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Convergence of Alexandrov spaces and spectrum of Laplacian

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Abstract. Denote by $\mathscr{A}(n)$ the family of isometry classes of compact n-dimensional Alexandrov spaces with curvature ≥ -1, and $\lambda_k(M)$ the k^{th} eigenvalue of the Laplacian on $M \in \mathscr{A}(n)$. We prove the continuity of $\lambda_k : \mathscr{A}(n) \to \boldsymbol{R}$ with respect to the Gromov-Hausdorff topology for each $k, n \in \boldsymbol{N}$, and moreover that the spectral topology introduced by Kasue-Kumura [**7**], [**8**] coincides with the Gromov-Hausdorff topology on $\mathscr{A}(n)$.

1. Introduction.

For $n \in \boldsymbol{N}$ and $D > 0$, let $\mathscr{M}_{\mathrm{Ric}}(n, D)$ denote the family of isometry classes of closed n-dimensional Riemannian manifolds with Ricci curvature $\mathrm{Ric}_M \geq -(n-1)$ and diameter $\mathrm{diam}(M) \leq D$, and $\mathscr{M}_{|\mathrm{sec}|}(n, D)$ the family of all $M \in \mathscr{M}_{\mathrm{Ric}}(n, D)$ whose sectional curvatures K_M satisfy $|K_M| \leq 1$. Fukaya [**5**] proved that, if we equip each $M \in \mathscr{M}_{|\mathrm{sec}|}(n, D)$ with normalized volume measure $d\,\mathrm{vol}_M/\mathrm{vol}(M)$, then for each $k \in \boldsymbol{N}$ the function λ_k which assigns to each $M \in \mathscr{M}_{|\mathrm{sec}|}(n, D)$ the k^{th} eigenvalue of the Laplacian on M extends to a continuous function on the measured Gromov-Hausdorff closure of $\mathscr{M}_{|\mathrm{sec}|}(n, D)$. After that, Bérard-Besson-Gallot [**1**] and Kasue-Kumura [**7**] each proved that the statement holds for $\mathscr{M}_{\mathrm{Ric}}(n, D)$ with some natural topologies different from (indeed finer than) the measured Gromov-Hausdorff topology. Precisely, Kasue-Kumura [**7**], [**8**] introduced a natural distance, called the spectral distance, between Riemannian manifolds by using the heat kernels, and proved the precompactness of $\mathscr{M}_{\mathrm{Ric}}(n, D)$ with respect to the spectral distance. The topology induced from the spectral distance is called the *spectral topology*. The spectral distance or topology completely expresses the closeness between the analytic structures on manifolds. Especially, the k^{th} eigenvalue of the Laplacian of a Riemannian manifold is continuous with respect to the spectral topology.

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In this paper, we consider the family, say $\mathscr{A}(n)$, of isometry classes of compact n-dimensional Alexandrov spaces of curvature ≥ -1. The family $\mathscr{A}(n)$ contains the d_{GH}-closure of the family, say $\mathscr{M}_{\mathrm{sec}}(n, D, v)$, consisting of all $M \in \mathscr{M}_{\mathrm{Ric}}(n, D)$ with sectional curvature $K_M \geq -1$ and volume $\mathrm{vol}(M) \geq v$ for any fixed $n \in \boldsymbol{N}$, $D, v > 0$, where d_{GH} denotes the Gromov-Hausdorff distance between compact metric spaces. In our previous paper [**10**], we gave a natural definition of the Laplacian Δ_M on $M \in \mathscr{A}(n)$. It has discrete spectrum consisting of eigenvalues

$$0 = \lambda_0(M) < \lambda_1(M) \leq \lambda_2(M) \leq \cdots \nearrow \infty.$$

We also proved the existence of Hölder continuous heat kernel on $M \in \mathscr{A}(n)$, so that the spectral distance is valid between Alexandrov spaces. We observe that almost all results of [**8**] hold for Alexandrov spaces instead of manifolds, without any change of their proofs. Our main theorem is stated as follows.

Theorem 1.1. *The spectral topology coincides with the Gromov-Hausdorff topology on $\mathscr{A}(n)$ for each $n \in \boldsymbol{N}$. In particular, the k^{th} eigenvalue $\lambda_k : \mathscr{A}(n) \to \boldsymbol{R}$ is continuous with respect to the Gromov-Hausdorff topology for each $n, k \in \boldsymbol{N}$.*

Denote by $\mathscr{A}(n, D, v)$ the family of $M \in \mathscr{A}(n)$ with $\mathrm{diam}(M) \leq D$ and the n-dimensional Hausdorff measure $\mathscr{H}^n(M) \geq v > 0$. Note that the d_{GH}-closure of $\mathscr{M}_{\mathrm{sec}}(n, D, v)$ is a proper subset of $\mathscr{A}(n, D, v)$, and that $\mathscr{A}(n, D, v)$ is d_{GH}-compact. As an immediate corollary to Theorem 1.1, we have an upper bound $c(n, k, D, v) > 0$ depending only on $n, k \in \boldsymbol{N}$, $D, v > 0$ for λ_k on $\mathscr{A}(n, D, v)$.

Let us mention the idea of the proof of Theorem 1.1. The proof relies on the non-smooth analysis based on the synthetic method of Alexandrov geometry, developed in our previous paper [**10**]. Assume that a sequence $M_i \in \mathscr{A}(n)$, $i = 1, 2, \ldots$, d_{GH}-converges to an $M \in \mathscr{A}(n)$. If the limit M has no singularities, then M_i converges to M with respect to the Lipschitz distance and then the proof is easy. However, this is not true in general. Nevertheless, we can still construct a bi-Lipschitz map between subsets $\Omega \subset M$ and $\Omega_i \subset M_i$ with bi-Lipschitz constant close to 1 for large i (see Theorem 3.1), where Ω is any compact subset of M such that $M \backslash \Omega$ is a small neighborhood of some 'pointed' singular set (precisely the set $\hat{S}_\delta$ defined in §2). This makes a correspondence between $W^{1,2}(\Omega_i)$ and $W^{1,2}(\Omega)$. For the theorem, it is essential to prove that for any fixed $k \in \boldsymbol{N}$ an eigenfunction for the k^{th} eigenvalue of Δ_{M_i} L^2-converges to some eigenfunction for the k^{th} eigenvalue of Δ_M on $\Omega(\approx \Omega_i)$ as $i \to \infty$, after taking some subsequence. Here, it is a crucial point to show no presence of the concentration of $W^{1,2}$-mass on $M_i \backslash \Omega_i$ of the eigenfunction in the convergence $M_i \to M$. Note that in the case of [**5**], i.e., when $M_i \in \mathscr{M}_{|\mathrm{sec}|}(n, D)$ and $\dim M \leq n$, then the singularities of M appears only by quotient of isometric

$O(n)$-action, and considering analysis on $O(n)$-invariant smooth functions keeps away from the difficulty. On the other hand, our singularities are not only caused by the quotient of group action, so that we need a different way to prove the theorem. Our idea is to take the mollifier of eigenfunctions of Δ_{M_i} to control the concentration of the L^2-mass on $M_i \backslash \Omega_i$ of the eigenfunctions (see Lemma 5.6). This also implies no presence of the energy concentration on $M_i \backslash \Omega_i$ of the eigenfunctions.

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2. Preliminaries.

We denote by $\theta(\delta)$ the symbol expressing a function such that $\theta(\delta) \to 0$ as $\delta \to 0$, and use it like Landau's symbols.

Let (X, d_X) and (Y, d_Y) be two metric spaces, $f : X \to Y$ a (not necessarily continuous) map, and $\varepsilon > 0$ a number. We call f an *ε-approximation* if the following (i) and (ii) hold:

$$\text{(i)} \qquad |d_Y(f(x), f(y)) - d_X(x, y)| < \varepsilon \quad \text{for any } x, y \in X,$$

$$\text{(ii)} \qquad Y \subset B(f(X), \varepsilon),$$

where $B(A, r)$ denotes the r-metric ball of a subset A of a metric space. Note that the Gromov-Hausdorff distance between X and Y is $d_{GH}(X, Y) < \theta(\varepsilon)$ if and only if a $\theta(\varepsilon)$-approximation from X to Y exists. The *dilatation* $\operatorname{dil}(f)$ *of* f is defined by

$$\operatorname{dil}(f) := \sup_{x \neq y \in X} \frac{d_Y(f(x), f(y))}{d_X(x, y)} \in [0, \infty].$$

Call f an *ε-almost isometry* if f is a bi-Lipschitz homeomorphism with $|\ln \operatorname{dil}(f)| + |\ln \operatorname{dil}(f^{-1})| < \varepsilon$.

In the following, we define some convention for Alexandrov spaces used in this paper. We refer to [**2**] for the basics for Alexandrov spaces. Let M be an n-dimensional Alexandrov space with curvature ≥ -1, i.e., a complete locally compact intrinsic metric space with curvature ≥ -1 in the sense of triangle comparison. For $\delta > 0$, the *δ-singular set of M* is defined to be

$$S_\delta := \{x \in M \mid \mathscr{H}^{n-1}(\Sigma_x) \leq \omega_{n-1} - \delta\},$$

where Σ_x is the space of directions at $x \in M$ and ω_{n-1} the volume of the unit $(n-1)$-sphere. It is known that, for $0 < \delta \ll 1/n$, $M \backslash S_\delta$ is a (incomplete)

Lipschitz-Riemannian manifold whose Riemannian metric is compatible with the original distance function ([**12**]). A *singular point of* M is defined to be a point where the space of directions is not isometric the unit n-sphere. The set of singular points of M, say the *singular set* S_M, coincides with $\bigcup_{\delta>0} S_\delta$. The Hausdorff dimension of S_M is $\leq n-1$ ([**2**], [**12**]). A *boundary point of* M is defined inductively as follows. If $\dim M = 1$, then M is a one-dimensional Riemannian manifold and the boundary is naturally defined. When $\dim M \geq 2$, a point $x \in M$ is a boundary point if and only if Σ_x has a boundary point. Denote by ∂M the set of boundary points of M. The *double* $\mathrm{dbl}(M)$ *of* M is obtained by gluing two copies of M along their boundaries. The double of an Alexandrov space with nonempty boundary is an Alexandrov space without boundary and of the same lower bound of curvature ([**2**], [**13**]). There is a natural isometric embedding of M into $\mathrm{dbl}(M)$. We denote by $\hat{S}_\delta$ the δ-singular points of $\mathrm{dbl}(M)$ contained in M. Then, the Hausdorff dimension of $\hat{S}_\delta$ is $\leq n-2$ ([**2**]).

We here define some convention in analysis on Alexandrov spaces. Refer to [**12**], [**10**] for the details. Assume from now on that M is compact. Recalling that $M \backslash S_\delta$ for $0 < \delta \ll 1/n$ is a Lipschitz-Riemannian manifold, we can define the Sobolev space $W^{1,2}$ on it. For $\Omega \subset M$, a symmetric bi-linear form $\mathscr{E}_\Omega$ on $W^{1,2}(\Omega)$ $(:= W^{1,2}(\Omega \backslash S_\delta))$ is defined by

$$\mathscr{E}_\Omega(u,v) := \int_{\Omega^\circ \backslash S_\delta} \langle \nabla u, \nabla v \rangle \, d\mathscr{H}^n, \quad u, v \in W^{1,2}(\Omega),$$

where Ω° denotes the interior of Ω and $\langle \cdot , \cdot \rangle$ the Riemannian inner product. Here, we note that the Riemannian volume measure is known to coincide with the n-dimensional Hausdorff measure $\mathscr{H}^n$ ([**12**]). Set $\mathscr{E}(u) := \mathscr{E}_\Omega(u,u)$ for simplicity. Recall that the $W^{1,2}$-*Hilbert inner product of* $u, v \in W^{1,2}(\Omega)$ is given by

$$(u,v)_{W^{1,2}(\Omega)} := (u,v)_{L^2(\Omega)} + \mathscr{E}_\Omega(u,v),$$

and the associated $W^{1,2}$-*norm of* $u \in W^{1,2}(\Omega)$ by

$$\|u\|_{W^{1,2}(\Omega)} := \sqrt{(u,u)_{W^{1,2}(\Omega)}} = \sqrt{\|u\|^2_{L^2(\Omega)} + \mathscr{E}(u)}.$$

The *Laplacian* $\Delta_M : \mathscr{D}(\Delta_M) \subset W^{1,2}(M) \to L^2(M)$ is defined as the *generator* of the Dirichlet form $(\mathscr{E}, W^{1,2}(M))$, i.e., the self-adjoint operator such that $\mathscr{D}(\sqrt{\Delta_M}) = W^{1,2}(M)$ and $\mathscr{E}_M(u,v) = (\sqrt{\Delta_M}u, \sqrt{\Delta_M}v)_{L^2}$ for any $u, v \in W^{1,2}(M)$. We proved in [**10**] that $W^{1,2}(M)$ is compactly embedded into $L^2(M)$. Consequently, the Laplacian Δ_M has only discrete spectrum as mentioned in §1, and there exists a complete orthonormal basis $\{\varphi_k\}_{k=0}^\infty$ on $L^2(M)$ consisting of eigenfunctions φ_k for $\lambda_k(M)$ of Δ_M.

We now recall the definition of the spectral distance introduced by Kasue-Kumura [7], [**8**]. We proved in [**10**] that any Alexandrov space M admits Hölder continuous heat kernel $p_M : (0,\infty) \times M \times M \to \boldsymbol{R}$. Let $M, N \in \mathscr{A}(n)$. Two $\mathscr{H}^n$-measurable maps $f : M \to N$ and $h : N \to M$ are called *ε-spectral approximations*, $\varepsilon > 0$, if

$$e^{-(t+1/t)}|p_M(t,x,y) - p_N(t,f(x),f(y))| < \varepsilon,$$

$$e^{-(t+1/t)}|p_M(t,h(z),h(w)) - p_N(t,z,w)| < \varepsilon$$

for any $t > 0$, $x, y \in M$, and $z, w \in N$. The *spectral distance* $SD(M,N)$ *between* M *and* N is defined to be the infimum of $\varepsilon > 0$ such that both ε-spectral approximations $f : M \to N$ and $h : N \to M$ exist. We here agree that $SD(M,N) := \infty$ if one of the spectral approximations f and h does not exist.

3. Construction of almost isometries.

The purpose of this section is to prove the following:

THEOREM 3.1. *Assume that a sequence* $M_i \in \mathscr{A}(n)$ d_{GH}*-converges to an* $M \in \mathscr{A}(n)$. *Then, for any* δ *with* $0 < \delta \ll 1/n$, *there exist* ε_i*-approximations* $f_{\delta,i} : M \to M_i$ *for some* $\varepsilon_i \searrow 0$ *and compact subsets* $D_{\delta,i} \subset M \backslash \hat{S}_\delta$ *with* $\bigcup_{i=1}^{\infty} D_{\delta,i} = M \backslash \hat{S}_\delta$ *such that the restriction* $f_{\delta,i} : D_{\delta,i} \to f_{\delta,i}(D_{\delta,i})$ *is a* $\theta(\delta)$*-almost isometry, and*

$$\lim_{i\to\infty} \mathscr{H}^n(M_i \backslash f_{\delta,i}(D_{\delta,i})) = 0.$$

In particular, $(M_i, \mathscr{H}^n)$ *converges to* $(M, \mathscr{H}^n)$ *in the measured Gromov-Hausdorff topology.*

Here, refer to [**5**] for the definition of the measured Gromov-Hausdorff topology.

We need some definitions for the proof of the theorem. An *ε-net of a metric space* X, $\varepsilon > 0$, is defined to be a discrete subset of X whose ε-neighborhood contains X. Let M be an n-dimensional Alexandrov space with distance function d_M. We say that a point $x \in M$ is *δ-strained*, $\delta > 0$, if there is a sequence $\{p_i\}_{i=\pm1,\dots,\pm n}$, called a *δ-strainer at* x, such that for any i, j,

$$\tilde{\angle} p_i x p_j > \begin{cases} \pi/2 - \delta & \text{if } i \neq \pm j, \\ \pi - \delta & \text{if } i = -j, \end{cases}$$

where $\tilde{\angle} xyz$ for $x, y, z \in M$ denotes the angle of the comparison triangle of $\triangle xyz$ in the hyperbolic plane at the vertex corresponding to y. It follows that $x \in M \backslash S_{\theta(\delta)}$ if and only if $x \in M$ is a $\theta(\delta)$-strained point. Let $x \in M$ be a

δ-strained point with strainer $\{p_i\}_{i=\pm 1,\dots,\pm n}$, and let

$$\varphi(y) := (d_M(p_i, y) - d_M(p_{-i}, y))_{i=1,\dots,n} \in \boldsymbol{R}^n, \quad y \in M.$$

Then, there exists an open neighborhood U of x such that the restriction $\varphi : U \to \varphi(U)$ is a $\theta(\delta)$-almost isometry and the image $\varphi(U)$ is an open subset of $\boldsymbol{R}^n$ (see [**2**]).

Proof of Theorem 3.1. When $\partial M = \emptyset$, we already proved the theorem in §3 of [**15**]. Since the basic strategy of the proof here is similar to that in [**15**], we omit some details.

Let r and t be small enough numbers such that $0 < t \ll r$. There exists a t-net $\{x_j\}_{j=1}^m$ of $M \backslash B(\hat{S}_\delta, r)$ such that $\{x_j\}_{j=1}^m \cap \partial M$ is a t-net of $\partial M \backslash B(\hat{S}_\delta, r)$ and that $d_M(x_j, \partial M) > t$ for $x_j \notin \partial M$. We embed M into its double $\mathrm{dbl}(M)$. Then, there exists a $\theta(\delta)$-strainer $\{p_{jk}\}_{k=\pm 1,\dots,\pm n}$ at each x_j on $\mathrm{dbl}(M)$ such that the map

$$\varphi_j := (d_{\mathrm{dbl}(M)}(p_{jk}, \cdot) - d_{\mathrm{dbl}(M)}(p_{j,-k}, \cdot))_{k=1,\dots,n}$$

from $U_j := B(x_j, 2t)$ to $\varphi_j(U_j) \subset \boldsymbol{R}^n$ is a $\theta(\delta)$-almost isometry. By an approximation $\mathrm{dbl}(M) \to \mathrm{dbl}(M_i)$ we project all the points x_j, p_{jk} to points in $\mathrm{dbl}(M_i)$, say y_j, q_{jk}, such that $\{x_j\} \cap \partial M = \{y_j\} \cap \partial M_i$. Supposing i is large enough, we may assume that the map

$$\psi_j := (d_{\mathrm{dbl}(M)}(q_{jk}, \cdot) - d_{\mathrm{dbl}(M)}(q_{j,-k}, \cdot))_{k=1,\dots,n}$$

from $V_j := B(y_j, 5t)$ to $\psi_j(V_j) \subset \boldsymbol{R}^n$ is also a $\theta(\delta)$-almost isometry. We also assume that if $x_j \in \partial M$, then p_{j1}, q_{j1} are symmetric to $p_{j,-1}$, $q_{j,-1}$ in $\mathrm{dbl}(M)$, $\mathrm{dbl}(M_i)$ respectively and $p_{jk} \in \partial M$, $q_{jk} \in \partial M_i$ for every k with $|k| \ge 2$, so that $\varphi_j = -\varphi_j \circ \iota$ and $\psi_j = -\psi_j \circ \iota_i$ for $x_j \in \partial M$, where $\iota : \mathrm{dbl}(M) \to \mathrm{dbl}(M)$ and $\iota_i : \mathrm{dbl}(M_i) \to \mathrm{dbl}(M_i)$ denote the symmetric isometries. Note that for $x_j \in \partial M$, $U_j \cap \partial M = \varphi_j^{-1}(\{x_1 = 0\})$ and $V_j \cap \partial M_i = \psi_j^{-1}(\{x_1 = 0\})$. Set $U_j' := B(x_j, t)$. There are Lipschitz functions $\chi_j : M \to [0, 1]$ such that

$$\mathrm{supp}[\chi_j] \subset U_j, \quad \chi_j = 1 \quad \text{on } U_j', \quad M \subset \bigcup_j \mathrm{supp}[\chi_j]^\circ,$$

$$\mathrm{supp}[\chi_j] \cap \partial M = \emptyset \quad \text{for } x_j \notin \partial M.$$

We inductively define a sequence of functions $f_{\delta,i}^j : \bigcup_{\ell=1}^j U_\ell' \to M_i$ by

$$f_{\delta,i}^1 := \psi_1^{-1} \circ \varphi_1 : U_1' \to M_i$$

and for $j \ge 2$,

$$f_{\delta,i}^j := \begin{cases} f_{\delta,i}^{j-1} & \text{on } \bigcup_{\ell=1}^j U_\ell' \backslash U_j, \\ \psi_j^{-1} \circ ((1-\chi_j)\psi_j \circ f_{\delta,i}^{j-1} + \chi_j \varphi_j) & \text{on } \bigcup_{\ell=1}^j U_\ell' \cap U_j. \end{cases}$$

Note that each $f_{\delta,i}^j$ (and also its inverse) maps boundary points to boundary points. The same discussion as in the proof of Theorem 3.1 of [**15**] yields that $f_{\delta,i} := f_{\delta,i}^m : M \backslash B(\hat{S}_\delta, r) \to M_i$ is a $\theta(\delta)$-almost isometry onto its image for i large enough against $r > 0$. Taking a sequence of positive numbers r_i slowly tending to zero, we obtain the desired $\theta(\delta)$-almost isometry $f_{\delta,i} : D_{\delta,i} := M \backslash B(\hat{S}_\delta, r_i) \to f_{\delta,i}(D_{\delta,i})$. We have $\lim_{i\to\infty} \mathscr{H}^n(M_i \backslash f_{\delta,i}(D_{\delta,i})) = 0$ in the same way as in the proof of Lemma 3.4 of [**15**]. This completes the proof. □

4. *SD*-precompactness of $\mathscr{A}(n, D, v)$.

In this section, we shall prove the *SD*-precompactness of $\mathscr{A}(n, D, v)$. Let $M \in \mathscr{A}(n)$. Take a C^∞-function $\rho_n : [0, \infty) \to [0, \infty)$ such that ρ_n is a positive constant around 0, $\rho_n(r) = 0$ for $r \geq 1$, and

$$\int_{\boldsymbol{R}^n} \rho_n(|\boldsymbol{x}|)\, d\boldsymbol{x} = 1,$$

where $d\boldsymbol{x}$ denotes the Lebesgue measure on $\boldsymbol{R}^n$ with respect to the variable $\boldsymbol{x} \in \boldsymbol{R}^n$. For $h > 0$, we define the *h-mollifier* $u_h : M \to \boldsymbol{R}$ *of a function* $u \in L^2(M)$ by

$$u_h(x) := \frac{\int_{y \in M} \rho_n(d(x, y)/h) u(y)\, d\mathscr{H}^n}{\int_{y \in M} \rho_n(d(x, y)/h)\, d\mathscr{H}^n}.$$

Set, for $R > 0$,

$$a(M) := \inf_{x \in M} \int_{y \in M} \rho_n(d(x, y))\, d\mathscr{H}^n,$$

$$b(M, R) := \inf_{\substack{x \in M \\ 0 < r \leq R}} \frac{r^n}{\mathscr{H}^n(B(x, r))}.$$

We proved in [**10**] (see Lemmas 4.3, 4.4, 7.1, and Theorems 4.1, 7.2 of [**10**] and their proofs) that for any $0 < h < 1$, $0 < r \leq R$, and $u \in W^{1,2}(M)$,

$$\sup_M |u_h| \leq \frac{c_n}{a(M)h} \sqrt{\mathscr{E}(u)}, \tag{4.1}$$

$$\|u_h - u\|_{L^2} \leq \frac{c_n h}{a(M)} \sqrt{\mathscr{E}(u)}, \tag{4.2}$$

$$\|u - \bar{u}_{B(x,r)}\|_{L^2(B(x,r))} \leq c_n b(M, R) r \|\nabla u\|_{L^2(B(x,3r))}, \tag{4.3}$$

where c_n is a constant depending only on n and

$$\bar{u}_{B(x,r)} := \frac{1}{\mathscr{H}^n(B(x,r))} \int_{B(x,r)} u\, d\mathscr{H}^n.$$

We also have some uniform estimate of the dilatation of u_h. The compactness of the embedding $W^{1,2}(M) \subset L^2(M)$ (Theorem 1.2 of [**10**]) is in fact obtained by using the mollifier defined here.

Lemma 4.1. *For any $n \in \mathbf{N}$ and $D, R, v > 0$, we have*

$$\inf_{M \in \mathscr{A}(n,D,v)} a(M) > 0 \quad \textit{and} \quad \sup_{M \in \mathscr{A}(n,D,v)} b(M,R) < \infty.$$

Here, $\mathscr{A}(n,D,v)$ is defined in §1.

Proof. Since ρ_n is positive on a neighborhood of 0, for the first estimate it suffices to show that

$$\inf_{M \in \mathscr{A}(n,D,v)} \inf_{x \in M} \mathscr{H}^n(B(x,r_0)) > 0 \quad \text{for a small } r_0 > 0.$$

This in fact follows from the Bishop-Gromov inequality ([**18**]).

The second estimate is also implied by the Bishop-Gromov inequality. □

Let us recall the concept of spectral embedding (cf. [**1**], [**7**], [**8**]). Consider the Hilbert space $\ell_2 := \{(a_k)_{k=0}^{\infty} \mid \sum_{k=0}^{\infty} a_k^2 < \infty\}$ and denote by $C_\infty([0,\infty),\ell_2)$ the space of continuous curves $\gamma : [0,\infty) \to \ell_2$ such that the ℓ_2-norm $|\gamma(t)|_{\ell_2}$ of $\gamma(t)$ tends to zero as $t \to \infty$. We equip $C_\infty([0,\infty),\ell_2)$ with the distance function d_∞ defined by

$$d_\infty(\gamma,\sigma) := \sup_{t>0} |\gamma(t) - \sigma(t)|_{\ell_2}, \quad \gamma,\sigma \in C_\infty([0,\infty),\ell_2).$$

Let $M \in \mathscr{A}(n)$ be an Alexandrov space and $\Phi = \{\varphi_k\}_{k=1}^{\infty}$ a complete orthonormal basis on $L^2(M)$ consisting of eigenfunctions φ_k for $\lambda_k := \lambda_k(M)$ of Δ_M. Define a curve $F_\Phi[x]$ in $C_\infty([0,\infty),\ell_2)$ for $x \in M$ by

$$F_\Phi[x](t) := (e^{-(t+1/t)/2} e^{-\lambda_k t/2} \varphi_k(x))_{k=0}^{\infty}, \quad t \geq 0.$$

Note here that every eigenfunction of Δ_M is Hölder continuous on M (see Theorem 1.4 of [**10**]). Then, the map F_Φ gives an embedding of M into $C_\infty([0,\infty),\ell_2)$ and is called the *spectral embedding of M with respect to the basis Φ*. Denote by $\mathscr{F}M$ the set of such bases Φ on $L^2(M)$, and by $\mathscr{F}\mathscr{A}(n)$ the set of pairs (M,Φ) of $M \in \mathscr{A}(n)$ and $\Phi \in \mathscr{F}M$. For $(M,\Phi),(N,\Psi) \in \mathscr{F}\mathscr{A}(n)$, we define $SD^*((M,\Phi),(N,\Psi))$ to be the Hausdorff distance between $F_\Phi(M)$ and $F_\Psi(N)$ in $C_\infty([0,\infty),\ell_2)$, i.e.,

$$SD^*((M,\Phi),(N,\Psi)) := \inf\{\varepsilon > 0 \mid F_\Phi(M) \subset B(F_\Psi(N),\varepsilon), F_\Psi(N) \subset B(F_\Phi(M),\varepsilon)\}.$$

Lemma 4.2. *The family $\mathscr{F}\mathscr{A}(n,D,v)$ is SD^*-precompact and $\mathscr{A}(n,D,v)$ SD-precompact. The natural projection $\pi : (\mathscr{F}\mathscr{A}(n), SD^*) \to (\mathscr{A}(n), SD)$ is Lipschitz continuous.*

Proof. Note that the Bishop-Gromov inequality implies the uniform boundedness of the doubling constant:

$$\sup_{\substack{x \in M \in \mathscr{A}(n,D,v) \\ 0<r\le D}} \frac{\mathscr{H}^n(B(x,2r))}{\mathscr{H}^n(B(x,r))} < \infty. \tag{4.4}$$

Combining (4.4), (4.3), Lemma 4.1 of this paper, and Theorem 2.4 of [**16**] (see also [**14**] and Remark 7.1 of [**10**]) yields that the heat kernel $p_M : (0,\infty)\times M \times M \to \boldsymbol{R}$ on each $M \in \mathscr{A}(n,D,v)$ satisfies

$$p_M(t,x,x) \le c_0 t^{-\nu/2} \quad \text{for any } t>0 \text{ and } x \in M,$$

where c_0 and ν are positive constants depending only on n, D, v. Thus, the discussions in §1.3 and §2.2 of [**8**] complete the proof. □

5. Proof of main theorem.

Let a sequence $\{M_i\}_{i=1,2,\dots}$ in $\mathscr{A}(n)$ d_{GH}-converge to an $M \in \mathscr{A}(n)$, let $\delta_\ell \searrow 0$ be a sequence of numbers with $\delta_1 \ll 1/n$, and let $f_{\delta,i}$ be as in Theorem 3.1. Define a linear map $F_{\ell,i} : L^2(M_i) \to L^2(M)$ by

$$F_{\ell,i}(u) := I_{D_{\delta_\ell,i}} \cdot u \circ f_{\delta_\ell,i}, \quad u \in L^2(M_i),$$

where I_A denotes the *indicator function of a set A*, i.e., $I_A(x) := 1$ for $x \in A$, and $I_A(x) := 0$ for $x \notin A$. Set for simplicity $\lambda_k^i := \lambda_k(M_i)$ and $\lambda_k := \lambda_k(M)$. We take a basis $\Phi_i = \{\varphi_k^i\}_{k=0}^\infty \in \mathscr{F}M_i$ for each i. The following is a key to the proof of the main theorem.

Lemma 5.1. *For each k we have*

$$\lim_{i\to\infty} \lambda_k^i = \lambda_k.$$

Moreover, by replacing with a subsequence of $\{M_i\}$, there exist a sequence $\ell(i) \to \infty$ and a basis $\Phi = \{\varphi_k\}_{k=1}^\infty \in \mathscr{F}M$ such that $F_{\ell(i),i}(\varphi_k^i)$ L^2-converges to φ_k as $i \to \infty$ for each fixed k.

Proof of Theorem 1.1 under assuming Lemma 5.1. We first prove that the spectral topology is not stronger than the Gromov-Hausdorff topology on $\mathscr{A}(n)$. Suppose the contrary, so that we find a sequence $M_i \in \mathscr{A}(n)$ d_{GH}-converges to $M \in \mathscr{A}(n)$ as $i \to \infty$, but there exists a constant $\varepsilon_0 > 0$ such that $SD(M_i, M) \ge$

ε_0 for all i. Note that Theorem 3.1 says that $(M_i, \mathscr{H}^n)$ converges to $(M, \mathscr{H}^n)$ with respect to the measured Gromov-Hausdorff topology. Since the diameters of M_i are uniformly bounded and the $\mathscr{H}^n$-volumes of M_i bounded away from zero, Lemma 4.2 implies the SD^*-precompactness of $\{(M_i, \Phi_i)\}_i$, $\Phi_i \in \mathscr{F}M_i$. Therefore, by replacing with a subsequence of $\{i\}$, the embedded image $F_{\Phi_i}[M_i]$ converges to a subset $FX \subset C_\infty([0,\infty), \ell_2)$ with respect to the Hausdorff distance. By Lemma 5.1 and the continuity of eigenfunctions, we obtain $FX = F_\Phi[M]$ for some $\Phi \in \mathscr{F}M$, namely $SD^*((M_i, \Phi_i), (M, \Phi)) \to 0$ as $i \to \infty$, which contradicts $SD(M_i, M) \geq \varepsilon_0 > 0$.

We next prove that the spectral topology is not weaker than the Gromov-Hausdorff topology on $\mathscr{A}(n)$. Assume that a sequence $M_i \in \mathscr{A}(n)$ SD-converges to an $M \in \mathscr{A}(n)$ as $i \to \infty$. By Theorem 3.1(iii) of [**8**] and $\mathscr{H}^n(M) > 0$, we have $\inf_i \mathscr{H}^n(M_i) > 0$. By Lemma 2.5 of [**8**], the diameters of M_i are uniformly bounded. Therefore, $\{M_i\}$ is a d_{GH}-relatively compact subset of $\mathscr{A}(n)$. By the first part of the proof, any d_{GH}-limit of $\{M_i\}$ must coincide with the SD-limit M. Thus, M_i d_{GH}-converges to M. This completes the proof of the main theorem. □

The rest of the paper is devoted to prove Lemma 5.1. With the notations defined in the top of this section, for a fixed integer $k \geq -1$ and for every $j = 0, \ldots, k$, let φ_j be an eigenfunction of Δ_M associated with eigenvalue λ_j such that $\{\varphi_j\}_{j=0}^k$ is L^2-orthonormal, and let $i(\ell) \to \infty$ be a sequence of positive integers. Consider the following:

Assumption 5.1. For each ℓ and $j = 0, \ldots, k$,

(a) $$\sup_{i \geq i(\ell)} \|F_{\ell,i}(\varphi_j^i) - \varphi_j\|_{L^2} \leq \theta(\delta_\ell),$$

(b) $$\lim_{i \to \infty} \lambda_j^i = \lambda_j.$$

If $k = -1$, then Assumption 5.1 says the empty statement and is trivially true.

To obtain Lemma 5.1, it suffices to prove the following:

Claim 5.1. *Under Assumption* 5.1, *by replacing with subsequences of* $\{i\}$, $\{\delta_\ell\}$, *and* $\{i(\ell)\}$ *if necessary, there exists an eigenfunction* $\varphi_{k+1} \in W^{1,2}(M)$ *of* Δ_M *for* λ_{k+1} *such that* $\{\varphi_j\}_{j=0}^{k+1}$ *is* L^2*-orthonormal and that* (a) *and* (b) *both hold for every* ℓ *and* $j = 0, \ldots, k+1$.

We suppose Assumption 5.1 and shall prove the claim.

Lemma 5.2. *Let* Ω *be a domain of a Lipschitz-Riemannian manifold such that* $W^{1,2}(\Omega)$ *is compactly embedded into* $L^2(\Omega)$, *and let* $\{u_i\} \subset W^{1,2}(\Omega)$ *be a sequence* *satisfying*

$$\varliminf_{i\to\infty} \|u_i\|_{W^{1,2}(\Omega)} < \infty.$$

Then, there exists a subsequence $\{u_{j(i)}\}$ *of* $\{u_i\}$ *converging to a function* $u \in W^{1,2}(\Omega)$ L^2*-strongly and* $W^{1,2}$*-weakly such that*

$$\|u\|_{W^{1,2}(\Omega)} \le \varliminf_{i\to\infty} \|u_i\|_{W^{1,2}(\Omega)}.$$

Proof. A standard discussion in functional analysis. □

Lemma 5.3. *Let U and V be two domains of Lipschitz-Riemannian manifolds and* $f : U \to V$ *a* δ*-almost isometry,* $\delta > 0$. *Then, we have* $u \circ f^{-1} \in W^{1,2}(V)$ *for any* $u \in W^{1,2}(U)$, *and the map* $W^{1,2}(U) \ni u \mapsto u \circ f^{-1} \in W^{1,2}(V)$ *is a* $\theta(\delta)$*-almost isometry with respect to* L^2 *and* $W^{1,2}$*-norm.*

Proof. See, for example, Theorem 4(ii) in 4.2.2 of [**4**] and its Remark. □

Lemma 5.4. *We have*

$$\varlimsup_{i\to\infty} \lambda_{k+1}^i \le \lambda_{k+1}.$$

Proof. There exists an eigenfunction $\psi \in W^{1,2}(M)$ for λ_{k+1} such that $\{\varphi_1, \dots, \varphi_k, \psi\}$ is L^2-orthonormal. By Theorem 1.1 of [**10**], for any $\delta > 0$ there exists an approximation $\psi_\delta \in W^{1,2}(M)$ of ψ such that $\mathrm{supp}[\psi_\delta] \cap \hat{S}_\delta = \emptyset$ and

$$\|\psi_\delta - \psi\|_{W^{1,2}} < \delta. \tag{5.1}$$

For i large enough, we have $\mathrm{supp}[\psi_\delta] \subset D_{\delta,i}$ and set

$$\psi_\delta^i := \begin{cases} \psi_\delta \circ f_{\delta,i}^{-1} & \text{on } f_{\delta,i}(D_{\delta,i}), \\ 0 & \text{on } M_i \setminus f_{\delta,i}(D_{\delta,i}). \end{cases}$$

Lemma 5.3 implies $\psi_\delta^i \in W^{1,2}(M_i)$. In what follows, we assume that i is sufficiently large against k and a fixed δ. Then, by Lemma 5.3 and $\mathscr{E}(\psi) = \lambda_{k+1}$, we have

$$\mathscr{E}(\psi_\delta^i) = (1 + \theta(\delta))\mathscr{E}(\psi_\delta) = (1 + \theta(\delta))(\lambda_{k+1} + \theta(\delta)) = \lambda_{k+1} + \theta(\delta).$$

Remark here that $\theta(\delta)$ is independent of i. By Lemma 5.3 and (5.1),

$$\|\psi_\delta^i\|_{L^2} = (1 + \theta(\delta))\|\psi_\delta\|_{L^2} = (1 + \theta(\delta))(\|\psi\|_{L^2} + \theta(\delta)) = 1 + \theta(\delta)$$

and also, in the same way, $(\psi_\delta^i, \varphi_j^i)_{L^2} = \theta(\delta)$ for each $j = 1, \dots, k$. Using the Gram-Schmidt method, we find a function $\hat{\psi}_\delta^i \in W^{1,2}(M_i)$ such that $(\hat{\psi}_\delta^i, \varphi_j^i)_{L^2} =$

0, $\|\hat{\psi}_\delta^i\|_{L^2} = 1$, and $\|\hat{\psi}_\delta^i - \psi_\delta^i\|_{W^{1,2}} \le \theta(\delta)$. Therefore,

$$\varlimsup_{i\to\infty} \lambda_{k+1}^i \le \varlimsup_{i\to\infty} \frac{\mathscr{E}(\hat{\psi}_\delta^i)}{\|\hat{\psi}_\delta^i\|_{L^2}} \le \lambda_{k+1} + \theta(\delta).$$

By taking $\delta \to 0$, this completes the proof. □

Lemma 5.5. *For any fixed ℓ, there exists a subsequence $I(\ell) \subset \{i\}$ such that as $I(\ell) \ni i \to \infty$, $F_{\ell,i}(\varphi_{k+1}^i)$ converges to a function $\varphi_{k+1,\ell} \in W^{1,2}(M)$ in the $L^2(\Omega)$-strong and $W^{1,2}(\Omega)$-weak topologies for any compact subset $\Omega \subset M \backslash \hat{S}_{\delta_\ell}$, and the limit satisfies*

$$\|\varphi_{k+1,\ell}\|_{W^{1,2}}^2 \le (1+\theta(\delta_\ell))\left(1 + \varlimsup_{i\to\infty} \lambda_{k+1}^i\right).$$

Proof. Let $\Omega \subset M \backslash \hat{S}_{\delta_\ell}$ be any compact domain with Lipschitz boundary and set $\Omega_i := f_{\delta_\ell,i}(\Omega)$ for large enough i. Note that Ω is a compact Lipschitz-Riemannian manifold with Lipschitz boundary, so that a standard discussion shows that $W^{1,2}(\Omega)$ is compactly embedded in $L^2(\Omega)$. Since $\|\varphi_{k+1}^i\|_{W^{1,2}(\Omega_i)}^2 \le \|\varphi_{k+1}^i\|_{W^{1,2}}^2 = 1 + \lambda_{k+1}^i$, we have, by Lemmas 5.3 and 5.4,

$$\varlimsup_{i\to\infty} \|F_{\ell,i}(\varphi_{k+1}^i)\|_{W^{1,2}(\Omega)}^2 \le (1+\theta(\delta_\ell))(1+\lambda_{k+1}).$$

Applying Lemma 5.2 yields that there exist a subsequence $I(\ell) \subset \{i\}$ and a function $\varphi_{k+1,\ell} \in W^{1,2}(\Omega)$ such that

(i) as $I(\ell) \ni i \to \infty$, $F_{\ell,i}(\varphi_{k+1}^i)|_\Omega$ converges to $\varphi_{k+1,\ell}$ $L^2(\Omega)$-strongly and $W^{1,2}(\Omega)$-weakly;

(ii) we have

$$\|\varphi_{k+1,\ell}\|_{W^{1,2}(\Omega)}^2 \le (1+\theta(\delta_\ell))\left(1 + \varlimsup_{i\to\infty} \lambda_{k+1}^i\right).$$

By taking a monotone increasing sequence of Ω tending to $M \backslash S_{\delta_\ell}$, the diagonal argument extends $\varphi_{k+1,\ell}$ to a function in $W_{\mathrm{loc}}^{1,2}(M)$ which satisfies (i) and (ii) for any compact subset $\Omega \subset M \backslash \hat{S}_\delta$. This completes the proof. □

Our next aim is to control the L^2-mass of the eigenfunctions φ_{k+1}^i around the δ-singular set of M.

Lemma 5.6. *For any fixed ℓ, let $\Omega \subset M \backslash \hat{S}_{\delta_\ell}$ be any compact subset such that $\mathscr{H}^n(M \backslash \Omega) \le \delta_\ell$, and set $\Omega_i := f_{\delta_\ell,i}(\Omega)$. Then we have*

$$\varlimsup_{i\to\infty} \|\varphi_{k+1}^i\|_{L^2(M_i \backslash \Omega_i)} \le \theta(\delta_\ell).$$

Proof. Since $\mathscr{H}^n(M_i) \to \mathscr{H}^n(M)$ and $\mathrm{diam}(M_i) \to \mathrm{diam}(M)$ as $i \to \infty$, Lemma 4.1 implies $\underline{\lim}_{i\to\infty} a(M_i) > 0$. Therefore, by (4.1), (4.2), and $\mathscr{E}(\varphi^i_{k+1}) = \lambda^i_{k+1}$, the h-mollifier $\tilde{\varphi}^i_{k+1}$ of φ^i_{k+1}, $0 < h < 1$, satisfies that for some constant $c > 0$ and for all sufficiently large i,

$$\sup_{M_i} |\tilde{\varphi}^i_{k+1}| \le \frac{c}{h} \quad \text{and} \quad \|\varphi^i_{k+1} - \tilde{\varphi}^i_{k+1}\|_{L^2} \le ch.$$

It then follows that

$$\begin{aligned}
\|\varphi^i_{k+1}\|_{L^2(M_i\setminus\Omega_i)} &\le \|\varphi^i_{k+1} - \tilde{\varphi}^i_{k+1}\|_{L^2} + \|\tilde{\varphi}^i_{k+1} - I_{\Omega_i}\tilde{\varphi}^i_{k+1}\|_{L^2} + \|I_{\Omega_i}\tilde{\varphi}^i_{k+1} - I_{\Omega_i}\varphi^i_{k+1}\|_{L^2} \\
&\le 2\|\varphi^i_{k+1} - \tilde{\varphi}^i_{k+1}\|_{L^2} + \|\tilde{\varphi}^i_{k+1}\|_{L^2(M_i\setminus\Omega_i)} \\
&\le 2ch + \frac{c\mathscr{H}^n(M_i\setminus\Omega_i)}{h}.
\end{aligned}$$

Here,

$$\begin{aligned}
\overline{\lim_{i\to\infty}}\, \mathscr{H}^n(M_i\setminus\Omega_i) &= \overline{\lim_{i\to\infty}}\, (\mathscr{H}^n(M_i) - \mathscr{H}^n(\Omega_i)) \\
&\le \mathscr{H}^n(M) - (1 - \theta(\delta_\ell))\mathscr{H}^n(\Omega) \\
&= \mathscr{H}^n(M\setminus\Omega) + \theta(\delta_\ell)\mathscr{H}^n(\Omega) \le \theta(\delta_\ell).
\end{aligned}$$

Therefore, if we set h to be the square root of the last $\theta(\delta_\ell)$ of the above formula, the proof is completed. □

Lemma 5.7. *For any fixed ℓ and for all sufficiently large $i \in I(\ell)$, we have*

(1) $$\|F_{\ell,i}(\varphi^i_{k+1})\|_{L^2} = 1 + \theta(\delta_\ell),$$

(2) $$\|\varphi_{k+1,\ell}\|_{L^2} = 1 + \theta(\delta_\ell),$$

(3) $$\|F_{\ell,i}(\varphi^i_{k+1}) - \varphi_{k+1,\ell}\|_{L^2} \le \theta(\delta_\ell),$$

(4) $$|(\varphi_{k+1,\ell}, \varphi_j)_{L^2}| \le \theta(\delta_\ell) \quad \textit{for any } j = 0, \ldots, k.$$

Proof. Let $\Omega \subset M\setminus\hat{S}_{\delta_\ell}$ be any compact subset such that $\mathscr{H}^n(M\setminus\Omega) \le \delta_\ell$, and set $\Omega_i := f_{\delta_\ell,i}(\Omega)$.

(1): Lemma 5.3 implies

$$\|F_{\ell,i}(\varphi^i_{k+1})\|_{L^2} \le (1 + \theta(\delta_\ell))\|\varphi^i_{k+1}\|_{L^2(f_{\delta_\ell,i}(D_{\delta_\ell,i}))} \le 1 + \theta(\delta_\ell).$$

On the other hand, by Lemmas 5.3 and 5.6,

$$\|F_{\ell,i}(\varphi^i_{k+1})\|_{L^2(\Omega)} \ge (1 - \theta(\delta_\ell))\|\varphi^i_{k+1}\|_{L^2(\Omega_i)} \ge 1 - \theta(\delta_\ell).$$

Thus we obtain (1).

(2): By Lemma 5.5 and the above,

$$\|\varphi_{k+1,\ell}\|_{L^2(\Omega)} = \lim_{I(\ell)\ni i\to\infty} \|F_{\ell,i}(\varphi_{k+1}^i)\|_{L^2(\Omega)} = 1 + \theta(\delta_\ell).$$

Taking $\Omega \to M\backslash\hat{S}_{\delta_\ell}$ we have (2).

(3): It follows from (1) that $F_{\ell,i}(\varphi_{k+1}^i)$ converges to $\varphi_{k+1,\ell}$ $L^2(M)$-weakly, which together with (1) and (2) shows (3).

(4): By $(\varphi_{k+1}^i, \varphi_j^i)_{L^2} = 0$, $\|\varphi_j^i\|_{L^2} = 1$, and by Lemma 5.6, we have, for all sufficiently large $i \in I(\ell)$,

$$\begin{aligned}|(\varphi_{k+1}^i, \varphi_j^i)_{L^2(\Omega_i)}| &= |(\varphi_{k+1}^i, \varphi_j^i)_{L^2} - (I_{M_i\backslash\Omega_i}\varphi_{k+1}^i, \varphi_j^i)_{L^2}| \\ &\le \|\varphi_{k+1}^i\|_{L^2(M_i\backslash\Omega_i)} \le \theta(\delta_\ell).\end{aligned}$$

Therefore, by (1), (2), Lemma 5.5, (a) of Assumption 5.1, and by Lemma 5.3,

$$\begin{aligned}|(\varphi_{k+1,\ell}, \varphi_j)_{L^2(\Omega)}| &\le \varlimsup_{I(\ell)\ni i\to\infty} |(F_{\ell,i}(\varphi_{k+1}^i), F_{\ell,i}(\varphi_j^i))_{L^2(\Omega)}| + \theta(\delta_\ell) \\ &\le (1+\theta(\delta_\ell)) \varlimsup_{I(\ell)\ni i\to\infty} |(\varphi_{k+1}^i, \varphi_j^i)_{L^2(\Omega_i)}| + \theta(\delta_\ell) \le \theta(\delta_\ell).\end{aligned}$$

This completes the proof. □

Proof of Claim 5.1. By Lemma 5.2, some subsequence of $\varphi_{k+1,\ell}$ converges to a function $\varphi_{k+1} \in W^{1,2}(M)$ L^2-strongly and $W^{1,2}$-weakly. (2), (4) of Lemma 5.7, and Lemma 5.5 respectively imply

$$\|\varphi_{k+1}\|_{L^2} = 1, \quad (\varphi_{k+1}, \varphi_j)_{L^2} = 0, \quad \|\varphi_{k+1}\|_{W^{1,2}}^2 \le 1 + \varliminf_{i\to\infty} \lambda_{k+1}^i.$$

We therefore obtain

$$\lambda_{k+1} \le \frac{\mathscr{E}(\varphi_{k+1})}{\|\varphi_{k+1}\|_{L^2}} = \mathscr{E}(\varphi_{k+1}) \le \varliminf_{i\to\infty} \lambda_{k+1}^i,$$

which together with Lemma 5.4 implies (b) for $j = k+1$ and that φ_{k+1} is an eigenfunction for eigenvalue λ_{k+1}.

We may assume that $I(\ell+1) \subset I(\ell)$ for every ℓ. Thus, if $i(\ell) \in I(\ell)$ is taken to be large enough for each ℓ, the diagonal argument together with Lemma 5.7(3) proves (a) by replacing with subsequences. This completes the proof.

□

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