J. Math. Soc. Japan
Vol. 55, No. 4, 2003

Complete non-compact spacelike hypersurfaces of constant mean curvature in de Sitter spaces

By Sebastián MONTIEL

(Received Apr. 9, 2001)
(Revised Apr. 16, 2002)

Abstract. We use the half-space model for the open set of a de Sitter space associated to the steady state space to obtain some sharp *a priori* estimates for the height and the slope of certain constant mean curvature spacelike graphs. These estimates allow us to prove some existence and uniqueness theorems about complete non-compact constant mean curvature spacelike hypersurfaces in de Sitter spaces with prescribed asymptotic future boundary. Their geometric properties are studied.

1. Introduction.

Constant mean curvature hypersurfaces are first order solutions to the isoperimetric problem both when the ambient space is a Riemannian manifold as well as it is a Lorentzian manifold. In the first case, they have been extensively studied. In the second case, they have also attracted a big amount of interest from a physical point of view (see [**C**], [**CB**], [**CFM**], [**Go1**], [**Go2**], [**MT**], [**St**]), because they are convenient initial data for the Cauchy problem corresponding to the Einstein equation and suitable tools in the study of gravitational waves.

From a mathematical point of view, they exhibit nice Bernstein type properties. In fact, some of the first papers dealing with existence and uniqueness of this type of hypersurfaces were written by Calabi [**C**] and Cheng and Yau [**CY**]. They discovered a Bernstein theorem for entire maximal (i.e. identically zero mean curvature) spacelike hypersurfaces in a Minkowski space. Later and by way of contrast, Treibergs [**Tr**] showed that there are a lot of entire spacelike graphs in the Minkowski space with non-zero constant mean curvature and he classified them according to their behaviours at the infinity.

In this paper we shall deal with spacelike hypersurfaces Σ^n of the *de Sitter space* $\boldsymbol{S}_1^{n+1}$, which can be thought of as the hyperquadric of the Minkowski $(n+2)$-dimensional space consisting of the unit spacelike vectors. This de Sitter space is the simplest example of globally hyperbolic and spatially closed Lorentzian manifold. It is 1-connected and has positive constant sectional curvature and, so, it is the Lorentzian analogue to the round sphere in Riemannian geometry.

The de Sitter space has a lot of complete spacelike hypersurfaces with constant mean curvature (see Section 3 below) with a good geometrical behaviour. The *umbilical* ones are obtained by intersecting $\boldsymbol{S}_1^{n+1}$ with hyperplanes. According to the causal

---

2000 *Mathematics Subject Classification.* 53A10, 53C42, 53C50, 35Q75, 83C99.
Key Words and Phrases. Constant mean curvature, spacelike hypersurface, de Sitter space.
Research partially supported by a spanish MCYT-FEDER grant BFM2001–2967.

character of the hyperplane, the corresponding hypersurface will be isometric to a sphere, a Euclidean space or a hyperbolic space. Goddard [**Go1**] had conjectured that those umbilical examples were the only complete spacelike hypersurfaces of $\boldsymbol{S}_1^{n+1}$ with constant mean curvature. The author [**Mo1**] solved this conjecture in the affirmative when the hypersurface Σ^n is compact (the case $n=2$ had been already studied by Akutagawa in [**Ak**] and the maximal case in [**CFM**]). In this way, an earlier theorem due to Akutagawa [**Ak**] and also in part to Ramanathan [**R**] was generalized. This result says that, if the constant mean curvature H of a complete spacelike hypersurface Σ^n of $\boldsymbol{S}_1^{n+1}$ satisfies $H^2 < 4(n-1)/n^2$, if $n>2$, or $H^2 \leq 1$, if $n=2$, then it is umbilical. In fact, under this assumption on the mean curvature, the Gauss equation for the hypersurface implies that its Ricci curvature is bounded from below by a positive constant. So, according to the Bonnet–Myers theorem, it must be compact and then one obtains the Akutagawa result via the above-mentioned theorem in [**Mo1**]. Later, Oliker showed in [**Ol**] that this Bernstein type property is stable with respect to perturbations of the data.

These results by Akutagawa and the author give the best possible sufficient condition for a complete spacelike hypersurface in $\boldsymbol{S}_1^{n+1}$ being umbilical. Indeed, non-umbilical examples with $n=2$ and $H^2>1$ were exhibited in [**Ak**], and the author pointed out in [**Mo1**] that, for $n>2$ and $H^2 \geq 4(n-1)/n^2$, there are also non-umbilical complete hypersurfaces. Precisely, these examples were the *hyperbolic cylinders* (cf. [**A1**], [**KKN**]) whose squared constant mean curvatures (depending only on its radius) take any value in $[4(n-1)/n^2, +\infty[$. Hence it seems natural to study complete (and *a fortiori* non-compact) spacelike hypersurfaces Σ^n in $\boldsymbol{S}_1^{n+1}$ with constant mean curvature H such that $H^2 \geq 4(n-1)/n^2$. Are there any examples besides the umbilical ones and the hyperbolic cylinders? The author gave in [**Mo2**] a partial uniqueness result for the boundary value $4(n-1)/n^2$. In this paper we shall see that this uniqueness fails to hold, at least when $H^2>1$. In fact, we shall construct complete non-compact spacelike hypersurfaces of the de Sitter space with constant mean curvature H such that $H^2>1$ and with prescribed *asymptotic future boundary* (see Section 7 for a definition). To do that, we shall consider the half $\mathscr{H}^{n+1}$ of the de Sitter space which models the so-called *steady state space* (cf. [**HE**], p. 127) and we shall use that this extendible (and so non-complete) space is isometric to the half-space $\boldsymbol{R}_+^{n+1} = \boldsymbol{R}^n \times \boldsymbol{R}^+$, endowed with the Lorentzian metric

$$ds^2_{(x,x_{n+1})} = \frac{1}{x_{n+1}^2}(|dx|^2 - (dx_{n+1})^2) \quad (x \in \boldsymbol{R}^n, x_{n+1} \in \boldsymbol{R}_+).$$

In this setting, the horizontal hyperplanes $x_{n+1} = t$, $t \in]0, +\infty[$, are umbilical spacelike hypersurfaces of constant mean curvature $H=1$ (for graphs we shall use the *upward* orientation) and we shall refer to them as *time slices* of the steady state space. The horizontal hyperplane $x_{n+1}=0$ which is the boundary of $\mathscr{H}^{n+1}$ represents its *future infinity* (see [**HE**], where it is usually denoted by $\mathscr{J}^+$).

By using this upper half-space model, we shall obtain some height and gradient estimates for spacelike graphs with constant mean curvature $H>1$ over certain compact domains in a given time slice. They allow us to solve on these domains the Dirichlet problem corresponding to the constant mean curvature equation, with zero boundary data. A careful control of the height and the slope of these solutions will lead us to find complete spacelike hypersurfaces with constant mean curvature $H > 1$ whose asymptotic future boundary is a given compact domain with mean convex boundary of the future infinity $\mathscr{J}^+$ of the steady state space. These hypersurfaces are umbilical only when the prescribed asymptotic boundary is chosen to be a round sphere in $\mathscr{J}^+$. Moreover, they are never hyperbolic cylinders, since this type of cylinders have asymptotic future boundaries with codimension bigger than one.

2. Halfspace model for the steady state space.

We denote by $\boldsymbol{R}_1^{n+2}$ the $(n+2)$-dimensional Minkowski space, which is nothing but the real vector space $\boldsymbol{R}^{n+2}$ endowed with the Lorentz metric defined by

$$\langle u, v \rangle = u_0 v_0 + \cdots + u_n v_n - u_{n+1} v_{n+1},$$

for all $u, v \in \boldsymbol{R}^{n+2}$. The one-sheeted hyperboloid

$$\boldsymbol{S}_1^{n+1} = \{p \in \boldsymbol{R}_1^{n+2} \mid \langle p, p \rangle = 1\}$$

consisting of all unit spacelike vectors in $\boldsymbol{R}_1^{n+2}$, equipped with the induced metric, is a geodesically complete Lorentzian manifold with constant curvature one, usually called *de Sitter space*, which has the topology $\boldsymbol{S}^n \times \boldsymbol{R}$. Take a non-zero null vector $a \in \boldsymbol{R}_1^{n+2}$ in the past half of the null cone (with vertex at the origin), that is, $\langle a, a \rangle = 0$ and $\langle a, e_{n+1} \rangle > 0$, where $e_{n+1} = (0, \dots, 0, 1)$. Then the open region of the de Sitter space given by

$$\mathscr{H}^{n+1} = \{p \in \boldsymbol{S}_1^{n+1} \mid \langle p, a \rangle > 0\}$$

is the so-called *steady state space* (Figure 1).

Of course, $\mathscr{H}^{n+1}$ is extendible and, so, non-complete, being only half a de Sitter space. Its boundary, as a subset of $\boldsymbol{S}_1^{n+1}$, is the null hypersurface

$$\{p \in \boldsymbol{S}_1^{n+1} \mid \langle p, a \rangle = 0\},$$

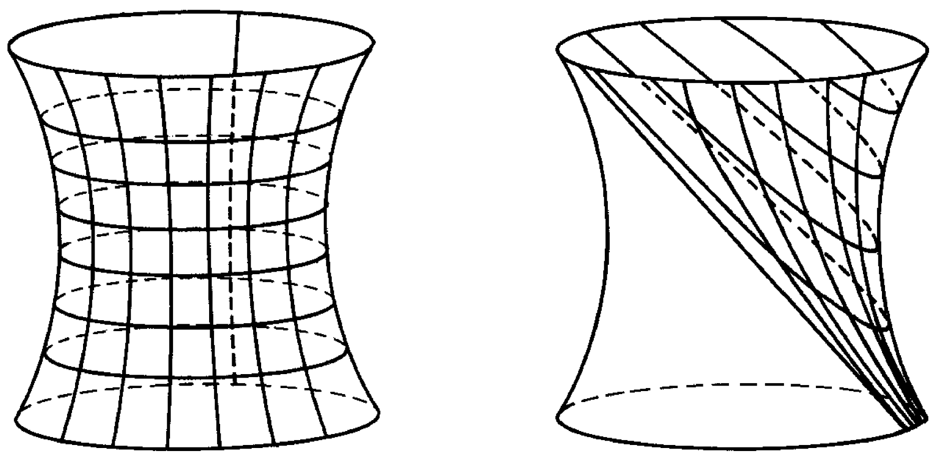

Figure 1. de Sitter and steady state spaces.

whose topology is $\boldsymbol{R} \times \boldsymbol{S}^{n-1}$. The characteristic property of this space is that it admits a foliation by means of the spacelike hypersurfaces

$$L^n(\tau) = \{p \in \boldsymbol{S}_1^{n+1} \mid \langle p, a \rangle = \tau\} \quad \tau \in]0, +\infty[,$$

which are (see [**Mo1**]) umbilical hypersurfaces of the de Sitter space having constant mean curvature one with respect to the unit past directed normal fields

$$N_\tau(p) = -p + \frac{1}{\tau} a \quad (p \in L^n(\tau)).$$

In the steady state model of the universe, matter is supposed to move along geodesics normal to these hypersurfaces. Then, they represent constant time slices and, since all of them are isometric to a Euclidean space $\boldsymbol{R}^n$, in this cosmological setting, the geometry of the spatial sections remains unchanged. It is convenient to notice that the hypersurfaces $L^n(\tau)$ approach to the boundary of $\mathscr{H}^{n+1}$ when τ tends to zero and that, when τ tends to $+\infty$, they approach to the spacelike future infinity for timelike and null lines of the de Sitter space, that, following [**HE**], we will denote by $\mathscr{J}^+$.

Now, we recall that there exists a well-known *upper half-space model* for $\mathscr{H}^{n+1}$ that we need to use. In fact, consider the map $\phi : \mathscr{H}^{n+1} \to \boldsymbol{R}^{n+1} = \boldsymbol{R}^n \times \boldsymbol{R}$ by

$$\phi(p) = \frac{1}{\langle p, a \rangle}(p - \langle p, a \rangle b - \langle p, b \rangle a, 1), \tag{1}$$

where $b \in \boldsymbol{R}_1^{n+2}$ is another null vector such that $\langle a, b \rangle = 1$ (and so b is in the future half of the null cone, that is, $\langle b, e_{n+1} \rangle < 0$) and where $\boldsymbol{R}^n$ stands for the orthogonal complement of the Lorentz plane spanned by a and b. Then, it is immediate to see that the image of ϕ lies in the half-space $\boldsymbol{R}_+^{n+1} = \boldsymbol{R}^n \times \boldsymbol{R}_+$ and that ϕ is a diffeomorphism from $\mathscr{H}^{n+1}$ to $\boldsymbol{R}_+^{n+1}$. Moreover, if $v \in T_p\mathscr{H}^{n+1} = T_p\boldsymbol{S}_1^{n+1}$, we have

$$(d\phi)_p(v) = \frac{1}{\langle p, a \rangle}(v - \langle v, a \rangle b - \langle v, b \rangle a, 0) - \frac{\langle v, a \rangle}{\langle p, a \rangle^2}(p - \langle p, a \rangle b - \langle p, b \rangle a, 1). \tag{2}$$

From this, it is straightforward to check that

$$\langle (d\phi)_p(v), (d\phi)_p(v) \rangle = \frac{1}{\langle p, a \rangle^2}\langle v, v \rangle.$$

Hence the map $\phi : \mathscr{H}^{n+1} \to \boldsymbol{R}_+^{n+1} = \boldsymbol{R}^n \times \boldsymbol{R}_+$ is an isometry provided that we put on the half-space $\boldsymbol{R}_+^{n+1}$ the Lorentz metric g given by

$$g_{(x, x_{n+1})}(u, v) = \frac{1}{x_{n+1}^2}\langle u, v \rangle \quad (x \in \boldsymbol{R}^n, x_{n+1} \in \boldsymbol{R}_+),$$

for all $u, v \in \boldsymbol{R}^{n+1}$, that is, the Minkowski metric divided by x_{n+1}^2. It is important to point out that the isometry ϕ inverts time orientation.

Then the half-space $\boldsymbol{R}_+^{n+1}$ endowed with that Lorentz metric

$$g_{(x, x_{n+1})} = \frac{1}{x_{n+1}^2}(|dx|^2 - (dx_{n+1})^2) \tag{3}$$

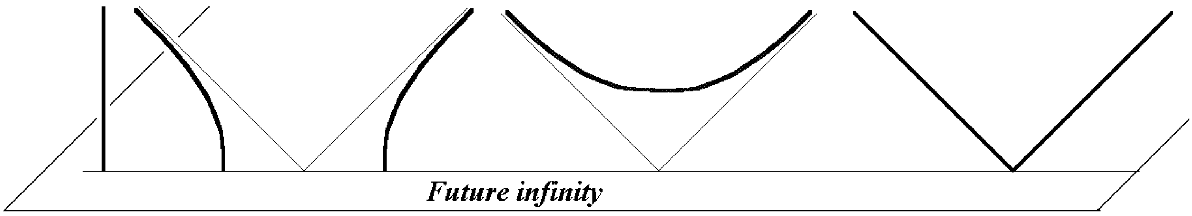

Figure 2. Geodesics in the upper half-space model.

is an *upper half-space model* for the steady state space. The existence of this upper half-space model leads us to think of $\mathscr{H}^{n+1}$ as the Lorentzian analogue to usual hyperbolic space and allows us to use suitable modifications of the Riemannian technics (for example in [**An1**], [**An2**], [**LM**], [**NS**], [**To**]) to study its constant mean curvature hypersurfaces. Taking into account that, according (1),

$$x_{n+1}(\phi(p)) = \frac{1}{\langle p, a\rangle} \quad (p \in \mathscr{H}^{n+1}), \tag{4}$$

the time slices $L^n(\tau)$ correspond to the horizontal hyperplanes $x_{n+1} = 1/\tau = t$, where $t \in]0, +\infty[$ that we shall denote by $L^n(t)$. All of them are umbilical spacelike hypersurfaces which have constant mean curvature one with respect to the upward orientation because

$$(d\phi)_p(N_\tau(p)) = \frac{1}{\tau}(0,1) \quad (p \in L^n(\tau)).$$

Notice that, in this model of the steady state space, the spacelike future infinity $\mathscr{J}^+$ is represented by the boundary hyperplane $x_{n+1} = 0$ and the unit geodesic flow normal to the hypersurfaces $L^n(\tau)$, with $0 < \tau$, corresponds to the vertical lines through that boundary hyperplane. This model also shows that the causal structures (see [**HE**], p. 127) of $\mathscr{H}^{n+1}$ and of the open half-space $\boldsymbol{R}_+^{n+1}$ of $\boldsymbol{R}_1^{n+1}$ are the same.

Also, using this upper half-space model for $\mathscr{H}^{n+1}$, we can easily visualize its isometries. They are, from (3), just the conformal maps of the Minkowski space $\boldsymbol{R}_1^{n+1}$ preserving the half-space $\boldsymbol{R}_+^{n+1}$, for example, horizontal translations or Euclidean homotheties with center in the future infinity $\mathscr{J}^+$.

Geodesics and totally geodesics submanifolds of $\mathscr{H}^{n+1}$, corresponding in the hyperquadric model to intersections of the de Sitter space $\boldsymbol{S}_1^{n+1}$ with affine subspaces of $\boldsymbol{R}_1^{n+2}$ passing through the origin, are represented in Figure 2 with respect to the upper half-space frame.

3. Umbilical hypersurfaces and hyperbolic cylinders.

We are interested in spacelike hypersurfaces of a de Sitter space with constant mean curvature. Basic examples are the *umbilical* ones, which are obtained by intersecting $\boldsymbol{S}_1^{n+1}$ with affine hyperplanes

$$\{p \in \boldsymbol{R}_1^{n+2} \mid \langle p, c\rangle = \sigma\} \quad (c \in \boldsymbol{R}_1^{n+2} - \{0\} \text{ and } \sigma^2 > |c|^2)$$

of the ambient Minkowski space. Choosing the orientation on these hypersurfaces so that the corresponding (constant) mean curvature is non-negative, we have that $H \in [0,1[$ when c is a timelike vector, $H = 1$ when it is null and $H \in]1,+\infty[$ when c is a spacelike vector. Moreover, the corresponding hypersurface is isometric to a sphere, or to a Euclidean space, or to two copies of a hyperbolic space, respectively (see [**Mo1**] for more details).

When the spacelike hypersurface of $\boldsymbol{S}_1^{n+1}$ with constant mean curvature is compact without boundary, we proved in [**Mo1**] that it must be one of these umbilical examples. We also know that, in the non-compact case, there are non-umbilical examples with a good geometrical behaviour: the *hyperbolic cylinders* given by

$$\{p \in \boldsymbol{S}_1^{n+1} \mid p_0^2 + \cdots + p_k^2 = \cosh^2 r\}$$

for $r > 0$ and $1 \le k \le n-1$. We pointed out in [**Mo1**] that, for a suitable orientation, they have constant mean curvatures, taking all the possible values in the interval $[2\sqrt{n-1}/n, +\infty[$. They have two connected components which are isometric to a product of an $(n-k)$-dimensional hyperbolic space and a k-dimensional sphere.

In order to construct other hypersurfaces of this type in the de Sitter space, we shall restrict ourselves to spacelike immersions $\psi : \Sigma^n \to \mathscr{H}^{n+1} \subset \boldsymbol{S}_1^{n+1}$. We shall start by using the upper half-space model for $\mathscr{H}^{n+1}$. So it will be identified with $\boldsymbol{R}_+^{n+1}$ with the conformal metric (3). Let N' be a unit (timelike) normal field for ψ with respect to the Minkowski metric. Hence

$$N = \psi_{n+1} N'$$

is a unit normal field for ψ with respect to the metric (3) of $\mathscr{H}^{n+1}$. Now we will denote by k_i' and k_i, $i = 1, \dots, n$, the principal curvatures of the immersion ψ computed respectively with respect to the Minkowski and the de Sitter metrics and the choices of unit normal fields. Analogously we will represent by H' and H the corresponding mean curvatures. By using the relation between the Levi-Civita connections of the two considered conformal metrics, we see that these principal and mean curvatures are related as follows

$$k_i = \psi_{n+1} k_i' + N_{n+1}' = -\langle k_i' \psi + N', e_{n+1} \rangle \quad i = 1, \dots, n, \tag{5}$$

$$H = -\langle H' \psi + N', e_{n+1} \rangle, \tag{6}$$

where $e_{n+1} = (0, \dots, 0, 1) \in \boldsymbol{R}^{n+1}$.

A consequence from (5) is that the umbilical hypersurfaces of $\mathscr{H}^{n+1}$ are just the umbilical hypersurfaces of the Minkowski space $\boldsymbol{R}_1^{n+1}$ which are contained in the half-space $\boldsymbol{R}_+^{n+1}$. That is, hyperplanes and one-sheeted and two-sheeted hyperboloids invariant under rotations whose axis is a vertical line. If we restrict ourselves to *spacelike* hypersurfaces, we have only spacelike hyperplanes and two-sheeted hyperboloids. Among them, only the hyperplanes and the lower sheets of hyperboloids are complete (notice that upper sheets intersect the boundary $\langle p, a \rangle = 0$ when one uses the original definition of $\mathscr{H}^{n+1}$). Using (6), one can see that, in the half-space model, the value of the constant mean curvature H of each hyperboloid

$$\{x \in \boldsymbol{R}_+^{n+1} \mid \langle x - c, x - c \rangle = -r^2\}$$

depends on the radius $r > 0$ and the height c_{n+1} of the center $c = (c_1, \dots, c_{n+1}) \in \boldsymbol{R}^{n+1}$. Let denote by $Q_+(r, c)$ the upper hyperboloid of radius r and center c, and denote the lower hyperboloid by $Q_-(r, c)$. Let $\Pi(u)$ be (the part included in the half-space model of) the hyperplane perpendicular to a unit upward timelike vector $u = (u_1, \dots, u_{n+1}) \in \boldsymbol{R}_1^{n+1}$. The following table is for the constant mean curvatures H of $Q_\pm(r, c)$ and $\Pi(u)$ with respect to the upward normal vector (see also Figure 3).

	$Q_+(r, c)$	$Q_-(r, c)$	$\Pi(u)$
	$H = -c_{n+1}/r$ (non-complete)	$H = c_{n+1}/r$ (complete)	$H = u_{n+1}$ (complete)
$H < 1$	$c_{n+1} > -r$	—	—
$H = 1$	$c_{n+1} = -r$	—	$u_{n+1} = 1$ horizontal hyperplane
$H > 1$	$c_{n+1} < -r$	$c_{n+1} > r$	$u_{n+1} > 1$

Also, it is not difficult to check that one of the components of each hyperbolic cylinder is contained in $\mathscr{H}^{n+1}$ (choosing the vector a as $(0, \dots, 0, 1, -1) \in \boldsymbol{R}^{n+2}$) and that it appears in the half-space model of $\mathscr{H}^{n+1}$ as the set

$$\{x = (x_1, \dots, x_{n+1}) \in \boldsymbol{R}_+^{n+1} \mid x_1^2 + \cdots + x_{k+1}^2 - x_{n+1}^2 \cosh^2 r = 0\}.$$

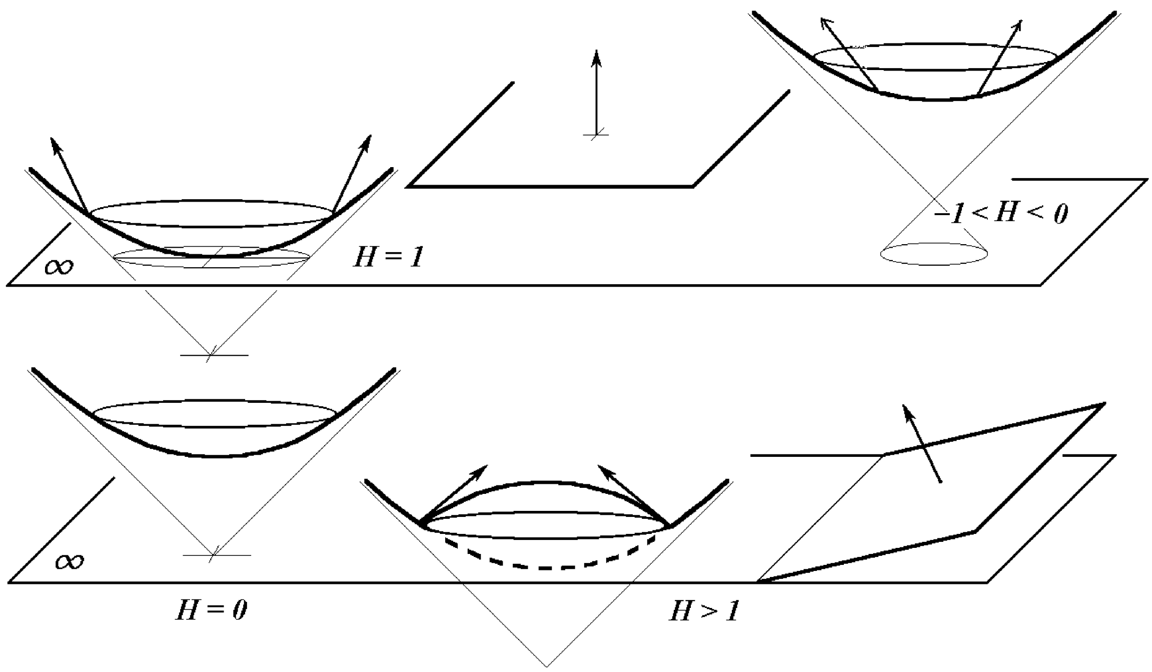

Figure 3. Spacelike umbilical hypersurfaces.

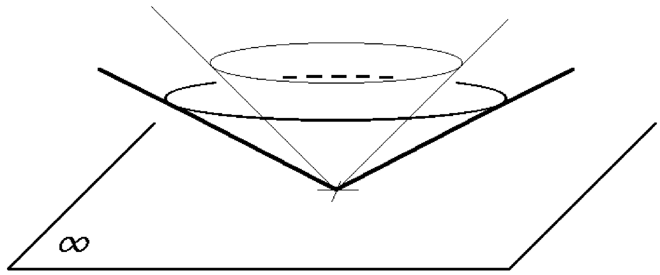

Figure 4. A hyperbolic cylinder.

When $k = n - 1$ this set is a half of a cone whose vertex is in the boundary of $\boldsymbol{R}^{n+1}_{+}$ (see Figure 4).

4. Constant mean curvature graphs and maximum principles.

Consider again a spacelike immersion $\psi : \Sigma^n \to \mathscr{H}^{n+1}$ in the upper half-space model for the steady state space. Since normal vectors at each point of Σ^n are timelike, the product $\langle N, e_{n+1} \rangle$ cannot vanish anywhere for any unit vector field N normal to the immersion. From now on we shall choose the orientation of our hypersurfaces so that

$$\langle N, e_{n+1} \rangle < 0,$$

that is, so that N points upwards. In this way N will be future directed in the Minkowski space but past directed in $\mathscr{H}^{n+1}$ (recall that the isometry ϕ changes time orientations).

Since each hyperplane $(d\psi)_p(T_p\Sigma^n)$, $p \in \Sigma^n$, is spacelike it projects one to one onto the future infinity $\mathscr{J}^+$ (the boundary hyperplane $x_{n+1} = 0$) and also onto any time slice $L^n(t)$ (the horizontal hyperplane $x_{n+1} = t$) with $t \in]0, +\infty[$. Then each spacelike immersion can be locally described as a graph on $\mathscr{J}^+$ or on any $L^n(t)$.

Suppose that Ω is a domain in $\boldsymbol{R}^n$, that f is a positive smooth function defined on Ω and that our immersion ψ is given by

$$\psi(x) = (x, f(x)) \quad (x \in \Omega \subset \boldsymbol{R}^n),$$

that is, ψ is the graph of the function f. Then

$$N' = \frac{1}{\sqrt{1 - |\nabla f|^2}} (\nabla f, 1)$$

is an upward unit normal field for the Minkowski metric (notice that $|\nabla f|^2 < 1$ is necessary because ψ is spacelike). Thus

$$-nH' = \operatorname{div}\left(\frac{\nabla f}{\sqrt{1 - |\nabla f|^2}} \right) \quad \text{on } \Omega.$$

Using (6) and the fact that, in our case, the immersion ψ is the graph of f, we have

$$H' = \frac{1}{f}\left(H - \frac{1}{\sqrt{1-|\nabla f|^2}}\right).$$

Combining these two equalities one can see that f is a solution to a second order partial differential equation.

Proposition 1. *Let Ω be a domain in $\boldsymbol{R}^n$ and $f \in C^2(\Omega)$ a positive function. Its graph $\psi(x) = (x, f(x)) \in \boldsymbol{R}_+^{n+1}$ is a spacelike hypersurface in the steady state space $\mathscr{H}^{n+1}$ with* (*not necessarily constant*) *mean curvature H with respect to the past directed normal field if and only if $|\nabla f|^2 < 1$ and f satisfies the following quasilinear elliptic equation*:

$$\operatorname{div}\left(\frac{\nabla f}{\sqrt{1-|\nabla f|^2}}\right) = -\frac{n}{f}\left(H - \frac{1}{\sqrt{1-|\nabla f|^2}}\right) \quad on\ \Omega, \tag{7}$$

where ∇ and div *are the Euclidean gradient and divergence operators on $\boldsymbol{R}^n$.*

In this paper, we shall solve the Dirichlet problems associated to this equation (7) with boundary conditions either $f = t$, $t \in]0, +\infty[$, that is, the boundary of the hypersurface is contained in a time slice, or $f = 0$, that is, its future asymptotic boundary (see Section 7 below for a precise definition) is prescribed.

Since this equation (7) is elliptic, the difference $f_1 - f_2$ of any two solutions f_1 and f_2 satisfies a *linear* elliptic equation on a neighborhood of each point. So the Hopf maximum principle for linear elliptic equations (see [**GT**], Theorem 9.2, or [**H**]) can be applied to this difference. Now, let Σ_1^n and Σ_2^n be the corresponding graphs, which are two spacelike hypersurfaces immersed in the steady state space $\mathscr{H}^{n+1}$ with the same constant mean curvature H. If Σ_1^n and Σ_2^n have a common point $p = (x, t)$ where they are tangent, we will say that Σ_1^n *lies above* Σ_2^n *near* p when $f_1 \geq f_2$ on a certain neighborhood of the point x. Hence, if Σ_1^n lies above Σ_2^n the difference $f_1 - f_2$ has a local minimum at x and the mentioned maximum principle implies that f_1 and f_2 and, so, the two hypersurfaces, coincide on a neighborhood of the point p. From that, it follows easily the following result.

Theorem 2 (Tangency principle). *Let Σ_1^n and Σ_2^n be two spacelike hypersurfaces immersed in the steady state space with the same constant mean curvature* (*with respect to the past directed or upward orientation*). *Suppose that they are tangent at a common non-boundary point p and that Σ_1^n lies above Σ_2^n near p. Then they coincide in a neighborhood of p.*

Remark 1. There is an analogous *boundary tangency principle* when the common point p is in $\partial\Sigma_1^n \cap \partial\Sigma_2^n$, provided that one also assumes that $\partial\Sigma_1^n$ and $\partial\Sigma_2^n$ are tangent at p.

Remark 2 (Mean curvature comparison). If the two hypersurfaces Σ_1^n and Σ_2^n are not supposed to have the same mean curvature and we represent by H_1 and H_2 their mean curvature functions, we have that, when Σ_1^n lies above Σ_2^n near a common point where they are tangent, then $H_1 \le H_2$ at that point (notice that in the Euclidean case we have the reverse inequality). This follows easily by comparing the Hessians of both height functions with respect to the e_{n+1} direction and taking into account that, in our case, height functions are the opposite of the true heights.

We shall use this tangency principle to see that a compact spacelike hypersurface with constant mean curvature whose non-empty boundary is contained in a time slice $L^n(t)$ must lie in either of the two domains of $\mathscr{H}^{n+1}$ determined by $L^n(t)$.

Proposition 3. *Let Σ^n be a compact spacelike hypersurface of the steady state space $\mathscr{H}^{n+1}$ with constant mean curvature H (upward orientation) and with boundary contained in the time slice $L^n(t)$ for some $t \in]0, +\infty[$. Then we have*

a) *$H \ge 1$ if and only if the hypersurface Σ^n is contained in $L^n(t)_+ = \{(x, x_{n+1}) \in \mathscr{H}^{n+1} \mid x_{n+1} \ge t\}$.*

b) *$H \le 1$ if and only if the hypersurface Σ^n is contained in $L^n(t)_- = \{(x, x_{n+1}) \in \mathscr{H}^{n+1} \mid x_{n+1} \le t\}$.*

Proof. We first see that Σ^n cannot have interior points in the two sides of $L^n(t)$, except when $\Sigma^n \subset L^n(t)$ and so $H = 1$. Otherwise, we would have that both the highest and the lowest points in Σ^n could be taken to be interior. Now take a time slice $L^n(s)$ with $s > t$ big enough to have $L^n(s) \cap \Sigma^n = \emptyset$ and decrease s until $L^n(s)$ touches Σ^n for the first time. So we obtain an interior tangency point. Since each time slice has constant mean curvature one (with respect to the upward orientation), the mean curvature comparison (Remark 2) implies $1 \le H$. Analogously, if we would have started with time slices $L^n(s)$ with $s < t$ small enough to have $L^n(s) \cap \Sigma^n = \emptyset$, we would have proved that $H \le 1$. As a consequence, we would have that $H = 1$ and so $\Sigma^n \subset L^n(t)$ by using the tangency principle of Theorem 2.

Thus either $\Sigma^n \subset L^n(t)_+$ or $\Sigma^n \subset L^n(t)_-$. In the first case, if Σ^n is not a domain in $L^n(t)$, using again the tangency principle at the highest point of Σ^n, we conclude that $1 < H$. Analogously, if $\Sigma^n \subset L^n(t)_-$ and Σ^n is not contained in $L^n(t)$, we have $1 > H$. $\square$

Another consequence of the tangency principle in Theorem 2 is the following result which generalizes the uniqueness for the Dirichlet problem corresponding to the mean curvature equation (7), with boundary condition in a time slice of $\mathscr{H}^{n+1}$.

Proposition 4 (Graphs monotonicity). *Let Ω_1 and Ω_2 be two compact domains in the time slice $L^n(t)$ of $\mathscr{H}^{n+1}$ such that $\Omega_1 \subset \Omega_2$. Consider two spacelike graphs Σ_1^n and Σ_2^n, corresponding to functions f_1 and f_2, with constant mean curvatures H_1 and H_2 (with respect to the upward orientation) such that $1 \le H_1 \le H_2$ and whose boundaries are $\partial\Sigma_1^n = \partial\Omega_1$ and $\partial\Sigma_2^n = \partial\Omega_2$. Then $f_1 \le f_2$ on Ω_1. In particular, if $\Omega_1 = \Omega_2$, then $f_1 \le f_2$.*

Proof. Using if necessary Euclidean horizontal translations, which are isometries of $\mathscr{H}^{n+1}$, we may suppose that $(0, t) \in \Omega_1 - \partial\Omega_1$. Consider the 1-parameter subgroup $\{h_s\}_{s \in \boldsymbol{R}}$ of isometries of $\mathscr{H}^{n+1}$ consisting of the following Euclidean homotheties

$$h_s(x, x_{n+1}) = e^s(x, x_{n+1}).$$

Choose $s_0 > 0$ big enough so that $h_{s_0}(\Sigma_2^n) \cap \Sigma_1^n = \emptyset$ and $h_{s_0}(\Sigma_2^n)$ is over Σ_1^n. Now decrease s from s_0 until a certain s_1 for which a point p of contact between $h_{s_1}(\Sigma_2^n)$ and Σ_1^n is reached for the first time. Let us see that $s_1 \leq 0$. If s_1 was positive, p would not belong to $\partial h_{s_1}(\Sigma_2^n) = h_{s_1}(\partial \Sigma_2^n)$ because $p \in \Sigma_1^n$ which is a graph over $\Omega_1 \subset \Omega_2$. Also we would have that $p \notin \partial \Sigma_1^n$ because $h_{s_1}(\Sigma_2^n)$ is over $L^n(t)$. Hence, if s_1 was positive, then the point p would be interior to both $h_{s_1}(\Sigma_2^n)$ and Σ_1^n. But then, the mean curvature comparison in Remark 2 would imply $H_2 \leq H_1$ because $h_{s_1}(\Sigma_2^n)$ is above Σ_1^n. From that and under our hypothesis, we deduce that $H_1 = H_2$ and the tangency principle in Theorem 2 would imply that $h_{s_1}(\Sigma_2^n) = \Sigma_1^n$, which is not possible because the boundaries of these two hypersurfaces are at different heights.

As a consequence we have that $s_1 \leq 0$, as claimed. But, in this case, it is clear that Σ_2^n is over Σ_1^n and so $f_1 \leq f_2$ on the smaller domain Ω_1. The case $\Omega_1 = \Omega_2$ follows immediately. $\square$

5. Height estimates.

The principal aim of this paper is to construct complete non-compact spacelike hypersurfaces in a de Sitter $\boldsymbol{S}_1^{n+1}$ with constant mean curvature $H > 1$ and prescribed asymptotic future boundary. These hypersurfaces will belong, by construction, in $\mathscr{H}^{n+1} \subset \boldsymbol{S}_1^{n+1}$ and will be obtained as a limit of constant mean curvature graphs over compact domains contained in time slices of $\mathscr{H}^{n+1}$. First we shall prove the existence of such graphs by using the continuity method. So we shall look for height and gradient estimates. Then higher order estimates will be obtained by using Schauder theory and the particular features of the equation (7).

Theorem 5 (Height estimates). *Let Σ^n be a compact spacelike hypersurface of $\mathscr{H}^{n+1}$ with constant mean curvature $H > 1$ and whose boundary is contained in a ball of radius $R > 0$ of the time slice $L^n(t)$, $t > 0$. Then*

$$t \leq x_{n+1|\Sigma^n} \leq \frac{Ht + \sqrt{(H^2-1)R^2 + t^2}}{H+1}.$$

Proof. The inequality on the left side follows from a) in Proposition 3. In order to prove the inequality on the right side, suppose that $(0, t)$ is the center of the ball B_R of radius R containing $\partial \Sigma^n$ (see the beginning of the proof of Proposition 4). Now, let us check that there is an umbilical two-sheeted hyperboloid with the same (constant) mean curvature H of Σ^n and whose lower sheet intersects the horizontal hyperplane $L^n(t)$ just in the sphere ∂B_R. In fact, it suffices to take a hyperboloid

$$|x|^2 - (x_{n+1} - a)^2 = -r^2$$

with center $(0, a)$, $a > t$, and radius $r > 0$, chosen so that

$$H = \frac{a}{r} \quad \text{and} \quad (t-a)^2 - r^2 = R^2.$$

If one writes r and a in terms of the data H and R, one has

$$r = \frac{Ht + \sqrt{(H^2-1)R^2 + t^2}}{H^2 - 1} \qquad a = H\frac{Ht + \sqrt{(H^2-1)R^2 + t^2}}{H^2 - 1}.$$

Notice that those choices are possible since we are assuming that $H > 1$. Let us denote by S the lower sheet of this hyperboloid. Consider again the family $\{h_s\}_{s \in \boldsymbol{R}}$ of isometries of $\mathscr{H}^{n+1}$ consisting of the Euclidean homotheties

$$h_s(x, x_{n+1}) = e^s(x, x_{n+1}) \quad (x \in \boldsymbol{R}^n, x_{n+1} > 0).$$

Pick $s_0 > 0$ big enough so that $h_{s_0}(S)$ does not touch Σ^n. Now we decrease s from s_0 until a certain value $s_1 \in \boldsymbol{R}$ such that $h_s(S)$ and Σ^n meet for the first time at some point p. We claim that $s_1 \leq 0$. In fact, if $s_1 > 0$, p cannot be a boundary point of Σ^n because $\partial\Sigma^n \subset B_R$ and $h_{s_1}(S) \cap L^n(t)$ is a sphere with radius bigger than R. Then the tangency principle in Theorem 2 can be applied to Σ^n and $h_{s_1}(S)$ to conclude that Σ^n is the piece of $h_{s_1}(S)$ above $L^n(t)$. But this contradicts the fact that the boundaries of these two hypersurfaces are different. Hence $s_1 \leq 0$ as we had claimed. Then it is not difficult to see that Σ^n is under the sheet S of our hyperboloid. As the Euclidean height of S is $a - r$, we deduce that

$$x_{n+1|\Sigma^n} \leq a - r = (H-1)r = \frac{Ht + \sqrt{(H^2-1)R^2 + t^2}}{H+1},$$

which was the required upper bound. $\square$

Remark 3. If we represent by $K = K(t, R, H)$ the upper bound provided by Theorem 5 above, it is straightforward to show that the derivatives

$$\frac{dK}{dt} \quad \text{and} \quad \frac{dK}{dH}$$

are positive. That is, the bound K is increasing with respect to the variables t and H. In particular, $K(t, R, H) \leq t + R$.

As a consequence of this height estimate for compact spacelike hypersurfaces of $\mathscr{H}^{n+1}$ with constant mean curvature $H > 1$, we obtain C^0-estimates for spacelike graphs of this type.

Corollary 6. *Let Ω be a compact domain in $\boldsymbol{R}^n$ and t a positive real number. Then, each smooth positive function f defined on Ω whose graph is a spacelike hypersurface in (the upper half-space model for) the steady state space $\mathscr{H}^{n+1}$ with constant mean curvature $H > 1$ and whose boundary is contained in the time slice $L^n(t)$, $0 < t$, satisfies*

$$t \leq f \leq t + \operatorname{diam}\Omega,$$

where $\operatorname{diam}\Omega$ *is the diameter of Ω in $\boldsymbol{R}^n$.*

6. Gradient estimates.

In order to find *a priori* estimates for the gradient of a solution to the mean curvature equation (7), we come back to the hyperquadric model for our space $\mathscr{H}^{n+1}$. That is, we set

$$\mathscr{H}^{n+1} = \{p \in \boldsymbol{S}_1^{n+1} \mid \langle p, a \rangle > 0\}$$

for a certain $a \in \boldsymbol{R}_1^{n+2} - \{0\}$ with $|a|^2 = 0$ and $\langle a, e_{n+1} \rangle > 0$.

In this setting a spacelike immersion ψ from a compact hypersurface Σ^n into the steady state space $\mathscr{H}^{n+1}$ can be thought of as an immersion

$$\psi : \Sigma^n \to \boldsymbol{S}_1^{n+1} \subset \boldsymbol{R}_1^{n+2} \quad \text{with } \langle \psi, a \rangle > 0.$$

Then any (timelike) unit vector field N normal to the immersion ψ can be viewed as a map

$$N : \Sigma^n \to \{p \in \boldsymbol{R}_1^{n+2} \mid \langle p, p \rangle = -1\},$$

where each one of the two sheets of the hyperboloid on the right side are isometric, with the induced metric, to the hyperbolic space $\boldsymbol{H}^{n+1}$ with constant sectional curvature -1. A direct computation leads to obtain the following two formulae (see, for example, [**Mo1**] for details), when the mean curvature H of the immersion ψ is constant,

$$\Delta\psi = -n\psi - nHN \quad \Delta N = |\sigma|^2 N + nH\psi, \tag{8}$$

where Δ is the Laplacian operator of the metric induced on the hypersurface Σ^n and σ is the second fundamental form of the immersion ψ.

We had chosen before the orientation of our hypersurfaces so that N was past directed (or upward in the half-space model). So, N must be in the same half of the null cone of $\boldsymbol{R}_1^{n+2}$ as a is. That is,

$$\langle N, a \rangle < 0.$$

With this choice of orientation, the umbilical hypersurfaces $L^n(\tau)$ which foliate $\mathscr{H}^{n+1}$ have constant mean curvature one, as we saw in Section 2, and the unit normal field N takes values in the lower sheet of the corresponding hyperboloid, which will be denoted by $\boldsymbol{H}^{n+1}$ (hyperbolic space). Further, since

$$\{x \in \boldsymbol{R}_1^{n+2} \mid \langle x, x \rangle = -1, \langle x, a \rangle = \rho\} \quad (\rho \in]-\infty, 0[)$$

is a foliation of $\boldsymbol{H}^{n+1}$ by means of parallel horospheres, the negative function $\langle N, a \rangle$ is a good estimate for the slope of the hypersurface because it measures the hyperbolic distance to the common infinity point of that family of horospheres. That is, to obtain a lower bound for $\langle N, a \rangle$ is equivalent to show that the image of N is contained in the domain of the hyperbolic space $\boldsymbol{H}^{n+1}$ determined by a horosphere for which the (constant) mean curvature of the horosphere is positive.

Theorem 7 (Gradient estimates). *Let $\psi : \Sigma^n \to \mathscr{H}^{n+1}$ be a spacelike immersion from a compact manifold Σ^n with non-empty boundary in the steady state space. Suppose that ψ has constant mean curvature $H > 1$ with respect to the past directed unit normal N and that ψ maps $\partial\Sigma^n$ into a time slice $L^n(\tau)$, for a certain $\tau > 0$. Then we can choose the unit normal field $\boldsymbol{n}$ for the immersion $\psi_{|\partial\Sigma^n} : \partial\Sigma^n \to L^n(\tau)$ such that $\langle N, \boldsymbol{n} \rangle > 0$. Assume that, with respect to this orientation $\boldsymbol{n}$ the mean curvature of $\psi_{|\partial\Sigma^n} : \partial\Sigma^n \to L^n(\tau)$ is non-negative. Then*

$$H\langle \psi, a \rangle + \langle N, a \rangle \geq 0$$

and so $\langle N, a \rangle \geq -H\tau$ on Σ^n.

Proof. As the mean curvature H is constant, we have from (8) that

$$\Delta(H\langle \psi, a \rangle + \langle N, a \rangle) = (|\sigma|^2 - nH^2)\langle N, a \rangle.$$

But the Schwarz inequality implies that $|\sigma|^2 - nH^2 \geq 0$, with equality at the umbilical points, and our choice of orientation was $\langle N, a\rangle < 0$. Hence, the function $H\langle\psi, a\rangle + \langle N, a\rangle$ is superharmonic on Σ^n and so it attains its minimum at a boundary point, say $q \in \partial\Sigma^n$. Denote by v the unit inner conormal field along $\partial\Sigma^n$. Then

$$0 \leq H\langle v_q, a\rangle + \langle (dN)_q(v_q), a\rangle = \langle v_q, a\rangle(H + \langle (dN)_q(v_q), v_q\rangle).$$

On the other hand, since $H > 1$ we know from Proposition 3 and (4) that $\langle\psi, a\rangle \leq \tau$ on Σ^n. Then, as we suppose that $\langle\psi, a\rangle = \tau$ along $\partial\Sigma^n$, we have

$$\langle v, a\rangle \leq 0$$

along the boundary $\partial\Sigma^n$. But, if the equality $\langle v, a\rangle = 0$ was attained at some point of $\partial\Sigma^n$, we could use the boundary versions of the mean curvature comparison and the tangency principle (Remarks 1 and 2) for $\psi(\Sigma^n)$ and $L^n(\tau)$ to conclude that $H = 1$. This is a contradiction. So we have

$$\langle v, a\rangle < 0$$

along $\partial\Sigma^n$. Now, if $\boldsymbol{n}$ is any unit normal vector for the restriction $\psi_{|\partial\Sigma^n}$ and we had $\langle N, \boldsymbol{n}\rangle = 0$, then $\boldsymbol{n}$ would be a non-null multiple of v and so $\langle \boldsymbol{n}, a\rangle \neq 0$, which is not possible because $\psi(\partial\Sigma^n) \subset L^n(\tau)$. So we can choose a unit normal field $\boldsymbol{n}$ so that $\langle N, \boldsymbol{n}\rangle > 0$. Using again that $\langle v, a\rangle < 0$ and the previous inequality we obtain

$$H + \langle (dN)_q(v_q), v_q\rangle \leq 0.$$

Then, as $nH = -\mathrm{trace}(dN)$, we have

$$(n-1)H \geq -\sum_{i=1}^{n-1} \langle (dN)_q(e_i), e_i\rangle,$$

where $\{e_1, \ldots, e_{n-1}\}$ is an orthonormal basis of $T_q\partial\Sigma^n$. Now, we may decompose the second fundamental form σ of the immersion ψ at any point of $\partial\Sigma^n$ as the sum of the second fundamental form σ^∂ of $\psi_{|\partial\Sigma^n} : \partial\Sigma^n \to L^n(\tau)$ and the second fundamental form of the umbilical hypersurface $L^n(\tau)$ in de Sitter space. Thus

$$-\langle (dN)_q(e_i), e_i\rangle = \langle \sigma_q(e_i, e_i), N(q)\rangle = \langle \sigma_q^\partial(e_i, e_i), N(q)\rangle - \frac{1}{\tau}\langle N(q), a\rangle.$$

Summing from $i = 1$ to $i = n-1$ and using the inequality above, we have

$$H \geq H^\partial(q)\langle \boldsymbol{n}(q), N(q)\rangle - \frac{1}{\tau}\langle N(q), a\rangle,$$

where H^∂ represents the mean curvature function of the immersion $\psi_{|\partial\Sigma^n} : \partial\Sigma^n \to L^n(\tau)$ with respect to the chosen orientation n. As we have that $H^\partial\langle \boldsymbol{n}, N\rangle \geq 0$, then

$$H\langle\psi(q), a\rangle + \langle N(q), a\rangle = H\tau + \langle N(q), a\rangle \geq 0.$$

This finishes the proof because q was the point of Σ^n where the function $H\langle\psi, a\rangle + \langle N, a\rangle$ attained its minimum. $\square$

REMARK 4. We are going to translate the statement of this Theorem 7 into the language of the upper half-space model for $\mathscr{H}^{n+1}$. In fact, let us call $\xi : \Sigma^n \to \boldsymbol{R}_+^{n+1}$ to the immersion $\phi \circ \psi$, where ϕ is the isometry between the two models for the steady state space that we defined in (1). It is clear from the definition of ϕ that

$$\langle \psi, a \rangle = \frac{1}{\xi_{n+1}}.$$

Moreover, if we denote by a^T the field tangent to $\boldsymbol{S}_1^{n+1}$ defined as the tangent part of the null vector a, we have

$$\langle N, a \rangle (p) = \langle N(p), a^T_{\psi(p)} \rangle = \xi_{n+1}^{-2} \langle (d\phi)_{\psi(p)}(N(p)), (d\phi)_{\psi(p)}(a^T(\psi(p))) \rangle,$$

for all $p \in \Sigma^n$. But $N'(p) = \xi_{n+1}^{-1}(d\phi)_{\psi(p)}(N(p))$ is a vector normal to the immersion ξ which is unit with respect to the Minkowski metric in $\boldsymbol{R}_1^{n+1}$ and, from (2), it can be shown that

$$(d\phi)_x(a^T(x)) = (0,1) = e_{n+1} \quad \text{for all } x \in \boldsymbol{S}_1^{n+1}.$$

Then the last inequality of Theorem 7 above becomes

$$H + \langle N', e_{n+1} \rangle = H - N'_{n+1} \geq 0.$$

When the considered hypersurface is a graph, this Theorem 7 yields a sharp C^1-estimate which will be important in the sequel to prove some existence results. In order to write the corresponding statement we need the following definition: an embedded closed hypersurface of a Euclidean space is said to be *mean convex* when its mean curvature with respect to the interior normal field is non-negative.

COROLLARY 8. *Let Ω be a compact domain in $\boldsymbol{R}^n$ whose boundary $\partial\Omega$ is mean convex. Suppose that f is a smooth positive function defined on Ω whose graph is a spacelike hypersurface in the steady state space $\mathscr{H}^{n+1}$ with constant mean curvature $H > 1$ and boundary contained in some time slice* (*horizontal hyperplane*). *Then*

$$|\nabla f|^2 \leq \frac{H^2 - 1}{H^2} < 1.$$

PROOF. As in Remark 4 above, we will denote by $\xi = \phi \circ \psi : \Omega \to \boldsymbol{R}_+^{n+1}$ the immersion

$$\xi(x) = (x, f(x)) \quad x \in \Omega.$$

Let $\boldsymbol{n}'(x) = \xi_{n+1}^{-1}(d\phi)_{\psi(x)}(\boldsymbol{n}(x))$, $x \in \partial\Omega$, where $\boldsymbol{n}$ is the unit field normal to $\psi_{|\partial\Omega}$ determined in Theorem 7, which will be a vector normal to the restricted immersion $\xi_{|\partial\Omega}$. We have

$$0 < \langle N, \boldsymbol{n} \rangle (x) = \langle N', \boldsymbol{n}' \rangle (x)$$

and hence the function $\langle N', \boldsymbol{n}' \rangle$ is also positive. But

$$0 = \langle \boldsymbol{n}, a \rangle (x) = \xi_{n+1}^{-1} \langle \boldsymbol{n}', e_{n+1} \rangle$$

and so the $(n+1)$-component of $\boldsymbol{n}'$ vanishes. Taking into account that, in our case,

$$N' = \frac{1}{\sqrt{1 - |\nabla f|^2}} (\nabla f, 1),$$

we deduce that $\langle \boldsymbol{n}'_H, \nabla f\rangle > 0$ along $\partial\Omega$, where we have put $\boldsymbol{n}' = (\boldsymbol{n}'_H, 0)$. But Corollary 6 asserts that f attains its minimum value at all the points of the boundary $\partial\Omega$. As a consequence the unit normal field $\boldsymbol{n}'_H$ is the *inner* unit normal field along $\partial\Omega \subset \boldsymbol{R}^n$ and so, from our assumptions, the mean curvature of ψ associated to $\boldsymbol{n}$ is non-negative. Then we can apply Theorem 7 and Remark 4 to get

$$H - N'_{n+1} = H - \frac{1}{\sqrt{1-|\nabla f|^2}} \geq 0$$

and the proof is completed. □

7. Existence results.

In Sections 5 and 6, we have obtained *a priori* height and gradient estimates for spacelike constant mean curvature graphs over compact domains in horizontal time slices, whose boundary is mean convex. These results will allow us to use the standard theory of existence for quasilinear elliptic equations. In fact, we want to solve the following Dirichlet problem corresponding to the mean curvature equation (7):

$$\begin{cases} \operatorname{div}\left(\dfrac{\nabla f}{\sqrt{1-|\nabla f|^2}}\right) + \dfrac{n}{f}\left(H - \dfrac{1}{\sqrt{1-|\nabla f|^2}}\right) = 0 \text{ and } |\nabla f|^2 < 1 \quad \text{on } \operatorname{int}\Omega, \\ f = t \quad \text{on } \partial\Omega, \end{cases} \tag{9}$$

where $H \geq 1$, Ω is a compact domain in $\boldsymbol{R}^n$ with mean convex boundary and where t is a positive number. From Proposition 1, we deduce that to find a smooth solution to (9) is equivalent to construct a spacelike graph over the domain $\Omega_t = \Omega \times \{t\} \subset L^n(t)$ in the steady state space $\mathscr{H}^{n+1}$, with constant mean curvature H and whose boundary is $\partial\Omega_t$.

Let us apply the continuity method in the following way. Define a set

$$J_t = \{H \in [1, +\infty[\,|\,\text{there exists an } H\text{-graph over } \Omega_t \text{ whose boundary is } \partial\Omega_t\}.$$

We want to show that each J_t is a non-empty, open and closed subset of $[1, +\infty[$. We shall do it in three steps:

1) $1 \in J_t$ because the domain Ω_t, being a domain in the umbilical hypersurface $L^n(t)$, is a graph of constant mean curvature 1. Hence $J_t \neq \emptyset$ for each $t > 0$.

2) Closedness of J_t follows from Corollaries 6 and 8. In fact, let $\{H_k\}_{k\in\boldsymbol{N}}$ be a sequence in J_t which converges to a number $H \geq 1$ and $\{f_k\}_{k\in\boldsymbol{N}}$ the sequence of corresponding solutions to (9). Corollary 6 gives us an uniform C^0-bound for the f_k. Corollary 8 implies that

$$|\nabla f_k|^2 \leq \frac{H_k^2 - 1}{H_k^2} \leq \frac{M^2-1}{M^2},$$

where M is any upper bound of the convergent sequence $\{H_k\}_{k\in\boldsymbol{N}}$. Thus, we have got C^0 and C^1 bounds for the sequence of the f_k. Then the properties of quasilinear elliptic equations of divergence type and Schauder's theory (see [**GT**]) provide us $C^{2,\alpha}$

bounds for our sequence. Hence there is a subsequence of $\{f_k\}_{k\in \boldsymbol{N}}$ converging to a solution f for the H-problem (9). Then $H \in J_t$ and J_t is closed.

3) We shall see that J_t is an open set by using the implicit function theorem. Take $H_0 \in J_t$ and let us see that the Dirichlet problem (9) can be solved for each H in a certain interval around H_0. Denote by Σ^n the graph associated to the solution f for H_0 and define a map

$$h : C_0^{2,\alpha}(\Sigma^n) \to C_0^{\alpha}(\Sigma^n)$$

mapping each $u \in C_0^{2,\alpha}(\Sigma^n)$ onto the mean curvature function of the normal graph on Σ^n corresponding to the function u. The differential of this map h at the point 0 is the linear Jacobi operator L of the hypersurface Σ^n, that is,

$$L = (dh)_0 = \Delta - |\sigma|^2 + n.$$

But this is a non-negative operator because, from the Schwarz inequality, we have $|\sigma|^2 \geq nH_0^2 \geq n$ and we suppose that $H_0 \geq 1$. Hence, L has a trivial kernel and it is a Fredholm operator of index zero and so an isomorphism. Thus the implicit function theorem assures that (9) can be solved for values of H near H_0.

As a consequence $J_t = [1, +\infty[$ for any $t > 0$ and we have a solution to (9) for any $H \geq 1$ and $t > 0$. Moreover these solutions are unique from Proposition 4. We gather the results that we have just proved in the following theorem.

Theorem 9. *Let Ω be a compact domain in $\boldsymbol{R}^n$ with mean convex boundary. There exists a unique solution $f_{H,t} \in C^\infty(\Omega)$ to the Dirichlet problem* (9) *satisfying*

$$t \leq f_{H,t} \leq t + \operatorname{diam}\Omega, \quad |\nabla f_{H,t}|^2 \leq \frac{H^2-1}{H^2}$$

for each $H \geq 1$ and $t > 0$. In particular, given a compact domain Γ on a time slice of the steady state space $\mathscr{H}^{n+1}$ with mean convex boundary and a real number $H \geq 1$, there exists a spacelike graph over Γ with constant mean curvature H and boundary $\partial\Gamma$.

The mere existence of these H-graphs, that is, the existence of the $f_{H,t}$, could have been deduced from [**Ge**], Theorem 5.1, where Gerhardt got *a priori* estimates under very much general conditions than ours. But, in our particular situation, we have obtained a sharp upper bound for the length of the gradient $\nabla f_{H,t}$ which is independent of t and which will allow us to consider the limit situation when t goes to zero.

Lemma 10. *Suppose that $\Omega \subset \boldsymbol{R}^n$ is a compact domain with mean convex boundary and fix a real number $H > 1$. For each point $x \in \operatorname{int}\Omega$ there exists a positive constant $C(x, H)$ depending only on x and H such that*

$$f_{H,t}(x) \geq C(x,H) > 0 \quad \textit{for any } t > 0,$$

where $f_{H,t}$ is the solution to the Dirichlet problem (9) *obtained in Theorem* 9 *above.*

Proof. In fact, take a point x_0 in the interior of the compact domain Ω and fix $R > 0$ so that the open ball $B_R(x_0)$ with radius R centered at x_0 is contained in Ω. Then, if $H > 1$ and $t > 0$, we can consider, as in the proof of Theorem 5, a two-sheeted hyperboloid whose lower sheet S intersects the horizontal hyperplane $x_{n+1} = t$ just in

$\partial(B_R(x_0) \times \{t\})$ and whose (constant) mean curvature is H. Then S has the same constant mean curvature as the graph G corresponding to the function $f_{H,t}$ and both hypersurfaces have their respective boundaries contained in $L^n(t)$. Moreover, since $B_R(x_0) \times \{t\} \subset \Omega \times \{t\}$, we may apply the graph monotonocity proved in Proposition 4 and deduce that the sheet S is under the graph G. In particular the point $(x_0, f_{H,t}(x_0))$ of G is higher than the highest point of S. So, if we recall the computations in Theorem 5, we have

$$f_{H,t}(x_0) \geq \frac{Ht + \sqrt{(H^2-1)R^2 + t^2}}{H+1}.$$

Remark 3 assures that the fraction on the right side is increasing in the variable $t > 0$. Thus

$$f_{H,t}(x_0) \geq \sqrt{\frac{H-1}{H+1}}R$$

for each $t > 0$, as we had asserted. □

Now, given a compact domain in $\boldsymbol{R}^n$ with mean convex boundary and a number $H > 1$, we choose a sequence $\{t_k\}_{k \in \boldsymbol{N}}$ of positive numbers with

$$\lim_{k \to \infty} t_k = 0.$$

Using Theorem 9, we have a solution $f_k = f_{H,t_k}$ to the corresponding Dirichlet problem (9), for each $k \in \boldsymbol{N}$. This same Theorem 9 implies that

$$t_k \leq f_k \leq M + \operatorname{diam} \Omega \quad |\nabla f_k|^2 \leq \frac{H^2-1}{H^2},$$

where M is any upper bound of the sequence $\{t_k\}_{k \in \boldsymbol{N}}$. Then, using Schauder's theory, we have that there is a subsequence which converges to a smooth function f defined on Ω with

$$0 \leq f \leq M + \operatorname{diam} \Omega \quad |\nabla f|^2 \leq \frac{H^2-1}{H^2}.$$

Since $f_k = t_k$ on $\partial\Omega$, one has $f = 0$ along that boundary. From Lemma 10, it is clear that f must be positive on the interior of Ω. Now, taking limits in (9), we see that the limit function f is a positive solution of the Dirichlet problem (9) with zero boundary condition. Hence we have the following result.

THEOREM 11. *Consider a compact domain $\Omega \subset \boldsymbol{R}^n$ with mean convex boundary and a real number $H > 1$. There exists a positive solution to the Dirichlet problem*

$$\begin{cases} \operatorname{div}\left(\dfrac{\nabla f}{\sqrt{1-|\nabla f|^2}}\right) + \dfrac{n}{f}\left(H - \dfrac{1}{\sqrt{1-|\nabla f|^2}}\right) = 0 \quad \text{on } \operatorname{int}\Omega, \\ f = 0 \quad \text{on } \partial\Omega, \end{cases}$$

which satisfies $|\nabla f|^2 \leq (H^2-1)/H^2$.

As a consequence, we obtain an existence result for complete non-compact spacelike hypersurfaces in the de Sitter space with constant mean curvature $H > 1$ and whose asymptotic future boundary is a prescribed mean convex embedded hypersurface of the future infinity $\mathcal{J}^+$ of the open set $\mathcal{H}^{n+1} \subset \boldsymbol{S}_1^{n+1}$. We shall say that Γ is the *asymptotic future boundary* of a hypersurface Σ^n of $\mathcal{H}^{n+1}$ when $\Gamma = \overline{\Sigma^n} \cap \mathcal{J}^+$ and the closure is taken in $\boldsymbol{R}_+^{n+1}$. We shall denote it by $\partial_\infty \Sigma^n$. On the other hand, we shall say, following [**NS**], that Γ is its *asymptotic future homological boundary* if, for each $t > 0$ sufficiently small, $\Sigma^n \cap L^n(t) = \Gamma(t)$, where

$$\Gamma(t) \to \Gamma \quad \text{as } t \to 0 \quad \text{and} \quad \Gamma(t) \text{ is null-homologous in } \Sigma^n.$$

Corollary 12. *Let $\mathcal{J}^+$ be the spacelike future infinity of the steady state space $\mathcal{H}^{n+1}$, $\Omega \subset \mathcal{J}^+$ a compact domain with mean convex boundary and a real number $H > 1$. There exists a complete spacelike hypersurface Σ^n embedded in $\mathcal{H}^{n+1}$ (in fact, a graph over Ω) with constant mean curvature H and with $\partial_\infty \Sigma^n = \partial\Omega$. Moreover, if Ω is star-shaped (with respect to an interior point), then Σ^n is the only one having constant mean curvature H and asymptotic future homological boundary $\partial\Omega$.*

Proof. All our assertions are immediate consequences from Theorem 11 above except the uniqueness and the fact that the hypersurface Σ^n graph of the solution f is complete. With respect to the completeness, notice that Σ^n is isometric to Ω endowed with the metric $\psi^* g$ induced from the metric g on $\mathcal{H}^{n+1}$, defined in (3), through the immersion given by $\psi(x) = (x, f(x))$ for each $x \in \Omega$. Then, using the gradient estimate in Theorem 11 and the Schwarz inequality, one has

$$(\psi^* g)_x = \frac{1}{f(x)^2}(|dx|^2 - \langle (\nabla f)_x, dx\rangle^2) \geq \frac{1}{H^2 - 1} \frac{\langle (\nabla f)_x, dx\rangle^2}{f(x)^2}.$$

Finally, we obtain

$$\psi^* g \geq \frac{1}{H^2 - 1} |d \log f|^2.$$

This inequality implies that the length of any curve in Σ^n reaching its asymptotic future boundary must be infinity because f vanishes along $\partial\Sigma^n$.

With respect to the uniqueness when Ω is star-shaped, let M^n be another spacelike hypersurface of $\mathcal{H}^{n+1}$ with the same constant mean curvature and such that $\partial\Omega$ is its asymptotic homological future boundary. Represent by x_0 the point of $\operatorname{int}\Omega$ with respect to which Ω is star-shaped and consider the 1-parameter subgroup $\{h_s\}_{s \in \boldsymbol{R}}$ of isometries of $\mathcal{H}^{n+1}$ given by

$$h_s(x, x_{n+1}) = (e^s(x - x_0), e^s x_{n+1}).$$

For s big enough, we have that $h_s(\Sigma^n) \cap M^n = \emptyset$ and that $h_s(\Sigma^n)$ is over M^n. So, if we decrease s until finding a first contact point, then, from the tangency principle, that contact point must be on $\partial\Omega$. Since we have $h_s(\partial\Omega) \cap \partial\Omega = \emptyset$ if $s \neq 0$ because Ω is star-shaped, that contact point must happen for $s = 0$. Thus Σ^n is above M^n. Analogously, if we start with an s small enough so that $h_s(\Sigma^n) \cap M^n = \emptyset$ and we deduce that Σ^n was below M^n. Then $M^n = \Sigma^n$. □

8. Geometric properties and Gauss maps of the solutions.

We already pointed out in Section 6 that the Gauss map of a spacelike hypersurface of the de Sitter space $\boldsymbol{S}_1^{n+1}$ can be thought of as a map

$$N : \Sigma^n \to \boldsymbol{H}^{n+1}$$

taking values in the hyperbolic space

$$\boldsymbol{H}^{n+1} = \{x \in \boldsymbol{R}_1^{n+2} \mid \langle x, x\rangle = -1, \langle x, a\rangle < 0\},$$

where a is any non-zero null vector in $\boldsymbol{R}_1^{n+2}$, for example that one that we chose in Section 2. When the hypersurface has constant mean curvature $H \geq 1$ we have certain restrictions on the image of the Gauss map. In fact, the following result gives for the de Sitter space a weaker analogue to a result already known when the ambient space is a Minkowski space (see [**A2**], [**P**]).

PROPOSITION 13. *Let Σ^n be a complete spacelike hypersurface of the de Sitter space $\boldsymbol{S}_1^{n+1}$ with constant mean curvature $H \geq 1$. Suppose that the image $N(\Sigma^n)$ of the Gauss map of Σ^n is contained in the closure of the interior domain enclosed by a horosphere. Then we have that $H = 1$. (When $n = 2$, this implies that Σ^2 is also an umbilical surface and the image of its Gauss map is exactly a horosphere.)*

PROOF. All the horospheres of $\boldsymbol{H}^{n+1}$ can be realized in the Minkowski model in the following way

$$\{x \in \boldsymbol{H}^{n+1} \mid \langle x, c\rangle = \rho\},$$

where $c \in \boldsymbol{R}_1^{n+2}$ is a non-zero null vector (which can be taken in the same half of the null cone as a) and ρ is a negative number. Then our hypothesis about the image of the Gauss map N can be expressed in this way

$$0 > \langle N(p), c\rangle \geq \rho \quad (p \in \Sigma^n),$$

for a certain non-zero null vector c and a negative number ρ. Hence we have that the infimum

$$\inf_{p \in \Sigma^n} \langle N(p), c\rangle$$

exists and is a negative number. On the other hand, from (8), we obtain

$$\Delta\langle N, c\rangle = |\sigma|^2\langle N, c\rangle + nH\langle \psi, c\rangle,$$

where ψ is the position vector of the hypersurface. As $\psi + N$ is a null vector at each point of Σ^n and $\langle \psi + N, N\rangle = -1 < 0$, we have that $\psi + N$ belongs to the same half of the null cone as N and, so, to the same one as c. Then

$$\langle \psi + N, c\rangle \leq 0 \quad \text{on } \Sigma^n.$$

Thus, we have the inequality

$$\Delta\langle N, c\rangle \leq nH(H - 1)\langle N, c\rangle.$$

Now, we may apply the generalized maximum principle for complete Riemannian manifolds due to Omori [**Om**] and Yau [**Y**] to the function $\langle N, c\rangle$. In fact, the Ricci tensor of Σ^n is given by (see, for example, [**Mo2**])

$$\mathrm{Ric} = (n-1)I - nHA + A^2, \tag{10}$$

where A is the Weingarten map of Σ^n and, so, we have that $\mathrm{Ric} \geq n - 1 - (n^2H^2/4)$, that is, Ric is bounded from below. Since the infimum of $\langle N, c\rangle$ was negative we have that $H = 1$. In the case $n = 2$ we can obtain the remaining assertions in [**Ak**] or [**R**]. □

Proposition 13 implies that $\overline{N(\Sigma^n)} \subset \boldsymbol{H}^{n+1}$ has more than one point at the infinity of the hyperbolic space, for each hypersurface Σ^n with constant mean curvature $H > 1$ constructed in Corollary 12 above. In fact, we can determine what the asymptotic behaviour of its Gauss map is.

Such Σ^n was constructed as the graph

$$\{(x, f(x)) \in \boldsymbol{R}_1^{n+1} = \boldsymbol{R}^n \times \boldsymbol{R}_+ \mid x \in \Omega\}$$

of a function $f \in C^\infty(\Omega)$ defined on a compact domain $\Omega \subset \boldsymbol{R}^n$ with mean convex boundary. We know that its Gauss map

$$N : \Sigma^n \to \boldsymbol{H}^{n+1}$$

takes values in the hyperbolic space. The same map ϕ defined in (1) that we used to identify the de Sitter and the upper half-space models for the steady state space $\mathscr{H}^{n+1}$ can be used again to identify the two usual models for hyperbolic space $\boldsymbol{H}^{n+1}$. In fact, the map $\phi : \boldsymbol{H}^{n+1} \to \boldsymbol{R}^{n+1} = \boldsymbol{R}^n \times \boldsymbol{R}$ given by

$$\phi(x) = \frac{1}{\langle x, a\rangle}(x - \langle x, a\rangle b - \langle x, b\rangle a, 1)$$

is an isometry from $\boldsymbol{H}^{n+1}$, viewed as a hyperquadric in the Minkowski space $\boldsymbol{R}_1^{n+2}$, to the *lower* half-space $\boldsymbol{R}_-^{n+1} = \boldsymbol{R}^n \times \boldsymbol{R}_-$ endowed with the Riemannian metric

$$ds^2_{(x, x_{n+1})} = \frac{1}{x_{n+1}^2}(|dx|^2 + (dx_{n+1})^2).$$

Now, we are going to look at the Gauss map N of the hypersurface Σ^n when we use the (lower) half-space model for $\boldsymbol{H}^{n+1}$. That is, we want to know what the map $\phi \circ N : \Sigma^n \to \boldsymbol{R}_-^{n+1}$ looks like. To do that, notice that, if we represent by

$$\psi : \Sigma^n \to \mathscr{H}^{n+1}$$

the immersion

$$\psi(x, f(x)) = \phi^{-1}(x, f(x)) \in \mathscr{H}^{n+1} \subset \boldsymbol{S}_1^{n+1} \quad (x \in \Omega),$$

then we have that

$$\frac{f(x)}{\sqrt{1 - |(\nabla f)_x|^2}}((\nabla f)_x, 1) = (d\phi)_{\psi(p)}(N_p) \quad (p = (x, f(x))\ \ x \in \Omega).$$

From (2), we deduce that

$$-\frac{\langle N,a\rangle}{\langle\psi,a\rangle^2}(p)=\frac{f}{\sqrt{1-|\nabla f|^2}}(x),\quad N'(p)=\frac{1}{\sqrt{1-|(\nabla f)_x|^2}}\left((\nabla f)_x-\frac{x}{f(x)}\right),$$

where $N'=N-\langle N,a\rangle b-\langle N,b\rangle a$. Hence, we obtain

$$(\phi\circ N)(x,f(x))=(x-f(x)(\nabla f)_x,-f(x)\sqrt{1-|(\nabla f)_x|^2}),$$

for each $x\in\Omega$. As a consequence, we can state the following result.

Proposition 14. *Let Σ^n be one of the complete spacelike hypersurfaces of the steady state space $\mathscr{H}^{n+1}$ constructed in Corollary* 12, *with constant mean curvature $H>1$ and whose prescribed asymptotic future boundary $\partial_\infty\Sigma^n=\partial\Omega$ is the boundary of a compact domain Ω of $\mathscr{J}^+$ with mean convex boundary. Then, the Gauss map $N:\Sigma^n\to\boldsymbol{H}^{n+1}$ of Σ^n extends smoothly to $\partial_\infty\Sigma^n$ as the identity map* (*if we consider the half-space model for hyperbolic space*). *In particular, $N(\partial_\infty\Sigma^n)=\partial\Omega$.*

Remark 5. Finally, we shall point out (without detailed proofs) that the geometry of complete non-compact spacelike hypersurfaces Σ^n of $\boldsymbol{S}_1^{n+1}$ with constant mean curvature is bounded. This fact was shown by Cheng and Yau [**CY**] and by Treibergs [**Tr**] when the corresponding ambient space was a Minkowski space. In our case, a suitable use of the formula for the Laplacian of the length of the second fundamental form of the hypersurface, such as it was computed in [**Mo2**] or in any of the references therein, and of the generalized maximum principle due to Omori and Yau ([**Om**], [**Y**]) leads to prove that $|\sigma|^2$ is bounded from above and that Σ^n has no umbilical points when $H>1$, provided that it is not totally umbilical, and that the principal curvatures $k_1,\dots,k_n$ of Σ^n satisfy

$$\frac{nH-\sqrt{n^2H^2-4(n-1)}}{2(n-1)}\le k_i\le\frac{nH+\sqrt{n^2H^2-4(n-1)}}{2}$$

for $i=1,\dots,n$. These inequalities make sense because the mean curvature H is bigger than or equal to the value $2\sqrt{n-1}/n$. If not, from (10), the Ricci curvature of the hypersurface would be bounded from below by a positive constant. Then, using the Bonnet-Myers theorem, the hypersurface would be compact and, in that case, we proved in [**Mo1**] that it would be umbilical. As a consequence, making use of (10), the Ricci curvature of Σ^n is bounded. Moreover, in the case $n=2$, we have that the Gauss curvature K of Σ^2 satisfies

$$1-H^2\le K\le 0,$$

the value $1-H^2$ being attained only when Σ^2 is umbilical and the value 0 only if Σ^2 is a hyperbolic cylinder.

Of course, all these considerations are applicable to the hypersurfaces that we constructed in Corollary 12 with $H>1$ and prescribed asymptotic future boundary $\partial\Omega$. *They have no umbilical points when the domain Ω is not a disc. All their principal curvature are positive and bounded and their Ricci curvatures are bounded. Thus, when $n=2$ the surfaces that we have constructed have negative Gauss curvature. Notice that* *the corresponding conformal structure must be hyperbolic*. In fact, if it was parabolic, then, since

$$\Delta\langle H\psi + N, a\rangle = (|\sigma|^2 - 2H^2)\langle N, a\rangle \leq 0,$$

from (8), and the function $\langle H\psi + N, a\rangle$ is non-negative, from Theorem 7, we would have that that function is constant and, so, $|\sigma|^2 - 2H^2 = 0$ and Σ^2 would be umbilical. But, in that case, Σ^2 would be isometric to a hyperbolic plane, which is a contradiction.

References

[A1] R. Aiyama, On complete spacelike surfaces with constant mean curvature in a Lorentzian 3-space form, Tsukuba J. Math., **15** (1991), 235–247.

[A2] R. Aiyama, On the Gauss map of complete spacelike hypersurfaces of constant mean curvature in Minkowski space, Tsukuba J. Math., **16** (1992), 353–361.

[Ak] K. Akutagawa, On spacelike hypersurfaces with constant mean curvature in the de Sitter space, Math. Z., **196** (1987), 13–19.

[An1] M. Anderson, Complete varieties in hyperbolic space, Invent. Math., **69** (1982), 477–494.

[An2] M. Anderson, Complete minimal hypersurfaces in hyperbolic n-manifolds, Comment. Math. Helv., **58** (1983), 264–290.

[C] E. Calabi, Examples of Bernstein problems for some non-linear equations, Proc. Sympos. Pure and Appl. Math., **15** (1970), 223–230.

[CY] S. Y. Cheng, S. T. Yau, Maximal spacelike hypersurfaces in the Lorentz-Minkowski space, Ann. of Math., **104** (1976), 407–419.

[CB] Y. Choquet-Bruhat, Maximal submanifolds and submanifolds of constant extrinsic curvature of a Lorentzian manifold, Ann. Scuola Norm. Sup. Pisa Cl. Sci., **3** (1976), 361–376.

[CFM] Y. Choquet-Bruhat, A. E. Fischer and J. E. Marsden, Maximal hypersurfaces and positivity of mass, Proc. of the E. Fermi Summer School of the Italian Physical Society, (ed. J. Ehlers), North-Holland, 1979.

[Ge] C. Gerhardt, H-surfaces in Lorentzian manifolds, Comm. Math. Phys., **89** (1983), 523–553.

[GT] D. Gilbarg, N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, Berlin, 1983.

[Go1] A. J. Goddard, Some remarks on the existence of spacelike hypersurfaces of constant mean curvature, Math. Proc. Cambridge Philos. Soc., **82** (1977), 489–495.

[Go2] A. J. Goddard, Foliations of space by spacelike hypersurfaces of constant mean curvature, Comm. Math. Phys., **54** (1977), 279–282.

[H] H. Hopf, Differential Geometry in the Large, Lecture Notes in Math., **1000**, Springer-Verlag, Berlin, 1983.

[HE] S. W. Hawking and G. F. R. Ellis, The Large Scale Structure of Spacetime, Cambridge Univ. Press, Cambridge, 1973.

[KKN] U. H. Ki, H. J. Kim and H. Nakagawa, On spacelike hypersurfaces with constant mean curvature of a Lorentz space form, Tokyo J. Math., **14** (1991), 205–216.

[LM] R. López and S. Montiel, Existence of constant mean curvature graphs in hyperbolic space, Calc. Var. Partial Differential Equations, **8** (1999), 177–190.

[MT] J. E. Marsden and F. J. Tipler, Maximal hypersurfaces and foliations of constant mean curvature in general relativity, Phys. Rep. C, **66** (1980), 109–139.

[Mo1] S. Montiel, An integral inequality for compact spacelike hypersurfaces in de Sitter space and applications to the case of constant mean curvature, Indiana Univ. Math. J., **37** (1988), 909–917.

[Mo2] S. Montiel, A characterization of hyperbolic cylinders in de Sitter space, Tohoku Math. J., **48** (1996), 23–31.

[NS] B. Nelli and J. Spruck, On the existence and uniqueness of constant mean curvature hypersurfaces in hyperbolic space, Geometric Analysis and the Calculus of Variations, (ed. J. Jost), International Press, Cambridge, 1996, 253–266.

[Ol] V. Oliker, A priori estimates of the principal curvatures of spacelike hypersurfaces in de Sitter space with applications to hypersurfaces in hyperbolic space, Amer. J. Math., **114** (1992), 605–626.

[Om] H. Omori, Isometric immersions of Riemannian manifolds, J. Math. Soc. Japan, **19** (1967), 205–214.

[P] B. Palmer, The Gauss map of a spacelike constant mean curvature hypersurface of Minkowski space, Comment. Math. Helv., **65** (1990), 52–57.

[R] J. Ramanathan, Complete spacelike hypersurfaces of constant mean curvature in de Sitter space, Indiana Univ. Math. J., **36** (1987), 349–359.

[St] S. Stumbles, Hypersurfaces of constant mean extrinsic curvature, Ann. Physics, **133** (1981), 28–56.

[To] Y. Tonegawa, Existence and regularity of constant mean curvature hypersurfaces in hyperbolic space, Math. Z., **221** (1996), 591–615.

[Tr] A. Treibergs, Entire spacelike hypersurfaces of constant mean curvature in Minkowski space, Invent. Math., **66** (1982), 39–56.

[Y] S. T. Yau, Harmonic functions on complete Riemannian manifolds, Comm. Pure Appl. Math., **28** (1975), 201–228.

Sebastián MONTIEL

Departamento de Geometría y Topología
Universidad de Granada
E-18071 Granada
Spain
E-mail: smontiel@goliat.ugr.es