}(x_2)$ or both of them. In particular we have* $\dim V_{\lambda,\mu} \leqq 2$.

PROOF. Let σ be the reflection with respect to the closed geodesic L given in Corollary 3.4. Since σ preserves the operators $\triangle$ and $\Box$, it also preserves $V_{\lambda,\mu}$. Let $V_{\lambda,\mu}^{\pm}$ be the subspaces of $V_{\lambda,\mu}$ on which $\sigma^* = \pm$ identity respectively. Let $u \in V_{\lambda,\mu}^{+}$. Then as the function on $T(\alpha_1, \alpha_2)$, we have $u(-x_1, x_2) = u(x_1, -x_2) = u(x_1, x_2)$. Hence from the equation (6.2) it follows that

$$u(x_1, x_2) = c v_{1,+}(x_1) v_{2,+}(x_2)$$

for some $c \in \boldsymbol{R}$. If $u \in V_{\lambda,\mu}^{-}$, then in the same way we have

$$u(x_1, x_2) = c v_{1,-}(x_1) v_{2,-}(x_2).$$

Conversely the functions $v_{1,+}(x_1)v_{2,+}(x_2)$ and $v_{1,-}(x_1)v_{2,-}(x_2)$ clearly satisfy the equation (6.2). If they have the double periods $(\alpha_1, 0)$ and $(0, \alpha_2)$, then they are C^∞ functions on $T(\alpha_1, \alpha_2)$, and invariant under the mapping $(x_1, x_2) \mapsto (-x_1, -x_2)$. Moreover it easily follows from the conditions (3.1) that they are C^∞ functions of

$$((x_1-t_1)^2-(x_2-t_2)^2,\ 2(x_1-t_1)(x_2-t_2))$$

around the point $(x_1, x_2)=(t_1, t_2)$, where (t_1, t_2) is one of the four points $(0, 0)$, $(\alpha_1/2, 0)$, $(0, \alpha_2/2)$, and $(\alpha_1/2, \alpha_2/2)$. These facts show that the above functions are in fact C^∞ functions on S, and hence are elements of $V_{\lambda,\mu}$. ■

Now we consider the case where (S^2, g, F) is an analytic Liouville surface. Let $(\alpha_1, \alpha_2, f_1, f_2)$ be one of the corresponding quadruples. Then as we have seen in Corollary 3.6, $f_1(x_1)$ is extended as a holomorphic function on a neighborhood of

$$\Lambda = \left\{z \in \boldsymbol{C} \mid \operatorname{Re} z = \frac{n\alpha_1}{2}, \text{ or } \operatorname{Im} z = \frac{n\alpha_2}{2},\ n \in \boldsymbol{Z}\right\}$$

and $f_2(t) = -f_1(\sqrt{-1}t)$, $t \in \boldsymbol{R}$. Hence in this case the condition for $V_{\lambda,\mu} \neq 0$ turns into the existence of a doubly periodic solution $v(z)$ of

$$\frac{d^2v}{dz^2} = (-\lambda f_1(z)+\mu)v \tag{6.4}$$

with double periods α_1 and $\sqrt{-1}\alpha_2$.

As an example, let $g=g_0$ be the standard metric on S^2 given by (4.6). Then

$$f_1(x_1) = (1-\beta)\sin^2\theta_1, \qquad x_1 = \int_0^{\theta_1} \frac{ds}{\sqrt{\beta\cos^2 s+\sin^2 s}},$$

and we have

$$f_1(z) = -\wp\left(z - \frac{\alpha_1+\sqrt{-1}\alpha_2}{4}\right) + \frac{1-2\beta}{3},$$

where $\wp(z)$ is the $\wp$ function of Weierstrass associated with the lattice $\Gamma(\alpha_1/2, \alpha_2/2)$, and

$$\frac{\alpha_1}{2} = \int_0^\pi \frac{ds}{\sqrt{\beta\cos^2 s+\sin^2 s}}, \qquad \frac{\alpha_2}{2} = \int_0^\pi \frac{ds}{\sqrt{(1-\beta)\cos^2 s+\sin^2 s}}.$$

In this case the equation (6.4) is essentially the same as Lamé's equation (cf. [**5**] Chapter 11).

References

[1] V. I. Arnold, Mathematical methods of classical mechanics, Springer-Verlag, New York, 1978.

[2] A. Besse, Manifolds all of whose geodesics are closed, Springer-Verlag, Berlin-Heidelberg, 1978.
[3] G. Darboux, Leçons sur la théorie générale des surfaces, troisième partie, Chelsea Publishing Company, New York, 1972.
[4] V. Guillemin, The Radon transform on Zoll surfaces, Adv. in Math., **22** (1976), 85-119.
[5] E. Hobson, The theory of spherical and ellipsoidal harmonics, Cambridge University Press, 1931.
[6] K. Kiyohara, On geodesic flows and spectral rigidity for 2-sphere and real projective plane with S^1-action of isometries, MPI preprint 88-51, Max-Planck-Institut für Mathematik.
[7] W. Klingenberg, Riemannian Geometry, Walter de Gruyter, Berlin-New York, 1982.

Kazuyoshi KIYOHARA
Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo 060
Japan

Note added in proof. Recently the author was informed by Professor M. Tanaka of the work of H. Viesel; Math. Ann., **166** (1966), 175-186. There a similar result to Theorem 2.1 was shown for analytic Liouville surfaces.