J. Math. Soc. Japan
Vol. 44, No. 1, 1992

# Automorphisms of algebraic K3 susfaces which act trivially on Picard groups

By Shigeyuki KONDŌ

(Received July 5, 1989)
(Revised Dec. 14, 1990)

## § 0. Introduction.

In this paper we give a correction and a proof of the result announced in [2]. Let $X$ be an algebraic K3 surface defined over $\boldsymbol{C}$. The second cohomology group $\mathrm{H}^2(X, \boldsymbol{Z})$ has a canonical structure of a lattice of rank 22 induced from the cup product. Let $S_X$ be the Picard group of $X$. Then $S_X$ admits a structure of sublattice of $\mathrm{H}^2(X, \boldsymbol{Z})$. Let $T_X$ be the orthogonal complement of $S_X$ in $\mathrm{H}^2(X, \boldsymbol{Z})$ which is called a *transcendental lattice* of $X$. Put $H_X = \mathrm{Ker}\{\mathrm{Aut}(X) \to O(S_X)\}$, where $O(S_X)$ is the group of isometries of the lattice $S_X$. Nikulin [3] proved that $H_X$ is a finite cyclic group of order $m$ and $\varphi(m)$ is a divisor of the rank of $T_X$, where $\varphi$ is the Euler function. We now give a correction of the result in [2] as follows:

THEOREM. *Let $X$ be an algebraic K3 surface and $m_X$ the order of $H_X$. Assume that the lattice $T_X$ is unimodular (i.e. $det(T_X) = \pm 1$). Then*

(i) *$m_X$ is a divisor of 66, 44, 42, 36, 28 or 12.*

(ii) *Suppose that $\varphi(m_X) = rank(T_X)$. Then $m_X$ is equal to either 66, 44, 42, 36, 28 or 12. Moreover for $m = 66, 44, 42, 36, 28$ or 12, there exists a unique (up to isomorphisms) K3 surface with $m_X = m$.*

In [2], on page 358, line 9, the statement "the order of the restriction ⋯" is false, and the Vorontsov's result [12] is correct. In [12], Vorontsov proved the result (i) of the above Theorem. In case $T_X$ is non unimodular, he also proved a similar result as the above theorem (see Corollary 6.2). His method is based on the theory of a cyclotomic field $\boldsymbol{Q}(m)$. Here we use mainly the theory of elliptic surfaces due to Kodaira [1] and Nikulin's results on finite automorphisms of K3 surfaces [3], [4]. Also we give examples of such K3 surfaces. Some of them are independently constructed by I. Dolgachev, K. Saito [6], T. Shioda, and the author.

In Section 1, we recall the result of Nikulin [3] on automorphisms of K3 surfaces. Section 2 is devoted to some remarks on elliptic pencils on K3 surfaces. In Section 3, we give examples of algebraic K3 surfaces as mentioned in Theorem, (ii). Sections 4 and 5 are devoted to a proof of Theorem. In Section 6, using the theory of elliptic surfaces, we give an another proof of Vorontsov's result on non unimodular case, and in Section 7, we also give examples of algebraic K3 surfaces with non unimodular transcendental lattices and $\varphi(m_X)=\text{rank}(T_X)$.

ACKNOWLEDGEMENT. I would like to thank to the referee who gave me an idea of a simple proof of the uniqueness in the main theorem.

## §1. Automorphisms of K3 surfaces.

A *lattice* $L$ is a free $\boldsymbol{Z}$-module of finite rank endowed with a integral symmetric bilinear form $\langle\,,\rangle$. By $L_1 \oplus L_2$, we denote the orthogonal direct sum of lattices of $L_1$ and $L_2$. An isomorphism of lattices preserving the bilinear form is called an *isometry*. For a lattice $L$, we denote by $O(L)$ the group of isometries of $L$. A lattice $L$ is *even* if $\langle x, x\rangle$ is even for each $x \in L$. A lattice $L$ is *non-degenerate* if the determinant $\det(L)$ of the matrix of its bilinear form is non-zero, and *unimodular* if $\det(L)=\pm 1$. We denote by $U$ the hyperbolic lattice $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ which is an even unimodular lattice of signature (1, 1), and by $A_k$, $D_l$ or $E_m$ an even negative definite lattice of rank $k$, $l$ or $m$ associated to the Dynkin matrix of type $A_k$, $D_l$ or $E_m$ ($m=6, 7, 8$) respectively. Note that $E_8$ is unimodular. For a lattice $L$ and an integer $m$, we denote by $L(m)$ the lattice whose quadratic form is the one on $L$ multiplied by $m$.

Let $L$ be a non-degenerate lattice. Then the bilinear form of $L$ determines a canonical embedding $L \subset L^* = \text{Hom}(L, \boldsymbol{Z})$. We denote by $A_L$ the factor group $L^*/L$ which is a finite abelian group. It follows from definitions that $L$ is unimodular if and only if $A_L$ is trivial. We denote by $l(L)$ the number of minimal generators of $A_L$. A lattice $L$ is *2-elementary* if $A_L$ is a 2-elementary abelian group. For a 2-elementary lattice $L$, define

$$\delta(L) = \begin{cases} 0 & \text{if } \langle t, t\rangle \in \boldsymbol{Z} \text{ for any } t \in L^* \\ 1 & \text{otherwise} \end{cases}$$

where the form $\langle\,,\rangle$ on $L^*$ is defined by extending the form of $L$ on $L^*$ under the above embedding $L \subset L^*$. It is known that an isomorphism class of 2-elementary lattice of signature $(1, r)$ is determined by the invariants $(\text{rank}(L), l(L), \delta(L))$ ([4], Theorem 4.3.2).

A compact connected complex surface $X$ is called a *K3 surface* if its canonical line bundle is trivial and $\dim H^1(X, \mathcal{O}_X)=0$. The second cohomology group $H^2(X, \boldsymbol{Z})$ admits a canonical structure of a lattice induced from the cup product $\langle\ ,\ \rangle$. It is even, unimodular and signature (3, 19), and hence isomorphic to $U\oplus U\oplus U\oplus E_8\oplus E_8$. Let $S_X$ be the Picard group of $X$. Then $S_X$ has a structure of sublattice of $\mathrm{H}^2(X, \boldsymbol{Z})$. We call $S_X$ the Picard lattice of $X$. Let $T_X$ be the orthogonal complement of $S_X$ in $\mathrm{H}^2(X, \boldsymbol{Z})$ which is called a *transcendental lattice* of $X$. The group $\mathrm{Aut}(X)$ of automorphisms of $X$ naturally acts on the lattices $S_X$ and $T_X$.

PROPOSITION 1.1. *The representation $Aut(X)$ on $S_X\oplus T_X$ is faithful, i.e. the induced map $Aut(X)\rightarrow O(S_X)\times O(T_X)$ is injective.*

PROOF. Let $g$ be an automorphism of $X$ which acts trivially on $S_X$ and $T_X$. Since $S_X\oplus T_X$ is of finite index in $\mathrm{H}^2(X, \boldsymbol{Z})$, $g$ also acts trivially on $\mathrm{H}^2(X, \boldsymbol{Z})$. It now follows from the Torelli theorem for K3 surfaces [**5**] that $g$ is the identity.

Let $\omega_X$ be a non trivial holomorphic 2-form on $X$. If $g\in\mathrm{Aut}(X)$, $g^*(\omega_X)=\alpha(g)\omega_X$ for some $\alpha(g)\in\boldsymbol{C}^*$. Hence we have a representation $\alpha:\mathrm{Aut}(X)\rightarrow\boldsymbol{C}^*$.

PROPOSITION 1.2 ([**3**], Theorem 3.1). *If $X$ is algebraic, then $\alpha(Aut(X))$ is a finite cyclic group. Moreover if $m=|\alpha(Aut(X))|>1$ and $\alpha(Aut(X))=\langle g\rangle$, then $\alpha(g)$ is a primitive $m$-th root of 1 and $T_X\otimes\boldsymbol{Q}$ is a direct sum of representations over $\boldsymbol{Q}$ of the cyclic group of order $m$ having maximal rank. In particular, $\varphi(m)\,|\,rank(T_X)$.*

Put $H_X=\mathrm{Ker}\{\mathrm{Aut}(X)\rightarrow O(S_X)\}$ and $m_X=|H_X|$. By Propositions 1.1, 1.2 and the fact $\mathrm{rank}(T_X)\leqq 21$, we have

COROLLARY 1.3 ([**3**], Corollary 3.3). *If $X$ is algebraic, then $H_X$ is a finite cyclic group. The non-trivial elements of $H_X$ act non-trivially on $\omega_X$. Moreover $m_X\leqq 66$.*

LEMMA 1.4. *Let $\sigma\in H_X$ and let $R$ be a smooth rational curve on $X$. Then $\sigma(R)=R$.*

PROOF. Since $\sigma\in H_X$, $\sigma^*[R]=[R]$. On the other hand, $R^2=-2$. Hence $\sigma(R)=R$.

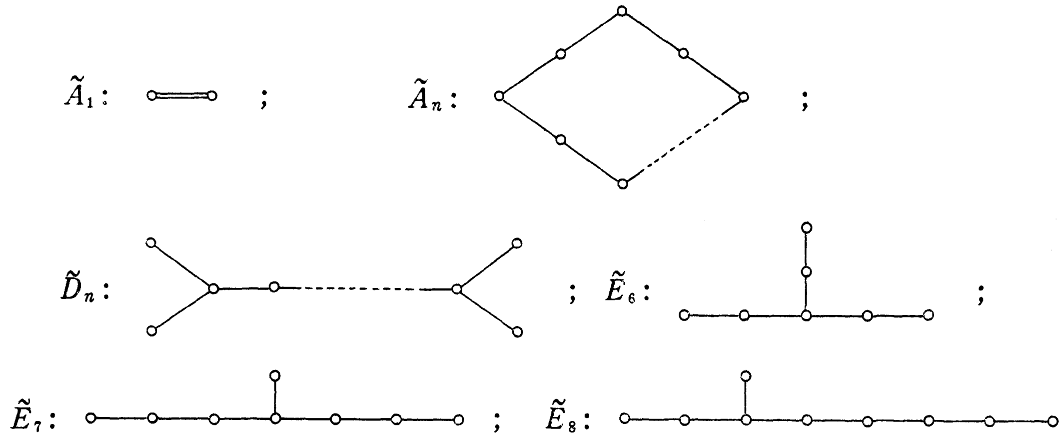
## §2. Elliptic pencils on K3 surfaces.

Let $X$ be an algebraic K3 surface. An elliptic pencil $\pi: X\rightarrow\boldsymbol{P}^1$ is a holomorphic map $\pi$ from $X$ to $\boldsymbol{P}^1$ whose general fibres are smooth elliptic curves. Let $F$ be a reducible singular fibre of $\pi$. Then every component of $F$ is a smooth rational curve. The dual graph $\Gamma$ of $F$ is defined as follows: (i) The vertices of $\Gamma$ correspond to components of $F$. (ii) Two vertices $C$ and $C'$ ($C$

and $C'$ are components of $F$) are joined by $m$-tuple lines if and only if $C\cdot C'=m$. It is known that the dual graph of reducible singular fibres are as follows ([1]):

| singular fibres (Kodaira's notation) | $\mathrm{I}_2$, III | $\mathrm{I}_3$, IV | $\mathrm{I}_{n+1}$ $(n\geq 3)$ | $\mathrm{I}^*_{n-4}$ $(n\geq 4)$ | IV* | III* | II* |
|---|---|---|---|---|---|---|---|
| dual graph | $\tilde{A}_1$ | $\tilde{A}_2$ | $\tilde{A}_n$ | $\tilde{D}_n$ | $\tilde{E}_6$ | $\tilde{E}_7$ | $\tilde{E}_8$ |

where




If the dual graph of $F$ is of type $\tilde{K}=\tilde{A}_m$, $\tilde{D}_n$ or $\tilde{E}_l$, we denote by the same letter $\tilde{K}$ $(=\tilde{A}_m$, $\tilde{D}_n$ or $\tilde{E}_l$ respectively) the sublattice of $S_X$ generated by components of $F$. Then $\tilde{K}$ is contained in the orthogonal complement $[F]^{\perp}$ of the class $[F]$ in $S_X$ and $\tilde{K}/\boldsymbol{Z}[F]$ is isomorphic to a lattice $K=A_m$, $D_n$ or $E_l$ respectively.

LEMMA 2.1. *Let $X$ be an algebraic K3 surface and let $S_X$ be the Picard lattice of $X$. Assume $S_X=U\oplus K$ where $K$ is a negative definite even lattice. Then* (i) *$X$ has an elliptic pencil $\pi$ with a section.* (ii) *Denote by $F$ a fibre of $\pi$. Then $[F]^{\perp}/\boldsymbol{Z}[F]\cong K$.*

PROOF. Let $\{e, f\}$ be a basis of $U$ with $\langle e, e\rangle=\langle f, f\rangle=0$ and $\langle e, f\rangle=1$. If necessary replacing $e$ by $\varphi(e)$, where $\varphi$ $(\in O(S_X))$ is a composition of reflections induced from smooth rational curves on $X$, we may assume that $e$ is represented by the class of a smooth elliptic curve $F$ and the linear system $|F|$ defines an elliptic pencil $\pi: X\to \boldsymbol{P}^1$ (See [**5**], §3, Proof of Corollary 3). Let $R$ be a divisor which represents the class $f-e$. Then $R^2=-2$. By the Riemann-Roch theorem, either $R$ or $-R$ is effective. Since $R\cdot F=1$, $R$ is effective. Put $R=\sum_{i=1}^{m} a_iR_i$, where $R_i$ is an irreducible component of $R$ and $a_i$ $(1\leqq i\leqq m)$

is a positive integer. Since $F\cdot R_i \geqq 0$ $(1\leqq i\leqq m)$ and $F\cdot R=1$, there exists an unique component $R_k$ such that $a_k=1$, $F\cdot R_k=1$ and $F\cdot R_i=0$ for any $i\neq k$. If $R_k^2\geqq 0$, then $R_k$ is not isomorphic to a smooth rational curve. On the other hand, the map $\pi|_{R_k}: R_k\to \boldsymbol{P}^1$ is of degree 1 because $F\cdot R_k=1$, which is impossible. Therefore $R_k^2=-2$, and hence $R_k$ is a section of $\pi$. Since $[F]=\varphi(e)$, the assertion (ii) is obvious.

LEMMA 2.2. *Let $X$ be an algebraic K3 surface. Assume that $X$ has an elliptic pencil $\pi: X\to \boldsymbol{P}^1$ and $[F]^{\perp}/\boldsymbol{Z}[F]\cong K_1\oplus\cdots\oplus K_r$, where $F$ is a fibre of $\pi$ and $K_i$ $(1\leqq i\leqq r)$ is a lattice isomorphic to $A_m$, $D_n$ or $E_l$ $(m\geqq 1,\ n\geqq 4$ and $l=6, 7, 8)$. Then $\pi$ has a reducible singular fibre $F_i$ $(1\leqq i\leqq r)$ whose dual graph is of type $\tilde{K}_i$ $(1\leqq i\leqq r)$.*

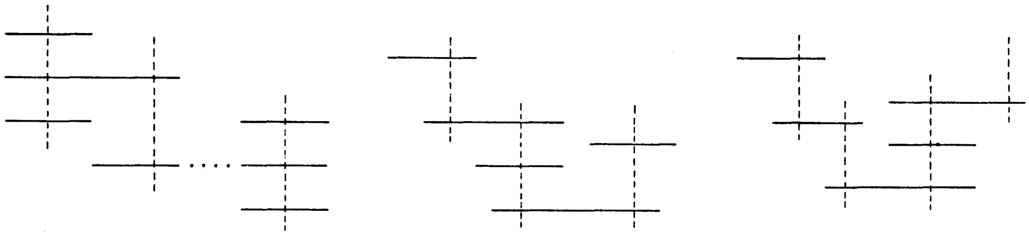
PROOF. Let $e_i\in[F]^{\perp}$ $(1\leqq i\leqq k)$ such that $\langle e_i, e_i\rangle=-2$, $\langle e_i, e_j\rangle\geqq 0$ $(i\neq j)$ and $\{e_i \bmod [F]\}$ is a base of $[F]^{\perp}/\boldsymbol{Z}[F]$. By the Riemann-Roch theorem, we may assume that all $e_i$ are represented by effective divisors. Put $e_i=\sum_j a_{i,j}[R_{i,j}]$ where $R_{i,j}$ is an irreducible curve and $a_{i,j}$ is a positive integer $(1\leqq i\leqq k)$. Since $F\cdot R_{i,j}\geqq 0$ and $F\cdot e_i=0$, we have $F\cdot R_{i,j}=0$. This means that $R_{i,j}$ is linearly equivalent to $F$ or $R_{i,j}$ is a component of some reducible singular fibre of $\pi$. Hence $[F]^{\perp}$ is generated by components of fibres of $\pi$. Let $F_1, \cdots, F_t$ be all reducible singular fibres of $\pi$ and let $\tilde{S}_i$ be the lattice generated by components of $F_i$ $(1\leqq i\leqq t)$. Put $S_i=\tilde{S}_i/\boldsymbol{Z}[F]$. Then $S_i$ is isomorphic to $A_m$, $D_n$ or $E_l$ $(m\geqq 1,\ n\geqq 4,\ l=6, 7, 8)$. We have seen that $[F]^{\perp}/\boldsymbol{Z}[F]=S_1\oplus\cdots\oplus S_t$. Note that $A_m$, $D_n$ and $E_l$ are indecomposable, that is, $A_m$, $D_n$ and $E_l$ are not isomorphic to the direct sum of two lattices of type $A_{m'}$, $D_{n'}$ or $E_{l'}$. Hence $t=r$ and $S_i\cong K_{\sigma(i)}$, $1\leqq i\leqq r$, where $\sigma$ is a permutation of $\{1, \cdots, r\}$. Since the dual graph of $F_i$ is determined by $S_i$, we have the desired result.

Let us assume that $S_X=U(m)\oplus K_1\oplus\cdots\oplus K_r$ where $m=1$ or 2, and $K_i$ $(1\leqq i\leqq r)$ is a lattice isomorphic to $A_1$, $D_{4n}$ $(n\geqq 1)$, $E_7$ or $E_8$. Note that $S_X$ is a 2-elementary lattice. It follows from the result of [4], §4-2 that there exists an automorphism $\sigma$ of $X$ of order 2 with $\sigma^*|S_X=1_{S_X}$ and $\sigma^*|T_X=-1_{T_X}$. Then by the same proof as that of Lemma 2.1, there exists an elliptic pencil $\pi=|F|: X\to \boldsymbol{P}^1$ with $[F]^{\perp}/\boldsymbol{Z}[F]\cong K_1\oplus\cdots\oplus K_r$. By Lemma 2.2, $\pi$ has a reducible singular fibre $F_i$ $(1\leqq i\leqq r)$ whose dual graph is of type $\tilde{K}_i$ $(1\leqq i\leqq r)$. Since $\sigma\in H_X$, $\sigma$ preserves the structure of $\pi$.

LEMMA 2.3. *Under the above situation, the followings hold:*

(i) *In case that the dual graph of singular fibre $F_0$ is of type $\tilde{D}_{4n}$, $\tilde{E}_7$ or $\tilde{E}_8$, $\sigma$ acts on $F_0$ as follows:*

$F_0$: *type* $\tilde{D}_{4n}$; $F_0$: *type* $\tilde{E}_7$; $\tilde{F}_0$: *type* $\tilde{E}_8$



*where* $\sigma$ *acts on dotted lines identically and acts on simple lines as an automorphism of order* 2.

(ii) *Assume that* $\pi$ *has a section* $R$ *and has at least one reducible singular fibre with the dual graph of type* $\tilde{D}_{4n}$, $\tilde{E}_7$ *or* $\tilde{E}_8$. *Then* $R$ *is a fixed curve of* $\sigma$.

PROOF. (i) By Lemma 1.4, $\sigma$ preserves each component of $F_0$ and each section invariant. Note that there exists a component $C_0$ of $F_0$ which meets three other component. This means that $\sigma$ is an automorphism of $C_0$ with three fixed points. Hence $C_0$ is a fixed curve of $\sigma$. On the other hand, $\sigma^*\omega_X = -\omega_X$ where $\omega_X$ is a nowhere vanishing holomorphic 2-form on $X$. Therefore the set of fixed points of $\sigma$ is either empty or a smooth curve. It now follows from this fact that the dotted lines in the above figures are exactly the set of fixed points of $\sigma | F_0$.

(ii) Let $F_0$ be a reducible singular fibre with the dual graph of type $\tilde{D}_{4n}$, $\tilde{E}_7$ or $\tilde{E}_8$. Then the assertion follows from the following two facts:

(a) $\sigma$ acts on any simple components of $F_0$ as a non-trivial automorphism of order 2; (b) $R$ meets exactly one simple component of $F_0$ transversally and $\sigma$ has no isolated fixed points.

## § 3. Examples.

In the following, we denote by $e_\nu$ a primitive $\nu$-th root of 1.

(3.0) For $m=66$ or 42, we gave an example of K3 surface with $m_X=m$ in [2]. Here we give affine equations of such K3 surfaces:

(3.0.1) $$m = 66: \quad X: y^2 = x^3 + t\prod_{i=1}^{11}(t-e_{11}^i),$$
$$g: (x, y, t) \longrightarrow (e_{66}^2 \cdot x,\ e_{66}^3 \cdot y,\ e_{66}^6 \cdot t),$$
$$S_X \cong U, \quad T_X \cong U \oplus U \oplus E_8 \oplus E_8.$$

(3.0.2) $\quad m=42: \quad X: y^2 = x^3+t^5\prod_{i=1}^{7}(t-e_7^i)$,

$$g:(x, y, t)\longrightarrow(e_{42}^2\cdot x,\ e_{42}^3\cdot y,\ e_{42}^{18}\cdot t),$$

$$S_X\cong U\oplus E_8,\quad T_X\cong U\oplus U\oplus E_8.$$

(3.1) $m=44$. We consider the elliptic curve $E$ over the function field $\boldsymbol{C}(t)$ defined by the equation:

$$y^2 = x^3+x+t^{11}.$$

Let $X$ be the Kodaira-Néron model of $E$ over $\boldsymbol{C}(t)$, which is a nonsingular projective surface having an elliptic pencil $\pi$. $X$ is also constructed by the following way: let $(x:y:z)$ be a system of a homogeneous coordinate of $\boldsymbol{P}^2$. We take two copies $W_0=\boldsymbol{P}^2\times\boldsymbol{C}_0$ and $W_1=\boldsymbol{P}^2\times\boldsymbol{C}_1$ of the cartesian product $\boldsymbol{P}^2\times\boldsymbol{C}$ and form their union $W=W_0\cup W_1$ by identifying $(x:y:z,t)\in W_0$ with $(x_1:y_1:z_1,t_1)\in W_1$ if and only if $t\cdot t_1=1$, $x=t^4\cdot x_1$, $y=t^6\cdot y_1$ and $z=z_1$. Then $X$ is given by the following equations:

$$f = zy^2-x^3-xz^2-z^3t^{11} = 0,$$

$$f_1 = z_1y_1^2-x_1^3-t_1^8x_1z_1^2-t_1z_1^3 = 0.$$

By the theory of elliptic surfaces [**1**], [8], [**9**], we can see that $\pi$ has a singular fibre of type II over $t=\infty$ and 22 singular fibres of type $\mathrm{I}_1$ over $t^{22}=-4/27$. Since $\pi$ has a singular fibre and the base curve of the elliptic pencil $\pi$ is $\boldsymbol{P}^1$, $b_1(X)=0$. Moreover $\omega=dt\wedge(zdx-xdz)/(\partial f/\partial y)$ defines a non-vanishing holomorphic 2-form on $X$. Hence $X$ is a K3 surface. We define an automorphism $g$ induced from the following automorphism of $E$:

$$(x, y, t)\longrightarrow(e_{44}^{22}x,\ e_{44}^{11}y,\ e_{44}^{2}t).$$

Then $g^*\omega=e_{44}^{13}\cdot\omega$. By Proposition 1.2, $\mathrm{rank}(T_X)=20$ and hence $\mathrm{rank}(S_X)=2$. On the other hand, $S_X$ contains classes of a section $\{x=z=0\}$ and a fibre which generates the unimodular lattice $U$. Therefore $S_X\cong U$. Since $|\det(S_X)|=|\det(T_X)|$, $T_X$ is also unimodular. By the classification of even indefinite unimodular lattices (cf. [7], Chap. 5), we have $T_X\cong U\oplus U\oplus E_8\oplus E_8$. Obviously $g$ acts on $S_X$ trivially.

(3.2) $m=36$. Let $X$ be the Kodaira-Néron model of the following elliptic curve over $\boldsymbol{C}(t)$:

$$y^2 = x^3-t^5\cdot\prod_{i=1}^{6}(t-e_6^i).$$

By the same way as in (3.1), the elliptic surface $X$ is a K3 surface. $X$ has a singular fibre of type II* over $t=0$ and 7 singular fibres of type II over $t=\infty$

and $t^6=1$. $X$ has an automorphism $g$ induced from the following automorphism of the elliptic curve over $\boldsymbol{C}(t)$:

$$(x, y, t) \longrightarrow (e_{36}^{2}\cdot x,\ e_{36}^{3}\cdot y,\ e_{36}^{30}\cdot t).$$

By the same reason as in (3.1), we have that $\operatorname{rank}(S_X)=22-\operatorname{rank}(T_X)=10$. On the other hand, $S_X$ contains classes of a section, a fibre and components of singular fibres which generate the unimodular lattice $U\oplus E_8$. Hence $S_X\cong U\oplus E_8$ and $T_X\cong U\oplus U\oplus E_8$. Since a singular fibre of type II* has no symmetry, $g$ acts on $S_X$ trivially.

(3.3) $m=28$. Let $X$ be the Kodaira-Néron model of the following elliptic curve over $\boldsymbol{C}(t)$:

$$y^2 = x^3+x+t^7.$$

By the same way as in (3.1), the elliptic surface $X$ is a K3 surface. $X$ has a singular fibre of type II* over $t=\infty$ and 14 singular fibres of type $\mathrm{I}_1$ over $t^{14}=-4/27$. $X$ has an automorphism $g$ induced from following automorphism of the elliptic curve over $\boldsymbol{C}(t)$:

$$(x, y, t) \longrightarrow (e_{28}^{14}\cdot x,\ e_{28}^{7}\cdot y,\ e_{28}^{2}\cdot t).$$

By the same reason as in (3.2), $S_X\cong U\oplus E_8$, $T_X\cong U\oplus U\oplus E_8$ and $g$ acts on $S_X$ trivially.

(3.4) $m=12$. Let $X$ be the Kodaira-Néron model of the following elliptic curve over $\boldsymbol{C}(t)$:

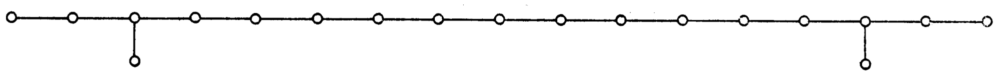
$$y^2 = x^3-t^5(t-1)(t+1).$$

By the same way as in (3.1), $X$ is a K3 surface. $X$ has 2 singular fibres of type II* over $t=0$, $\infty$ and 2 singular fibres of type II over $t=\pm 1$. $X$ has an automorphism $g$ induced from the following automorphism of the elliptic curve over $\boldsymbol{C}(t)$:

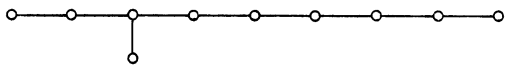
$$(x, y, t) \longrightarrow (e_{12}^{2}\cdot x,\ e_{12}^{3}\cdot y,\ -t).$$

By the same reason as in (3.2), $S_X\cong U\oplus E_8\oplus E_8$, $T_X\cong U\oplus U$ and $g$ acts on $S_X$ trivially.

REMARK 3.5. We note here the following fact which is not used in the proof of our Theorem. The above elliptic K3 surfaces have a unique section. Moreover the set of all smooth rational curves on $X$ is the set consisting of components of reducible singular fibres and the section. This follows from the Vinberg's result in [**11**]. The dual graph of all smooth rational curves on $X$ is as follows:



$S_X = U \oplus E_8 \oplus E_8$



$S_X = U \oplus E_8$



$S_X = U$

where a vertex "○" corresponds to a smooth rational curve and two vertices $E$ and $E'$ are joined by $n$-tuple line if and only if the intersection number $E \cdot E'$ is equal to $n$.

## §4. Automorphisms which act trivially on Picard groups.

In this section, we shall prove the first part of the theorem.

LEMMA 4.1. *If $T_X$ is unimodular, then $S_X \cong U$, $U \oplus E_8$ or $U \oplus E_8 \oplus E_8$.*

PROOF. Since $|\det(T_X)| = |\det(S_X)|$, $S_X$ is also an even unimodular lattice. By the Hodge index theorem, the signature of $S_X$ is $(1, \rho(X)-1)$. Hence the assersion follows from [7], Chap. 5.

LEMMA 4.2. *If $T_X$ is unimodular, then $X$ has an elliptic pencil with a section. Moreover its reducible singular fibre (if exists) is of type* II*.

PROOF. This follows from Lemmas 2.1, 2.2 and 4.1.

The first part of the theorem follows from the following:

THEOREM 4.3. (i) *If $S_X \cong U$, then $m_X | 66$, 44 or 12.*
(ii) *If $S_X \cong U \oplus E_8$, then $m_X | 42$, 36 or 28.*
(iii) *If $S_X \cong U \oplus E_8 \oplus E_8$, then $m_X | 12$.*
(iv) *If $\varphi(m_X) = \mathrm{rank}(T_X)$, then $m_X = 66$, 44, 42, 36, 28 or 12.*

PROOF. In the following, we mean by an automorphism of an elliptic curve $F$ an automorphism preserving a group structure of $F$. Let $\pi : X \to \boldsymbol{P}^1$ be an elliptic pencil as in Lemma 4.2. Recall that an irreducible singular fibre of $\pi$ is either of type $\mathrm{I}_1$ or of type II ([1]). Let $r$ (resp. $s$) be the number of singular fibres of type $\mathrm{I}_1$ (resp. type II). It is known that

$$\sum_{F:\ \text{singular fibre}} e(F) = e(X) = 24\,,$$

where $e(M)$ is the Euler number of a topological space $M$. Since the Euler number of a singular fibre of type $\mathrm{I}_1$, II or II* is 1, 2 or 10, respectively, we have $2s + r = 24 - 10 \cdot k$, where $k$ is the number of singular fibres of type II*.

Case (i): $S_X=U$. In this case, $2s+r=24$. By Proposition 1.2, it suffices to see that $5 \nmid m_X$ and $8 \nmid m_X$.

Lemma 4.4. $5 \nmid m_X$.

Proof. Let $g \in H_X$ and assume $|g|=5$. Then $g$ preserves a section $E$ of the elliptic pencil $\pi$ (Lemma 1.4). On the other hand, if $g^k|E=1$, $g^k$ acts on a fibre $F$ as an automorphism. Note that $|g^k|$ is a divisor of 4 or 6. Hence $g$ acts on $E$ as an automorphism of order 5. Since $2s+r=24$, the pair $(s, r)$ is equal to (12, 0), (7, 10) or (2, 20). In any case, the set of fixed points of $g$ lies on the singular fibres of type II. Therefore, if $g$ has a fixed curve, then its Euler number is non negative. It now follows from the Lefschetz fixed point formula (cf. [**10**], Lemma 1.6) and Proposition 1.2 that:

$$\begin{aligned} 0 &\leqq \#\{\text{isolated fixed points of } g\} + \sum_{C:\ \text{fixed curve of } g} e(C) \\ &= \sum_i \operatorname{trace} g^* | H^i(X, \boldsymbol{Q}) = 2 + \operatorname{trace} g^* | S_X \otimes \boldsymbol{Q} + \operatorname{trace} g^* | T_X \otimes \boldsymbol{Q} \\ &= 2+2+(-1)\times 5 = -1. \end{aligned}$$

Thus we have a contradiction.

Lemma 4.5. $8 \nmid m_X$.

Proof. Assume that $8|m_X$ and let $g \in H_X$ with $|g|=8$. If $g$ acts on a section $E$ as an automorphism of order 2, then $g^2$ acts on each fibre as an automorphism of order 4. Therefore the functional invariant of the elliptic pencil $\pi$ is equal to the constant 1728 (cf. [**1**]). However this is impossible since each singular fibre is either of type $\mathrm{I}_1$ or of type II. Hence $|g|E|=4$ or 8, and the pair $(s, r)$ is one of the following: (12, 0), (10, 4), (8, 8), (6, 12), (4, 16), (2, 20), (0, 24).

Claim. *$g$ acts on $E$ as an automorphism of order* 4.

Proof of Claim. If $g$ acts on $E$ as an automorphism of order 8, then $(s, r)$ is either (8, 8) or (0, 24). Moreover two $g$-invariant fibres are smooth and the set of fixed points of $g^4$ lies on these two fibres. A similar argument as in the proof of Lemma 4.4 shows that this case contradicts the Lefschetz fixed point formula.

In case $(s, r)=(12, 0)$, (8, 8), (4, 16) or (0, 24): In this case, two $g$-invariant fibres $F_1$ and $F_2$ are smooth. Recall that $g^*\omega_X=\alpha(g)\omega_X$, where $\alpha(g)$ is a primitive 8-th root of 1. Let $p$ be the intersection of the section $E$ and $F_1$. By considering the action of $g$ on the tangent space of $p$, $g$ acts on $F_1$ as an automorphism of order 8 because $g$ acts on $E$ of order 4. This contradicts the fact that no smooth fibres have an automorphism of order 8.

In case $(s, r)=(10, 4)$, $(6, 12)$ or $(2, 20)$: Set $\iota=g^4$. Then $\iota^*|S_X=1_{S_X}$ and $\iota^*|T_X=-1_{T_X}$. It follows from [**4**], Theorem 4.2.2 that the set of fixed points of $\iota$ is the smooth curve $C+E'$, where $C$ is a smooth curve of genus 10 and $E'$ is a smooth rational curve. By the above claim, $E'=E$. Since $g$ acts on $E$ as an automorphism of order 4 and $C\cdot F>0$, $|g|C|=4$. Let $F_1$ and $F_2$ be singular fibres of type II invariant under $g$. Then the set of fixed points of $g$ and $g^2$ on $C$ is contained in $C\cap(F_1\cup F_2)$. Since $F_j$ $(1\leqq j\leqq 2)$ is not a fixed curve of $\iota$, $C$ passes through the singular points of $F_1$ and $F_2$, and both $g$ and $g^2$ have exactly two fixed points on $F_j$ which are $E\cap F_j$ and $C\cap F_j$ (=the singular point of $F_j$) $(j=1, 2)$ (see Figure 1).

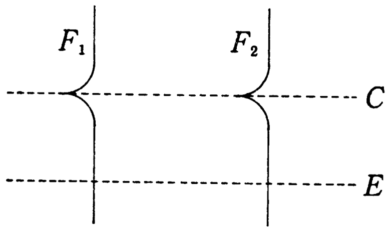



Figure 1.

By the Hurwitz formula, we have

$$4(2\cdot\mathrm{genus}(C/\langle g\rangle)-2)+2(4-1)=2\cdot\mathrm{genus}(C)-2=18\,.$$

Hence

$$8(\mathrm{genus}(C/\langle g\rangle)-1)=12\,.$$

This is a contradiction. Thus we now have proved Theorem 4.3, (i).

CASE (ii): $S_X=U\oplus E_8$. In this case, $2s+r=14$. It suffices to prove $m_X\neq 26, 13, 10, 8, 5$. Note that $\pi$ has a singular fibre of type II* invariant under the action of $g$. By the relation $2s+r=14$, $g$ does not act on the base of order 5 or 10, and hence $m_X\neq 5, 10$.

LEMMA 4.6. $m_X\neq 26, 13$.

PROOF. Assume $13|m_X$ and let $g\in H_X$ be of order 13. First note that any smooth elliptic curve has no automorphism of order 13. Hence $g$ acts on a section $E$ as an automorphism of order 13, $(s, r)=(0, 14)$ and $g$ preserves one singular fibre $F$ of type $\mathrm{I}_1$. Moreover $g$ fixes the intersection point $F\cap E$ and the singular point of $F$, and hence $F$ is a fixed curve of $g$. Hence $g$ acts on the tangent space at the singular point of $F$ trivially. This contradicts the fact $g^*\omega_X=e_{13}\cdot\omega_X$.

LEMMA 4.7. $m_X \neq 8$.

PROOF. If $m_X=8$, then $g$ acts on a section $E$ as an automorphism of order 4 or 8 because the functional invariant of $\pi$ is not equal to the constant 1728. Hence by $2s+r=14$, $(s, r)=(5, 4)$ or $(1, 12)$ and $g$ acts on the base of order 4. Note that there exists exactly one singular fibre $F$ of type II invariant under $g$. Since $g^4|S_X=1$ and $g^4|T_X=-1$, the set of fixed points of $g^4$ is equal to the smooth curve $C+E_1+\cdots+E_5$, where $C$ is a smooth curve of genus 6, $E_i$ $(1 \leqq i \leqq 5)$ are smooth rational curves ([4], Theorem 4.2.2). By Lemma 2.3, we may assume that $E=E_1$ and $E_i$ $(2 \leqq i \leqq 5)$ are some components of the singular fibre $F'$ of type II*. Then $|g|C|=4$ because $g$ acts on the base as an automorphism of order 4 and $C \cdot F>0$. Denote by $D$ the component with multiplicity 3 of $F'$ which intersects the component with multiplicity 6. Then $g^4$ acts on $D$ as an involution (Lemma 2.3), and hence $C$ intersects $D$ transversally and does not meet other components of $F'$. Also $F$ is not a fixed curve of $g^4$, $C$ passes through the singular points of $F$. Thus both $g$ and $g^2$ have exactly two fixed points $C \cap F$ and $C \cap D$ on $C$ (see Figure 2).

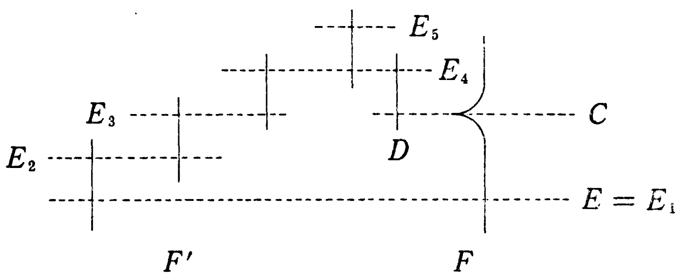



Figure 2 (the dotted lines are the fixed curves of $g^4$).

By the Hurwitz formula, we have

$$10 = 2 \cdot \text{genus}(C)-2 = 4(2 \cdot \text{genus}(C/\langle g \rangle)-2)+3 \cdot 2\,.$$

This is a contradiction. Thus we have proved Theorem 3.3, (ii).

CASE (iii): $S_X=U \oplus E_8 \oplus E_8$. In this case, $2s+r=4$. It suffices to show that $m_X \neq 10$, 5, 8. By the relation $2s+r=4$, $m_X \neq 10$, 5. We denote by $F_1$, $F_2$ the singular fibres of $\pi$ of type II* which are invariant under the action of $g$.

LEMMA 4.8. $m_X \neq 8$.

PROOF. If $m_X=8$, then by $2s+r=4$, $(s, r)=(0, 4)$ and $g$ (resp. $g^4$) acts on the section $E$ (resp. on fibres) as an automorphism of order 4 (resp. of order 2). Since $g^4|S_X=1$ and $g^4|T_X=-1$, the set of fixed points of $g^4$ is equal to the smooth curve $C+E_1+\cdots+E_9$, where $C$ is a smooth curve of genus 2 and $E_i$,

$1 \leqq i \leqq 9$, are smooth rational curves ([**4**], Theorem 4.2.2). By Lemma 2.3, we may assume that $E_1 = E$ and $E_j$, $2 \leqq j \leqq 9$, are some components of $F_1$ and $F_2$. Then $|g|C|=4$ because $C \cdot F > 0$ and $g$ acts on the base as an automorphism of order 4. Denote by $D_i$ the component with multiplicity 3 of $F_i$ which intersects the component with multiplicity 6 of $F_i$ $(i=1, 2)$. Then $g^4$ acts on $D_i$ as an involution (Lemma 2.3), and hence $C$ intersects $D_i$ transversally $(i=1, 2)$ and does not meet other components of $F_1$ and $F_2$. Thus both $g$ and $g^2$ have exactly two fixed points $C \cap D_i$ $(i=1, 2)$ on $C$ (see Figure 3).

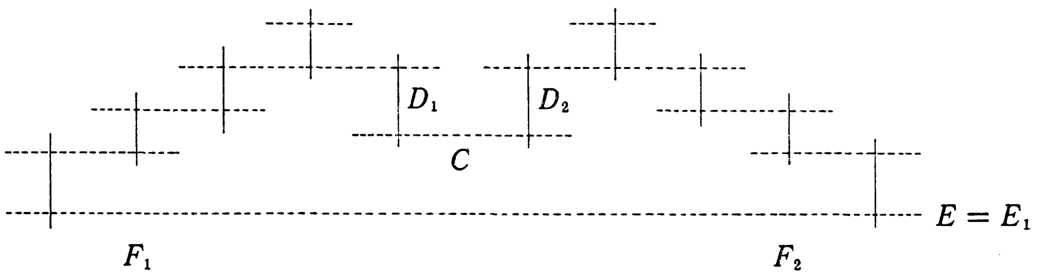



Figure 3 (the dotted lines are the fixed curves of $g^4$).

By the Hurwitz formula, we have

$$2 = 2 \cdot \mathrm{genus}(C) - 2 = 4(2 \cdot \mathrm{genus}(C/\langle g \rangle) - 2) + 3 \cdot 2 .$$

This is a contradiction. Thus we have proved Theorem 4.3, (iii).

Now the last assertion (iv) follows from the following:

LEMMA 4.9. $2 \mid |H_X|$.

PROOF. First note that $\mathrm{H}^2(X, \boldsymbol{Z}) \cong S_X \oplus T_X$ since $S_X$ is unimodular. Let $\iota$ be an involution of the lattice $H^2(X, \boldsymbol{Z}) = S_X \oplus T_X$ such that $\iota | S_X = 1$ and $\iota | T_X = -1$. Obviously $\iota$ preserves a holomorphic 2-form and effective cycles on $X$. Hence by the Torelli theorem for K3 surfaces [**5**], there exists an automorphism $\sigma$ of order 2 with $\sigma^* = \iota$. Thus we have now finished the proof of Theorem 4.3.

PROPOSITION 4.10. *Assume $\varphi(m_X) = rank(T_X)$. Let $n$ be the order of the action of $H_X$ on the base of $\pi$. Then the pair $(s, r)$ and $n$ are uniquely determined by $m_X$ as in the following table:*

| $m_X$ | 66 | 44 | 42 | 36 | 28 | 12 |
|---|---|---|---|---|---|---|
| $(s, r)$ | (12, 0) | (1, 22) | (7, 0) | (7, 0) | (0, 14) | (2, 0) |
| $n$ | 11 | 22 | 7 | 7 | 14 | 2 |

PROOF. If $m_X=66$, 42, 36 or 12, then by the relation $2s+r=24-10k$, general fibres have an automorphism of order 3, and hence the functional invariant of $\pi$ is equal to the constant 0. Therefore there are no singular fibres of type $\mathrm{I}_1$. If $m_X=44$ (resp. 28), then $g$ acts on a section as an automorphism of order 22 (resp. 14), because general fibres have no automorphisms of order 4. Hence $(s, r)=(1, 22)$ or $(0, 24)$ (resp. $(s, r)=(0, 14)$). If $g$ is of order 44 and $(s, r)=(0, 24)$, then two $g$-invariant fibres are of type $\mathrm{I}_1$. By the same argument as in the proof of Lemma 4.6, we have a contradiction.

## § 5. Uniqueness.

In this section, we shall prove the second assertion of the theorem. The idea of our proof is due to the referee. Let $X$ be an algebraic K3 surface. Assume that $\varphi(m_X)=\mathrm{rank}\, T_X$ and $m_X=66$, 44, 42, 36, 28 or 12. Let $\pi: X\to \boldsymbol{P}^1$ be an elliptic pencil with a section mentioned in § 4. The type of singular fibres of $\pi$ is completely determined by $m_X$ (Lemma 4.2, Proposition 4.10). Let $X_\eta$ a generic fibre of $\pi$. Then $X_\eta$ is an elliptic curve over the function field $\boldsymbol{C}(t)$ of $\boldsymbol{P}^1$ with a rational point and $\pi: X\to \boldsymbol{P}^1$ is the minimal model of $X_\eta$. Let

$$y^2 = x^3+a(t)x+b(t)$$

be the Weierstrass model of $X_\eta$. The discriminant $\Delta(t)$ and the functional invariant $j(t)$ are defined by the formula

$$\Delta(t) = 4a(t)^3+27b(t)^2 \quad \text{and} \quad j(t) = a(t)^3/\Delta(t).$$

Let $\nu_t\equiv\mathrm{ord}_t(\Delta) \pmod{12}$, $\gamma_t=\mathrm{ord}_t(j(t))$. Then the type of a singular fibre depends on $\nu_t$, $\gamma_t$. In our case, we have the following table (c.f. [**1**], p. 604, Table 1, [**9**], § 5):

Table 1.

| Type | smooth | $\mathrm{I}_1$ | II | II* |
|---|---|---|---|---|
| $\nu_t$ | 0 | 1 | 2 | 10 |
| $\gamma_t$ | $\geqq 0$ | $-1$ | $\geqq 1$ | $\geqq 1$ |

In case $m_X=66$, 42, 36 or 12, general fibres of $\pi$ have an automorphism of order 3. Hence $j(t)\equiv 0$ and hence $a(t)\equiv 0$.

(5.1) In case $m_X=66$, $\pi$ has exactly 12 singular fibres of type II (Proposition 4.10). We may assume that $\pi$ has singular fibres over $t=0, \xi_1, \cdots, \xi_{11}$ $(\neq\infty)$, $g$ preserves the fibre $\pi^{-1}(0)$ invariant and $g$ acts on the base as $g(t)=e_{11}\cdot t$ where $e_{11}$ is a primitive 11-th root of unity. Then by the above Table 1, $\nu_0=\nu_{\xi_1}=\cdots=\nu_{\xi_{11}}=2$ and $\nu_t=0$ for any $t\neq 0, \xi_1, \cdots, \xi_{11}$. Therefore

$$\Delta(t)=u(t)^{12}\cdot\left\{t^2\prod_{i=1}^{11}(t-\xi_i)^2\right\}$$

for some $u(t)\in \boldsymbol{C}(t)$. After a change of coordinate

$$(x, y, t)\longrightarrow(u(t)^2\cdot x,\ u(t)^3\cdot y,\ t),$$

we may assume $b(t)=t\prod_{i=1}^{11}(t-\xi_i)$. Thus $\pi: X\to \boldsymbol{P}^1$ is isomorphic to the example (3.0.1).

(5.2) In case $m_X=42$, 36 or 12, the same way as in (5.1) shows the elliptic K3 surface $\pi: X\to\boldsymbol{P}^1$ is isomorphic to the examples (3.0.2), (3.2) or (3.4), respectively.

(5.3) In case $m_X=44$, $\pi$ has exactly one singular fibre of type II and 22 singular fibres of type $\mathrm{I}_1$ (Proposition 4.10). Moreover $g$ preserves the singular fibre of type II invariant and acts on the set of singular fibres of type $\mathrm{I}_1$ as a permutation of order 22. By a change of coordinate of the base, we may assume that $\pi$ has the singular fibre of type II over $t=\infty$, has the singular fibres of type $\mathrm{I}_1$ over $t=(-4/2)^{1/22}\cdot e_{22}^k$ $(1\leqq k\leqq 22)$, and $g$ acts on $\boldsymbol{P}^1$ by $g(t)=e_{22}t$ where $e_{22}$ is a primitive 22-th root of unity. Since $j(t)$ has exactly 22 poles of order 1 at $t=(-4/27)^{1/22}\cdot e_{22}^k$ $(1\leqq k\leqq 22)$ (see Table 1), we get $j(t)=1/(4+27t^{22})$. Then $b(t)^2=a(t)^3\cdot t^{22}$, and hence $a(t)^{1/2}\in\boldsymbol{C}(t)$. Thus we have

$$y^2=x^3+c(t)^2\cdot x+c(t)^3\cdot t^{11}\qquad (c(t)=a(t)^{1/2}).$$

By our assumption on singular fibres,

$$\Delta(t)=c(t)^6(4+27t^{22})=u(t)^{12}\cdot(4+27t^{22})$$

for some $u(t)\in\boldsymbol{C}(t)$. After a change of coordinate

$$(x, y, t)\longrightarrow(u(t)^2\cdot x,\ u(t)^3\cdot y,\ t),$$

we have an equation $y^2=x^3+x+t^{11}$. Thus $\pi: X\to\boldsymbol{P}^1$ is isomorphic to the example (3.1).

(5.4) In case $m_X=28$, the same way as in (5.3) shows that $\pi: X\to\boldsymbol{P}^1$ is isomorphic to the example (3.3).

## § 6. Non unimodular case.

First of all we give a proof of the following basic result due to Vorontsov [**12**].

THEOREM 6.1 ([**12**], Theorem 4). *Let $X$ be an algebraic $K3$ surface with $|H_X|=m_X>1$. Assume that the transcendental lattice $T_X$ is non unimodular, i.e. $A_{T_X}\neq\{0\}$. Then*

(i) *$m_X=p^k$ for some prime number $p$,*

(ii) *$A_{T_X}$ is a $p$-elementary abelian group.*

PROOF. Let $p$ be a prime number with $p|m_X$ and let $g\in H_X$ with $|g|=p$. Then, for any $x^*\in T_X^*=\mathrm{Hom}(T_X, \boldsymbol{Z})$,

$$\tilde{x} = \sum_{\nu=1}^{p} (g^*)^{\nu}(x^*)$$

is a $g^*$-invariant vector in $T_X\otimes\boldsymbol{Q}$. Therefore, $\alpha(g)\langle\omega_X, \tilde{x}\rangle=\langle g^*\omega_X, g^*(\tilde{x})\rangle=\langle\omega_X, \tilde{x}\rangle$, where $\omega_X$ is a non-zero holomorphic 2-form on $X$. Since $\alpha(g)\neq 1$, $\langle\omega_X, \tilde{x}\rangle=0$, and hence $\tilde{x}\in(T_X\cap S_X)\otimes\boldsymbol{Q}=\{0\}$. Thus $\tilde{x}=0$. On the other hand, $g^*$ acts trivially on $A_{S_X}\cong A_{T_X}$, and hence $\tilde{x}\equiv px^*\ (\mathrm{mod}\ T_X)$. Therefore $px^*\equiv 0\ (\mathrm{mod}\ T_X)$.

COROLLARY 6.2 ([**12**], Theorem 7). *We keep the assumption of Theorem 6.1. Then $m_X=2^k$ $(1\leqq k\leqq 4)$, $3^l$ $(1\leqq l\leqq 3)$, $5^n$ $(n=1, 2)$, 7, 11, 13, 17 or 19.*

PROOF. Since $\varphi(m_X)|\mathrm{rank}(T_X)$ and $\mathrm{rank}(T_X)\leqq 21$, it follows from Theorem 6.1 that $m_X$ is one of the above list or $m_X=2^5$.

LEMMA 6.3. $m_X\neq 2^5$.

PROOF. If $m_X=2^5$, then $\mathrm{rank}(T_X)=\varphi(m_X)=2^4$ and $\mathrm{rank}(S_X)=6$. By Theorem 6.1, (ii), $S_X$ is a 2-elementary even indefinite lattice of rank 6. Such lattices are completely described by the Nikulin's theorem ([**4**], Theorem 4.3.2) as follows: we use the same notation as in [4]. Since $\mathrm{rank}(S_X)\equiv l(S_X)\ (\mathrm{mod}\ 2)$ ([**4**], Theorem 4.3.2, (2)) and $l(S_X)\leqq\mathrm{rank}(S_X)$, $l(S_X)=2$, 4 or 6. By [4], Theorem 4.3.2, (6), (7), if $l(S_X)=2$ (resp. $l(S_X)=6$), then $\delta(S_X)=0$ (resp. $\delta(S_X)=1$). Hence $(\mathrm{rank}(S_X), l(S_X), \delta(S_X))=(6, 2, 0), (6, 4, 0), (6, 4, 1)$ or $(6, 6, 1)$. Thus we have that $S_X\cong U\oplus D_4$, $U(2)\oplus D_4$, $U\oplus A_1^4$ or $U(2)\oplus A_1^4$, where $A_1^4=A_1\oplus A_1\oplus A_1\oplus A_1$. By the same proof as that of Lemma 2.1, there exists an elliptic pencil $\pi=|F|$: $X\to\boldsymbol{P}^1$ with $[F]^{\perp}/\boldsymbol{Z}[F]\cong D_4$ or $A_1^4$. By Lemma 2.2, $\pi$ has a singular fibre of type $\mathrm{I}_2$, III or $\mathrm{I}_0^*$.

CLAIM 1. *The cases $S_X \cong U \oplus A_1^4$ and $U(2) \oplus A_1^4$ do not occur.*

PROOF OF CLAIM 1. If $S_X \cong U \oplus A_1^4$ or $U(2) \oplus A_1^4$, then $g$ acts on the base of $\pi$ identically because $\pi$ has four reducible sidgular fibres of type $\mathrm{I}_2$ (or III) and $g \in H_X$. In case $S_X \cong U \oplus A_1^4$, this is impossible because $\pi$ has a section (Lemma 2.1) and the functional invariant of $\pi$ is not equal to the constant 1728. In case $S_X \cong U(2) \oplus A_1^4$, the set of fixed points of $g^{16}$ is a smooth irreducible curve $C$ of genus 5 ([4], Theorem 4.2.2). It is easy to see that $C$ meets transversally each component of a reducible singular fibre of $\pi$ at two points. Thus $C \cdot F = 4$ where $F$ is a general fibre. Hence $g^4$ has four fixed points $C \cap F$ on $F$. This contradicts the fact that no smooth elliptic curves have an automorphism of order 8.

CLAIM 2. *In case $S_X \cong U \oplus D_4$ or $U(2) \oplus D_4$, $g$ acts on the base as an automorphism of order* 16.

PROOF OF CLAIM 2. In case $S_X \cong U \oplus D_4$, $\pi$ has a section (Lemma 2.1). Hence the assertion follows from the formula $\sum_{F:\,\mathrm{fibre}} e(F) = 24$ and the fact that the functional invariant of $\pi$ is not equal to the constant 1728. Also, in case $S_X \cong U(2) \oplus D_4$, the above formula implies $|g|\boldsymbol{P}^1| \neq 32$. Now we assume $|g|\boldsymbol{P}^1| \leqq 8$. By [4], Theorem 4.2.2, the set of fixed points of $g^{16}$ is a smooth curve $C+E$, where $C$ is a smooth curve of genus 6 and $E$ is a multiple component of the singular fibre $F_1$ of type $\mathrm{I}_0^*$ (Lemma 2.3). It follows from Lemma 2.3 that $C$ meets transversally each simple component of $F_1$ at one point. Hence $C \cdot F_1 = 4$. If $g^8|C$ is trivial, then a general fibre $F$ of $\pi$ has an automorphism $g^8|F$ of order 4 which fixes $C \cap F$. This is a contradiction, and hence $g^8$ acts on $C$ as an involution. If there exists a singular fibre $F'$ of type $\mathrm{I}_1$, then $C$ meets $F'$ at the singular point and other two points of $F'$. On the other hand, $g^8$ acts on $F'$ as an automorphism of order 4 which is impossible. Hence it now follows from the formula $\sum_{F:\,\mathrm{fibre}} e(F) = 24$ that $\pi$ has exactly one singular fibre $F_1$ of type $\mathrm{I}_0^*$ and 9 singular fibres $G_i$ of type II ($1 \leqq i \leqq 9$). Since $C \cdot F_1 = 4$, $C$ meets each $G_i$ at the singular point and a smooth point of $G_i$. The involution $g^8|C$ has at least 22 fixed points on $C$ which are $C \cap F_1$ and $C \cap G_i$. This contradicts the Hurwitz formula.

It follows from Claim 2, Lemma 2.2 and the formula $\sum_{\mathrm{fibre}} e(F) = 24$ that $\pi$ has exactly one singular fibre $F_1$ of type $\mathrm{I}_0^*$, one singular fibre $F_2$ of type II and 16 singular fibres of type $\mathrm{I}_1$. By [4], Theorem 4.2.2, the set of fixed points of the involution $g^{16}$ is the following smooth reducible curve:

(a) In case $S_X \cong U \oplus D_4$, $C+E_1+E_2$ where $C$ is a smooth curve of genus 7, $E_1$ and $E_2$ are smooth rational curves. By Lemma 2.3, we may assume that $E_1$ is a section of $\pi$ and $E_2$ is the multiple component of $F_1$.

(b) In case $S_X \cong U(2) \oplus D_4$, $C+E$ where $C$ is a smooth curve of genus 6 and $E$ is the multiple component of $F_1$ (Lemma 2.3).

In either case $g$ acts on $C$ as an automorphism of order 16 because $C \cdot F > 0$ and $g$ acts on the base as an automorphism of order 16. Let $F_i = L_1 + L_2 + L_3 + L_4 + 2L_5$ be the irreducible decomposition of $F_1$. By Lemma 2.3, $g^{16}$ acts on $L_i$ as an involution $(1 \leqq i \leqq 4)$.

In case (a), we may assume that $L_1$ meets $E_1$. Since $F_2$ and $L_j$ are not fixed curves of $g^{16}$, $C \cdot L_j = 1$ $(2 \leqq j \leqq 4)$ and $C$ passes through the singular point of $F_2$. Hence $g^k$ $(k=2, 4, 8)$ has exactly four fixed points on $C$ which are $C \cap L_j$ $(2 \leqq j \leqq 4)$ and the singular point of $F_2$ (see Figure 4). Hence by the Hurwitz formula, $16(2g(C/\langle g \rangle)-2)+15 \cdot 4 = 2g(C)-2 = 12$, which is a contradiction.

In case (b), $C \cdot L_j = 1$ $(1 \leqq j \leqq 4)$ and $C$ passes through the singular point and a smooth point of $F_2$ (see Figure 5). By the same way as in the case (a), we have a contradiction. Thus we have proved Lemma 6.3 and Corollary 6.2.

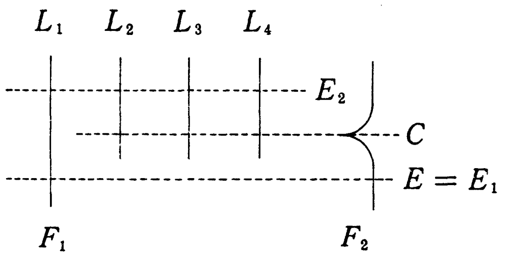



Figure 4

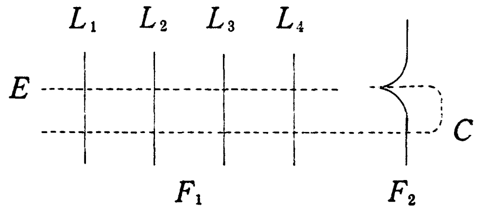



Figure 5

(the dotted lines are the fixed curves of $g^{16}$).

THEOREM 6.4 ([**12**], Theorem 7). *We keep the assumption of Theorem* 6.1. *Suppose that rank* $(T_X) = \varphi(m_X)$. *Then* $m_X = 3^k$ $(1 \leqq k \leqq 3)$, $5^n$ $(n=1, 2)$, 7, 11, 13, 17 *or* 19. *Moreover if m is one of these, then there exists an algebraic* $K3$ *surface with* $m_X = m$ *and rank*$(T_X) = \varphi(m)$.

PROOF. For existence, we shall give examples of such K3 surfaces in §7. By Corollary 6.2, we only need to see that there exist no algebraic K3 surfaces with $m_X = 2^k$ $(1 \leqq k \leqq 4)$ and $\mathrm{rank}(T_X) = \varphi(m_X)$. Since $\mathrm{rank}(T_X) \geqq 2$, the case $m_X = 2$ does not occur.

CLAIM 1. $m_X \neq 4$.

PROOF OF CLAIM 1. If $m_X = 4$, then $\mathrm{rank}(T_X) = 2$ and $S_X$ is an even 2-elementary indefinite lattice of rank 20 (Theorem 6.1, (ii)). By the fact $l(S_X) = l(T_X) \leqq \mathrm{rank}(T_X)$ and [**4**], Theorem 4.3.2, (2), (3), we have $(\mathrm{rank}(S_X), l(S_X),$

$\delta(S_X))=(20, 2, 1)$, i.e. $S_X \cong U \oplus E_8 \oplus E_8 \oplus A_1 \oplus A_1$. Consider the elliptic pencil $\pi$ with a section defined by an element $x \in U$ with $x^2=0$ (Lemma 2.1). Then $\pi$ has two singular fibres of type II* and two singular fibres of type $I_2$ (or III) (Lemma 2.2). This implies that the functional invariant of $\pi$ is not constant. Since $g \in H_X$ and $\pi$ has 4 reducible singular fibres, $g$ acts on the base of $\pi$ identically. Hence $g$ acts on general fibres as an automorphism of order 4, which is a contradiction.

CLAIM 2. $m_X \neq 8$.

PROOF OF CLAIM 2. If $m_X=8$, then $\mathrm{rank}(T_X)=4$, $\mathrm{rank}(S_X)=18$ and $l(S_X) =l(T_X) \leqq \mathrm{rank}(T_X)$. It follows from [**4**], Theorem 4.3.2 that (rank $S_X$, $l(S_X)$, $\delta(S_X))=(18, 2, 0)$, $(18, 2, 1)$, $(18, 4, 0)$ or $(18, 4, 1)$, i.e. $S_X \cong U \oplus E_8 \oplus D_8$, $U \oplus E_8 \oplus E_7 \oplus A_1$, $U \oplus D_8 \oplus D_8$ or $U \oplus E_7 \oplus E_7 \oplus A_1 \oplus A_1$, respectively. Consider the elliptic pencil $\pi$ with a section $E$ defined by an element $x \in U$ with $x^2=0$ (Lemma 2.1).

In case $S_X \cong U \oplus E_8 \oplus E_7 \oplus A_1$ or $U \oplus E_7 \oplus E_7 \oplus A_1 \oplus A_1$, $\pi$ has at least three reducible singular fibres (Lemma 2.2). By the same argument as in the Proof of Claim 1, we have a contradiction.

If $S_X \cong U \oplus E_8 \oplus D_8$ or $U \oplus D_8 \oplus D_8$, $g$ acts on the base as an automorphism of order at least 4 because the functional invariant of $\pi$ is not constant. It follows from the formula $\sum_{F:\mathrm{fibre}} e(F)=24$ that $\pi$ has 4 irreducible singular fibres of type $I_1$ and $g$ acts on the base as an automorphism of order 4.

In case $S_X \cong U \oplus E_8 \oplus D_8$, $\pi$ has a singular fibre $F_1$ of type II* and a singular fibre $F_2$ of type $I_4^*$ (Lemma 2.2). The set of fixed points of the involution $g^4$ is $C+\sum_{i=1}^{8} E_i$ ([**4**], Theorem 4.3.2), where $C$ is a smooth elliptic curve and $E_i$ $(1 \leqq i \leqq 8)$ are smooth rational curves. By Lemma 2.3, we may assume that $E_1=E$ and $E_i$, $2 \leqq i \leqq 8$, are components of $F_1$, $F_2$. Recall that the set of fixed points of $g^4$ is a smooth curve (c.f. § 2). If $C$ is a fibre of $\pi$, then $C \cdot E=1$ which contradicts the above remark. Hence $C \cdot F_1>0$. Therefore $g$ acts on $C$ as an automorphism of order 4 because $|g|E|=4$. Denote by $D_1$ the component with multiplicity 3 of $F_1$ which intersects the component with multiplicity 6 and by $L_i$ $(1 \leqq i \leqq 3)$ the simple components of $F_2$ with $L_i \cdot E=0$ $(1 \leqq i \leqq 3)$. Then by Lemma 2.3, $L_i$ $(1 \leqq i \leqq 3)$ and $D_1$ are not fixed curves of $g^4$, and hence $C \cdot L_i =C \cdot D_1=1$ and $C$ does not meet any other components of $F_1$ and $F_2$. Thus both $g$ and $g^2$ have exactly 4 fixed points $C \cap D_1$, $C \cap L_i$ $(1 \leqq i \leqq 3)$ on $C$ (see Figure 6).

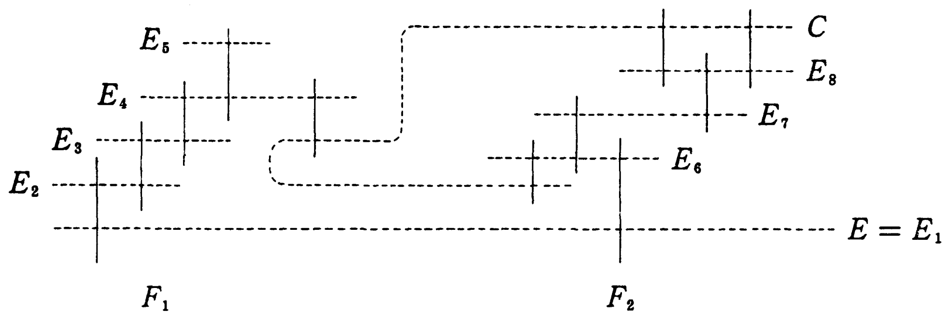



Figure 6 (the dotted lines are fixed curves of $g^4$).

By the Hurwitz formula, we have

$$0 = 2g(C)-2 = 4(2g(C/\langle g\rangle)-2)+12\,.$$

This is a contradiction.

In case $S_X \cong U \oplus D_8 \oplus D_8$, $\pi$ has two reducible singular fibres $F_1$, $F_2$ of type $I_4^*$ (Lemma 2.2). The set of fixed points of the involution $g^4$ is $\sum_{i=1}^{8} E_i$, where $E_i$ $(1 \leqq i \leqq 8)$ are smooth rational curves. By Lemma 2.3, we may assume that $E_1 = E$ and $E_i$ $(2 \leqq i \leqq 7)$ are components of $F_1$, $F_2$. Let $L$ be a simple component of $F_1$ which does not meet the section $E$. By Lemma 2.3, $g$ has a fixed point $p$ on $L$ which is not the intersection point of $L$ and other component. Since $g$ has no isolated fixed points, the curve $E_8$ passes through $p$. Thus $E_8$ meets 6 simple components of $F_1$ and $F_2$ not meeting $E$ (see Figure 7). On the other hand, $|g|E_8|=4$ because $F \cdot E_8 > 0$ and $g$ acts on the base as an automorphism of order 4. This contradicts the Hurwitz formula.

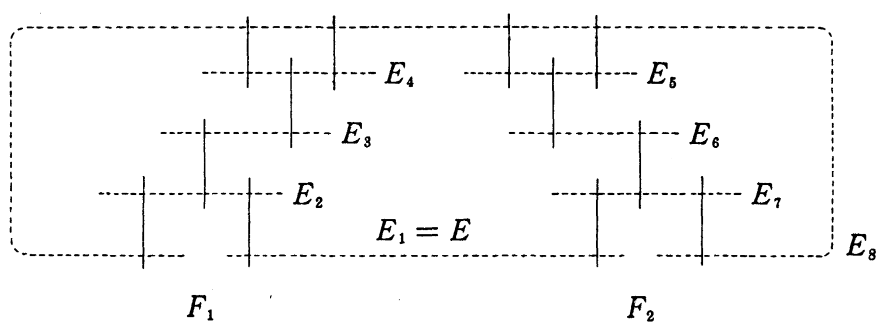



Figure 7 (the dotted lines are the fixed curves of $g^4$).

CLAIM 3. $m_X \neq 16$.

PROOF OF CLAIM 3. If $m_X = 16$, then $\mathrm{rank}(T_X) = 8$, $\mathrm{rank}(S_X) = 14$ and $l(S_X) = l(T_X) \leqq 8$. It follows from [4], Theorem 4.3.2 that $(\mathrm{rank}(S_X),\ l(S_X),\ \delta(S_X)) =$

(14, 2, 0), (14, 4, 0), (14, 4, 1), (14, 6, 0), (14, 6, 1), (14, 8, 0) or (14, 8, 1), i.e. $S_X \cong U \oplus E_8 \oplus D_4$, $U \oplus D_8 \oplus D_4$, $U \oplus E_8 \oplus A_1^4$, $U \oplus D_4^3$, $U \oplus D_8 \oplus A_1^4$, $U(2) \oplus D_4^3$ or $U \oplus D_4^2 \oplus A_1^4$. By the same proof as that of Lemma 2.1, there exists an elliptic pencil $\pi = |F| : X \to \boldsymbol{P}^1$ with $[F]^{\perp}/\boldsymbol{Z}[F] \cong U^{\perp}$ in $S_X$.

If $\pi$ has a section and has at least three reducible singular fibres, then by the same argument as in the proof of the above Claim 1, we have a contradiction.

In case $S_X \cong U(2) \oplus D_4^3$, $g$ acts on the base as identity and the set of fixed points of $g^8$ is $\sum_{i=1}^4 E_i$, where $E_i$ is a smooth rational curve. By Lemma 2.3, we may assume that $E_i$ $(1 \leqq i \leqq 3)$ are the multiple components of the singular fibres of type $\mathrm{I}_0^*$. It is easy to see that $E_4$ meets a general fibre at 4 points. This implies that a general fibre has an automorphism $g^4$ of order 4, and hence the local functional invariant of $\pi$ is equal to the constant 1728. This is a contradiction.

Hence we may assume that $S_X \cong U \oplus E_8 \oplus D_4$ or $U \oplus D_8 \oplus D_4$. In these cases, it is easy to see that $\pi$ has exactly 8 irreducible singular fibres of type $\mathrm{I}_1$ and $g$ acts on the base as an automorphism of order 8. By the similar proof of Claim 2, we can see that these cases do not occur. Thus we have proved Theorem 6.4.

## §7. Examples (non unimodular case)

In this section we give examples of algebraic K3 surfaces which have non unimodular transcendental lattice $T_X$ and $\varphi(|H_X|) = \mathrm{rank}(T_X)$ (see Theorem 6.4). By the result of Vorontsov ([**12**], Theorem 7), such a K3 surface is isomorphic to one of these examples. Our examples are elliptic K3 surfaces except $m_X = 25$. In the following we give affine equations of these elliptic K3 surfaces.

(7.1) $m_X = 19$. $X: y^2 = x^3 + t^7 x + t$,

$g: (x, y, t) \longrightarrow (e_{19}^7 \cdot x, e_{19} \cdot y, e_{19}^2 \cdot t)$.

(7.2) $m_X = 17$. $X: y^2 = x^3 + t^7 x + t^2$,

$g: (x, y, t) \longrightarrow (e_{17}^7 \cdot x, e_{17}^2 \cdot y, e_{17}^2 \cdot t)$.

(7.3) $m_X = 13$. $X: y^2 = x^3 + t^5 x + t$,

$g: (x, y, t) \longrightarrow (e_{13}^5 \cdot x, e_{13} \cdot y, e_{13}^2 \cdot t)$.

(7.4) $m_X = 11$. $X: y^2 = x^3 + t^5 x + t^2$,

$g: (x, y, t) \longrightarrow (e_{11}^5 \cdot x, e_{11}^2 \cdot y, e_{11}^2 \cdot t)$.

(7.5) $m_X=7$. $X: y^2=x^3+t^3x+t^8$,

$g:(x, y, t) \longrightarrow (e_7^3 \cdot x, e_7 \cdot y, e_7^2 \cdot t)$.

(7.6) $m_X=5$. $X: y^2=x^3+t^3x+t^7$,

$g:(x, y, t) \longrightarrow (e_5^3 \cdot x, e_5^2 \cdot y, e_5^2 \cdot t)$.

(7.7) $m_X=27$. $X: y^2=x^3+t \cdot \prod_{\nu=1}^{9}(t-e_{27}^{3\nu})$,

$g:(x, y, t) \longrightarrow (e_{27}^2 \cdot x, e_{27}^3 \cdot y, e_{27}^6 \cdot t)$.

(7.8) $m_X=9$. $X: y^2=x^3-t^5 \cdot \prod_{\nu=1}^{3}(t-e_9^{3\nu})$,

$g:(x, y, t) \longrightarrow (e_9^2 \cdot x, e_9^3 \cdot y, e_9^3 \cdot t)$.

(7.9) $m_X=3$. $X: y^2=x^3-t^5(t-1)^5(t+1)^2$,

$g:(x, y, t) \longrightarrow (e_3 \cdot x, y, t)$.

REMARK 7.10. Let $\pi: \tilde{X} \to \boldsymbol{P}^1$ be the Kodaira-Néron model of $X$. Let $r$ be the rank of the Mordell-Weil group of $\pi$. Then it follows from [**9**], §5 that $r$ and the singular fibres of $\pi$ are as follows:

Table 2.

| $m_X$ | $r$ | singular fibres |
|---|---|---|
| 19 | 1 | II, III, $\mathrm{I}_1\times 19$ |
| 17 | 1 | III, IV, $\mathrm{I}_1\times 17$ |
| 13 | 1 | II, III*, $\mathrm{I}_1\times 13$ |
| 11 | 1 | IV, III*, $\mathrm{I}_1\times 11$ |
| 7 | 1 | III*, IV*, $\mathrm{I}_1\times 7$ |
| 5 | 1 | II*, III*, $\mathrm{I}_1\times 5$ |
| 27 | 0 | IV, $\mathrm{II}\times 10$ |
| 9 | 0 | II*, IV*, $\mathrm{II}\times 3$ |
| 3 | 0 | IV, $\mathrm{II}^*\times 2$ |

By the formula $\mathrm{rank}(S_X)=2+r+\sum_{F:\text{ fibre}}[\#\{\text{components of } F\}-1]$ ([8]), we can see $\varphi(m_X)=\mathrm{rank}(T_X)$.

REMARK 7.11. In the above examples, $g\in H_X$. In fact, in case of $m_X=5$, 7, 11, 13, 17 or 19, any reducible singular fibres have no symmetries of order

$m_X$. In particular, $g$ preserves each component of them. In case of $m_X=3$, 9, or 27, by definition of $(X, g)$, $g$ preserves at least one section, and hence $g$ preserves each component of reducible singular fibres. It is known that $S_X$ is generated by sections and components of singular fibres ([8]). On the other hand, $g^*|S_X\otimes \boldsymbol{Q}$ is a representation of the cyclic group of order $m_X$ over $\boldsymbol{Q}$. Since $r\leqq 1$, we have the desired result.

Note that all above elliptic K3 surfaces have an automorphism $\iota:(x, y, t)\to(x, -y, t)$. However this involution acts on $S_X$ nontrivially.

(7.12) $m_X=25$. In this case, $\mathrm{rank}(S_X)=22-\mathrm{rank}(T_X)=2$. By the theory of reductions of indefinite bilinear forms of rank 2, there are no elements $x\in S_X$ for which $x^2=0$. In particular, $X$ does not have a structure of elliptic surfaces ([**5**], §3, Corollary 3). We construct a K3 surface with $m_X=25$ as follows:

Let $C$ be a non singular sextic curve in $\boldsymbol{P}^2$ defined by the following equation: $C=\{x_0^6+x_0x_1^5+x_1x_2^5=0\}$. Let $\varphi$ be a transformation defined by $\varphi(x_0:x_1:x_2)=(x_0:e_{25}^5\cdot x_1:e_{25}^4\cdot x_2)$. Then $C$ is invariant under $\varphi$. Denote by $X$ the double covering of $\boldsymbol{P}^2$ ramified at $C$. Then $X$ is a K3 surface. Let $g$ be an automorphism induced from $\varphi$ so that the order of $g^*|T_X$ is odd. Note that an affine equation of $X$ is given by $z^2=1+x^5+xy^5$ and $\omega_X=(dx\wedge dy)/z$ defines a nowhere vanishing holomorphic 2-form on $X$. Then $g^*\omega_X=e_{25}^9\omega_X$. Since $\varphi(25)|\mathrm{rank}(T_X)$, $\mathrm{rank}(S_X)=2$. Moreover $g|S_X\otimes\boldsymbol{Q}$ is representation of $\boldsymbol{Z}/25$ over $\boldsymbol{Q}$, and hence $g$ acts trivially on $S_X$.

## References


[1] K. Kodaira, On compact analytic surfaces II; III, Ann. Math., **77** (1963), 563-626; **78** (1963), 1-40.

[2] S. Kondō, On automorphisms of algebraic K3 surfaces which act trivially on Picard groups, Proc. Japan Acad., **62** (1986), 356-359.

[3] V. V. Nikulin, Finite automorphism groups of Kähler surfaces of type K3, Proc. Moscow Math. Soc., **38** (1979), 75-137.

[4] ———, On the quotient groups of the automorphism group of hyperbolic forms by the subgroups generated by 2-reflections, J. Soviet Math., **22** (1983), 1401-1476.

[5] I. Piatetskii-Shapiro and I. R. Shafarevich, A Torelli theorem for algebraic surfaces of type K3, Math. USSR Izv., **35** (1971), 530-572.

[6] K. Saito, Algebraic surfaces for regular systems of weights, Algebraic Geometry and Commutative Algebra in Honor of M. Nagata, Kinokuniya, 1988, 517-614.

[7] J. P. Serre, A course in arithmetic, Springer, Berlin-Heidelberg-New York, 1973.

[8] T. Shioda, On elliptic modular surfaces, J. Math. Soc. Japan, **24** (1972), 20-59.

[9] ———, An explicit algorithm for computing the Picard number of certain algebraic surfaces, Amer. J. Math., **108** (1986), 415-432.

[10] K. Ueno, A remark on automorphisms of Enriques surfaces, J. Fac. Sci. Univ. Tokyo, **23** (1976), 149-165.

[11] E. B. Vinberg, Some arithmetic groups in Lobachevskii spaces, Discrete subgroups

of Lie groups and applications to Moduli, Tata-Oxford, 1975, 323-348.
[12] S. P. Vorontsov, Automorphisms of even lattices that arise in connection with automorphisms of algebraic K3 surfaces, Vestnik Mosk. Univ. Mathematika, **38** (1983), 19-21.

Shigeyuki KONDŌ
Department of Mathematics
Faculty of Sciences
Saitama University
Shimo-Okubo 255, Urawa
Saitama, 358
Japan