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The $W^{k,p}$-continuity of wave operators for Schrödinger operators

Dedicated to Professor S. T. Kuroda on his sixtieth birthday

By Kenji YAJIMA

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1. Introduction, Theorems.

For the pair of Schrödinger operators $H_0 = D_1^2 + \cdots + D_m^2$ and $H = H_0 + V$, where $D_j = -i\partial/\partial x_j$, $j=1, \cdots, m$, and V is the multiplication operator with the real valued function $V(x)$, the wave operators $W_\pm = W_\pm(H, H_0)$ are defined by

$$W_\pm = s-\lim_{t\to\pm\infty} e^{itH}e^{-itH_0}, \tag{1.1}$$

where $s-$indicates the strong limit in $L^2(\boldsymbol{R}^m)$. In this paper, we prove under suitable conditions on $V(x)$ that $W_\pm$ are bounded in the Sobolev spaces $W^{k,p}(\boldsymbol{R}^m)$ for any $1 \leqq p \leqq \infty$ and $k=0, 1, \cdots, l$. The merit of the wave operators is that they intertwine the part H_c of H on the continuous spectral subspace $L_c^2(H)$ and H_0: $H_c = W_\pm H_0 W_\pm^*$ on $L_c^2(H)$. Hence the $W^{k,p}(\boldsymbol{R}^m)$-boundedness of $W_\pm$ implies that the functions $f(H_0)$ and $f(H)P_c(H)$, $P_c(H)$ being the orthogonal projection onto $L_c^2(H)$, have equivalent operator norms from $W^{k,p}(\boldsymbol{R}^m)$ to $W^{k',q}(\boldsymbol{R}^m)$ for any $1 \leqq p, q \leqq \infty$ and $k, k' = 0, 1, \cdots, l$:

$$\begin{aligned} C_1\|f(H_0)\|_{B(W^{k,p}, W^{k',q})} &\leqq \|f(H)P_c(H)\|_{B(W^{k,p}, W^{k',q})} \\ &\leqq C_2\|f(H_0)\|_{B(W^{k,p}, W^{k',q})}, \end{aligned} \tag{1.2}$$

where the constants are independent of f. We shall apply (1.2) to obtain, among others, the $L^p - L^q$ estimates for the propagators of the time dependent Schrödinger equations $i\partial u/\partial t = Hu$ and of the wave or Klein-Gordon equations with potentials $\partial^2 u/\partial t^2 + Hu + \mu^2 u = 0$, and the "Fourier multiplier theorems" for the generalized eigenfunction expansions associated with H.

We assume that $V(x)$ satisfies the following assumption, where $\mathcal{F}$ is the Fourier transform, $\langle x\rangle = (1+|x|^2)^{1/2}$, $l \geqq 0$ is a fixed integer, and $m_* = (m-1)\cdot/(m-2)$. For multi-indices $\alpha = (\alpha_1, \cdots, \alpha_m)$, $D^\alpha = D_1^{\alpha_1}\cdots D_m^{\alpha_m}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_m$.

ASSUMPTION 1.1. *$V(x)$ is a real valued function on $\boldsymbol{R}^m$, $m \geqq 3$, such that for any $|\alpha| \leqq l$ $\mathcal{F}(\langle x\rangle^\sigma D^\alpha V) \in L^{m_*}(\boldsymbol{R}^m)$ for some $\sigma > 2/m_*$ and satisfies one of the*

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following conditions:

1. $\|\mathcal{F}(\langle x\rangle^{\sigma}V)\|_{L^{m*}(\boldsymbol{R}^m)}$ *is sufficiently small.*

2. *The spatial dimension* $m=2m'-1$ *is odd. For any* $|\alpha|\leqq\max\{l, l+m'-4\}$ *there exists a constant* $C_\alpha>0$ *such that*

$$|(\partial^\alpha V/\partial x^\alpha)(x)| \leqq C_\alpha\langle x\rangle^{-\delta}, \qquad \delta > \max(m+2,\, 3m/2-2).$$

The assumption $\mathcal{F}(\langle x\rangle^{\sigma}V)\in L^{m*}(\boldsymbol{R}^m)$ implies $\langle x\rangle^{\sigma}V\in L^{m-1}(\boldsymbol{R}^m)$ and V is short-range in the sense of Agmon [**2**]. It follows (cf. Agmon [**2**] or Kuroda [**17**]) that H and H_0 with the domain $W^{2,2}(\boldsymbol{R}^m)$, are selfadjoint in $L^2(\boldsymbol{R}^m)$; the wave operators (1.1) exist and are complete:

$$\text{Image}\, W_\pm = L^2_{ac}(H) = \text{the absolutely continuous subspace for } H; \tag{1.3}$$

and that the singular continuous spectrum of H is absent:

$$L^2_{ac}(H) = L^2_c(H) = \text{the continuous subspace for } H. \tag{1.4}$$

The wave operators $W_\pm$ are partial isometries and a fortiori bounded in $L^2(\boldsymbol{R}^m)$. The completeness (1.3) and the absence of singular continuous spectrum (1.4) imply that the limits

$$Z_\pm = W_\pm(H_0, H) = s-\lim_{t\to\pm\infty} e^{itH_0}e^{-itH}P_c(H)$$

also exist and $Z_\pm=W_\pm^*$.

The main result in this paper is the following theorem that $W_\pm$ and $Z_\pm$ are in fact bounded in $W^{k,p}(\boldsymbol{R}^m)$ for any $0\leqq k\leqq l$ and $1\leqq p\leqq\infty$.

THEOREM 1.1. *Let V satisfy Assumption* 1.1 *and let* 0 *be neither eigenvalue nor resonance of H. Then, for any* $k=0,\cdots,l$ *and* $1\leqq p\leqq\infty$, $W_\pm$ *and* $Z_\pm$ *originally defined on* $L^2(\boldsymbol{R}^m)\cap W^{k,p}(\boldsymbol{R}^m)$ *can be extended to bounded operators in* $W^{k,p}(\boldsymbol{R}^m)$ *and*

$$\begin{aligned} \|W_\pm f\|_{W^{k,p}} &\leqq C_p\|f\|_{W^{k,p}}, \qquad f\in L^2(\boldsymbol{R}^m)\cap W^{k,p}(\boldsymbol{R}^m);\\ \|Z_\pm f\|_{W^{k,p}} &\leqq C_p\|f\|_{W^{k,p}}, \qquad f\in L^2(\boldsymbol{R}^m)\cap W^{k,p}(\boldsymbol{R}^m). \end{aligned} \tag{1.5}$$

REMARK 1.1. 0 is said to be resonance of H if there exists a solution u of $-\Delta u(x)+V(x)u(x)=0$ such that $\langle x\rangle^{-\gamma}u(x)\in L^2(\boldsymbol{R}^m)$ for any $\gamma>1/2$ but not for $\gamma=0$. Under the Assumption 1.1, it is well known ([8], [**19**]) that 0 can never be a resonance of H if $m\geqq5$; and that 0 is neither eigenvalue nor resonance of H if $\|\mathcal{F}(\langle x\rangle^{\sigma}V)\|_{L^{m*}}$ is sufficiently small.

REMARK 1.2. If 0 is resonance of H, (1.5) does not hold. If 0 is eigenvalue of H, then (1.5) does not hold in general. This can be seen by comparing the results of Jensen-Kato ([8]) or Murata ([**19**]) with Corollary 1.1 below.

We list some immediate consequences of Theorem 1.1. For Banach spaces

$\mathfrak{X}$ and $\mathfrak{Y}$ we write $\boldsymbol{B}(\mathfrak{X}, \mathfrak{Y})$ for the space of bounded operators from $\mathfrak{X}$ to $\mathfrak{Y}$, $\boldsymbol{B}(\mathfrak{X})=\boldsymbol{B}(\mathfrak{X}, \mathfrak{X})$.

THEOREM 1.2. *Let the assumption of Theorem* 1.1 *be satisfied,* $0\leqq k$, $k'\leqq l$, *and* $1\leqq p$, $q\leqq\infty$. *Then there exists a constant* $C>0$ *such that for Borel functions* f *on* $\boldsymbol{R}^1$ *we have*

$$\begin{aligned} C^{-1}\|f(H_0)\|_{\boldsymbol{B}(W^{k,p},W^{k',q})} &\leqq \|f(H)P_c(H)\|_{\boldsymbol{B}(W^{k,p},W^{k',q})} \\ &\leqq C\|f(H_0)\|_{\boldsymbol{B}(W^{k,p},W^{k',q})}. \end{aligned} \tag{1.6}$$

PROOF. In virtue of (1.3) and (1.4), $W_\pm$ are unitary operators from $L^2(\boldsymbol{R}^m)$ onto $L_c^2(H)$. Since

$$e^{-itH}W_\pm = W_\pm e^{-itH_0}, \qquad -\infty < t < \infty \tag{1.7}$$

as is easily seen, we have the unitary equivalence $H_c=W_\pm H_0 W_\pm^*$ on $L_c^2(H)$. It follows for Borel functions f that $f(H_c)=W_\pm f(H_0)W_\pm^*$ and $f(H_0)=W_\pm^* f(H_c)W_\pm$ on $L_c^2(H)$, or

$$f(H)P_c(H) = W_\pm f(H_0)W_\pm^*, \qquad f(H_0)=W_\pm^* f(H)P_c(H)W_\pm \tag{1.8}$$

on $L^2(\boldsymbol{R}^m)$. Applying (1.5) to (1.8), we immediately obtain (1.6). (Q.E.D.)

REMARK 1.3. The intermediate results that follow from (1.8) and Theorem 1.1:

$$\begin{aligned} &\|f(H)P_c(H)u\|_{W^{k,p}} \leqq C\|f(H_0)W_\pm^* u\|_{W^{k,p}}, \\ &\|f(H_0)u\|_{W^{k,p}} \leqq C\|f(H)P_c(H)W_\pm u\|_{W^{k,p}} \end{aligned} \tag{1.9}$$

will also be used in what follows, where the constant C is independent of Borel f or u.

We should mention here the works of Melin [**18**] and Jensen-Nakamura [**10**]. The wave operators are in fact not the only operators which satisfy the interwining property (1.8) and the $W^{k,p}(\boldsymbol{R}^m)$ continuity. Indeed, Melin [**18**] has constructed a family of such operators A_θ, $\theta\in S^{m-1}$ when *m is odd and V is smooth and small.* Thus, his A_θ may as well be used to obtain the estimates (1.2) for such case. It is not clear to us, however, whether his results immediately lead to the boundedness of $W_\pm$. Jensen-Nakamura [**10**] have shown the boundedness of $f(H)$ from L^p to L^q and its extensions to Besov spaces for more general Schrödinger operators $H=H_0+V(x)$ including the case when $|V(x)|$ increases at infinity. Their results are different from ours in the respect, among others, that their q is strictly bigger than p except for the obvious case $p=q=2$.

As an immediate corollary of Theorem 1.2 we obtain the following L^p–L^q estimate for the propagator e^{-itH} for the time dependent Schrödinger equation which has been recently proved by Journe-Soffer-Sogge [**11**] under a slightly different condition on V.

THEOREM 1.3. *Let the assumption of Theorem* 1.1 *be satisfied. Then, for any* $k=0, \cdots, l$, $2\leqq p\leqq\infty$, *and* $1/p+1/q=1$, *there exists a constant* C_{pk} *such that for all* $t\neq 0$

$$\|e^{-itH}P_c(H)f\|_{W^{k,p}}\leqq C_{pk}|t|^{m(1/p-1/2)}\|f\|_{W^{k,q}},\qquad f\in L^2\cap W^{k,q}. \tag{1.10}$$

PROOF. For e^{-itH_0} (1.10) is well known (cf. Kato [**13**]). (1.6) implies (1.10) for H. (Q.E.D.)

The estimates (1.6) or (1.9) can be applied not only to Schrödinger equations: For example, we can apply them to the wave and Klein-Gordon equations with potential

$$\frac{\partial^2 u}{\partial t^2}+Hu(t, x)+\mu^2 u=0,\qquad u(0, x)=\phi(x),\qquad u_t(0, x)=\psi(x) \tag{1.11}$$

and obtain the following generalization of L^p-L^q estimates which are well-known for the free wave and Klein-Gordon equations (cf. Strichartz [**24**], Pecher [**20**]). See also Beals and Strauss [**3**].

THEOREM 1.4. *Let the assumption of Theorem* 1.1 *be satisfied. Then, for any* $k=0, \cdots, l$, $2\leqq p\leqq 2(m+1)/(m-1)$ *and* $1/p+1/q=1$, *there exists a constant* $C>0$ *such that for any* ϕ, $\psi\in L_c^2(H)\cap W^{k,q}$ *with* $\sqrt{H+\mu^2}\phi\in W^{k,q}$, *the solution* $u(t, x)$ *of* (1.11) *satisfies*

$$\|u(t, x)\|_{W^{k,p}(R_x^m)}\leqq C|t|^{1+m(1/p-1/q)}(\|\sqrt{H+\mu^2}\phi\|_{W^{k,q}}+\|\psi\|_{W^{k,q}}),\qquad |t|\geqq 1. \tag{1.12}$$

If $k\leqq l-1$, *the condition* $\sqrt{H+\mu^2}\phi\in W^{k,q}$ *may be replaced by* $\phi\in W^{k+1,q}$ *and* $\|\sqrt{H+\mu^2}\phi\|_{W^{k,q}}$ *in* (1.12) *by* $\|\phi\|_{W^{k+1,q}}$.

PROOF. The solution $u(t)$ of (1.11) can be written in the form

$$u(t)=\cos(t\sqrt{H+\mu^2})P_c(H)\phi+\frac{\sin(t\sqrt{H+\mu^2})}{\sqrt{H+\mu^2}}P_c(H)\psi. \tag{1.13}$$

Let $M(t)=|t|^{1+m(1/p-1/q)}$. It is well known ([**24**]) that for any $k=0, 1, \cdots$ and for those p, q as in the theorem, we have for $|t|\geqq 1$

$$\left\|\frac{\cos(t\sqrt{H_0+\mu^2})}{\sqrt{H_0+\mu^2}}\phi\right\|_{W^{k,q}}\leqq CM(t)\|\phi\|_{W^{k,p}},$$

$$\left\|\frac{\sin(t\sqrt{H_0+\mu^2})}{\sqrt{H_0+\mu^2}}\psi\right\|_{W^{k,q}}\leqq CM(t)\|\psi\|_{W^{k,p}}.$$

Applying (1.6), we immediately obtain (1.12). When $k\leqq l-1$, we have

$$\|\sqrt{H+\mu^2}\phi\|_{W^{k,q}} \leqq C\|\sqrt{H_0+\mu^2}\phi\|_{W^{k,q}} \leqq C'\|\phi\|_{W^{k+1,q}}$$

again by (1.6), which completes the proof of the theorem. (Q.E.D.)

It is well known that estimates (1.10) or (1.12) lead to various space-time integrability properties of the propagators of corresponding equations which are important in non-linear analysis. We omit here, however, the detailed discussion into such direction and content ourselves by showing an inequality of Strichartz type [**25**] as a prototype (cf. Ginibre-Velo [**7**], Yajima [**28**], Pecher [**21**] and Brenner [**5**]).

THEOREM 1.5. *Let the assumption of Theorem* 1.1 *be satisfied and let* ϕ, $\psi \in L_c^2(H)$ *be such that* $(H+\mu^2)^{1/4}\phi \in L^2(\boldsymbol{R}^m)$ *and* $(H+\mu^2)^{-1/4}\psi \in L^2(\boldsymbol{R}^m)$. *Then the solution* u *of* (1.11) *belongs to* $L^p(\boldsymbol{R}_{t,x}^{m+1})$ *with* $p=2(m+1)/(m-1)$ *and*

$$\|u\|_{L^p(\boldsymbol{R}_{t,x}^{m+1})} \leqq C(\|(H+\mu^2)^{1/4}\phi\|_{L^2}+\|(H+\mu^2)^{-1/4}\psi\|_{L^2}). \tag{1.14}$$

PROOF. Write $B_0=\sqrt{H_0+\mu^2}$ and $B=\sqrt{H+\mu^2}$. By Strichartz's inequality (cf. [**25**], Corollary 2), we have

$$\|\{\cos(tB_0)/B_0^{1/2}\}\phi\|_{L^p(\boldsymbol{R}^{m+1})}+\|\{\sin(tB_0)/B_0^{1/2}\}\phi\|_{L^p(\boldsymbol{R}^{m+1})} \leqq C\|\phi\|_{L^2(\boldsymbol{R}^m)}.$$

Applying (1.9) to $\{\cos(tB)/B^{1/2}\}P_c(H)$ at every fixed t, we have

$$\|\{\cos(tB)/B^{1/2}\}P_c(H)\phi\|_{L^p(\boldsymbol{R}^{m+1})}^p = \int_{-\infty}^{\infty}\|\{\cos(tB)/B^{1/2}\}P_c(H)\phi\|_{L^p(\boldsymbol{R}_x^m)}^p dt$$

$$\leqq C_p^p\int_{-\infty}^{\infty}\|\{\cos(tB_0)/B_0^{1/2}\}W_\pm^*\phi\|_{L^p(\boldsymbol{R}_x^m)}^p dt \leqq C_p^p C^p\|\phi\|_{L^2}^p.$$

This implies $\|\cos(tB)P_c(H)\phi\|_{L^p(\boldsymbol{R}^{m+1})} \leqq C_p\|B^{1/2}\phi\|_{L^2}$. $\|\sin(tB)B^{-1}P_c(H)\psi\|_{L^p(\boldsymbol{R}^{m+1})}$ can be estimated in a similar fashion. Applying these estimates to (1.13), we obtain (1.14). (Q.E.D.)

REMARK 1.4. When $l \geqq 1$, the conditions $(H+\mu^2)^{1/4}\phi \in L^2$ and $(H+\mu^2)^{-1/4}\psi \in L^2$ are respectively equivalent to $\phi \in W^{1/2,2}$ and $\psi \in W^{-1/2,2}$ and the norms $\|(H+\mu^2)^{1/4}\phi\|_{L^2}$ and $\|(H+\mu^2)^{-1/4}\psi\|_{L^2}$ in (1.14) may be respectively replaced by $\|\psi\|_{W^{1/2,2}}$ and $\|\phi\|_{W^{-1/2,2}}$. This can be shown as in the proof of Theorem 1.4.

We can make Theorem 1.2 more precise in such a way that, when H admits the generalized eigenfunction expansions, this will give the 'Fourier multiplier theorem' for the generalized Fourier transform. For $j=1, \cdots, m$, define $D_j^\pm = W_\pm D_j W_\pm^*$, where $D_j=-i\partial/\partial x_j$. We call $D^\pm=(D_1^\pm, \cdots, D_m^\pm)$ the asymptotic momentum operators. $D_j^\pm$ are commuting selfadjoint operators in $L^2(\boldsymbol{R}^m)$ and, for any Borel function f on $\boldsymbol{R}^m$, $f(D^\pm)$ can be defined by functional calculus. We have $f(D^{\pm})P_c(H)=W_{\pm}f(D)W_{\pm}^*$ as before and the application of Theorem 1.1 yields the following

THEOREM 1.6. *Let the assumption of Theorem* 1.1 *be satisfied, k, $k'=0, \cdots, l$, and $1\leqq p, q\leqq\infty$. Then, there exists a constant C independent of u and Borel functions f such that*

$$\|f(D^{\pm})P_c(H)u\|_{W^{k,p}} \leqq C\|f(D)W_{\pm}^*u\|_{W^{k,p}},$$

$$\|f(D)u\|_{W^{k,p}} \leqq C\|f(D^{\pm})P_c(H)W_{\pm}u\|_{W^{k,p}}, \tag{1.15}$$

$$C^{-1}\|f(D)\|_{B(W^{k,p},W^{k',q})} \leqq \|f(D^{\pm})P_c(H)\|_{B(W^{k,p},W^{k',q})} \leqq C\|f(D)\|_{B(W^{k,p},W^{k',q})}. \tag{1.16}$$

REMARK 1.5. When $f(\xi)$ is a function of $|\xi|^2$, $f(\xi)=\tilde{f}(|\xi|^2)$, we have $f(D)=\tilde{f}(H_0)$ and $f(D^{\pm})P_c(H)=f(H)P_c(H)$. Thus Theorem 1.6 can be regarded as a precision of Theorem 1.2.

We relate Theorem 1.6 with the Fourier multiplier theorem for the generalized Fourier transform associated with H. For simplicity, we assume (2) of Assumption 1.1 and that 0 is neither resonance nor eigenvalue of H. We write $\phi_0(x, \xi)=e^{ix\cdot\xi}$ and $\hat{u}(\xi)=\mathcal{F}u(\xi)$. M_γ is the multiplication operator with $\langle x\rangle^{-\gamma}$ and $R(z)=(H-z)^{-1}$, $R_0(z)=(H_0-z)^{-1}$ are resolvents.

Kato and Kuroda [14] have (essentially) shown the followings: For $\gamma>1$, $\boldsymbol{B}(L^2(\boldsymbol{R}^m))$-valued function $M_\gamma R(z)M_\gamma$ of $z\in\boldsymbol{C}^1\setminus[0, \infty)$ has continuous boundary values $M_\gamma R(\lambda\pm i0)M_\gamma$ on $[0, \infty)$; and the functions defined by $\phi_{\pm}(\cdot, \xi)=(1-R(\xi^2\pm i0)V)\phi_0(\cdot, \xi)$ are outgoing (incoming) generalized eigenfunctions of H in the sense that they are solutions of $(-\Delta+V(x))\phi_{\pm}(x, \xi)=|\xi|^2\phi_{\pm}(x, \xi)$ satisfying the outgoing (incoming) radiation condition:

$$\phi_{\pm}(x, \xi)=\phi_0(x, \xi)+\frac{e^{\pm i|x||\xi|}}{|x|^{(m-1)/2}}(f(\hat{x}, \xi)+O(|x|^{-1})) \tag{1.17}$$

as $|x|\to\infty$ with fixed $\hat{x}=x/|x|$. Define the generalized Fourier transform by

$$\mathcal{F}_{\pm,H}f(\xi)=(2\pi)^{-m/2}\int_{\boldsymbol{R}^m}\overline{\phi_{\pm}(x, \xi)}f(x)dx. \tag{1.18}$$

Then:

1. $\mathcal{F}_{\pm,H}$ are unitary from $L_c^2(H)$ onto $L^2(\boldsymbol{R}^m)$ and vanish on the point spectral subspace $L_p^2(H)$ for H.

2. $\mathcal{F}_{\pm,H}$ diagonalize H_c in the sense that

$$\mathcal{F}_{\pm,H}H_c\mathcal{F}_{\pm,H}^*g(\xi)=|\xi|^2g(\xi). \tag{1.19}$$

3. The wave operators $W_{\pm}$ can be expressed in terms of $\mathcal{F}_{\pm,H}$:

$$W_{\pm}f(x) = \mathcal{F}_{\pm,H}^{*}\mathcal{F}f(x) = (2\pi)^{-m/2}\int_{\boldsymbol{R}^m}\phi_{\pm}(x,\xi)\hat{f}(\xi)d\xi. \tag{1.20}$$

Note that the unitarity of $\mathcal{F}_{\pm,H}$ implies the generalized eigenfunction expansions:

$$f(x) = (2\pi)^{-m/2}\int_{\boldsymbol{R}^m}\phi_{\pm}(x,\xi)\mathcal{F}_{\pm,H}f(\xi)d\xi, \qquad f \in L_c^2(H). \tag{1.21}$$

For a function f write M_f for the multiplication operator with $f(\xi)$ (this is a little abuse of notation but should not cause any confusion). In virtue of (1.20), we have

$$f(D^{\pm})P_c(H) = W_{\pm}f(D)W_{\pm}^{*} = \mathcal{F}_{\pm,H}^{*}\mathcal{F}f(D)\mathcal{F}^{*}\mathcal{F}_{\pm,H} = \mathcal{F}_{\pm,H}^{*}M_f\mathcal{F}_{\pm,H},$$

that is, $f(D^{\pm})$ is nothing but the Fourier multiplier M_f for the generalized Fourier transform $\mathcal{F}_{\pm,H}$. Thus Theorem 1.6 immediately implies the following

THEOREM 1.7. *Let V satisfy the condition* (2) *of Assumption* 1.1 *and let* 0 *be neither eigenvalue of H nor resonance. Let k, $k'=0, \cdots, l$ and $1 \leqq p, q \leqq \infty$. Then there exists a constant $C>0$ independent of Borel functions f on $\boldsymbol{R}^m$ such that*

$$\begin{aligned} C^{-1}\|f(D)\|_{B(W^{k,p},W^{k',q})} &\leqq \|\mathcal{F}_{\pm,H}^{*}M_f\mathcal{F}_{\pm,H}\|_{B(W^{k,p},W^{k',q})} \\ &\leqq C\|f(D)\|_{B(W^{k,p},W^{k',q})}. \end{aligned} \tag{1.22}$$

REMARK 1.6. The argument above shows that estimate (1.22) remains valid if $\mathcal{F}_{\pm,H}^{*}M_f\mathcal{F}_{\pm,H}$ is replaced by more general $\mathcal{F}_{\pm,H_2}^{*}M_f\mathcal{F}_{\pm,H_1}$, where $H_j=H_0+V_j$, $j=1, 2$, are two Schrödinger operators satisfying the condition of the Theorem 1.7.

Combining Theorem 1.7 with the well known (ordinary) Fourier multiplier theorem, we obtain the following L^p boundedness theorem for the multiplier for the generalized Fourier transform.

COROLLARY 1.1. *Let V satisfy the condition* (2) *of Assumption* 1.1 *and let* 0 *be neither eigenvalue of H nor resonance. Suppose that $P(\xi)$ is such that*

$$\sup_{R>0} R^{-m}\int_{R<|\xi|<2R} \left| |\xi|^{|\alpha|}\partial_{\xi}^{\alpha}P(\xi)\right|^2 d\xi < \infty, \qquad |\alpha| \leqq [m/2]+1, \tag{1.23}$$

where $[m/2]$ is the greatest integer $\leqq m/2$. Then $\mathcal{F}_{\pm,H}^{*}M_P\mathcal{F}_{\pm,H} \in \boldsymbol{B}(W^{k,p})$ for any $k=0, \cdots, l$ and $1<p<\infty$. In particular, if $P(\xi)=f(\xi^2)$ satisfies (1.23), *$f(H)P_c(H) \in \boldsymbol{B}(W^{k,p})$.*

PROOF. It is well known that $P(D)$ is bounded in $W^{k,p}(\boldsymbol{R}^m)$ under the condition (1.23) (cf. Taylor [**26**]). Hence, Corollary 1.1 follows immediately from Theorem 1.7. (Q.E.D.)

REMARK 1.7. As in the preceding remark, Corollary 1.1 remains valid for

$\mathcal{F}^*_{\pm,H_2}M_P\mathcal{F}_{\pm,H_1}$ as well, where $H_j=H_0+V_j$, $j=1, 2$, are two Schrödinger operators satisfying the condition of the corollary.

The rest of the paper is devoted to the proof Theorem 1.1. Explaining the plan of the paper, we outline the proof for W_+, using slightly sloppy notation. $\hat{V}(k)=\mathcal{F}V(k)$ and $R_0(z)=(H_0-z)^{-1}$ and $R(z)=(H-z)^{-1}$ are the resolvents of H_0 and H, respectively.

We use the stationary representation formula of the wave operator:

$$W_+f = f-\frac{1}{2\pi i}\int_{-\infty}^{\infty} R(\lambda-i0)VR_0(\lambda+i0)fd\lambda. \tag{1.24}$$

Expanding $R(z)=\sum_{n=0}^{\infty}(-1)^n R_0(z)(VR_0(z))^n$, $z=\lambda-i0$, in (1.24) yields the formal expansion

$$W_+f = f+\sum_{n=1}^{\infty}(-1)^n W_n f, \qquad W_n f = \frac{1}{2\pi i}\int_{-\infty}^{\infty}(R_0(\lambda-i0)V)^n R_0(\lambda+i0)fd\lambda. \tag{1.25}$$

In Section 2, we show that W_n, $n=1, 2, \cdots$, are bounded in L^p and

$$\|W_n\|_{B(L^p)} \leqq C_1(C_2\|\mathcal{F}(\langle x\rangle^\sigma V)\|_{L^{m*}})^n \tag{1.26}$$

as follows. We estimate the adjoint operator

$$W_n^* = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} R_0(\lambda-i0)(VR_0(\lambda+i0))^n d\lambda.$$

Taking the Fourier transform the adjoint of the integral of (1.25) and performing the λ integration, we have

$$\mathcal{F}W_n^*f(\xi) = \lim_{\varepsilon\downarrow 0}(2\pi)^{-mn/2}\int_{\boldsymbol{R}^{mn}} \frac{\{\prod_{j=1}^n\hat{V}_j(k_j-k_{j-1})\}\hat{f}(\xi-k_n)dk_1\cdots dk_n}{\prod_{j=1}^n\{k_j^2-2k_j\cdot\xi-i\varepsilon\}}. \tag{1.27}$$

Set $K_n(k_1, \cdots, k_n)=i^n(2\pi)^{-nm/2}2^{-n}\prod_{j=1}^n\hat{V}(k_j-k_{j-1})$, $k_0=0$, and

$$\begin{aligned}&\hat{K}_n(t_1, \cdots, t_n, \omega_1, \cdots, \omega_n)\\ &\quad= \int_{[0,\infty)^n} e^{-i\sum_{j=1}^n t_j s_j/2}(s_1\cdots s_n)^{m-2}K(s_1\omega_1, \cdots, s_n\omega_n)ds_1\cdots ds_n,\end{aligned}$$

where $(t_1, \cdots, t_n)\in\boldsymbol{R}^n$ and $(\omega_1, \cdots, \omega_n)\in\Sigma^n$, Σ being the unit sphere of $\boldsymbol{R}^m$. Taking the inverse Fourier transform of (1.27) leads to

$$\begin{aligned}&W_n^*f(x)\\ &= \int_{[0,\infty)^{n-1}\times I\times\Sigma^n}\hat{K}_n(t_1, \cdots, t_{n-1}, \tau, \omega_1, \cdots, \omega_n)f(\bar{x}+\rho)dt_1\cdots dt_{n-1}d\tau d\omega_1\cdots d\omega_n\end{aligned} \tag{1.28}$$

where $I=(-\infty, -\sigma)$, $\sigma=2\omega_n(x+t_1\omega_1+\cdots+t_{n-1}\omega_{n-1})$, is the range of the integration by the variable τ, $\bar{y}=y-2(\omega_n\cdot y)\omega_n$, and $\rho=t_1\bar{\omega}_1+\cdots+t_{n-1}\bar{\omega}_{n-1}-\tau\omega_n$. Since

$x \to \tilde{x}$ is an isometry, it follows by Minkowski's inequality that

$$\|W_n f\|_{L^p} \leqq 2\|\hat{K}_n\|_{L^1([0,\infty)^n \times \Sigma^n)} \|f\|_{L^p}, \qquad 1 \leqq p \leqq \infty. \tag{1.29}$$

On the other hand, the interpolation inequality implies

$$\|\hat{K}_n\|_{L^1([0,\infty)^n, L^1(\Sigma^n))} \leqq (C_2 \|\mathcal{F}(\langle x\rangle^\sigma V)\|_{L^{m*}})^n, \tag{1.30}$$

and by combining (1.29) with (1.30), we obtain (1.26). Thus, the series in (1.25) converges in the operator norm of $B(L^p)$ and W_+ is bounded in $L^p(\boldsymbol{R}^m)$, if $\|\mathcal{F}(\langle x\rangle^\sigma V)\|_{L^{m*}} < C_2^{-1}$.

In Section 3, we prove that W_+ is bounded in L^p when $\|\mathcal{F}(\langle x\rangle^\sigma V)\|_{L^{m*}}$ is not necessary small. In this case (1.25) no longer converges in norm and we stop the expansion at the m-th stage. We write $W_+ f = f - W_1 f + \cdots + (-1)^m W_m f + (-1)^{m+1} L f$. Here W_j, $j=1, \cdots, m$, are as above (hence are bounded in $L^p(\boldsymbol{R}^m)$) and L is given by

$$L = \frac{1}{\pi i}\int_0^\infty R_0(k^2 - i0) V N_{m'-1}(k) V \{R_0(k^2+i0) - R_0(k^2-i0)\} k\,dk, \tag{1.31}$$

where $N_{m'-1}(k) = \{R_0(k^2-i0)V\}^{m'-1} R(k^2-i0) \{V R_0(k^2-i0)\}^{m'-1}$ (we deal with the odd case $m=2m'-1$ only). Let $G_\pm(x, k)$ be the outgoing (incoming) fundamental solutions of $-\Delta - k^2$ and $G_{\pm,x,k}(y) = G_\pm(x-y, k)$. Then the integral kernel $T(x, y, k)$ of

$$R_0(k^2-i0) V N_{m'-1}(k) V \{R_0(k^2+i0) - R_0(k^2-i0)\}$$

is given by $(N_{m'-1}(k) V (G_{+,y,k} - G_{-,y,k}), V G_{+,x,k})$. We shall show by using the mapping properties of resolvents $R_0(k^2 \pm i0)$ and $R(k^2 \pm i0)$ that $T(x, y, k)$ is continuous in (x, y, k), $m'+1$ times differentiable in k and satisfies the estimates

$$|(\partial/\partial k)^j T(x, y, k)| \leqq C\langle k\rangle^{-3} \langle x\rangle^{-(m-1)/2} \langle y\rangle^{-(m-1)/2}.$$

It follows that L also has the continuous integral kernel

$$L(x, y) = \frac{1}{\pi}\int_0^\infty T(x, y, k) k\,dk, \tag{1.32}$$

which is bounded by $C\langle x\rangle^{-(m-1)/2} \langle y\rangle^{-(m-2)/2}$. We apply the integrations by parts by the variable k in (1.32) for gaining the extra decay property of $L(x, y)$ in the variable $|x| \pm |y|$ and show that $L(x, y)$ satisfies the well known criterion for the $L^p(\boldsymbol{R}^m)$-boundedness:

$$\sup_{y \in \boldsymbol{R}^m} \int_{\boldsymbol{R}^m} |L(x, y)|\,dx < \infty, \qquad \sup_{x \in \boldsymbol{R}^m} \int_{\boldsymbol{R}^m} |L(x, y)|\,dy < \infty. \tag{1.33}$$

It is crucial here (at least with our method) that the Hankel functions $r^{(m-2)/2} H^{(j)}_{(m-2)/2}(r)$ are $e^{\pm ir}$ times polynomials of degree $m'-1$ for odd $m=2m'-1$ and we need the condition that m is odd.

In Section 4, we give the argument which is necessary to prove that W_+ is in fact bounded in $W^{k,p}$, $k=1, \cdots, l$, when $l \geqq 1$. We differentiate the integral in (1.25) by D_j and express the commutator $[D_j, W_n]$ as the sum of n integrals each of which has exactly the same form as that of (1.25) except that one of V's is replaced by D_jV. These integrals can be estimated by using exactly the same method as in Section 2 and we obtain

$$\|W_n f\|_{W^{1,p}} \leqq C(C_2\|\mathcal{F}(\langle x\rangle^\sigma V)\|_{L^{m_*}})^{n-1}(\|\mathcal{F}(\langle x\rangle^\sigma V)\|_{L^*}+\|\mathcal{F}(\langle x\rangle^\sigma DV)\|_{L^{m_*}})\|f\|_{W^{1,p}},$$

where C_2 is the same constant as in (1.26). Thus, the series in (1.25) in fact converges in the norm of $\boldsymbol{B}(W^{1,p})$ and $W_+ \in \boldsymbol{B}(W^{1,p})$ provided that $\|\mathcal{F}(\langle x\rangle^\sigma V)\|_{L^*} < C_2^{-1}$. Similarly, we can estimate $[D_j, L]$ by using the method of Section 3 and we conclude that $W_+ \in \boldsymbol{B}(W^{1,p})$ under the condition of Theorem 1.1. The cases $k \geqq 2$ may be proved by repeating the argument above.

The résumé of this paper is announced in [**29**].

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2. L^p boundedness of $W_\pm$ for small potentials.

In this section, we first record some preliminary results which are valid under the assumption of Theorem 1.1 and then proceed to the proof of Theorem 1.1 in the case that $\|\mathcal{F}(\langle x\rangle^\sigma V)\|_{L^{m_*}(\boldsymbol{R}^m)}$, $\sigma > 2/m_*$, is small and $l=0$.

We begin with recalling some basic facts from the stationary theory of scattering (cf. Kato [**13**], Kato-Yajima [**15**] and Kuroda [**16**]). For a Banach space $\mathfrak{X}$, $\boldsymbol{B}_\infty(\mathfrak{X})$ is the space of compact operators in $\mathfrak{X}$. For a closable operator T whose closure is bounded, we denote its closure by the same symbol T.

LEMMA 2.1. *Let $A(x), B(x) \in L^m(\boldsymbol{R}^m)$ be real valued and A and B be the multiplications by $A(x)$ and $B(x)$, respectively. Let $Q_0(z)=AR_0(z)B^*$. Then:*

1. *$Q_0(z)$ is a $B_\infty(L^2)$-valued uniformly bounded analytic function of $z \in \boldsymbol{C}^1 \setminus [0, \infty)$:*

$$\|Q_0(z)\| \leqq C\|A\|_{L^m(\boldsymbol{R}^m)}\|B\|_{L^m(\boldsymbol{R}^m)}, \qquad z \in \boldsymbol{C}^1 \setminus [0, \infty), \tag{2.1}$$

and $\|Q_0(z)\|\to 0$ *as* $|z|\to\infty$.

2. *It has continuous boundary values* $Q_0(\lambda\pm i0)=AR_0(\lambda\pm i0)B^*$ *on* $[0,\infty)$ *which are locally Hölder continuous when* $\lambda>0$.

3. *For any* $f\in L^2(\boldsymbol{R}^m)$, $AR_0(\lambda\pm i\varepsilon)f\in L^2(\boldsymbol{R}, L^2(\boldsymbol{R}^m), d\lambda)$ *and, as* $\varepsilon\to+0$, *it converges to* $AR_0(\lambda\pm i0)f$ *in that space. Moreover,*

$$\sup_{\varepsilon>0}\int_{-\infty}^{\infty}\|AR_0(\lambda\pm i\varepsilon)f\|_{L^2}^2 d\lambda=\int_{-\infty}^{\infty}\|AR_0(\lambda\pm i0)f\|_{L^2}^2 d\lambda\leqq C\|f\|_{L^2}^2\|A\|_m^2. \tag{2.2}$$

Similar statements hold for $BR_0(\lambda\pm i\varepsilon)f$.

PROOF. We have only to show the two points that (a): $\|Q_0(z)\|\to 0$ as $|z|\to\infty$; and that (b): $Q_0(\lambda\pm i0)$ exist and are continuous near $\lambda=0$, since other statements are well-known ([**15**]). We take $A_n, B_n\in L^{m+\varepsilon}\cap L^{m-\varepsilon}$, $\varepsilon>0$ such that $\lim_{n\to\infty}(\|A_n-A\|_{L^m}+\|B_n-B\|_{L^m})=0$ and set $Q_n(z)=A_nR_0(z)B_n$. Then both (a) and (b) hold for $Q_n(z)$ ([**13**]) and (2.1) implies that $\lim_{n\to\infty}\|Q_n(z)-Q_0(z)\|_{B(L^2)}=0$ uniformly on $\boldsymbol{C}^1\setminus[0,\infty)$. Thus they hold for $A, B\in L^m$ as well.

(Q.E.D.)

The following is a result of Lemma 2.1 and the standard argument of scattering theory.

LEMMA 2.2. *Let A and B be as in Lemma* 2.1 *and* $V=B^*A$. *Let, in addition,* $V(x)\in L^2(\boldsymbol{R}^m)$ *if* $m=3$. *Then,* $H=H_0+V$ *is selfadjoint with the domain* $D(H)=D(H_0)$ *and*

$$R(z)=R_0(z)-R_0(z)B^*(1+Q_0(z))^{-1}AR_0(z),\qquad \operatorname{Im} z\neq 0. \tag{2.3}$$

Set $Q(z)\equiv AR(z)B^*$. *Then*:

1. $Q(z)$ *is* $\boldsymbol{B}_\infty(L^2)$*-valued meromorphic in* $z\in\boldsymbol{C}^1\setminus[0,\infty)$ *and* $I-Q(z)=(I+Q_0(z))^{-1}$.

2. *If H has no non-negative eigenvalues and* 0 *is not resonance of H, then* $Q(z)$ *has the norm continuous boundary values* $Q(\lambda\pm i0)$ *on* $\boldsymbol{R}^+=[0,\infty)$. *They satisfy*

$$\|Q(\lambda\pm i0)\|_{B(L^2)}\leqq C\|A\|_{L^m}\|B\|_{L^m}; \tag{2.4}$$

for any $f\in L^2(\boldsymbol{R}^m)$, $AR(\lambda\pm i\varepsilon)f\in L^2(\boldsymbol{R}^+, L^2(\boldsymbol{R}^m), d\lambda)$; *They converge to* $AR(\lambda\pm i0)f$ *in* $L^2(\boldsymbol{R}^+, L^2(\boldsymbol{R}^m), d\lambda)$ *as* $\varepsilon\to+0$;

$$\sup_{\varepsilon>0}\int_0^{\infty}\|AR(\lambda\pm i\varepsilon)f\|_{L^2}^2 d\lambda=\int_0^{\infty}\|AR(\lambda\pm i0)f\|_{L^2}^2 d\lambda\leqq C\|f\|_{L^2}^2\|V\|_{L^{m/2}}. \tag{2.5}$$

Similar statements hold for $BR(\lambda\pm i\varepsilon)f$.

3. *The wave operators* $W_\pm$ *exist and are complete. Under the assumption of statement* (2), *we have for* $f, g\in L^2(\boldsymbol{R}^m)$:

$$(W_\pm f, g) = (f, g) - \frac{1}{2\pi i}\int_0^\infty (A\{R_0(\lambda \pm i0) - R_0(\lambda \mp i0)\} f, BR(\lambda \pm i0)g)d\lambda. \quad (2.6)$$

Let $\mathcal{F}(\langle x\rangle^\sigma V) \in L^{m*}(\boldsymbol{R}^m)$ and set $A(x) = |V(x)|^{1/2}$ and $B(x) = \mathrm{sgn}V(x)\cdot|V(x)|^{1/2}$ so that $V = B^*A$. By Hausdorff-Young's inequality we have $\langle x\rangle^\sigma V \in L^{m-1}$ and by Hölder's inequality $\|V\|_{L^{m/2}} \leqq C\|\mathcal{F}(\langle x\rangle^\sigma V)\|_{L^{m}_*}$. Hence $A(x)$ and $B(x)$ satisfy the main assumption of Lemma 2.1 and 2.2. If we further assume that $\|\mathcal{F}(\langle x\rangle^\sigma V)\|_{L^m_*}$ is small or $|V(x)| \leqq C\langle x\rangle^{-1-\varepsilon}$, $\varepsilon > 0$, then positive eigenvalues are absent from H (cf. [**12**]). Thus, under the condition of Theorem 1.1 which will be assumed in the sequel, the additional condition in (2), (3) of Lemma 2.2 is also satisfied, and all the results of Lemma 2.1 and 2.2 hold. In what follows, we shall deal with W_+ only.

Write the integral in (2.6) in the form

$$\lim_{\varepsilon \downarrow 0}\int_0^\infty ([R(\lambda - i\varepsilon)B^*]A\{R_0(\lambda + i\varepsilon) - R_0(\lambda - i\varepsilon)\} f, g)d\lambda, \quad (2.7)$$

and replace $R(\lambda - i\varepsilon)B^*$ by the N-iteration of (2.3)

$$\begin{aligned} R(z)B^* &= R_0(z)B^*(1+Q_0(z))^{-1} \\ &= R_0(z)B^*\sum_{n=0}^{N-1}(-1)^n Q_0(z)^n + (-1)^N R(z)B^*Q_0(z)^N, \end{aligned}$$

with $z = \lambda - i\varepsilon$. Then, in virtue of Lemma 2.1 and 2.2, we obtain

$$(W_+ f, g) = \sum_{n=0}^{N}(-1)^n(W_n f, g) + (-1)^{N+1}(L_N f, g), \quad (2.8)$$

where $W_0 = I$ is the identity operator and for $n = 1, \cdots,$

$$(W_n f, g) = \frac{1}{2\pi i}\int_0^\infty (Q_0(\lambda - i0)^{n-1}A\{R_0(\lambda + i0) - R_0(\lambda - i0)\} f, BR_0(\lambda + i0)g)d\lambda; \quad (2.9)$$

$$(L_N f, g) = \frac{1}{2\pi i}\int_0^\infty (Q_0(\lambda - i0)^N A\{R_0(\lambda + i0) - R_0(\lambda - i0)\} f, BR(\lambda + i0)g)d\lambda. \quad (2.10)$$

PROPOSITION 2.1. *For $n = 1, 2, \cdots$, W_n can be written in the form*

$$(W_n f, g) = \frac{1}{2\pi i}\int_{-\infty}^\infty (Q_0(\lambda - i0)^{n-1}AR_0(\lambda + i0) f, BR_0(\lambda + i0)g)d\lambda, \quad (2.11)$$

and the following statements are satisfied:

1. (2.11) *defines a bounded operator W_n in L^2 and $\|W_n\|_{B(L^2)} \leqq C_1(C_2\|V\|_{L^{m/2}})^n$, where the constants are independent of V.*

2. *Let $V_j \in L^{m/2}$ be such that $\|V_j - V\|_{L^{m/2}} \to 0$ as $j \to \infty$, then the operators $W_n^{(j)}$ corresponding to V_j converge to W_n in the norm of $\boldsymbol{B}(L^2)$.*

3. *There exists a constant τ_0 such that for $\|V\|_{L^{m/2}} \leqq \tau_0$ the series*

$\sum_{n=0}^{\infty}(-1)^n W_n$ converges to W_+ in the norm of $\boldsymbol{B}(L^2)$.

PROOF. Since $R_0(\lambda+i0)=R_0(\lambda-i0)$ for $\lambda<0$, the region of integration in (2.9) may be replaced by the whole line. On the other hand $I(\lambda)=Q_0(\lambda-i0)^{n-1}AR_0(\lambda-i0)f$ (resp. $J(\lambda)=BR_0(\lambda+i0)g$) is the boundary value of the L^2-valued analytic function $I(z)=Q_0(z)^{n-1}AR_0(z)f$ (resp. $J(z)=BR_0(z)g$) in the lower (resp. upper) half plane which satisfies

$$\sup_{\mu>0}\int_{-\infty}^{\infty}\|I(\lambda-i\mu)\|_{L^2}^2 d\lambda<\infty \qquad \left(\text{resp. } \sup_{\mu>0}\int_{-\infty}^{\infty}\|J(\lambda+i\mu)\|_{L^2}^2 d\lambda<\infty\right).$$

It follows from the orthogonality property of Hardy functions in the upper and lower half planes that $I(\lambda)$ and $J(\lambda)$ are orthogonal to each other,

$$\int_{-\infty}^{\infty}(I(\lambda),\ J(\lambda))_{L^2}d\lambda=0,$$

and we obtain (2.11). Applying (2.1) and (2.2) and Schwarz inequality to (2.11), we have

$$|(W_n f,\ g)| \leqq C_1(C\|V\|_{L^{m/2}})^n\|f\|_{L^2}\|g\|_{L^2}.$$

This implies statement (1).

(2) Let A_j, B_j and $Q_{0j}(z)$ be respectively A, B and $Q_0(z)$ corresponding to $V_j(x)$. We have

$$\|A_j-A\|_{L^m}\leqq\|V_j-V\|_{L^{m/2}}^{1/2}, \qquad \|B_j-B\|_{L^m}\leqq 2\|V_j-V\|_{L^{m/2}}^{1/2}$$

and (2.1) and (2.2) imply that, as $j\to\infty$, $\|Q_{j0}(\lambda-i0)-Q_0(\lambda-i0)\|_{B(L^2(R^m))}\to 0$ uniformly in $\lambda\in R$, and the operator $f\to A_jR_0(\lambda+i0)f$ and $g\to B_jR_0(\lambda+i0)g$ converge respectively to $f\to AR_0(\lambda+i0)f$ and $g\to BR_0(\lambda+i0)g$ in $\boldsymbol{B}(L^2(\boldsymbol{R}^m),\ L^2(\boldsymbol{R},\ L^2(\boldsymbol{R}^m),\ d\lambda))$. Thus statement (2) follows from (2.11).

(3) Using (2.1), (2.2), (2.5) and Schwarz inequality, we estimate:

$$\begin{aligned}\left|\left(\left(W_+-\sum_{n=0}^{N}(-1)^nW_n\right)f,\ g\right)\right| &= |(L_Nf,\ g)|\\ &\leqq\frac{1}{2\pi}\left|\int_0^{\infty}(Q_0(\lambda-i0)^NA\{R_0(\lambda+i0)-R_0(\lambda-i0)\}f,\ BR(\lambda+i0)g)d\lambda\right|\\ &\leqq C_1(C_2\|V\|_{L^{m/2}})^{N+1}\|f\|_{L^2}\|g\|_{L^2}.\end{aligned}$$

Thus $\|W_+-\sum_{n=0}^{N}(-1)^nW_n\|_{B(L^2)}\leqq C_1(C_2\|V\|_{L^{m/2}})^{N+1}\to 0$ as $N\to\infty$, if $C_2\|V\|_{L^{m/2}}<1$. (Q.E.D.)

We now proceed to the proof of Theorem 1.1 in the case that $l=0$ and $\|\mathcal{F}(\langle x\rangle^{\sigma}V)\|_{L^{m_*}}$ is small. We wish to show that the series $\sum_{n=0}^{\infty}(-1)^nW_n$ converges in the norm of $\boldsymbol{B}(L^p)$ for any $1\leqq p\leqq\infty$. For this, it suffices to prove the following

THEOREM 2.1. *There exist constants C_1 and C_2 such that for any $1\leqq p\leqq\infty$,*

$$\|W_n f\|_{L^p} \leqq C_1(C_2\|\mathcal{F}(\langle x\rangle^\sigma V)\|_{L^{m_*}})^n\|f\|_{L^p}, \qquad f \in L^2 \cap L^p. \tag{2.12}$$

As it will slightly simplify the notation, we prove (2.12) for the adjoint operator W_n^* though entirely similar argument works for W_n as well. For proving Theorem 2.1, it suffices to show the following

PROPOSITION 2.2. *Let $V_1, \cdots, V_n$ be such that $\hat{V}_1, \cdots, \hat{V}_n \in C_0^\infty$, A_j and B_j be the operators A and B corresponding to V_j and $Q_1(z)=I$, $Q_j(z)=A_jR_0(z)B_{j-1}^*$, $j=2, \cdots, n$. Let Z_n be defined for $f, g \in \mathcal{S}$, the space of rapidly decreasing functions, by*

$$(Z_n f, g) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\Big(\Big(\prod_{j=2}^{n} Q_j(\lambda-i0)^*\Big)B_nR_0(\lambda+i0)f,\ A_1R_0(\lambda+i0)g\Big)d\lambda \tag{2.13}$$

where the product should be taken from the left to the right. Then, for any $1\leqq p\leqq\infty$, Z_n can be extended to a bounded operator in L^p and it satisfies the following estimate with constants C_1 and C_2 independent of f and V_j, $j=1, \cdots, n$:

$$\|Z_n f\|_{L^p} \leqq C_1\Big(\prod_{j=1}^{n} C_2\|\mathcal{F}(\langle x\rangle^\sigma V_j)\|_{L^{m_*}}\Big)\|f\|_{L^p}, \qquad f \in L^p. \tag{2.14}$$

To see that Proposition 2.2 indeed implies Theorem 2.1, we note that $\mathcal{F}(\langle x\rangle^\sigma V)\in L^{m*}$ is equivalent to $\hat{V}\in H^\sigma_{m*}$, the generalized Sobolev space (cf. [1]) of order σ, and C_0^∞ is dense in H^σ_{m*}. Take a sequence $V_j \in \mathcal{S}$ such that $\hat{V}_j\in C_0^\infty$ and $\|\mathcal{F}(\langle x\rangle^\sigma(V-V_j))\|_{L^{m_*}}\to 0$ as $j\to\infty$, and define $W_n^{(j)}$ by (2.11) with V_j replacing V. In virtue of Proposition 2.2, $\|W_n^{(j)}-W_n\|_{\boldsymbol{B}(L^2)}\to 0$. On the other hand, (2.14) implies that $W_n^{(j)}$ is convergent in the norm of $\boldsymbol{B}(L^p)$ as $j\to\infty$. It follows that W_n is in fact bounded in L^p and (2.12) is satisfied.

For proving Proposition 2.2 we prepare two lemmas. We define the integral operator $T_{k,\varepsilon}$ depending on the parameters $k\in\boldsymbol{R}^m\setminus\{0\}$ and $\varepsilon>0$ by

$$T_{k,\varepsilon}f(x) = \mathcal{F}^*\Big(\frac{\hat{f}(\xi)}{k^2-2k\cdot\xi-i\varepsilon}\Big)(x) = \frac{i}{2|k|}\int_0^\infty e^{-it(|k|-i\varepsilon|k|^{-1})/2}f(x+t\omega)dt, \tag{2.15}$$

where $\omega=k/|k|\in\Sigma$, Σ being the unit sphere, and

$$T_{k,0}f(x) = T_kf(x) = \frac{i}{2|k|}\int_0^\infty e^{-it|k|/2}f(x+t\omega)dt. \tag{2.16}$$

LEMMA 2.3. *For $\varepsilon\geqq 0$ and $f(k, x)\in C_0^\infty(\boldsymbol{R}_k^m, L^p(\boldsymbol{R}_x^m))$ define*

$$G_\varepsilon f = \int_{\boldsymbol{R}^m} T_{k,\varepsilon}f(k, \cdot)dk, \qquad \tilde{G}_\varepsilon f = \int_{\boldsymbol{R}^m} T_{k,\varepsilon}f_k(k, \cdot)dk, \tag{2.17}$$

where $f_k(k, x)=e^{ik\cdot x}f(k, x)$. Then there exists a constant $C>0$ independent of f and $\varepsilon>0$ such that

$$\|G_\varepsilon f\|_{L^p(\boldsymbol{R}^m)} \leqq C\sum_{|\alpha|\leqq 2}\int_{\boldsymbol{R}^m}\langle k\rangle^{m-2}\|D_k^\alpha f(k, \cdot)\|_{L^p(\boldsymbol{R}^m)}\frac{dk}{|k|^{m-1}}, \tag{2.18}$$

$$\|\tilde{G}_\varepsilon f\|_{L^p(\boldsymbol{R}^m)} \leqq C \sum_{|\alpha|\leqq 2} \int_{\boldsymbol{R}^m} \langle k\rangle^{m-2} \|D_k^\alpha f(k, \cdot)\|_{L^p(\boldsymbol{R}^m)} \frac{dk}{|k|^{m-1}}. \tag{2.19}$$

As $\varepsilon \to 0$, $G_\varepsilon f$ *and* $\tilde{G}_\varepsilon f$ *respectively converge to* $G_0 f$ *and* $\tilde{G}_0 f$ *in* $L^p(\boldsymbol{R}^m)$.

PROOF. We prove the lemma for G_ε first. Using the polar coordinates $k = s\omega$ and changing the order of integration, we write

$$G_\varepsilon f(x) = \frac{i}{2} \int_\Sigma d\omega \int_0^\infty dt \Big(\int_0^\infty e^{-it(s-i\varepsilon s^{-1})/2} s^{m-2} f(s\omega, x+t\omega) ds \Big). \tag{2.20}$$

We estimate the inner most integral

$$\tilde{f}_\varepsilon(t, x, \omega) \equiv \int_0^\infty e^{-it(s-i\varepsilon s^{-1})/2} s^{m-2} f(s\omega, x+t\omega) ds.$$

Integrating by parts twice using the identity

$$2(-it(1+i\varepsilon/s^2))^{-1} (\partial/\partial s) e^{-it(s-i\varepsilon s^{-1})/2} = e^{-it(s-i\varepsilon s^{-1})/2}, \tag{2.21}$$

we obtain

$$\tilde{f}_\varepsilon(t, x, \omega) = \sum_{j=0}^{2} t^{-2} \int_0^\infty e^{-it(s-i\varepsilon s^{-1})/2} g_j(s, \varepsilon) (\partial/\partial s)^j f(s\omega, x+t\omega) ds. \tag{2.22}$$

Here $g_j(s, \varepsilon)$ satisfies $|g_j(s, \varepsilon)| \leqq C s^{m-4+j}$, $j = 0, 1, 2$, $m \geqq 3$; and when $m=3$, $g_0(s, \varepsilon)$ can be written in the form $g_0(s, \varepsilon) = g_{01}(\varepsilon/s^2)(1/s)$ with $g_{01}(1/s^2)(1/s)$ integrable on $(0, \infty)$. Hence, for $m \geqq 4$ we clearly have

$$|\tilde{f}_\varepsilon(t, x, \omega)| \leqq C \langle t\rangle^{-2} \int_0^\infty \sum_{j=0}^{2} \langle s\rangle^{m-2} |(\partial/\partial s)^j f(s\omega, x+t\omega)| ds, \qquad t \in \boldsymbol{R}^1. \tag{2.23}$$

When $m=3$, the summands in (2.22) with $j=1, 2$ can be estimated as above and the one with $j=0$ as

$$\begin{aligned} \Big| \int_0^\infty e^{-it(s-i\varepsilon s^{-1})/2} g_{01}(\varepsilon/s^2)(1/s) f(s\omega, x) ds \Big| &\leqq \int_0^\infty |g_{01}(1/s^2)(1/s) f(\sqrt{\varepsilon} s\omega, x)| ds \\ &\leqq \sup_s |f(s\omega, x)| \int_0^\infty |g_{01}(1/s^2)(1/s)| ds \leqq C \int_0^\infty |(\partial f/\partial s)(s\omega, x)| ds. \end{aligned} \tag{2.24}$$

Thus (2.23) holds for $m=3$ as well. Take the $L^p(\boldsymbol{R}_x^m)$ norm in both side of (2.20) and apply the Minkowski inequality. We have in virtue of (2.23) that

$$\begin{aligned} \|G_\varepsilon f\|_{L^p} &\leqq C \int_\Sigma d\omega \int_0^\infty dt \langle t\rangle^{-2} \sum_{j=0}^{2} \int_0^\infty \langle s\rangle^{m-2} \|(\partial/\partial s)^j f(s\omega, \cdot)\|_{L^p} ds \\ &\leqq C \sum_{j=0}^{2} \int_\Sigma d\omega \int_0^\infty \langle s\rangle^{m-2} \|(\partial/\partial s)^j f(s\omega, \cdot)\|_{L^p} ds \end{aligned} \tag{2.25}$$

which is bounded by the RHS of (2.18). $G_\varepsilon f(x)$ obviously converges to $G_0 f(x)$ for every fixed x and $G_\varepsilon f(x)$ is bounded by the integral of the RHS of (2.23) by the variables (t, ω) which is just shown to be in $L^p(\boldsymbol{R}_x^m)$ and is independent of ε. Hence, by the Lebesgue dominated convergence theorem, $G_\varepsilon f$ converges to $G_0 f$ in L^p. This completes the proof for $G_\varepsilon f$.

The proof for $\tilde{G}_\varepsilon f$ is essentially the same. As we did for $G_\varepsilon f(x)$, we write it in the form

$$\tilde{G}_\varepsilon f(x) = \frac{i}{2}\int_\Sigma d\omega \int_0^\infty dt\Big(\int_0^\infty e^{its/2-\varepsilon t/2s+isx\cdot\omega} s^{m-2} f(s\omega,\ x+t\omega)ds\Big), \tag{2.26}$$

and estimate the s-integral

$$f_\varepsilon^*(t,\ x,\ \omega) \equiv \int_0^\infty e^{its/2-\varepsilon t/2s+isx\cdot\omega} s^{m-2} f(s\omega,\ x+t\omega)ds$$

via integrations by parts. It is obvious that for any $(t,\ x,\ \omega)$

$$|f_\varepsilon^*(t,\ x,\ \omega)| \leqq \int_0^\infty \langle s\rangle^{m-2} |f(s\omega,\ x+t\omega)|\,ds\,. \tag{2.27}$$

When $|t+2x\cdot\omega| \geqq 1$, we estimate it by integrating by parts twice using the identity

$$\frac{2}{i(t+2x\cdot\omega)(1+i\varepsilon'/s^2)}(\partial/\partial s)e^{its/2-\varepsilon t/2s+isx\cdot\omega} = e^{its/2-\varepsilon t/2s+isx\cdot\omega}, \tag{2.28}$$

where $\varepsilon' = -\varepsilon t/(t+2x\cdot\omega)$, which has the same form as (2.21) except that ε and t are replaced by ε' and $-(t+2x\cdot\omega)$, respectively. Thus the argument used for proving (2.23) and (2.24) gives

$$|f_\varepsilon^*(t,\ x,\ \omega)| \leqq C\langle t+2x\cdot\omega\rangle^{-2}\int_0^\infty \sum_{j=0}^{2} \langle s\rangle^{m-2} |(\partial/\partial s)^j f(s\omega,\ x+t\omega)|\,ds\,. \tag{2.29}$$

We integrate both sides of (2.29) by the variable $t\in(0,\ \infty)$. Changing t by $t-2x\cdot\omega$ and writing $\bar{x}_\omega$ for the reflection $x-2(x\cdot\omega)\omega$ of x along the ω axis, we see that right hand side becomes

$$C\int_{2x\cdot\omega}^\infty \langle t\rangle^{-2}\int_0^\infty \sum_{j=0}^{2} \langle s\rangle^{m-2} |(\partial/\partial s)^j f(s\omega,\ \bar{x}_\omega+t\omega)|\,ds.$$

Extending the region of integration by t to the whole line and using Minkowski's inequality, we have

$$\Big\|\int_0^\infty |f_\varepsilon^*(t,\ x,\ \omega)|\,dt\Big\|_{L^p(\boldsymbol{R}_x^m)} \leqq C\int_{-\infty}^\infty \sum_{j=0}^{2} \langle s\rangle^{m-2} \|(\partial/\partial s)^j f(s\omega,\ x)\|_{L^p(\boldsymbol{R}_x^m)}ds\,. \tag{2.30}$$

Integration of the last inequality by ω yields the desired estimate for $\tilde{G}_\varepsilon f$. The convergence of $\tilde{G}_\varepsilon f$ to $\tilde{G}_0 f$ can be proved exactly in the same way as of $G_\varepsilon f$. (Q.E.D.)

Setting $\varepsilon = 0$ in (2.20) and (2.26), we obtain the following.

LEMMA 2.4. *For $L^p(\boldsymbol{R}_x^m)$-valued $C_0^\infty(\boldsymbol{R}_k^m)$ function $f(k,\ x)$, set*

$$\hat{f}(t,\ \omega,\ x) = \frac{i}{2}\int_0^\infty e^{-its/2} f(s\omega,\ x)s^{m-2}ds, \qquad t\in\boldsymbol{R},\quad \omega\in\Sigma \tag{2.31}$$

and $f_k(k, x)=e^{ik\cdot x}f(k, x)$. *Then*:

$$G_0f(x)=\int_{\boldsymbol{R}^m}T_kf(k, x)dk=\int_{\Sigma}d\omega\left\{\int_0^{\infty}\hat{f}(t, \omega, x+t\omega)dt\right\}, \tag{2.32}$$

$$\tilde{G}_0f(x)=\int_{\boldsymbol{R}^m}T_kf_k(k, x)dk=\int_{\Sigma}d\omega\left\{\int_{-\infty}^{-2x\cdot\omega}\hat{f}(t, \omega, \bar{x}_\omega-t\omega)dt\right\}, \tag{2.33}$$

where $\bar{x}_\omega=x-(2x\cdot\omega)\omega$ *is the reflection of* x *along the* ω *axis.*

PROOF OF PROPOSITION 2.2. In virtue of Lemma 2.1, Z_n is bounded in L^2 and

$$Z_nf=\lim_{\varepsilon_1\downarrow 0}\cdots\lim_{\varepsilon_n\downarrow 0}\lim_{\varepsilon_0\downarrow 0}\frac{1}{2\pi i}\int_{-\infty}^{\infty}R_0(\lambda-i\varepsilon_0)\left(\prod_{j=1}^{n}V_jR_0(\lambda+i\varepsilon_j)\right)fd\lambda, \tag{2.34}$$

as a weak limit. Let $f\in\mathcal{S}$. As $\hat{V}_j\in C_0^\infty$, $j=1\sim n$, the Fourier transform of the function under the limit signs is clearly in $\mathcal{S}$ and can be written as

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\left\{(2\pi)^{-nm/2}\int_{\boldsymbol{R}^{mn}}\frac{\prod_{j=1}^{n}\hat{V}_j(k_j)\hat{f}(\xi-\sum_{j=1}^{n}k_j)dk_1\cdots dk_n}{\{\xi^2-\lambda+i\varepsilon_0\}\prod_{j=1}^{n}\{(\xi-\sum_{l=1}^{j}k_l)^2-\lambda-i\varepsilon_j\}}\right\}d\lambda$$
$$=(2\pi)^{-mn/2}\int_{\boldsymbol{R}^{mn}}\frac{\{\prod_{j=1}^{n}\hat{V}_j(k_j-k_{j-1})\}\hat{f}(\xi-k_n)dk_1\cdots dk_n}{\prod_{j=1}^{n}\{k_j^2-2k_j\cdot\xi-i(\varepsilon_j+\varepsilon_0)\}}.$$

Here we changed the variables $(k_1, \cdots, k_n)$ by $(k_1-k_0, \cdots, k_n-k_{n-1})$, $k_0=0$ and performed the λ-integration using the residue theorem. After taking the limit $\varepsilon_0\to 0$, we obtain

$$\mathcal{F}Z_nf(\xi)=\lim_{\varepsilon_1\downarrow 0}\cdots\lim_{\varepsilon_n\downarrow 0}(2\pi)^{-mn/2}\int_{\boldsymbol{R}^{mn}}\frac{\{\prod_{j=1}^{n}\hat{V}_j(k_j-k_{j-1})\}\hat{f}(\xi-k_n)dk_1\cdots dk_n}{\prod_{j=1}^{n}\{k_j^2-2k_j\cdot\xi-i\varepsilon_j\}}. \tag{2.35}$$

Note here that the mapping

$$f\longrightarrow(2\pi)^{-mn/2}\int_{\boldsymbol{R}^{mn}}\frac{\{\prod_{j=1}^{n}\hat{V}_j(k_j-k_{j-1})\}\hat{f}(\xi-k_n)dk_1\cdots dk_n}{\prod_{j=1}^{n}\{k_j^2-2k_j\cdot\xi-i\varepsilon_j\}},$$

is not only continuous in $\mathcal{S}$ but also continuous in $\mathcal{S}'$. Hence $Z_nf\in\mathcal{S}'$ can be defined for $f\in\mathcal{S}'$ as long as the limit (2.35) exists in $\mathcal{S}'$. We show that the limit does exist in the strong topology of $L^p(\boldsymbol{R}^m)$ for any $f\in L^p(\boldsymbol{R}^m)$.

Let $f\in L^p(\boldsymbol{R}^m)$ and set $\tilde{K}_n(k_1, \cdots, k_n)=(2\pi)^{-mn/2}\prod_{j=1}^{n}\hat{V}_j(k_j-k_{j-1})$. Applying the Fourier inversion formula to (2.35) and using (2.15), we see that Z_nf is the $\lim_{\varepsilon_1\downarrow 0}\cdots\lim_{\varepsilon_n\downarrow 0}$ of

$$\int_{\boldsymbol{R}^m}T_{k_1,\varepsilon_1}\left\{\int_{\boldsymbol{R}^m}T_{k_2,\varepsilon_2}\left\{\cdots\left\{\int_{\boldsymbol{R}^m}\tilde{K}_n(k_1, \cdots, k_n)T_{k_n,\varepsilon_n}f_{k_n}dk_n\right\}\cdots\right\}dk_2\right\}dk_1, \tag{2.36}$$

where $f_k(x)=e^{ik\cdot x}f(x)$. Since $\hat{V}_j\in C_0^\infty(\boldsymbol{R}^m)$ by the assumption, $\tilde{K}_n(k_1, \cdots, k_n)f(x)$ is an $L^p(\boldsymbol{R}_x^m)$-valued $C_0^\infty(\boldsymbol{R}^{mn})$ function of $(k_1, \cdots, k_n)$. It follows from the $\tilde{G}_\varepsilon$ part of Lemma 2.3 that as $\varepsilon_n\to 0$

$$\int_{\boldsymbol{R}^m}\tilde{K}_n(k_1, \cdots, k_n)T_{k_n, \varepsilon_n}f_{k_n}(x)dk_n$$

converges in the topology of $C_0^\infty(\boldsymbol{R}^{m(n-1)}_{k_1,\cdots,k_{n-1}}, L^p(\boldsymbol{R}^m_x))$ to

$$\int_{\boldsymbol{R}^m}\tilde{K}_n(k_1, \cdots, k_n)T_{k_n}f_{k_n}(x)dk_n \tag{2.37}$$

and, from the G_ε part of Lemma 2.3 that $Z_nf(x)$ is the $\lim_{\varepsilon_1\downarrow 0}\cdots\lim_{\varepsilon_{n-1}\downarrow 0}$ of

$$\int_{\boldsymbol{R}^m}T_{k_1,\varepsilon_1}\Big\{\cdots\Big\{\int_{\boldsymbol{R}^m}T_{k_{n-1},\varepsilon_{n-1}}\Big\{\int_{\boldsymbol{R}^m}\tilde{K}_n(k_1, \cdots, k_n)T_{k_n}f_{k_n}(x)dk_n\Big\}dk_{n-1}\Big\}\cdots\Big\}dk_1.$$

Then, the repeated use of the G_ε part of Lemma 2.3 implies that $Z_nf\in L^p$ and Z_n can be expressed as the superposition of $T_{k_1}\cdots T_{k_n}M_{e^{ix\cdot k_n}}$ over $(k_1, \cdots, k_n)\in \boldsymbol{R}^{nm}$, $M_{e^{ix\cdot k_n}}$ being the multiplication operator by $e^{ix\cdot k_n}$:

$$Z_nf(x)=\int_{\boldsymbol{R}^m}T_{k_1}\Big\{\int_{\boldsymbol{R}^m}T_{k_2}\Big\{\cdots\Big\{\int_{\boldsymbol{R}^m}\tilde{K}_n(k_1, \cdots, k_n)T_{k_n}f_{k_n}dk_n\Big\}\cdots\Big\}dk_2\Big\}dk_1. \tag{2.38}$$

We compute the last integral using Lemma 2.4. By (2.33) we have

$$\int_{\boldsymbol{R}^m}\tilde{K}_n(k_1, \cdots, k_n)T_{k_n}f_{k_n}dk_n$$
$$=\int_\Sigma d\omega_n\int_{-\infty}^{-2x\cdot\omega_n}F_n(k_1, \cdots, k_{n-1}, t_n, \omega_n)f(\bar{x}-t_n\omega_n)dt_n,$$

where $\bar{y}=y-2(y\cdot\omega_n)$ is the reflection of y along the ω_n axis and

$$F_n(k_1, \cdots, k_{n-1}, t_n, \omega_n)=\frac{i}{2}\int_0^\infty e^{-it_ns_n/2}\tilde{K}_n(k_1, \cdots, k_{n-1}, s_n\omega_n)s_n^{m-2}ds_n.$$

It is clear that $F_n\in C_0^\infty(\boldsymbol{R}^{(n-1)m}, C^\infty(\boldsymbol{R}^1\times\Sigma))\cap L^1(\boldsymbol{R}^{(n-1)m}\times\boldsymbol{R}^1\times\Sigma)$ and (2.38) is a C_0^∞ function of $(k_1, \cdots, k_{n-1})$ with values in $L^p(\boldsymbol{R}^m_x)$. Next we apply (2.32) to (2.38) to compute

$$\int_{\boldsymbol{R}^m}T_{k_{n-1}}\Big\{\int_{\boldsymbol{R}^m}\tilde{K}_n(k_1, \cdots, k_n)T_{k_n}f_{k_n}dk_n\Big\}dk_{n-1}.$$

Setting $F_{n-1}(k_1, \cdots, k_{n-2}, t_{n-1}, t_n, \omega_{n-1}, \omega_n)$ by

$$F_{n-1}=\Big(\frac{i}{2}\Big)^2\int_{[0,\infty)^2}e^{-i(t_{n-1}s_{n-1}+t_ns_n)/2}$$
$$\times\tilde{K}_n(k_1, \cdots, k_{n-2}, s_{n-1}\omega_{n-1}, s_n\omega_n)(s_{n-1}s_n)^{m-2}ds_{n-1}ds_n,$$

we see it can be written as

$$\int_{\Sigma^2}\int_0^\infty\int_{-\infty}^{\sigma_2}F_{n-1}(k_1, \cdots, k_{n-2}, t_{n-1}, t_n, \omega_{n-1}, \omega_n)$$
$$\times f(\bar{x}+t_{n-1}\bar{\omega}_{n-1}-t_n\omega_n)d\omega_{n-1}d\omega_ndt_{n-1}dt_n,$$

where $\sigma_2=-2(x+t\omega_{n-1})\cdot\omega_n$. Repeating this procedure $n-2$ more times, we finally arrive at the expression (1.28):

$$Z_n f(x) = \int_{[0,\infty)^{n-1}\times I\times\Sigma^n} \hat{K}_n(t_1, \cdots, t_{n-1}, \tau, \omega_1, \cdots, \omega_n) \times f(\bar{x}+\rho)dt_1\cdots dt_{n-1}d\tau d\omega_1\cdots d\omega_n . \tag{2.39}$$

Here $K_n(k_1, \cdots, k_n) = i^n(2\pi)^{-nm/2}2^{-n}\prod_{j=1}^n \hat{V}(k_j-k_{j-1})$, $k_0=0$ and

$$\hat{K}_n(t_1, \cdots, t_n, \omega_1, \cdots, \omega_n) = \int_{[0,\infty)^n} e^{-i\Sigma_{j=1}^n t_j s_j/2}(s_1\cdots s_n)^{m-2}K(s_1\omega_1, \cdots, s_n\omega_n)ds_1\cdots ds_n, \tag{2.40}$$

$\rho=t_1\bar{\omega}_1+\cdots+t_{n-1}\bar{\omega}_{n-1}-\tau\omega_n$, and $I=(-\infty, -\sigma)$, $\sigma=2\omega_n(x+t_1\omega_1+\cdots+t_{n-1}\omega_{n-1})$, is the range of the integration by the variable τ. Taking the absolute value in both sides of (2.39) and extending the region of integration by the variable τ to the whole line lead to

$$|Z_n f(x)| \leqq \int_{[0,\infty)^{n-1}\times R^1\times\Sigma^n} |\hat{K}_n(t_1, \cdots, t_{n-1}, \tau, \omega_1, \cdots, \omega_n) \times f(\bar{x}+\rho)|dt_1 \cdots dt_{n-1}d\tau d\omega_1 \cdots d\omega_n .$$

Note that $x\to\bar{x}$ is an isometry and ρ is independent of x. We apply Minkowski's inequality to the last inequality and obtain for any $1\leqq p\leqq\infty$ that

$$\|Z_n f\|_{L^p} \leqq \|\hat{K}_n\|_{L^1([0,\infty)^{n-1}\times R^1\times\Sigma^n)}\|f\|_{L^p} \leqq 2\|\hat{K}_n\|_{L^1([0,\infty)^n\times\Sigma^n)}\|f\|_{L^p} .$$

Thus the proof of Proposition 2.2 will be completed by the following lemma.

LEMMA 2.5. *Let $\sigma>2/m_*$. Then, there exists a constant C_2 such that*

$$\|\hat{K}_n\|_{L^1([0,\infty)^n, L^1(\Sigma^n))} \leqq (C_2\|\mathcal{F}(\langle x\rangle^\sigma V)\|_{L^{m_*}})^n, \qquad n=1, 2, \cdots . \tag{2.41}$$

PROOF. As was remarked above, $\mathcal{F}(\langle x\rangle^\sigma V)\in L^{m_*}$ if and only if $\hat{V}\in H^\sigma_{m_*}$, the generalized Sobolev space. It is easy to see that

$$\|K_n\|_{L^{m_*}(R^{nm})} \leqq 2^{-n}(2\pi)^{mn/2}\prod_{j=1}^n \|\hat{V}_j\|_{L^{m_*}} \tag{2.42}$$

and, for $|\alpha_j|\leqq 1$,

$$\|(\partial/\partial k_1)^{\alpha_1}\cdots(\partial/\partial k_n)^{\alpha_n}K_n\|_{L^{m_*}(R^{nm})} \leqq (2\pi)^{mn/2}\prod_{j=1}^n \|\hat{V}_j\|_{W^{2,m_*}} . \tag{2.43}$$

By Hardy's inequality, we also have for $a_j=0, 1$, $j=1, \cdots, n$, that

$$\||k_1|^{-a_1}\cdots|k_n|^{-a_n}K_n\|_{L^{m_*}(R^{nm})} \leqq C^n \sum_{|\alpha_1|\leqq a_1,\cdots,|\alpha_n|\leqq a_n} \|(\partial/\partial k_1)^{\alpha_1}\cdots(\partial/\partial k_n)^{\alpha_n}K_n\|_{L^{m_*}(R^{nm})} . \tag{2.44}$$

On the other hand, we have by Hausdorff-Young's inequality,

$$\left(\int_{[0,\infty)^n}|\hat{K}_n(t_1, \cdots, t_n, \omega_1, \cdots, \omega_n)|^{m-1}dt_1\cdots dt_n\right)^{1/(m-1)}$$
$$\leqq C^n\left(\int_{[0,\infty)^n}|K_n(s_1\omega_1, \cdots, s_n\omega_n)|^{m_*}(s_1\cdots s_n)^{m-1}ds_1\cdots ds_n\right)^{1/m_*}.$$

Integrating both sides of the inequality above by $(\omega_1, \cdots, \omega_n)\in\Sigma^n$ and using the Hölder's inequality, we obtain by (2.42) that

$$\|\hat{K}_n\|_{L^1(\Sigma^n, L^{m-1}([0,\infty)^n))} \leqq C^n\|K_n\|_{L^{m_*}(R^{mn})} \leqq C_1^n\prod_{j=1}^n\|\hat{V}_j\|_{L^{m_*}}. \tag{2.45}$$

Integrating by parts by the variables s_j in the defining equation (2.40) of $\hat{K}_n$ and arguing similarly as above using (2.43) and (2.44), we also obtain

$$\left\|\left(\prod_{j=1}^n\langle t_j\rangle\right)\hat{K}_n\right\|_{L^1(\Sigma^n, L^{m-1}([0,\infty)^n))} \leqq C_3^n\prod_{j=1}^n\|\hat{V}_j\|_{W^{2,m_*}}. \tag{2.46}$$

It follows from (2.45), (2.46) and the multi-linear complex interpolation theorem ([**4**], p. 96) that

$$\left\|\left(\prod_{j=1}^n\langle t_j\rangle^{\sigma/2}\right)\hat{K}_n\right\|_{L^1(\Sigma^n, L^{m-1}([0,\infty)^n))} \leqq C_4^n\prod_{j=1}^n\|\hat{V}_j\|_{H^\sigma_{m_*}(R^m)} = C_4^n\prod_{j=1}^n\|\mathcal{F}(\langle x\rangle^\sigma V_j)\|_{L^{m_*}}.$$

The estimate (2.41) follows by combining this inequality with the following

$$\|\hat{K}_n\|_{L^1(\Sigma^n, L^1([0,\infty)^n))} \leqq C_5^n\left\|\left(\prod_{j=1}^n\langle t_j\rangle^{\sigma/2}\right)\hat{K}_n\right\|_{L^1(\Sigma^n, L^{m-1}([0,\infty)^n))},$$

which is a consequence of Hölder's inequality since $\sigma>2/m_*$. (Q.E.D.)

3. L^p boundedness of W_+ for large potentials V.

In this section, we prove that W_+ is bounded in L^p assuming that V satisfies Assumption 1.1 (2) (with $l=0$) and 0 is neither resonance nor eigenvalue of H, in particular $m=2m'-1$ is odd. As in the previous section, we begin with the representation formula (2.6). Here we rewrite it in a way slightly different from the one in the previous section: We replace $R(\lambda-i\varepsilon)B^*$ in (2.7) by the right hand side of the identity:

$$R(z)B^* = R_0(z)B^*\sum_{j=0}^{m-1}(-1)^jQ_0(z)^j+(-1)^mR_0(z)B^*Q_0(z)^{m'-1}Q(z)Q_0(z)^{m'-1},$$
$$z=\lambda-i\varepsilon,$$

which produces:

$$(W_+f, g) = \sum_{n=0}^m(-1)^n(W_nf, g)+(-1)^{m+1}(L_mf, g), \tag{3.1}$$

where W_0 is the identity operator, W_n, $n=1, \cdots, m$, are the same as in the previous section (hence are bounded in $L^p(\boldsymbol{R}^m)$) and

$$(L_m f, g) = \frac{1}{2\pi i}\int_0^\infty (F(\lambda-i0)A\{R_0(\lambda+i0)-R_0(\lambda-i0)\}f,\ BR_0(\lambda+i0)g)d\lambda, \quad (3.2)$$

where $F(z)=A(R_0(z)V)^{m'-1}R(z)(VR_0(z))^{m'-1}B^*$. The operator $L=L_m$ defined by (3.2) is of course bounded in $L^2(\boldsymbol{R}^m)$. Changing the variable λ by k^2, we write L in the form:

$$L = \frac{1}{\pi i}\int_0^\infty R_0(k^2-i0)VN(k)V\{R_0(k^2+i0)-R_0(k^2-i0)\}kdk, \quad (3.3)$$

where $N(k)=N_{m'-1}(k)=\{R_0(k^2-i0)V\}^{m'-1}R(k^2-i0)\{VR_0(k^2-i0)\}^{m'-1}$. We prove that L is bounded in $L^p(\boldsymbol{R}^m)$ by showing that its integral kernel $L(x, y)$ satisfies the well known criterion for the L^p-boundedness:

$$\sup_{y\in\boldsymbol{R}^m}\int_{\boldsymbol{R}^m}|L(x, y)|dx < \infty, \qquad \sup_{x\in\boldsymbol{R}^m}\int_{\boldsymbol{R}^m}|L(x, y)|dy < \infty. \quad (3.4)$$

As is well known ([**23**]), $R_0(k^2\pm i0)$ is the convolution with ($j=1$ for $+$ and $j=2$ for $-$):

$$G_\pm(x, k) = \frac{\pm i}{4}\left(\frac{k}{2\pi|x|}\right)^\nu H_\nu^{(j)}(k|x|) = \frac{\pm i}{4(2\pi)^\nu|x|^{m-2}}(k|x|)^\nu H_\nu^{(j)}(k|x|),$$
$$\nu = (m-2)/2,$$

where $H_\nu^{(j)}(r)$ is the Hankel function of j-th kind. Expanding $(z\pm(it/2))^{\nu-1/2}$ by the binomial theorem in Hankel's formula ([**27**]):

$$z^\nu H_\nu^{(j)}(z) = \frac{\sqrt{2}e^{\pm i(z-(2\nu+1)\pi/4)}}{\sqrt{\pi}\Gamma(\nu+(1/2))}\int_0^\infty e^{-t}t^{\nu-1/2}\left(z\pm\frac{it}{2}\right)^{\nu-1/2}dt, \quad (3.5)$$

we see that $z^\nu H_\nu^{(j)}(z)$ is $e^{\pm iz}$ times a polynomial of order $m'-2$ for odd $m=2m'-1$ and

$$G_\pm(x, k) = \frac{e^{\pm ik|x|}}{|x|^{m-2}}\sum_{j=0}^{m'-2}(\pm 1)^j C_j(k|x|)^j \quad (3.6)$$

where the constants C_j are independent of the signs $\pm$. We write $G_{\pm,x,k}(y)=G_\pm(x-y, k)$.

In virtue of (3.3), the distribution kernel $L(x, y)$ of L is at least formally given by

$$L(x, y) = \frac{1}{\pi i}\int_0^\infty (N(k)V(G_{+,y,k}-G_{-,y,k}),\ VG_{+,x,k})kdk, \quad (3.7)$$

where $(\cdot, \cdot)$ should be considered as a coupling between suitable function spaces. We denote the integrand of (3.7) by $T(x, y, k)=(N(k)V(G_{+,y,k}-G_{-,y,k}), VG_{+,x,k})$. For making its oscillation property explicit, we set $G_{\pm,x,k}=e^{\pm ik|x|}\tilde{G}_{\pm,x,k}$, that is,

$$\tilde{G}_{\pm,x,k}(y) = \sum_{j=0}^{m'-2}(\pm 1)^j C_j\frac{k^j e^{\pm ik\phi(x,y)}}{|x-y|^{m-2-j}}, \qquad \phi(x, y) = |x-y|-|x|, \quad (3.8)$$

and define

$$T_{\pm}(x, y, k)=(N(k)V\tilde{G}_{\pm,y,k}, V\tilde{G}_{+,x,k}) \tag{3.9}$$

and

$$L_{\pm}(x, y)=\int_{0}^{\infty}e^{-ik(|x|\mp|y|)}T_{\pm}(x, y, k)kdk \tag{3.10}$$

so that $L(x, y)=(L_{+}(x, y)-L_{-}(x, y))/i\pi$.

In what follows we write d/dk for the partial derivative w.r.t. k.

For showing that (3.7) is well defined and for estimating $L_{\pm}(x, y)$, we shall use the following lemmas. The first two are concerned with the mapping property and the decay of the derivatives of the resolvents $R_{0}(k^{2}\pm i0)$ and $R(k^{2}\pm i0)$. The proof can be found in Murata [19], Jensen [9] and Jensen-Kato [8]. Recall that $|V(x)|\leqq C\langle x\rangle^{-\delta}$ with $\delta>\max(m+2, 3m/2-2)$ by (2) of Assumption 1.1 which we are assuming in this section.

LEMMA 3.1. *Let $j=0, 1, \cdots, \gamma, \gamma'>j+1/2$, and $\gamma+\gamma'>2$ when $j=0$. Let $t\in\boldsymbol{R}$ and $0\leqq s\leqq 2$. Then, $\langle x\rangle^{-\gamma}R_{0}(k^{2}\pm i0)\langle x\rangle^{-\gamma'}$ are $\boldsymbol{B}(H^{t}(\boldsymbol{R}^{m}), H^{t+2}(\boldsymbol{R}^{m}))$ valued C^{j} functions of $k\in[0, \infty)$ and satisfy the estimates*

$$\|(d/dk)^{j}\langle x\rangle^{-\gamma}R_{0}(k^{2}\pm i0)\langle x\rangle^{-\gamma'}\|_{B(H^{t}(R^{m}),H^{t+s}(R^{m}))}\leqq C\langle k\rangle^{-1+s}. \tag{3.11}$$

LEMMA 3.2. *Let $0\leqq j\leqq\delta-1, \gamma, \gamma'>j+1/2$, and $\gamma+\gamma'>2$ when $j=0$. Let $-2\leqq t\leqq 0$ and $0\leqq s\leqq 2$. Then, $\langle x\rangle^{-\gamma}R(k^{2}\pm i0)\langle x\rangle^{-\gamma'}$ are $\boldsymbol{B}(H^{t}(\boldsymbol{R}^{m}), H^{t+2}(\boldsymbol{R}^{m}))$ valued C^{j} functions of $k\in[0, \infty)$ and satisfy the estimates*

$$\|(d/dk)^{j}\langle x\rangle^{-\gamma}R(k^{2}\pm i0)\langle x\rangle^{-\gamma'}\|_{B(H^{t}(R^{m}),H^{t+s}(R^{m}))}\leqq C\langle k\rangle^{-1+s}. \tag{3.12}$$

The next lemma is elementary but plays an important role in our theory.

LEMMA 3.3. *Let $1\leqq q<m/(m-t)$, $t>0$ and $\rho>(m/q)-(m-t)$. Then*

$$\left(\int_{R^{m}}\left(\frac{\langle y\rangle^{-\rho}}{|x-y|^{m-t}}\right)^{q}dy\right)^{1/q}\leqq C\begin{cases}\langle x\rangle^{-(m-t)} \text{ if } \rho q>m;\\ \langle x\rangle^{-(m-t)}(\log(1+\langle x\rangle))^{1/q} \text{ if } \rho q=m;\\ \langle x\rangle^{(m/q)-(m-t)-\rho} \text{ if } \rho q<m.\end{cases}$$

PROOF. For $\langle x\rangle\leqq 10$, the lemma is obvious. When $\langle x\rangle\geqq 10$, we split the region of integration into three parts: $\Omega_{1}=\{y:|x-y|<|x|/2\}$, $\Omega_{2}=\{y:|x-y|\geqq|x|/2, |y|>1\}$, $\Omega_{3}=\{y:|x-y|\geqq|x|/2, |y|\leqq 1\}$. The contribution from Ω_{3} is obviously bounded by $C\langle x\rangle^{-(m-t)}$. If we bound $\langle y\rangle^{-\rho}$ by $C\langle x\rangle^{-\rho}$, it is easy to see that the contribution from Ω_{1} is bounded by $C\langle x\rangle^{(m/q)-(m-t)-\rho}$. For estimating the integral over the region Ω_{2}, we bound $\langle y\rangle^{-\rho}$ by $|y|^{-\rho}$, change the variables y by $|x|y$ and then split the region of integration into two parts: $1/|x|\leqq|y|\leqq 1/2$ and $|y|\geqq 1/2$. Then:

$$\int_{\Omega_2}\left(\frac{\langle y\rangle^{-\rho}}{|x-y|^{m-t}}\right)^q dy \leqq C\left(\int_{|y|\geqq 1/2}+\int_{1/|x|\leqq|y|\leqq 1/2}\right)\frac{|x|^{m-(m-t)q-\rho q}dy}{|y|^{\rho q}|\hat{x}-y|^{(m-t)q}}$$
$$\leqq C|x|^{m-(m-t)q-\rho q}\left(1+\int_{1/|x|\leqq|y|\leqq 1/2}\frac{dy}{|y|^{\rho q}}\right).$$

Summing up these three contributions, we obtain the lemma. (Q.E.D.)

In what follows Lemma 3.1, 3.2 and 3.3 will be used in the following forms. We write M_γ for the multiplication operator with the function $\langle x\rangle^{-\gamma}$. Choose and fix $0<\varepsilon<1/2$ such that $\delta>\max(m+2+\varepsilon, 3m/2-2+\varepsilon)$.

LEMMA 3.4. *Let $j=0, 1, \cdots, m'+1$, $\gamma, \gamma'>j+1/2$ and $0\leqq\tau, \tau'\leqq m'-2$. Then $M_\gamma N_{m'-1}(k)M_{\gamma'}$ is a $\boldsymbol{B}(H^{-\tau'}(\boldsymbol{R}^m), H^\tau(\boldsymbol{R}^m))$-valued C^j-function of k and*

$$\|(d/dk)^j M_\gamma N_{m'-1}(k)M_{\gamma'}\|_{B(H^{-\tau'}, H^\tau)} \leqq C\langle k\rangle^{-(m-\tau-\tau')}. \tag{3.13}$$

PROOF. As a prototype, we prove the lemma only for the case that $j\geqq 1$, $m=9\,(m'=5)$ and $m'-3<\tau, \tau'\leqq m'-2$. Differentiating with Leibniz' formula shows that $(d/dk)^j M_\gamma N_{m'-1}(k)M_{\gamma'}$ is a sum over the indices $j_1, \cdots, j_m$ with $j_1+\cdots+j_m=j$ of

$$C_{j_1,\cdots,j_m}M_\gamma R_0^{(j_m)}VR_0^{(j_{m-1})}V\cdots VR^{(j_{m'})}V\cdots VR_0^{(j_1)}M_{\gamma'}, \tag{3.14}$$

where $C_{j_1,\cdots,j_m}$ is a constant and $R_0^{(j)}=(d/dk)^jR_0(k^2-i0)$ and etc.. We let $\kappa=(1+\varepsilon)/2$. Using the assumption $\gamma, \gamma'>j+1/2$ and (3.11) with $s=2$, we estimate the first and last resolvents as follows gaining the smoothness by the order 2:

$$\|M_{j_1+\kappa}R_0^{(j_1)}M_{\gamma'}\|_{B(H^{-\tau'}, H^{2-\tau'})} \leqq C\langle k\rangle,$$
$$\|M_\gamma R_0^{(j_m)}M_{j_m+\kappa}\|_{B(H^{\tau-2}, H^\tau)} \leqq C\langle k\rangle. \tag{3.15}$$

We estimate the second terms from the left and right as follows using the assumption $|\partial_x^\alpha V(x)|\leqq C\langle x\rangle^{-(m+2+\varepsilon)}$, $|\alpha|\leqq 1$ and $m+2+\varepsilon\geqq j_i+j_{i+1}+2\kappa+1$, and the estimate (3.11) of Lemma 3.1 with $0<s=\tau'-2\leqq 1$ (resp. $0<s=\tau-2\leqq 1$):

$$\|M_{j_2+\kappa}R_0^{(j_2)}VM_{j_1+\kappa}^{-1}\|_{B(H^{2-\tau'}, L^2)} \leqq C\langle k\rangle^{\tau'-3},$$
$$\|M_{j_m+\kappa}^{-1}VR_0^{(j_{m-1})}M_{j_{m-1}+\kappa}\|_{B(L^2, H^{\tau-2})} \leqq C\langle k\rangle^{\tau-3}. \tag{3.16}$$

We estimate other factors as $\|M_{j_{i+1}+\kappa}^{-1}VR_0^{(j_i)}M_{j_i+\kappa}\|_{B(L^2)}\leqq C\langle k\rangle^{-1}$, $i=3, \cdots, m'-1, m'+1, \cdots, m-2$, and $\|M_{j_{m'}+\kappa}^{-1}VR^{(j_{m'})}VM_{j_{m'}+\kappa}^{-1}\|\leqq C\langle k\rangle^{-1}$ by using (3.11) and (3.12) with $s=t=0$ which produces

$$\|M_{j_{m-1}+\kappa}^{-1}VR_0^{(j_{m-2})}V\cdots VR^{(j_{m'})}V\cdots VR_0^{(j_3)}VM_{j_2+\kappa}^{-1}\|_{B(L^2)} \leqq C\langle k\rangle^{4-m}. \tag{3.17}$$

Combining (3.15), (3.16) and (3.17), we see that each summand (3.14) is bounded from $H^{-\tau'}(\boldsymbol{R}^m)$ to $H^\tau(\boldsymbol{R}^m)$ and its operator norm is bounded by $C\langle k\rangle^{\tau+\tau'-m}$. This proves (3.13). (Q.E.D.)

We write

$$(\partial/\partial k)^s \tilde{G}_{\pm,x,k}(y) = \sum_{j=0}^{m'-2} (\pm 1)^j C_j F^{\pm}_{s,j,k,x}(y), \tag{3.18}$$

by setting for $j=0, \cdots, m'-2$,

$$F^{\pm}_{s,j,k,x}(y) = \left(\frac{\partial}{\partial k}\right)^s \left(\frac{k^j e^{\pm ik\phi(x,y)}}{|x-y|^{m-2-j}}\right).$$

When $s<j$, we clearly have $F^{\pm}_{s,j,0,x}=0$, and for $s \geqq j$

$$F^{\pm}_{s,j,0,x}(y) = \frac{s!(\pm 1)^{s-j}\phi(x, y)^{s-j}}{(s-j)!|x-y|^{m-2-j}}. \tag{3.19}$$

Recall that $\phi(x, y)=|x-y|-|x|$.

LEMMA 3.5. *Let $0<\varepsilon<1/2$, $m/2<\rho<(m+\varepsilon)/2$ and*

$$0 < \frac{1}{q_j} - \frac{1}{2} - \frac{1}{m}\left(\frac{m}{2} - 2 - j\right) < \frac{\varepsilon}{2m}, \qquad \text{for} \quad 0 \leqq j \leqq m'-3,$$

and $q_j=2$ for $j=m'-2$. Then, for any $s=0, 1, \cdots$ the following statements are satisfied:

1. *For $j=0, 1, \cdots, m'-2$, $M_{s+\rho}F^{\pm}_{s,j,k,x}$ is $L^{q_j}(\boldsymbol{R}^m)$ valued continuous in (k, x) and*

$$\|M_{s+\rho}F^{\pm}_{s,j,k,x}\|_{L^{q_j}(\boldsymbol{R}^m)} \leqq C\langle x\rangle^{-(m-1)/2}\langle k\rangle^j.$$

2. *For $j=0, 1, \cdots, m'-3$, $M_{m-2+s-2j+\varepsilon}F^{\pm}_{s,j,0,x}$ is $L^{q_j}(\boldsymbol{R}^m)$ valued continuous in x and*

$$\|M_{m-2+s-2j+\varepsilon}F^{\pm}_{s,j,0,x}\|_{L^{q_j}(\boldsymbol{R}^m)} \leqq C\langle x\rangle^{2+j-m}.$$

3. *For $j=m'-2$, $M_{s+\rho-j}F^{\pm}_{s,j,0,x}$ is $L^2(\boldsymbol{R}^m)$ valued continuous in x and*

$$\|M_{s+\rho-j}F^{\pm}_{s,j,0,x}\|_{L^2(\boldsymbol{R}^m)} \leqq C\langle x\rangle^{-(m-1)/2}.$$

PROOF. (1) Since $|\phi(x, y)| \leqq |y|$, we have $\langle y\rangle^{-s-\rho}|F^{\pm}_{s,j,k,x}(y)| \leqq C\langle k\rangle^j \cdot\langle y\rangle^{-\rho}|x-y|^{2+j-m}$. On the other hand, by the choice of q_j, $\rho > (m/q_j)-(m-2-j)$ for any $0 \leqq j \leqq m'-2$; $m > \rho q_j$ for $0 \leqq j \leqq m'-3$, and $m<\rho q_j$ for $j=m'-2$. Hence (1) follows immediately from Lemma 3.3.

(2) We may assume $j \leqq s$. We have $\langle y\rangle^{-(m-2+s-2j+\varepsilon)}|F^{\pm}_{s,j,0,x}(y)| \leqq C \cdot\langle y\rangle^{-(m-j-2+\varepsilon)}|x-y|^{2+j-m}$ and $q_j(m-j-2+\varepsilon) > m$ by the choice of q_j. Hence Lemma 3.3 implies (2).

(3) We have $\langle y\rangle^{-(s+\rho-m'+2)}|F^{\pm}_{s,m'-2,0,x}(y)| \leqq C\langle y\rangle^{-\rho}|x-y|^{m'-m}$. Thus (3) follows as in the proof of (1). (Q.E.D.)

PROPOSITION 3.1. *$T_{\pm}(x, y, k)$ is well defined continuous function of (x, y, k). It is $C^{m'+1}$ in the variable k and its derivatives are continuous in the all variables (x, y, k). Moreover,*

$$|(d/dk)^l T_\pm(x, y, k)| \leqq C\langle x\rangle^{-(m-1)/2}\langle y\rangle^{-(m-1)/2}\langle k\rangle^{-3}, \quad l=0, \cdots, m'+1. \tag{3.20}$$

$L_\pm(x, y)$ is continuous in (x, y) and satisfies the estimate

$$|L_\pm(x, y)| \leqq C\langle x\rangle^{-(m-1)/2}\langle y\rangle^{-(m-1)/2}. \tag{3.21}$$

PROOF. Computing the derivative $(d/dk)^{m'+1}$ of the right hand side of (3.9) with Leibniz' formula, we see that $(d/dk)^{m'+1}T_\pm(x, y, k)$ is a sum of $C_{\alpha\beta\gamma jj'}(VN^{(\alpha)}(k)VF^{\pm}_{\beta,j,y,k}, F^{+}_{\gamma,j',x,k})$ over the indices (α, β, γ) and j, j' such that $\alpha+\beta+\gamma=m'+1$ and $0\leqq j, j'\leqq m'-2$. Taking $\kappa=(1+\varepsilon)/2$, we write this in the form

$$C_{\alpha\beta\gamma jj'}(M_{\alpha+\kappa}N^{(\alpha)}(k)M_{\alpha+\kappa}\cdot M^{-1}_{\alpha+\kappa}VF^{\pm}_{\beta,j,y,k},\ M^{-1}_{\alpha+\kappa}VF^{+}_{\gamma,j',x,k}),$$

where $N^{(\alpha)}(k)=(d/dk)^\alpha N(k)$. We have $\langle x\rangle^{\alpha+\kappa}|V(x)| \leqq C\langle x\rangle^{-\max(\beta,\gamma)-\rho}$, $m/2<\rho<(m+\varepsilon)/2$, by the assumption $|V(x)|\leqq C\langle x\rangle^{-(m+2+\varepsilon)}$. Hence, if we take q_j, $q_{j'}$ which satisfy the condition for q in Lemma 3.5, we have

$$\begin{aligned}&\|M^{-1}_{\alpha+\kappa}VF^{\pm}_{\beta,j,y,k}\|_{L^{q_j}} \leqq C\langle k\rangle^{j}\langle y\rangle^{-(m-1)/2},\\ &\|M^{-1}_{\alpha+\kappa}VF^{\pm}_{\gamma,j',x,k}\|_{L^{q_{j'}}} \leqq C\langle k\rangle^{j'}\langle x\rangle^{-(m-1)/2}.\end{aligned} \tag{3.22}$$

Since $L^{q_j}(\boldsymbol{R}^m)\subset H^{2+j-(m+\varepsilon/2)}$, $0\leqq j\leqq m'-3$, by Sobolev's embedding theorem, Lemma 3.4 implies

$$\|M_{\alpha+\kappa}N^{(\alpha)}(k)M_{\alpha+\kappa}\|_{B(L^{q_j},L^{q_{j'}})} \leqq \begin{cases} C\langle k\rangle^{-4-j'-j'+\varepsilon}, & 0\leqq j, j'\leqq m'-3;\\ C\langle k\rangle^{-((m-\varepsilon)/2+2+j)}, & 0\leqq j'\leqq m'-3, \quad j=m'-2;\\ C\langle k\rangle^{-((m-\varepsilon)/2+2+j')}, & 0\leqq j\leqq m'-3, \quad j'=m'-2;\\ C\langle k\rangle^{-m}, & j=j'=m'-2. \end{cases} \tag{3.23}$$

Combining (3.22) and (3.23), we obtain (3.20). The statement for $L_\pm(x, y)$ follows by integration. (Q.E.D.)

The estimate (3.21) is of course not enough to conclude that $L(x, y)$ satisfies the criterion (3.4). We dig up the extra decay property in the variable $|x|\pm|y|$ by performing the integration by parts by the variable k in (3.10):

$$\begin{aligned}(-i(|x|\mp|y|))^{m'+1}L_\pm(x, y) &= \int_0^\infty \{(d/dk)^{m'+1}e^{-ik(|x|\mp|y|)}\}T_\pm(x, y, k)kdk\\ &= \sum_{l=1}^{m'}(-1)^{l+1}\{(d/dk)^{m'-l}e^{-ik(|x|\mp|y|)}\}(d/dk)^l(T_\pm(x, y, k)k)|_{k=0} \qquad (3.24)\\ &\quad +(-1)^{m'+1}\int_0^\infty e^{-ik(|x|\mp|y|)}(d/dk)^{m'+1}(T_\pm(x, y, k)k)dk \equiv \sum_{l=1}^{m'+1}L_{\pm,l}(x, y),\end{aligned}$$

where, for $l=1, \cdots, m'$

$$L_{\pm,l}(x, y)=(-1)^{l+1}l(-i(|x|\mp|y|))^{m'-l}(d/dk)^{l-1}T_{\pm}(x, y, k)|_{k=0} \quad (3.25)$$

are boundary terms and $L_{\pm,m'+1}(x, y)$ are the integral terms. Adding $L_{\pm}(x, y)$ to both sides of (3.24) and dividing the resulting equation by $1+(-i(|x|\mp|y|))^{m'+1}$, we obtain

$$L_{\pm}(x, y)=\frac{L_{\pm}(x, y)+\sum_{l=1}^{m'+1}L_{\pm,l}(x, y)}{1+(-i(|x|\mp|y|))^{m'+1}} \equiv \sum_{l=0}^{m'+1}Z_{\pm,l}(x, y), \quad (3.26)$$

where $Z_{\pm,l}(x, y)=\{1+(-i(|x|\mp|y|))^{m'+1}\}^{-1}L_{\pm,l}(x, y)$, $l=0, \cdots, m'+1$, and $L_{\pm,0}(x, y)=L_{\pm}(x, y)$.

PROPOSITION 3.2. *$Z_{\pm,0}(x, y)$ and $Z_{\pm,m'+1}(x, y)$ satisfy the criterion* (3.4) *and are integral kernels of bounded operators in $L^p(\boldsymbol{R}^m)$.*

For the proof, we use the following elementary lemma whose proof is omitted here.

LEMMA 3.6. *Let $1<l$ and let $j+j'\leqq m+1$ and $j, j'<l$. Then*

$$\sup_{y\in\boldsymbol{R}^m}\int_{\boldsymbol{R}^m}\langle|x|\mp|y|\rangle^{-l}\langle x\rangle^{j-m}\langle y\rangle^{j'-m}dx<\infty,$$

$$\sup_{x\in\boldsymbol{R}^m}\int_{\boldsymbol{R}^m}\langle|x|\mp|y|\rangle^{-l}\langle x\rangle^{j-m}\langle y\rangle^{j'-m}dy<\infty.$$

PROOF OF PROPOSITION 3.2. Recall $m'=(m+1)/2$. In virtue of (3.21) and (3.26), we have

$$|Z_{\pm,0}(x, y)|\leqq C\langle|x|\mp|y|\rangle^{-(m'+1)}\langle x\rangle^{-(m-1)/2}\langle y\rangle^{-(m-1)/2}. \quad (3.27)$$

Lemma 3.6 implies that $Z_{\pm,0}(x, y)$ satisfy (3.4). On the other hand, (3.20) yields that

$$\begin{aligned}|L_{\pm,m'+1}(x, y)| &= C\left|\int_0^\infty e^{-ik(|x|\mp|y|)}(d/dk)^{m'+1}(T_{\pm}(x, y, k)k)dk\right| \\ &\leqq C\langle x\rangle^{-(m-1)/2}\langle y\rangle^{-(m-1)/2},\end{aligned}$$

and $|Z_{\pm,m+1}(x, y)|$ are bounded by the right hand side of (3.27) as well. Thus $Z_{\pm,m+1}(x, y)$ likewise satisfy (3.4). (Q.E.D.)

We next examine $Z_{\pm,l}(x, y)$ for $1\leqq l\leqq m'$. Recall (3.9) and compute $(d/dk)^{l-1}T_{\pm}(x, y, k)|_{k=0}$ by Leibniz's formula. Plugging the result into (3.25), we write $Z_{\pm,l}(x, y)$ in the form:

$$\begin{aligned}&\frac{(-1)^{l+1}(-i(|x|\pm|y|))^{m'-l}}{1+(-i(|x|\mp|y|))^{m'+1}}\sum_{\alpha+\beta+\gamma=l-1}\sum_{j,j'=0}^{m'-2}\frac{(\pm1)^j l!}{\alpha!\beta!\gamma!} \\ &\quad\times C_jC_{j'}\langle N^{(\alpha)}(0)VF^{\pm}_{\beta,j,0,y}, VF^{+}_{\gamma,j',0,x}\rangle. \quad (3.28)\end{aligned}$$

We show that most terms of $Z_{\pm,l}(x, y)$ in (3.28) satisfy the criterion (3.4) using Lemma 3.5, 3.6 and the following

LEMMA 3.7. *Let $\alpha+\beta+\gamma\leqq m'-1$ and $0\leqq j, j'\leqq m'-2$. Then*:

$$|\langle N^{(\alpha)}(0)VF^{\pm}_{\beta,j,0,y}, VF^{+}_{\gamma,j',0,x}\rangle| \leqq C\langle y\rangle^{2+j-m}\langle x\rangle^{2+j'-m}. \tag{3.29}$$

PROOF. We may assume $0\leqq j\leqq\beta$ and $0\leqq j'\leqq\gamma$ as otherwise the lemma is obvious. The proof is similar to that of Proposition 3.1 but here we use the bound $|V(x)|\leqq C\langle x\rangle^{-(3m/2)+2-\varepsilon}$ as well. Take and fix ρ and q_j, $j=0, \cdots, m'-2$, as in Lemma 3.5. We write

$$\begin{aligned}(&N^{(\alpha)}(0)VF^{\pm}_{\beta,j,0,y}, VF^{+}_{\gamma,j',0,x})\\ &=(M_{\rho-\gamma}N^{(\alpha)}(0)M_{\rho-\beta}\cdot M^{-1}_{\rho-\beta}VF^{\pm}_{\beta,j,0,y}, M^{-1}_{\rho-\gamma}VF^{+}_{\gamma,j',0,x}).\end{aligned}$$

Since $|\langle x\rangle^{\rho-\beta}V(x)|\leqq C\min\{\langle x\rangle^{2-m-\beta-(\varepsilon/2)}, \langle x\rangle^{-(m/2)-2-\beta-(\varepsilon/2)}\}$ by the assumption, we have by Lemma 3.5

$$\|M^{-1}_{\rho-\beta}F^{\pm}_{\beta,j,0,y}\|_{L^{q_j}}\leqq C\langle y\rangle^{2+j-m}, \qquad \|M^{-1}_{\rho-\gamma}VF^{+}_{\gamma,j',0,x}\|_{L^{q_{j'}}}\leqq C\langle x\rangle^{2+j'-m}.$$

On the other hand, $\rho-\beta$, $\rho-\gamma>\alpha+(1/2)$ implies as in the proof of Proposition 3.1 that $M_{\rho-\gamma}N^{(\alpha)}(0)M_{\rho-\beta}$ is bounded from L^{q_j} to the dual space $(L^{q_{j'}})^*$ of $L^{q_{j'}}$. Combining these two estimates, we obtain the lemma. (Q.E.D.)

PROPOSITION 3.3. *Let $1\leqq l\leqq m'$. Then, all summands in* (3.28) *for $Z_{\pm,l}(x, y)$ except two, one with $\alpha=\beta=j=0$, $\gamma=j'=l-1$ and the other with $\alpha=\gamma=j'=0$, $\beta=j=l-1$, satisfy the criterion* (3.4) *and are kernels of $L^p(\boldsymbol{R}^m)$-bounded operators. In particular all summands for $Z_{\pm,m'}(x, y)$ satisfy* (3.4).

PROOF. By Lemma 3.7, all terms in (3.28) are bounded by $C\langle|x|\mp|y|\rangle^{-l-1}\cdot\langle x\rangle^{2+j-m}\langle y\rangle^{2+j'-m}$. Note that $j+j'+4\leqq 2m'=m+1$ for non-vanishing terms. Hence, in virtue of Lemma 3.6, all terms except those with either $l-1\leqq j\leqq m'-2$ or $l-1\leqq j'\leqq m'-2$ satisfy the criterion (3.4). Since $\alpha+\beta+\gamma=l-1$ and $F_{\beta,j,0,x}=0$ unless $\beta\geqq j$, the exceptional terms are exactly those mentioned in the proposition. (Q.E.D.)

Finally we estimate the contribution to $L(x, y)=(L_+(x, y)-L_-(x, y))/i\pi$ of the exceptional terms mentioned in Proposition 3.3. Note that we need to deal with the terms with $1\leqq l\leqq m'-1$ and that $F^{\pm}_{0,0,0,x}(y)=|x-y|^{2-m}$ and $F^{\pm}_{l-1,l-1,0,x}(y)=(l-1)!|x-y|^{1+l-m}$ do not depend on the sign $\pm$. Hence, if we define $K_{\pm,l}(x, y)$ by

$$K_{+,l}(x, y)=(-1)^{l+1}l!\,C_0C_{l-1}\langle N(0)V|y-\cdot|^{2-m}, V|x-\cdot|^{1+l-m}\rangle,$$

$$K_{-,l}(x, y)=(-1)^{l+1}l!\,C_0C_{l-1}\langle N(0)V|y-\cdot|^{1+l-m}, V|x-\cdot|^{2-m}\rangle.$$

then, their contribution to $L_+(x, y)-L_-(x, y)$ is given explicitly by

$$\sum_{l=1}^{m'-1}\left\{\frac{(-i(|x|-|y|))^{m'-l}}{1+(-i(|x|-|y|))^{m'+1}}-\frac{(-i(|x|+|y|))^{m'-l}}{1+(-i(|x|+|y|))^{m'+1}}\right\}K_{+,l}(x, y)$$
$$+\sum_{l=1}^{m'-1}\left\{\frac{(-i(|x|-|y|))^{m'-l}}{1+(-i(|x|-|y|))^{m'+1}}-(-1)^{l-1}\frac{(-i(|x|+|y|))^{m'-l}}{1+(-i(|x|+|y|))^{m'+1}}\right\}K_{-,l}(x, y). \tag{3.30}$$

Thus the proof of the L^p boundedness of L will be completed if we show the following

PROPOSITION 3.4. *The function* (3.30) *satisfies the criterion* (3.4).

PROOF. We denote the functions in the braces in front of $K_{\pm,l}(x, y)$ by $J_{\pm,l}(x, y)$. In virtue of Lemma 3.7, we have $|K_{+,l}(x, y)|\leqq C\langle x\rangle^{1+l-m}\langle y\rangle^{2-m}$ and $|K_{-,l}(x, y)|\leqq C\langle x\rangle^{2-m}\langle y\rangle^{1+l-m}$. Elementary estimations imply $|J_{\pm,1}(x, y)| \leqq C\langle |x|-|y|\rangle^{-2}\langle |x|+|y|\rangle^{-2}\langle x\rangle\langle y\rangle$ and

$$|J_{\pm,l}(x, y)| \leqq C\left\{\begin{matrix}\langle y\rangle\\ \langle x\rangle\end{matrix}\right\}\langle |x|-|y|\rangle^{-l-1}\langle |x|+|y|\rangle^{-1}, \qquad 2\leqq l\leqq m'-1.$$

It follows that $|J_{\pm,1}(x, y)K_{\pm,1}(x, y)|\leqq C\min\{\langle x\rangle^{1-m}\langle y\rangle^{3-m}, \langle x\rangle^{3-m}\langle y\rangle^{1-m}\}\langle |x|-|y|\rangle^{-2}$ and for $2\leqq l\leqq m'-1$:

$$|J_{+,l}(x, y)K_{+,l}(x, y)| \leqq C\langle x\rangle^{(1/2)+l-m}\langle y\rangle^{(5/2)-m}\langle |x|-|y|\rangle^{-l-1},$$
$$|J_{-,l}(x, y)K_{-,l}(x, y)| \leqq C\langle y\rangle^{(1/2)+l-m}\langle x\rangle^{(5/2)-m}\langle |x|-|y|\rangle^{-l-1}.$$

$J_{\pm,1}(x, y)K_{\pm,1}(x, y)$ satisfies the criterion (3.4) since

$$\sup_y\int_{R^m}\frac{dx}{\langle x\rangle^{m-1}\langle y\rangle^{m-3}\langle |x|-|y|\rangle^2}=\sup_x\int_{R^m}\frac{dy}{\langle x\rangle^{m-3}\langle y\rangle^{m-1}\langle |x|-|y|\rangle^2}<\infty,$$

and so does $J_{\pm,l}(x, y)K_{\pm,l}(x, y)$, $2\leqq l\leqq m'-1$, in virtue of Lemma 3.6. Hence the function (3.30) satisfies (3.4) as well and the proof is completed. (Q.E.D.)

4. $W^{k,p}$-boundedness of $W_\pm$.

In this section, we give the extra argument which is necessary to prove that $W_\pm$ is in fact bounded in $W^{k,p}$, $k=1, \cdots, l$, if the assumption of Theorem 1 is satisfied with $l\geqq 1$. We prove $W_+\in \boldsymbol{B}(W^{1,p})$ only as the other cases may be proved by repeating the following argument.

We begin with the case that $\|\mathcal{F}(\langle x\rangle^\sigma V)\|_{L^{m_*}}$ is small. In what follows C_2 is the constant appeared in Theorem 2.1.

PROPOSITION 4.1. *For* $n=1, \cdots, k=1, \cdots, n$ *and* $j=1, \cdots, m$, *let* X_{nkj} be *defined by*

$$X_{nkj}f = \frac{1}{2\pi i}\int_{-\infty}^{\infty}(R_0(\lambda-i0)V)^{k-1}R_0(\lambda-i0)(D_jV)R_0(\lambda-i0)(VR_0(\lambda+i0))^{n-k}fd\lambda. \tag{4.1}$$

Then, X_{nkj} is bounded in L^p for any $1\leqq p\leqq\infty$ and

$$\|X_{nkj}f\|_{L^p} \leqq C_1(C_2\|\mathcal{F}(\langle x\rangle^\sigma V)\|_{L^{m_*}})^{n-1}\|\mathcal{F}(\langle x\rangle^\sigma DV)\|_{L^{m_*}}\|f\|_{L^p} \tag{4.2}$$

where C_1 and C_2 are the constants appeared in (2.12).

PROOF. When $\hat{V}\in C_0^\infty$, Proposition 2.2 implies (4.2). For general V we approximate it by a sequence $V_i\in\mathcal{S}$ such that $\hat{V}_i\in C_0^\infty$ and $\sum_{|\alpha|\leqq l}\|\mathcal{F}(\langle x\rangle^\sigma D^\alpha(V-V_i))\|_{L^{m_*}}\to 0$ as $i\to\infty$. (4.2) follows as in the proof of Theorem 2.1. (Q.E.D.)

LEMMA 4.1. *Let $f\in W^{1,p}$. Then $W_nf\in W^{1,p}$ and $D_jW_nf-W_nD_jf=\sum_{k=1}^n X_{nkj}f$.*

PROOF. Take $V_i\in\mathcal{S}$ as in the proof of Proposition 4.1 and define $W_n^{(i)}$ by (2.11) with V_i in place of V. We have, for $g\in C_0^\infty$,

$$(W_n^{(i)}f, D_jg) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\langle V_i(R_0(\lambda-i0)V_i)^{n-1}R_0(\lambda+i0)f,\ R_0(\lambda+i0)D_jg\rangle d\lambda,$$

where $\langle\cdot,\cdot\rangle$ should be considered as the coupling between suitable function spaces. Then, since $R_0(\lambda\pm i0)$ and D_j commute, $V_i\in\mathcal{S}$ and $[D_j, V_i]=D_jV_i$, it is easy to see that

$$D_jW_n^{(i)}f = \sum_{k=1}^n X_{nkj}^{(i)}f+W_n^{(i)}D_jf\,, \tag{4.3}$$

where $X_{nkj}^{(i)}$ is X_{nkj} with V_i in place of V_i. In virtue of Proposition 4.1 and Theorem 2.1, we have for any $1\leqq p\leqq\infty$ that

$$\lim_{i\to\infty}\|X_{nkj}^{(i)}-X_{nkj}\|_{B(L^p)} = 0, \qquad \lim_{i\to\infty}\|W_n^{(i)}-W_n\|_{B(L^p)} = 0.$$

It follows by taking the limit $i\to\infty$ in (4.3) that $D_jW_nf=\sum_{k=1}^n X_{nkj}f+W_nD_jf\in L^p$. This proves the lemma. (Q.E.D.)

Combining Proposition 4.1 and Lemma 4.1, we have the following.

THEOREM 4.1. *For $n=1,\cdots,W_n$ is bounded in $W^{1,p}$, $1\leqq p\leqq\infty$ and*

$$\|W_n\|_{B(W^{1,p})} \leqq C(C_2\|\mathcal{F}(\langle x\rangle^\sigma V)\|_{L^{m_*}})^{n-1}(\|\mathcal{F}(\langle x\rangle^\sigma V)\|_{L^{m_*}}+\|\mathcal{F}(\langle x\rangle^\sigma DV)\|_{L^{m_*}}), \tag{4.4}$$

where the constants are independent of n.

It clearly follows from Theorem 4.1 that, if $\|\mathcal{F}(\langle x\rangle^\sigma V)\|_{L^{m_*}}<C_2^{-1}$, the series $\sum_{n=0}^\infty W_n$ in fact converges in the operator norm of $\boldsymbol{B}(W^{1,p})$ and that $W_+\in\boldsymbol{B}(W^{1,p})$.

When $\|\mathcal{F}(\langle x\rangle^\sigma V)\|_{L^{m_*}}$ is not small, it suffices to show that $L=L_m$ of

Section 3 is bounded in $W^{1,p}$ if V satisfies Assumption 1.1, (2) with $l\geqq 1$ and 0 is not an eigenvalue or resonance of H. Recall that

$$(L_m f, g)=\frac{1}{2\pi i}\int_0^\infty (V(R_0^- V)^{m'-1}R^-(VR_0^-)^{m'-1}V\{R_0^+-R_0^-\}f, R_0^+ g)d\lambda, \quad (4.5)$$

where we used the notation $R_0^\pm=R_0(\lambda\pm i0)$ and $R^-=R(\lambda-i0)$ for the brevity. Let $f\in W^{1,p}$. Then following the argument in the proof Lemma 4.1, we have that

$$D_j Lf-LD_j f=Y_{0j}f+\sum_{k=1}^{m+1}Y_{kj}f, \qquad j=1,\cdots,m, \quad (4.6)$$

where $(Y_{kj}f, g)$, $k=1,\cdots,m$, is equal to the RHS of (4.5) with D_jV in place of k-th V; and $(Y_{0j}f, g)$ is equal to the RHS of (4.5) with $-R^-D_jVR^-$ in place of R^-. It is clear that the proof of L^p boundedness of L in Section 3 implies that Y_{kj}, $k=1,\cdots$ is bounded in L^p. On the other hand, it is easy to see that Lemma 3.4 remains valid when $M_\gamma N_{m'-1}(k)M_{\gamma'}$ is replaced by

$$M_\gamma\{R_0(k^2-i0)V\}^{m'-1}R(k^2-i0)(D_jV)R(k^2-i0)\{VR_0(k^2-i0)\}^{m'-1}M_{\gamma'}.$$

Hence, the argument in Section 3 implies that Y_{0j} also is bounded in L^p. Thus, (4.6) implies L is bounded in $W^{1,p}$. This completes the proof of Theorem 1.1.

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Kenji YAJIMA
Department of Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Meguro-ku, Tokyo 153
Japan

Note Added in Proof. The author has recently succeed in generalizing Theorem 1.1 to the case when the spatial dimension is even $m \geqq 4$ and V is not necessarily small. The details will appear in "The $W^{k,p}$-continuity of wave opertors for Schrödinger operators III, Even dimensional Cases $m \geqq 4$", Journal of Mathematical Sciences, University of Tokyo.