JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 11, Number 2, Summer 1999

A COLLOCATION METHOD FOR THE SOLUTION OF THE FIRST BOUNDARY VALUE PROBLEM OF ELASTICITY IN A POLYGONAL DOMAIN IN $\boldsymbol{R}^2$

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Dedicated to the memory of Professor Dr. Siegfried Prössdorf
(*January 2, 1939–July 19, 1998*)

ABSTRACT. We present a spline collocation method for the numerical solution of a system of integral equations on a polygon in $\mathbf{R}^2$. This integral equation arises if one solves the first boundary value problem for the Lamé equation with a double layer potential. The derivation and the analysis of the integral equation is given in detail. The optimal order of the spline collocation method is proved for sufficiently graded meshes.

1. Introduction. In this paper we consider a collocation method for the approximate solution of a boundary integral equation for the first boundary value problem for the Lamé equation in $\Omega \subset \mathbf{R}^2$, see [**15**]. We assume that the domain Ω has a polygonal boundary Γ.

To derive the integral equation of the second kind we use a double layer potential and the pseudostress tensor, see [**14, 12**]. The resulting integral equation takes the form, see Section 2,

$$\mathcal{B}\vec{u} := (I+\mathcal{K})\vec{u} = \vec{f}, \tag{1.1}$$

where the elastic double layer potential operator $\mathcal{K}$ is given by

$$\begin{aligned}\mathcal{K}\vec{u}(x_0) = -\frac{1}{\pi}\int_\Gamma \bigg[\frac{(x_0-y)\cdot n_y}{\|x_0-y\|^2}\bigg(&(1-\bar{\omega})I_{2\times 2}\\ &+ 2\bar{\omega}\frac{(x_0-y)(x_0-y)^T}{\|x_0-y\|^2}\bigg)\vec{u}(y)\bigg]\,ds_y, \quad x_0\in\Gamma.\end{aligned} \tag{1.2}$$

Received by the editors on October 20, 1997, and in revised form on August 13, 1998.
AMS 1991 *Mathematics Subject Classification.* 45E10, 65N38, 65R20, 73C02.
Key words and phrases. Lamé equation, boundary integral equation, spline collocation, solvability in weighted Sobolev spaces.

Here n_y denotes the exterior normal to Γ at the point y, $I_{2\times 2}$ is the 2×2 unit matrix, and

$$\bar{\omega} = (\lambda+\mu)/(\lambda+3\mu), \tag{1.3}$$

λ, μ being the Lamé constants. Note that the kernel of $\mathcal{K}$ has only fixed singularities at the corners. Local to each corner, the operator (1.2) may be interpreted as a 4×4 system of Mellin convolution operators, see Section 3. The integral operator (1.2) is not compact in the case of a polygonal boundary so the standard theory for collocation methods does not apply here. To the authors' knowledge this special boundary integral equation is used here for the first time to approximate the solution of the Lamé equation in polygonal domains. In order to prove that the boundary integral equation always has a solution we first have to show some properties of the double layer potential, and here we imitate the proofs of Costabel [**4**].

To analyze the integral equation we first localize the integral operator (1.2) around each corner and show the Fredholm property and the existence of the inverse for these localized operators. Related results were obtained in the papers [**19, 20**] and [**16**]. With this result we can prove the unique solvability of the integral equation on the polygon. We also prove a regularity result for the solutions of the integral equations, and this shows that the use of higher order splines makes sense.

We use continuous splines of any order and graded meshes to get the optimal order of convergence. In order to show the stability of our method we have to modify the spline space in the vicinity of each corner. This technique is well known. In [**5, 17**] it is used for the solution of integral equations of the second kind with noncompact integral operators, and in [**3, 7**] this technique is applied to the solution of the Laplace equation in polygonal domains. The proof of stability relies on the stability of the finite section method for systems of Wiener–Hopf operators [**10**]. However, in contrast to the corresponding scalar integral equation for the Laplacian, the analysis is complicated by the fact that the second kind operator (1.1) need not be strongly elliptic in L^2.

The outline of the paper is as follows: In Section 2 we derive the integral equation and prove some results for the double layer potential and some uniqueness results for weak solutions of the Lamé equation.

In Section 3 we first localize the integral operator around each corner and then we study the localized operators. We put these results together to prove that the integral equation on the boundary of Ω has a solution for every righthand side in $L^2(\partial\Omega)$. If the righthand side has a higher regularity, then the solution becomes more regular, i.e., belongs to certain weighted Sobolev spaces.

In Section 4 we define the meshes and the spline spaces which we use. Then we prove the stability of our method if the meshes fulfill some simple condition and if the spline space is suitably modified. A further approximation result then shows the order of convergence of our method.

2. The boundary value problem and the corresponding boundary integral equation. In this section we define the boundary value problem, which we will study, and we introduce the generalized stress operator, see [**12**]. We extend the trace operator and the generalized stress operator to a sufficiently large function space and formulate the first and second Green formula for our differential operator in Lemma 2.6. The mapping properties of the single and double layer operator are studied in Lemma 2.10 and the jump of the double layer potential across the boundary is derived in Lemma 2.12. At the end of this section we prove uniqueness for the exterior boundary value problem, where the generalized stress is prescribed at the boundary, and the boundary values of the double layer potential are given.

We follow closely the article [**4**] of Costabel.

Let $\Omega \subset R^2$ be an open bounded domain with polygonal boundary Γ. We denote by Ω^c the complement of $\overline{\Omega}$, $\Omega^c := \mathbf{R}^2 \setminus \overline{\Omega}$, and we assume that Ω is contained in some sufficiently large ball $B_{R_0}(0)$, $R_0 > 0$.

For functions $\vec{u} = (u_1, u_2)^T \in (H^2(\Omega))^2$ the Lamé operator P is defined by

$$P\vec{u} := -\mu\Delta\vec{u} - (\lambda+\mu)\mathrm{grad}\,(\mathrm{div}\,\vec{u}), \quad \mu > 0, \lambda \geq 0. \tag{2.1}$$

It is the aim of Sections 2 and 3 to study the existence and the properties of the solution of the equation

$$\begin{aligned} (P\vec{u})(x) = 0, &\quad x \in \Omega \\ \vec{u}|_\Gamma = \vec{f}, &\ \vec{f} \in (H^{1/2}(\Gamma))^2 \end{aligned} \tag{2.2}$$

with the help of a corresponding boundary integral equation.

The operator P can be written in another way with the help of the following definitions, see [**15**] for the physical meaning of the terms,

$$\begin{aligned} \varepsilon_{i,j}(\vec{u}) &:= \frac{1}{2}(\partial_i u_j + \partial_j u_i) \\ \sigma_{i,j}(\vec{u}) &:= \lambda(\varepsilon_{1,1}(\vec{u}) + \varepsilon_{2,2}(\vec{u}))\delta_{i,j} + 2\mu\varepsilon_{i,j}(\vec{u}) \end{aligned}, \quad i,j = 1,2. \tag{2.3}$$

The ith component of $P\vec{u}$ can be written as

$$(P\vec{u})_i = -\sum_{j=1}^{2} \partial_j(\sigma_{i,j}(\vec{u})). \tag{2.4}$$

We further introduce the following notations

$$V^i := (H^i(\Omega))^2, \quad i = 0,1,2, \quad \text{and} \quad V_0^1 := (H_0^1(\Omega))^2. \tag{2.5}$$

Formula (2.4) and a partial integration give us the following relation (first Green formula) for functions $\vec{u} \in V^2$ and $\vec{v} \in V^1$:

$$\int_\Omega P\vec{u} \cdot \vec{v}\, dy = \Phi_\Omega(\vec{u}, \vec{v}) - \int_\Gamma \mathcal{T}_\mu(n_y)\vec{u} \cdot \vec{v}\, ds_y. \tag{2.6}$$

The symmetric bilinear form $\Phi_\Omega(\cdot,\cdot)$ on V^1 is given by

$$\begin{aligned} \Phi_\Omega(\vec{u}, \vec{v}) := \int_\Omega \Bigg(\lambda \sum_{j=1}^{2} \varepsilon_{jj}(\vec{u}) \sum_{j=1}^{2} \varepsilon_{jj}(\vec{v}) \\ + 2\mu \sum_{i,j=1}^{2} \varepsilon_{i,j}(\vec{u})\varepsilon_{i,j}(\vec{v}) \Bigg) dy. \end{aligned} \tag{2.7}$$

The generalized stress operator $\mathcal{T}_\kappa$, see [**12**], is defined in the following way

$$\mathcal{T}_\kappa(n)\vec{u} := \mathcal{T}_0(n)\vec{u} + \kappa \begin{pmatrix} n_2\partial_1 u_2 - n_1\partial_2 u_2 \\ n_1\partial_2 u_1 - n_2\partial_1 u_1 \end{pmatrix}, \tag{2.8}$$

$$\mathcal{T}_0(n)\vec{u} := \mu \begin{pmatrix} n \cdot \nabla u_1 \\ n \cdot \nabla u_2 \end{pmatrix} + (\lambda + \mu)\mathrm{div}\,(\vec{u})n, \tag{2.9}$$

with $\kappa \in \mathbf{R}$, $n = (n_1, n_2)^T \in \mathbf{R}^2$, $\vec{u} \in V^1$.

Because of the symmetry of $\Phi_\Omega(\cdot,\cdot)$ we get the second Green formula for $\vec{u}, \vec{v} \in V^2$:

$$\begin{aligned}\int_\Omega (P\vec{u}\cdot\vec{v} - P\vec{v}\cdot\vec{u})\,dy &= \int_\Gamma (\mathcal{T}_\mu(n_y)\vec{v}\cdot\vec{u} - \mathcal{T}_\mu(n_y)\vec{u}\cdot\vec{v})\,ds_y \\ &= \int_\Gamma (\mathcal{T}_{\mu+\omega}(n_y)\vec{v}\cdot\vec{u} - \mathcal{T}_{\mu+\omega}(n_y)\vec{u}\cdot\vec{v})\,ds_y, \\ &\qquad \omega \in \mathbf{R}.\end{aligned}$$

The second equality follows by Gauss's formula. So we finally get the second Green formula in the following form

$$\int_\Omega (P\vec{u}\cdot\vec{v} - P\vec{v}\cdot\vec{u})\,dy = \int_\Gamma (\mathcal{T}_\kappa(n_y)\vec{v}\cdot\vec{u} - \mathcal{T}_\kappa(n_y)\vec{u}\cdot\vec{v})\,ds_y \tag{2.10}$$

for $\vec{u}, \vec{v} \in V^2$, $\kappa \in \mathbf{R}$.

Let $\vec{f} \in V^0$ be given. The function $\vec{u} \in V_0^1$ is the weak solution of

$$\begin{aligned} P\vec{u} &= \vec{f} \\ u|_\Gamma &= 0 \end{aligned} \tag{2.11}$$

if and only if

$$\Phi_\Omega(\vec{u}, \vec{\phi}) = \int_\Omega \vec{f}\cdot\vec{\phi}\,dy, \quad \forall\,\vec{\phi} \in V_0^1. \tag{2.12}$$

Korn's inequality, see [**9**], says that there are constants $c_1, c_2 > 0$, which depend only on Ω, for which

$$c_1\|\vec{u}\|^2_{V_0^1} \le \Phi_\Omega(\vec{u},\vec{u}) \le c_2\|\vec{u}\|^2_{V_0^1}, \quad \forall\,\vec{u} \in V_0^1. \tag{2.13}$$

Equation (2.13) together with the Lax–Milgram lemma gives us the following result.

Corollary 2.1. *The equation* (2.11) *always has a uniquely determined weak solution.*

In the following we denote by γ_0 the trace operator

$$\gamma_0 \vec{u} := \vec{u}|_\Gamma. \tag{2.14}$$

Gagliardo's trace lemma, see [**4**], implies

$$\gamma_0 : H^s_{\mathrm{loc}}(\mathbf{R}^2) \longrightarrow H^{s-1/2}(\Gamma), s \in ((1/2), 1], \text{ is continuous} \tag{2.15}$$

and has a continuous right inverse γ_0^-

$$\gamma_0^- : H^{s-(1/2)}(\Gamma) \longrightarrow H^s_{\mathrm{loc}}(\mathbf{R}^2). \tag{2.16}$$

Using the map γ_0^-, the Lax–Milgram lemma and equation (2.13), one can prove the following lemma.

Lemma 2.2. *For every $\vec{v} \in (H^{1/2}(\Gamma))^2$ there exists a unique solution $T\vec{v}$ of the equation*

$$\begin{aligned} P\vec{u} &= 0 \\ \gamma_0 \vec{u} &= \vec{v} \end{aligned} \tag{2.17}$$

The mapping $\vec{v} \to T\vec{v}$ is linear and continuous, i.e., there exists a constant $c_T > 0$, such that

$$\|T\vec{v}\|_{V^1} \leq c_T \|\vec{v}\|_{(H^{1/2}(\Gamma))^2}. \tag{2.18}$$

We denote by $G(x,y)$ the fundamental solution for the operator P:

$$P_y G(x,y) = \delta(x-y) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =: \delta(x-y) I_{2\times 2}, \tag{2.19}$$

where the index y denotes the differentiation with respect to y. The function G is given by, see [**1**],

$$\begin{aligned} G(x,y) := \frac{1}{4\pi\mu(\lambda+2\mu)} \bigg(&- (\lambda+3\mu)\ln(r) I_{2\times 2} \\ &+ (\lambda+\mu) \frac{(x-y)(x-y)^T}{r^2} \bigg), \end{aligned} \tag{2.20}$$

$x, y \in \mathbf{R}^2$, $r := \|x - y\|$, $x \neq y$. G is the kernel for the Green operator for P. We will also denote the Green operator by G. If we substitute $G(x, y)$ for $\vec{u}(y)$ in the Green formula (2.10) then we get

$$\begin{aligned} \vec{v}(x) &= \int_\Omega G(x,y)^T (P\vec{v})(y)\,dy \\ &\quad + \int_\Gamma \Big(G(x,y)^T \mathcal{T}_{\kappa,y}(n_y)\vec{v}(y) - \Big(\mathcal{T}_{\kappa,y}(n_y) G(x,y)\Big)^T \vec{v}(y)\Big)\,ds_y, \\ &\qquad\qquad x \in \Omega, \quad \vec{v} \in V^2. \end{aligned} \tag{2.21}$$

By V_P^1 we denote the set of all functions $\vec{u} \in V^1$, for which the distribution $P\vec{u}$ belongs to V^0. The norm on V_P^1 is given by

$$\|\vec{u}\|^2_{V_P^1} := \|\vec{u}\|^2_{V^1} + \|P\vec{u}\|^2_{V^0}. \tag{2.22}$$

Now we extend the generalized stress operator to functions in V_P^1. First we recall the following lemma from [**11**, p. 113].

Lemma 2.3. *V^2 is dense in V_P^1.*

The next lemma is an easy consequence of our definitions.

Lemma 2.4. *Let $\vec{u} \in V_P^1$. The mapping*

$$\begin{aligned} \vec{\phi} &\longrightarrow \langle \gamma_1^{(\mu)} \vec{u}, \vec{\phi}\rangle \\ &:= \Phi_\Omega(\vec{u}, \gamma_0^- \vec{\phi}) - \int_\Omega (P\vec{u} \cdot \gamma_0^- \vec{\phi})\,dy \end{aligned} \tag{2.23}$$

is a continuous linear functional $\gamma_1^{(\mu)}\vec{u}$ on $(H^{1/2}(\Gamma))^2$, which coincides for $\vec{u} \in V^2$ with

$$\vec{\phi} \longrightarrow \int_\Gamma \mathcal{T}_\mu(n_y)\vec{u} \cdot \vec{\phi}\,ds_y. \tag{2.24}$$

The mapping

$$\gamma_1^{(\mu)} : V_P^1 \longrightarrow (H^{-1/2}(\Gamma))^2 \tag{2.25}$$

is continuous.

Remark. That $\gamma_1^{(\mu)}$ coincides with $\mathcal{T}_\mu(n_y)$ for functions in V^2 and the density result in Lemma 2.3 show that the definition of $\gamma_1^{(\mu)}$ is independent of the chosen operator γ_0^-. The operator γ_0^- is not unique.

As a next step we define $\mathcal{T}_0$ and then $\mathcal{T}_\kappa$, $\kappa \in [0,\mu]$, for functions in V_P^1. The starting point is formula (2.1) for P. For $\vec{u} \in V^2$ and $\vec{v} \in V^1$ we get by partial integration, cf. (2.6),

$$
\begin{aligned}
\int_\Omega P\vec{u}\cdot\vec{v}\,dy &= \int_\Omega \left(\mu \sum_{i,j=1}^2 \partial_i u_j \partial_i v_j + (\lambda+\mu)\mathrm{div}\,(\vec{u})\mathrm{div}\,(\vec{v})\right) dy \\
&\quad - \int_\Gamma \mathcal{T}_0(n_y)\vec{u}\cdot\vec{v}\,ds_y \\
&=: \widetilde{\Phi}_\Omega(\vec{u},\vec{v}) - \int_\Gamma \mathcal{T}_0(n_y)\vec{u}\cdot\vec{v}\,ds_y.
\end{aligned}
\tag{2.26}
$$

For $\tilde{\Phi}_\Omega$ we have the following properties

$$
\begin{aligned}
\tilde{\Phi}_\Omega(\vec{u},\vec{v}) &= \tilde{\Phi}_\Omega(\vec{v},\vec{u}), \quad \vec{u},\vec{v} \in V^1 \\
\mu \sum_{i,j=1}^2 \int_\Omega |\partial_j u_i|^2 dy &\le \tilde{\Phi}_\Omega(\vec{u},\vec{u}), \quad \vec{u} \in V^1
\end{aligned}
\tag{2.27}
$$

Now we extend the operator $\mathcal{T}_0$ to functions in V_P^1. By convex combination we then define $\mathcal{T}_\kappa$, $\kappa \in [0,\mu]$, for functions in V_P^1.

Lemma 2.5. (i) *Let $\vec{u} \in V_P^1$. The mapping*

$$
\vec{\phi} \longrightarrow \langle \gamma_1^{(0)}\vec{u}, \vec{\phi}\rangle := \tilde{\Phi}_\Omega(\vec{u}, \gamma_0^-\vec{\phi}) - \int_\Omega P\vec{u}\cdot\gamma_0^-\vec{\phi}\,dy
\tag{2.28}
$$

is a continuous functional $\gamma_1^{(0)}\vec{u}$ on $(H^{1/2}(\Gamma))^2$, which coincides for $\vec{u} \in V^2$ with the mapping

$$
\vec{\phi} \longrightarrow \int_\Gamma \mathcal{T}_0(n_y)\vec{u}\cdot\vec{\phi}\,ds_y.
$$

The mapping

$$\gamma_1^{(0)} : V_P^1 \longrightarrow (H^{-1/2}(\Gamma))^2$$

is continuous.

(ii) *For* $\kappa \in [0,\mu]$, $\kappa = \tilde{\lambda}\mu$, $\tilde{\lambda} \in [0,1]$, *we define*

$$\gamma_1^{(\kappa)} := \tilde{\lambda}\gamma_1^{(\mu)} + (1-\tilde{\lambda})\gamma_1^{(0)}. \tag{2.29}$$

The mapping $\gamma_1^{(\kappa)}$ *is continuous from* V_P^1 *into* $(H^{-1/2}(\Gamma))^2$. *On* V^2, $\gamma_1^{(\kappa)}$ *coincides with* $\mathcal{T}_\kappa(n_y)$. *We further get*

$$\langle \gamma_1^{(\kappa)}\vec{u}, \vec{\phi}\rangle = \Phi_\Omega^{(\kappa)}(\vec{u}, \gamma_0^- \vec{\phi}) - \int_\Omega P\vec{u} \cdot \gamma_0^- \vec{\phi}\, ds_y \tag{2.30}$$

where

$$\Phi_\Omega^{(\kappa)}(\cdot,\cdot) := \tilde{\lambda}\Phi_\Omega(\cdot,\cdot) + (1-\tilde{\lambda})\tilde{\Phi}_\Omega(\cdot,\cdot). \tag{2.31}$$

Because of (2.27) *the inequality of Korn* (2.13) *also holds for* $\Phi_\Omega^{(\kappa)}$, $\kappa \in [0,\mu]$.

For a function $\vec{u} \in (L^2(\mathbf{R}^2))^2$ with $\vec{u}|_\Omega \in V^1$ and $\vec{u}|_{\Omega^c} \in (H^1_{\mathrm{loc}}(\Omega^c))^2$, the traces $\gamma_0(\vec{u}|_\Omega)$ and $\gamma_0(\vec{u}|_{\Omega^c})$ are well defined. Let

$$[\gamma_0\vec{u}] := \gamma_0(\vec{u}|_\Omega) - \gamma_0(\vec{u}|_{\Omega^c}) \in (H^{1/2}(\Gamma))^2. \tag{2.32}$$

For a function $\vec{u} \in (H^1_{\mathrm{loc}}(\Omega^c))^2$ and $P\vec{u} \in (L^2_{loc}(\mathbf{R}^2))^2$ the operator $\gamma_{1,\Omega^c}^{(\kappa)}$, $\kappa \in [0,\mu]$, is given by (2.29) and (2.23), where Ω has to be replaced by Ω^c. Here we will assume that $\operatorname{supp}(\gamma_0^- \vec{v}) \subset B_{2R_0}(0)$, for all $\vec{v} \in (H^{1/2}(\Gamma))^2$. We will denote the set of all functions $\vec{u} \in (H^1_{\mathrm{loc}}(\Omega^c))^2$ with $P\vec{u} \in (L^2_{\mathrm{loc}}(\mathbf{R}^2))^2$ by $V_P^1(\Omega^c)$.

If $\vec{u} \in (L^2(\mathbf{R}^2))^2$ with $\vec{u}|_\Omega \in V_P^1$ and $\vec{u}|_{\Omega^c} \in V_P^1(\Omega^c)$, then we define

$$[\gamma_1^{(\kappa)}\vec{u}] := \gamma_1^{(\kappa)}(\vec{u}|_\Omega) - \gamma_{1,\Omega^c}^{(\kappa)}(\vec{u}|_{\Omega^c}). \tag{2.33}$$

The next lemma follows easily by Lemma 2.3 and the above definitions.

Lemma 2.6. *Let* $\kappa \in [0, \mu]$.

(i) *For* $\vec{u} \in V_P^1$ *and* $\vec{v} \in V^1$ *the first Green formula holds:*

$$\int_\Omega P\vec{u} \cdot \vec{v}\, dy = \Phi_\Omega^{(\kappa)}(\vec{u}, \vec{v}) - \langle \gamma_1^{(\kappa)} \vec{u}, \gamma_0 \vec{v} \rangle \tag{2.34}$$

(ii) *The second Green formula holds for all* $\vec{u}, \vec{v} \in V_P^1$.

$$\int_\Omega (\vec{u} \cdot P\vec{v} - \vec{v} \cdot P\vec{u})\, dy = \langle \gamma_1^{(\kappa)} \vec{u}, \gamma_0 \vec{v} \rangle - \langle \gamma_1^{(\kappa)} \vec{v}, \gamma_0 \vec{u} \rangle. \tag{2.35}$$

(iii) *Let* $\vec{u} \in (L^2(\mathbf{R}^2))^2$ *be given with*

$$\vec{u}|_\Omega \in V_P^1 \quad \textit{and} \quad \vec{u}|_{\Omega^c} \in V_P^1(\Omega^c).$$

Then

$$\begin{aligned} u(x) &= (GP\vec{u})(x) + \langle [\gamma_1^{(\kappa)} \vec{u}], G(x, \cdot) \rangle \\ &\quad - \int_\Gamma (\mathcal{T}_\kappa(n_y) G(x, y))^T [\gamma_0 \vec{u}]\, ds_y, \quad x \in \mathbf{R}^2 \setminus \Gamma. \end{aligned} \tag{2.36}$$

With the help of Lemma 2.6, Lemma 2.2 and Corollary 2.1, the next lemma is proved analogously to Lemma 3.5 of [**4**].

Lemma 2.7. *The trace mapping*

$$(\gamma_0, \gamma_1^{(\kappa)}) : \vec{\phi} \longrightarrow (\gamma_0 \vec{\phi}, \gamma_1^{(\kappa)} \vec{\phi})$$

maps $(C_0^\infty(\mathbf{R}^2))^2$ *onto a dense subset of* $(H^{1/2}(\Gamma))^2 \times (H^{-1/2}(\Gamma))^2$.

Lemma 2.8. *The trace operator*

$$\gamma_0 : u \longrightarrow u|_\Gamma : H^s_{\mathrm{loc}}(\mathbf{R}^2) \longrightarrow H^{s-1/2}(\Gamma)$$

is continuous for $s \in ((1/2), (3/2))$.

Proof. See [**4**, Lemma 3.6].

Lemma 2.9. *The Green operator G fulfills*

$$G : (H^s(\mathbf{R}^2))^2 \longrightarrow (H^{s+2}_{\text{loc}}(\mathbf{R}^2))^2.$$

Proof. Calculate the symbol matrix of P with the Fourier transform. □

Now we define the single layer operator K_0 and the double layer operator K_1:

$$(K_0\vec{v})(x) = \int_\Gamma G(x,y)\vec{v}(y)\,ds_y, \quad x \in \mathbf{R}^2 \setminus \Gamma \tag{2.37}$$

$$(K_1^{(\kappa)}\vec{v})(x) = \int_\Gamma (\mathcal{T}\kappa(n_y)G(x,y))^T\vec{v}(y)\,ds_y, \quad x \in \mathbf{R}^2 \setminus \Gamma. \tag{2.38}$$

The following lemma repeats the results of Theorem 1.(i), (ii) of [**4**] for the Lamé operator.

Lemma 2.10. (i) *The mapping $K_0 : (H^{-1/2+\sigma}(\Gamma))^2 \to (H^{1+\sigma}_{\text{loc}}(\mathbf{R}^2))^2$, $\sigma \in (-(1/2),(1/2))$, is continuous.*

(ii) *The mapping $K_1^{(\kappa)} : (H^{1/2}(\Gamma))^2 \to (H^1(\Omega))^2$, $\kappa \in [0,\mu]$, is continuous.*

Proof. (i) Let $\vec{v} \in (H^{-1/2+\sigma}(\Gamma))^2$. Then $\gamma_0'\vec{v}$ (where γ_0' is the adjoint of γ_0) is a distribution in $\mathbf{R}^2$ with compact support $\langle\gamma_0'\vec{v},\vec{\phi}\rangle := \langle\vec{v},\gamma_0\vec{\phi}\rangle$, for all $\vec{\phi} \in (C_0^\infty(\mathbf{R}^2))^2$. Now we have

$$K_0 = G \circ \gamma_0'$$

and Lemma 2.8 shows

$$\gamma_0' : (H^{(1/2)-s}(\Gamma))^2 \longrightarrow (H^{-s}_{\text{comp}}(\mathbf{R}^2))^2, \quad s \in ((1/2),(3/2)).$$

Lemma 2.9 finally implies

$$G \circ \gamma_0' : (H^{1/2-s}(\Gamma))^2 \longrightarrow (H^{2-s}_{\mathrm{loc}}(\mathbf{R}^2))^2.$$

Define $\sigma = 1 - s \in (-(1/2), (1/2))$. Then the above equation gives

$$K_0 : (H^{-(1/2)+\sigma}(\Gamma))^2 \longrightarrow (H^{1+\sigma}_{\mathrm{loc}}(\mathbf{R}^2))^2.$$

(ii) Let $\vec{v} \in (H^{1/2}(\Gamma))^2$ and $\vec{u} := T\vec{v} \in V_P^1$, where $T\vec{v}$ is the solution of (2.17), see Lemma 2.2. Formula (2.36), where we define $\vec{u}|_{\Omega^c} \equiv 0$, now gives us

$$\begin{aligned} T\vec{v} &= \int_\Omega G(x,y) PT\vec{v}(y)\, dy + \langle \gamma_1^{(\kappa)} T\vec{v}, G(\cdot, x)\rangle - K_1^{(\kappa)}\vec{v} \\ &= K_0 \gamma_1^{(\kappa)} T\vec{v} - K_1^{(\kappa)}\vec{v}. \end{aligned}$$

Therefore

$$\begin{aligned} K_1^{(\kappa)} &= K_0 \circ \gamma_1^{(\gamma)} \circ T - T \\ &= (K_0 \circ \gamma_1^{(\kappa)} - I) \circ T \end{aligned}$$

and we have

$$\begin{aligned} T &: (H^{1/2}(\Gamma))^2 \longrightarrow V_P^1 && \text{(Lemma 2.2)} \\ \gamma_1^{(\kappa)} &: V_P^1 \longrightarrow H^{-1/2}(\Gamma))^2 && \text{(Lemma 2.5)} \\ K_0 &: (H^{-1/2}(\Gamma))^2 \longrightarrow V^1 && \text{(part (i)).} \quad \square \end{aligned}$$

In the next lemma we collect some smoothness properties for the double layer potential, and for a special parameter $\bar{\kappa}$ we estimate the norm of $K_1^{(\bar{\kappa})}\vec{v}(x)$ and its derivatives.

Lemma 2.11. *For $\vec{v} \in (H^{1/2}(\Gamma))^2$ and*

$$\bar{\kappa} := \mu \frac{\lambda + \mu}{\lambda + 3\mu} \tag{2.39}$$

the following results hold:

(i) $K_1^{(\kappa)}\vec{v} \in (C^\infty(\mathbf{R}^2 \setminus \Gamma))^2$, $\kappa \in \mathbf{R}$.

(ii)

$$\|(K_1^{(\bar{\kappa})}\vec{v})(x)\| = \mathcal{O}_{\|x\|\to\infty}\left(\frac{1}{\|x\|}\right),$$
$$\|(\nabla K_1^{(\bar{\kappa})}\vec{v})(x)\| = \mathcal{O}_{\|x\|\to\infty}\left(\frac{1}{\|x\|^2}\right),$$

(iii) $K_1^{(\kappa)}\vec{v} \in V_P^1(\Omega^c)$, $\kappa \in [0,\mu]$.

Proof. (i) The function $G(x,y)$ is in C^∞ outside the diagonal in $\mathbf{R}^n \times \mathbf{R}^n$. This proves (i).

(ii) By a calculation we get

(2.40)
$$\mathcal{T}_{\bar{\kappa}}(n_y)G(x,y) = -\frac{1}{2\pi}\left((1-\bar{\omega})I_{2\times 2} + 2\bar{\omega}\frac{(y-x)(y-x)^T}{\|x-y\|^2}\right)\frac{(y-x)\cdot n_y}{\|x-y\|^2},$$

where $\bar{\omega}$ is defined in (1.3).

Property (ii) follows by (2.40) and the compactness of Γ.

(iii) Here the arguments of Lemma 2.5 and Lemma 2.10 for a domain $\tilde{\Omega} := (\mathbf{R}^2 \setminus \overline{\Omega}) \cap B_R(0)$, $R > R_0$ arbitrary, have to be repeated. This shows that $K_1^{(\kappa)}\vec{v} \in (H^1(\widetilde{\Omega}))^2$ and $P(K_1^{(\kappa)}\vec{v}) \in (L^2(\tilde{\Omega}))^2$. Together with property (i) this proves part (iii).□

Remark. Formula (2.40) holds, only for the special choice $\kappa = \bar{\kappa}$, where $\mathcal{T}_{\bar{\kappa}}$ is called the pseudostress operator. For $\kappa \neq \bar{\kappa}$ a further term appears in formula (2.40). This term has a stronger singularity for $x = y$, see [**14**] and is not covered by the analysis in Sections 3 and 4.

Lemma 2.12. *For $\vec{v} \in (H^{1/2}(\Gamma))^2$, $\kappa \in [0,\mu]$, we get*

$$[\gamma_0 K_1^{(\kappa)}\vec{v}] = -\vec{v}, \quad [\gamma_1^{(\kappa)} K_1^{(\kappa)}\vec{v}] = 0.$$

Proof. Let $\vec{v} \in (H^{1/2}(\Gamma))^2$, $\vec{\phi} \in (C_0^\infty(\mathbf{R}^2))^2$, $\vec{u} := K_1^{(\kappa)}\vec{v}$. By

Lemma 2.11 we can apply Lemma 2.6 and get

$$\int_{\Omega} \vec{u} \cdot P\vec{\phi}\,dy = \langle \gamma_1^{(\kappa)} \vec{u}|_{\Omega}, \gamma_0 \vec{\phi} \rangle - \langle \gamma_1^{(\kappa)} \vec{\phi}|_{\Omega}, \gamma_0 \vec{u}|_{\Omega} \rangle$$
$$\int_{\Omega^c} \vec{u} \cdot P\vec{\phi}\,dy = \langle -\gamma_1^{(\kappa)} \vec{u}|_{\Omega^c}, \gamma_0 \vec{\phi} \rangle + \langle \gamma_1^{(\kappa)} \vec{\phi}|_{\Omega^c}, \gamma_0 \vec{u}|_{\Omega^c} \rangle.$$

The different signs in the second formula are caused by the choice of the outer normal for Ω in the definition of $\gamma_1^{(\kappa)}$. This implies

$$\int_{\mathbf{R}^2} \vec{u} \cdot P\vec{\phi}\,dy = \langle [\gamma_1^{(\kappa)} \vec{u}], \gamma_0 \vec{\phi} \rangle - \langle \gamma_1^{(\kappa)} \vec{\phi}, [\gamma_0 \vec{u}] \rangle. \tag{2.41}$$

On the other hand we have

$$\vec{u} = K_1^{(\kappa)} \vec{v} = G((\gamma_1^{(\kappa)})' \vec{v}),$$

where the distribution $(\gamma_1^{(\kappa)})' \vec{v}$ with compact support is defined by

$$\langle (\gamma_1^{(\kappa)})' \vec{v}, \vec{\phi} \rangle = \langle \vec{v}, \gamma_1^{(\kappa)} \vec{\phi} \rangle, \quad \vec{\phi} \in (C_0^\infty(\mathbf{R}^2))^2.$$

Now the left side of equation (2.41) can be rewritten

$$\begin{aligned} \int_{\mathbf{R}^2} \vec{u} \cdot P\vec{\phi}\,dy &= \langle G \circ (\gamma_1^{(\kappa)})' \vec{v}, P\vec{\phi} \rangle \\ &= \langle (\gamma_1^{(\kappa)})' \vec{v}, \underbrace{G \circ P}_{=I} \vec{\phi} \rangle \\ &= \langle \vec{v}, \gamma_1^{(\kappa)} \vec{\phi} \rangle. \end{aligned} \tag{2.42}$$

Formulas (2.41) and (2.42) give us

$$\langle [\gamma_1^{(\kappa)} K_1^{(\kappa)} \vec{v}], \gamma_0 \vec{\phi} \rangle = \langle [\gamma_0 K_1^{(\kappa)} \vec{v}] + \vec{v}, \gamma_1^{(\kappa)} \vec{\phi} \rangle, \quad \forall\, \vec{\phi} \in (C_0^\infty(\mathbf{R}^2))^2.$$

Lemma 2.7 proves the lemma. $\square$

Now we can prove the uniqueness of the solution of the exterior Neumann problem, where instead of the normal derivative the pseudostress operator is used.

Lemma 2.13. *Let* $\kappa \in [0,\mu)$ *and* $\vec{u} \in (H^1_{\mathrm{loc}}(\Omega^c))^2$ *with*

(i) $P\vec{u} = 0$.

(ii) $\gamma_1^{(\kappa)}\vec{u} = 0$.

(iii) $\vec{u}|_{\mathbf{R}^2\setminus\overline{\Omega}} \in (C^\infty(\mathbf{R}^2 \setminus \overline{\Omega}))^2$ and

$$|\vec{u}(x)| = \mathcal{O}\bigg(\frac{1}{\|x\|}\bigg), \qquad |\nabla\vec{u}(x)| = \mathcal{O}\bigg(\frac{1}{\|x\|^2}\bigg).$$

Then we have $\vec{u} = 0$.

Proof. The first Green formula for $\vec{u}|_{\Omega^c\cap B_R(0)}$, $R > R_0$, and property (ii) give

$$\begin{aligned}0 &= \int_{\Omega^c\cap B_R(0)} \vec{u}\cdot P\vec{u}\,dy\\ &= \Phi^{(\kappa)}_{\Omega^c\cap B_R(0)}(\vec{u},\vec{u}) + \langle\underbrace{\gamma_1^{(\kappa)}\vec{u}}_{=0},\vec{u}\rangle - \int_{\partial B_R(0)} \mathcal{T}_\kappa\vec{u}\cdot\vec{u}\,ds_y.\end{aligned}$$

Because $\mathcal{T}_\kappa\vec{u}$ contains only first derivatives of $\vec{u}$, we get by the Cauchy–Schwarz inequality

$$|\mathcal{T}_\kappa\vec{u}(x)\cdot\vec{u}(x)| \le \frac{C}{\|x\|^3}.$$

Now we have

$$\Big|\Phi^{(\kappa)}_{\Omega^c\cap B_R(0)}(\vec{u},\vec{u})\Big| \underset{R\to\infty}{\le} C\,4\pi R\,\frac{1}{R^3}$$

On the other hand, we have that

$$\Phi^{(\kappa)}_{\Omega^c\cap B_R(0)}(\vec{u},\vec{u}) \ge 0$$

is a monotonically increasing function of R, and this finally implies

$$0 = \Phi^{(\kappa)}_{\Omega^c\cap B_R(0)}(\vec{u},\vec{u}), \quad \forall\, R \ge R_0.$$

Now inequality (2.27) implies that $\vec{u}$ is constant and property (iii) proves the lemma. $\square$

In the next section we will prove the injectivity of our boundary integral equation with the help of the last lemma. For the solution of the equation (2.2) we use the double layer potential (2.38) with $\kappa = \bar{\kappa}$ defined in (2.39) and finally we need the boundary values of the double layer potential. For $\vec{u} \in (C(\Gamma))^2$ we get by direct calculation from (2.38)–(2.40)

$$\lim_{\Omega \ni x \to x_0 \in \Gamma} (K^{(\bar{\kappa})}\vec{u})(x) = -\frac{1}{2}\vec{u}(x_0) + \frac{1}{2}(\mathcal{K}\vec{u})(x_0), \tag{2.43}$$

where x_0 is not a corner point of Γ and $\mathcal{K}$ is defined by (1.2).

3. Solvability and regularity results for the integral equation. In this section we study the operator $\mathcal{B} := I - \mathcal{K}$ which was defined in (1.1), (1.2). At the beginning the localization of the operator $\mathcal{B}$ around each corner is given and in Lemma 3.5 and Theorem 3.8 the Fredholm property and the existence of the inverse of these localized operators are proved. Lemma 3.10 and Lemma 3.11 contain some local regularity results for the solutions of our integral equation. In Theorem 3.15 the continuity of the solution of our integral equation is shown if the righthand side is continuous. This enables us to prove that $\mathcal{B}$ is an isomorphism in L^2 (but see also the remark after Theorem 3.17). We finally collect our regularity results in Theorem 3.18, which is important for the approximation results in Section 4.

We will assume that the polygon Γ is parametrized by $\gamma : [0,T] \to \mathbf{R}^2$ in the following way: Introduce $n+1$ points in $[0,T]$ by

$$0 = s_0 < s_1 < \cdots < s_n = T,$$

where $\xi_i := \gamma(s_i)$, $i = 0(1)n$, are the corners of Γ, $\xi_0 = \xi_n$ and

$$\gamma|_{[s_i,s_{i+1}]}(s) = \xi_i + (s - s_i)\zeta_i, \zeta_i := (\cos(\alpha_i), \sin(\alpha_i))^T. \tag{3.1}$$

The outer normal to Γ on $\gamma(s_i, s_{i+1})$ is given by $\eta_i := (\sin(\alpha_i), -\cos(\alpha_i))^T$.

In the following we will identify the functions on Γ and on $[0,T]$. So the study of (1.1) leads us to the study of the integral equation

$$\mathcal{B}^{(\omega)}\vec{u}(s) := \vec{u}(s) + \mathcal{K}^{(\omega)}\vec{u}(s) = \vec{f}(s), \quad s \in [0,T], \tag{3.2}$$

where the integral operator $\mathcal{K}^{(\omega)}$ is defined by

$$\mathcal{K}^{(\omega)}\vec{u}(s) := \int_0^T k^{(\omega)}(s,\tau)\vec{u}(\tau)\,d\tau, \tag{3.3}$$

and

$$\begin{aligned} k^{(\omega)}(s,\tau) := -\frac{1}{\pi}\frac{(\gamma(s)-\gamma(\tau))\cdot n(\tau)}{\|\gamma(s)-\gamma(\tau)\|^2} \\ \cdot\left((1-\omega)I_{2\times 2} + 2\omega\frac{(\gamma(s)-\gamma(\tau))(\gamma(s)-\gamma(\tau))^T}{\|\gamma(s)-\gamma(\tau)\|^2}\right), \end{aligned} \tag{3.4}$$

see (2.40), (2.43)). Finally we are only interested in $\mathcal{K} = \mathcal{K}^{(\bar{\omega})}$, see (2.43), (1.3), but we will study $\mathcal{K}^{(\omega)}$, $\omega \in [0,1]$. For $\omega = 0$ we get a decoupled system and the kernel is the well-known kernel of the double layer potential for the Laplace equation.

In the following we will split the operator $\mathcal{K}^{(\omega)}$ into a compact operator $\mathcal{K}^{(\omega,2)}$ and an operator $\mathcal{K}^{(\omega,1)}$ which acts locally around the corners.

For simplicity we will assume that

$$2 < \min_{i=0}^{n-1}\{s_{i+1} - s_i\}, \tag{3.5}$$

and introduce a subdivision of $[0,T]$ into the intervals

$$J_{3i+j} := \begin{cases} [s_i, s_i+1] & j=0, \\ [s_i+1, s_{i+1}-1] & j=1, \\ [s_{i+1}-1, s_{i+1}], & j=2, \end{cases} \quad i = 0(1)n-1. \tag{3.6}$$

The kernel of $\mathcal{K}^{(\omega,1)}$ is given by

$$k^{(\omega,1)}(s,\tau) := \sum_{i=0}^{n-1} \chi_i(s)k^{(\omega)}(s,\tau)\chi_i(\tau), \tag{3.7}$$

where χ_i, $i = 1(1)n-1$, is the characteristic function of $(s_i - 1, s_i + 1)$ and χ_0 of $[0,1) \cup (T-1,T]$.

The kernel of $\mathcal{K}^{(\omega,2)}$ is defined by

$$k^{(\omega,2)} := k^{(\omega)} - k^{(\omega,1)}, \tag{3.8}$$

and by construction we have

$$\mathcal{B}^{(\omega)} = I + \mathcal{K}^{(\omega,1)} + \mathcal{K}^{(\omega,2)}. \tag{3.9}$$

and

$$\begin{gathered}(I + \mathcal{K}^{(\omega,1)})\vec{u}(s) = \vec{u}(s),\\ s \in [s_i + 1, s_{i+1} - 1], i = 1(1)n-1,\end{gathered} \tag{3.10}$$

because $k^{(\omega,1)}(s,\tau) = 0$, $s \in [s_i + 1, s_{i+1} - 1]$, $i = 0(1)n-1$. Furthermore,

$$(I + \mathcal{K}^{(\omega,1)})\vec{u}(s) = 0, \quad s \in [s_i - 1, s_i + 1], \tag{3.11}$$

holds, if $\operatorname{supp}(\vec{u}) \cap [s_i - 1, s_i + 1] = \varnothing$.

Now we localize $I + \mathcal{K}^{(\omega,1)}$ around each corner and get the following equivalent operator for the i^{th} corner

$$\mathcal{B}_i^{(\omega)} := I + \mathcal{K}_i^{(\omega)}, \tag{3.12}$$

where $\mathcal{K}_i^{(\omega)}$ is defined by

$$(\mathcal{K}_i^{(\omega)} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix})(s) := \begin{pmatrix} \int_0^1 k_{i,1}^{(\omega)}(s/\tau) \begin{pmatrix} v_3 \\ v_4 \end{pmatrix}(\tau)\,(d\tau/\tau) \\ \int_0^1 k_{i,2}^{(\omega)}(s/\tau) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}(\tau)\,(d\tau/\tau) \end{pmatrix}, \quad s \in [0,1]. \tag{3.13}$$

The kernels are defined by

$$\begin{aligned} k_{i,1}^{(\omega)}(z) := \frac{1}{\pi} \frac{z \sin(\alpha_i)}{z^2 + 2z\cos(\alpha_i) + 1} \Bigg(&(1-\omega) I_{2\times 2} + \frac{2\omega}{z^2 + 2z\cos(\alpha_i) + 1} \\ &\begin{pmatrix} z^2 + 2z\cos(\alpha_i) + \cos^2(\alpha_i) & \sin(\alpha_i)(\cos(\alpha_i) + z) \\ \sin(\alpha_i)(\cos(\alpha_i) + z) & \sin^2(\alpha_i) \end{pmatrix} \Bigg), \end{aligned} \tag{3.14}$$

$$\begin{aligned} k_{i,2}^{(\omega)}(z) := \frac{1}{\pi} \frac{z \sin(\alpha_i)}{z^2 + 2z\cos(\alpha_i) + 1} \Bigg(&(1-\omega) I_{2\times 2} + \frac{2\omega}{z^2 + 2z\cos(\alpha_i) + 1} \\ &\begin{pmatrix} z^2\cos^2(\alpha_i) + 2z\cos(\alpha_i) + 1 & z\sin(\alpha_i)(z\cos(\alpha_i) + 1) \\ z\sin(\alpha_i)(z\cos(\alpha_i) + 1) & z^2\sin^2(\alpha_i) \end{pmatrix} \Bigg), \end{aligned} \tag{3.15}$$

and we have identified

(3.16)
$$\begin{pmatrix} v_1(s) \\ v_2(s) \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} u_1(s_i - s) \\ u_2(s_i - s) \end{pmatrix}, \quad \begin{pmatrix} v_3(s) \\ v_4(s) \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} u_1(s_i + s) \\ u_2(s_i + s) \end{pmatrix},$$
$$s \in [0,1].$$

We also assumed $\alpha_{i-1} = 0$, so that $\pi - \alpha_i$ is the interior angle at ξ_i, and omitted some computations.

In the next lemma we will collect some rather obvious mapping properties of $\mathcal{K}^{(\omega,2)}$ and $\mathcal{K}^{(\omega)}$; note that, for example, the kernel $k^{(\omega,2)}$, see (3.7), (3.8), is C^∞ on each set $J_i \times J_k$. Then we will study the properties of $I + \mathcal{K}^{(\omega,1)}$ with the help of $\mathcal{B}_i^{(\omega)}$, $i = 1(1)n$.

Lemma 3.1. *For $\omega \in [0,1]$ we have*

(i) $\mathcal{K}^{(\omega,2)} : (L^2(0,T))^2 \to (L^2(0,T))^2$ *is compact.*

(ii) $\vec{u} \in (L^2(0,T))^2$ *implies* $\mathcal{K}^{(\omega,2)}\vec{u}|_{J_i} \in (C^\infty(J_i))^2$, $i = 0(1)3n-1$.

(iii) *If* $\vec{u} \in (L^2(0,T))^2$ *then* $\mathcal{K}^{(\omega,2)}\vec{u}$ *is continuous at each point* s_i.

(iv) $\vec{u} \in (L^2(0,T))^2$ *implies* $\mathcal{K}^{(\omega)}\vec{u}|_{(s_i,s_{i+1})} \in (C^\infty(s_i,s_{i+1}))^2$, $i = 0(1)n-1$.

We now define four functions, which build up the functions $k_{i,1}^{(\omega)}$ and $k_{i,2}^{(\omega)}$:

(3.17)
$$\left.\begin{aligned} l_{i,1}(z) &:= \frac{z}{1 + 2\cos(\alpha_i)z + z^2} \\ l_{i,2}(z) &:= \frac{z}{(1 + 2\cos(\alpha_i)z + z^2)^2} \\ l_{i,3}(z) &:= \frac{z^2}{(1 + 2\cos(\alpha_i)z + z^2)^2} \\ l_{i,4}(z) &:= \frac{z^3}{(1 + 2\cos(\alpha_i)z + z^2)^2} \end{aligned}\right., \quad \alpha_i \in (-\pi,\pi)$$

The following properties are clear: $l_{i,j} \in C^\infty([0,\infty))$, $l_{i,j}(0) = 0$, $l_{i,j}(x) > 0$ if $x > 0$ and

(3.18)
$$\int_0^\infty x^q l_{i,j}(x)\,dx < \infty, \quad q \in (-2,0), \quad \forall\, i,j.$$

From now on we will omit the index i for $l_{i,j}$ and $k_{i,j}^{(\omega)}$. We get

$$
\begin{aligned}
k_1^{(\omega)}(z) &= (1-\omega)k_D(z)I_{2\times 2} + \omega \begin{pmatrix} k_1^{(1)}(z) & k_2^{(1)}(z) \\ k_2^{(1)}(z) & k_3^{(1)}(z) \end{pmatrix} \\
k_2^{(\omega)}(z) &= (1-\omega)k_D(z)I_{2\times 2} + \omega \begin{pmatrix} k_1^{(2)}(z) & k_2^{(2)}(z) \\ k_2^{(2)}(z) & k_3^{(2)}(z) \end{pmatrix}
\end{aligned} \tag{3.19}
$$

with

$$
\begin{aligned}
k_D(z) &= \frac{1}{\pi}\frac{z\sin(\alpha_i)}{z^2+2\cos(\alpha_i)z+1} \\
&= \frac{\sin(\alpha_i)}{\pi} l_1(z)
\end{aligned} \tag{3.20}
$$

$$
\begin{aligned}
k_3^{(1)}(z) &= \frac{2}{\pi}\frac{z\sin(\alpha_i)^3}{(z^2+2\cos(\alpha_i)z+1)^2} \\
&= \frac{2}{\pi}\sin(\alpha_i)^3 l_2(z) \\
k_1^{(1)}(z) &= \frac{2}{\pi} z\sin(\alpha_i)\frac{z^2+2\cos(\alpha_i)z+\cos(\alpha_i)^2}{(z^2+2\cos(\alpha_i)z+1)^2} \\
&= -\frac{2}{\pi}\frac{\sin(\alpha_i)^3 z}{(z^2+2\cos(\alpha_i)z+1)^2} + \frac{2}{\pi}\frac{z\sin(\alpha_i)}{z^2+2\cos(\alpha_i)z+1} \\
&= 2k_D(z) - k_3^{(1)}(z) \\
k_2^{(1)}(z) &= \frac{2}{\pi}\frac{z\sin(\alpha_i)^2(z+\cos(\alpha_i))}{(z^2+2\cos(\alpha_i)z+1)^2} \\
&= \frac{2}{\pi}\sin(\alpha_i)^2 l_3(z) + \frac{2}{\pi}\cos(\alpha_i)\sin(\alpha_i)^2 l_2(z)
\end{aligned} \tag{3.21}
$$

$$
\begin{aligned}
k_3^{(2)}(z) &= \frac{2}{\pi}\frac{z^3\sin(\alpha_i)^3}{(z^2+2\cos(\alpha_i)z+1)^2} \\
&= \frac{2}{\pi}\sin(\alpha_i)^3 l_4(z)
\end{aligned}
$$

$$
\begin{aligned}
k_1^{(2)}(z) &= \frac{2}{\pi} z \sin(\alpha_i) \frac{\cos(\alpha_i)^2 z^2 + 2\cos(\alpha_i) z + 1}{(z^2 + 2\cos(\alpha_i) z + 1)^2} \\
&= -\frac{2}{\pi} \frac{\sin(\alpha_i)^3 z^3}{(z^2 + 2\cos(\alpha_i) z + 1)^2} + \frac{2}{\pi} \frac{z \sin(\alpha_i)}{z^2 + 2\cos(\alpha_i) z + 1} \qquad (3.22) \\
&= 2k_D(z) - k_3^{(2)}(z) \\
k_2^{(2)}(z) &= \frac{2}{\pi} \frac{z^2 \sin(\alpha_i)^2 (z\cos(\alpha_i) + 1)}{(z^2 + 2\cos(\alpha_i) z + 1)^2} \\
&= \frac{2}{\pi} \sin(\alpha_i)^2 l_3(z) + \frac{2}{\pi} \cos(\alpha_i) \sin(\alpha_i)^2 l_4(z)
\end{aligned}
$$

Remark. The mapping properties of Mellin convolutions with kernel $l_j(z)$ also hold for Mellin convolutions with kernel $k_1^{(\omega)}$ or $k_2^{(\omega)}$. The kernel $k_D(z)$ is the kernel of the double layer potential.

Now we recall some definitions from [**5**].

Let $\rho \geq 0$, $l \in \mathbf{N}$, $p \in [1, \infty]$, be given. Then we define

$$X_\rho^{p,l}(0,1) := \{ u \in \mathcal{D}'(0,1) \mid x^{j-\rho} D^j u \in L^p(0,1), : j = 0(1)l \} \qquad (3.23)$$

with the norm

$$\|u\|_{p,l,\rho,(0,1)} := \sum_{0 \leq j \leq l} \|x^{j-\rho} D^j u\|_{L^p(0,1)}. \qquad (3.24)$$

In [**5**] the following two conditions also appear for functions g on $[0, \infty)$:

$$
\begin{aligned}
(H1^p) \int_0^\infty x^{(1/p)-1} |g(x)|\, dx &< \infty \\
(H1_\rho^{p,l}) \int_0^\infty x^{(1/p)-1-\rho} \big| x^j D^j g(x) \big|\, dx &< \infty, \quad j = 0(1)l.
\end{aligned}
$$

Formula (3.18) shows that all l_j, $j \in \{1, \dots, 4\}$ fulfill the conditions $(H1^p)$, $1 < p \leq \infty$, $(H1_1^{p,1})$, $1 \leq p < \infty$, and $(H1_\rho^{2,1})$, $\rho \in [0, 3/2)$. Formulas (3.19)–(3.22) now show that $\mathcal{B}_i^{(\omega)}$, see (3.12), maps $(L^p(0,1))^4$ continuously into $(L^p(0,1))^4$ for $1 < p \leq \infty$ and $(X_1^{p,0}(0,1))^4$ continuously into $(X_1^{p,0}(0,1))^4$ for $1 \leq p < \infty$, see [**5**, p. 275 and the proof of Theorem 1.10]. We have shown the following lemma.

Lemma 3.2. *For $\omega \in \mathbf{R}$, $i \in \{1, \dots, n\}$, the following mappings are continuous:*

$$\begin{aligned} \mathcal{B}_i^{(\omega)} &: (L^p(0,1))^4 \longrightarrow (L^p(0,1))^4, \quad 1 < p \leq \infty, \\ \mathcal{B}_i^{(\omega)} &: (X_1^{p,0}(0,1))^4 \longrightarrow (X_1^{p,0}(0,1))^4, \quad 1 \leq p < \infty, \\ \mathcal{B}_i^{(\omega)} &: (X_\rho^{2,0}(0,1))^4 \longrightarrow (X_\rho^{2,0}(0,1))^4, \quad 0 \leq \rho < 3/2. \end{aligned}$$

To calculate the Mellin symbol of the operator $\mathcal{B}_i^{(\omega)}$ we first collect the Mellin transformations $\hat{l}_i(s)$ of the $l_i(z)$, see [**8**]:

(3.25)

$$\begin{aligned} \hat{l}_1(s) &= \frac{\pi}{\sin(\alpha_i)} \frac{\sin(\alpha_i s)}{\sin(\pi s)} \\ \hat{l}_2(s) &= -\frac{\pi}{2\sin(\alpha_i)^3} \frac{1}{\sin(\pi s)} (s\cos(\alpha_i)\sin(\alpha_i)\cos(\alpha_i s) \\ &\qquad + s\sin(\alpha_i)^2 \sin(\alpha_i s) - \sin(\alpha_i s)) \\ \hat{l}_3(s) &= -\frac{\pi}{2\sin(\alpha_i)^3} \frac{1}{\sin(\pi s)} (-s\sin(\alpha_i)\cos(\alpha_i s) + \cos(\alpha_i)\sin(\alpha_i s)) \\ \hat{l}_4(s) &= -\frac{\pi}{2\sin(\alpha_i)^3} \frac{1}{\sin(\pi s)} (s\cos(\alpha_i)\sin(\alpha_i)\cos(\alpha_i s) \\ &\qquad - s\sin(\alpha_i)^2 \sin(\alpha_i s) - \sin(\alpha_i s)) \end{aligned}$$

In the next lemma we calculate the Mellin symbol matrix of the operator $\mathcal{B}_i^{(\omega)}$.

Lemma 3.3. *The Mellin symbol matrix $\widehat{\mathcal{B}}_i^{(\omega)}(s)$ of $\mathcal{B}_i^{(\omega)}$, $\omega \in \mathbf{R}$, is given by*

(3.26)

$$\widehat{\mathcal{B}}_i^{(\omega)}(s) = \begin{pmatrix} I_{2\times 2} & g_i(s) I_{2\times 2} + \omega h_i(s) \mathcal{S}_{i,1}(s) \\ g_i(s) I_{2\times 2} + \omega h_i(s) \mathcal{S}_{i,2}(s) & I_{2\times 2} \end{pmatrix}$$

$$g_i(s) = \frac{\sin(\alpha_i s)}{\sin(\pi s)}, \quad h_i(s) = \sin(\alpha_i) \frac{s}{\sin(\pi s)} \tag{3.27}$$

$$\mathcal{S}_{i,1}(s) = \begin{pmatrix} \cos(\alpha_i(s-1)) & -\sin(\alpha_i(s-1)) \\ -\sin(\alpha_i(s-1)) & -\cos(\alpha_i(s-1)) \end{pmatrix}$$
(3.28)
$$\mathcal{S}_{i,2}(s) = \begin{pmatrix} \cos(\alpha_i(s+1)) & \sin(\alpha_i(s+1)) \\ \sin(\alpha_i(s+1)) & -\cos(\alpha_i(s+1)) \end{pmatrix}$$

Proof. We substitute the formulas (3.25) into the formula (3.19) and use (3.20)–(3.22). □

Remark (*on reflection matrices*). The matrices $\mathcal{S}_{i,1}$ and $\mathcal{S}_{i,2}$, which appear in Lemma 3.2 can be viewed as reflection matrices. A reflection matrix S_β in $\mathbf{R}^2$, which describes the reflection at the straight line orthogonal to $(\cos(\beta), \sin(\beta))^T$, has the following form

$$S_\beta = \begin{pmatrix} -\cos(2\beta) & -\sin(2\beta) \\ -\sin(2\beta) & \cos(2\beta) \end{pmatrix},$$

which shows $S_\beta = S_\beta^T = S_\beta^{-1}$. Define

$$\beta_1(s) = \frac{\pi}{2} - \frac{(s-1)}{2}\alpha_i, \quad \beta_2(s) = \frac{\pi}{2} + \frac{(s+1)}{2}\alpha_i.$$

Then one obtains

$$\mathcal{S}_{i,1}(s) = S_{\beta_1(s)}, \quad \mathcal{S}_{i,2}(s) = S_{\beta_2(s)}.$$

This means that $\mathcal{S}_{i,1}(s)$ and $\mathcal{S}_{i,2}(s)$ are reflection matrices for real s. There is a further reflection matrix $\tilde{S}_i$ independent of s by which the matrices $\mathcal{S}_{i,1}(s)$ and $\mathcal{S}_{i,2}(s)$ are conjugated:

(3.29) $$\tilde{S}_i := S_{(\beta_1(s)+\beta_2(s))/2} = S_{(\pi+\alpha_i)/2} = \begin{pmatrix} \cos(\alpha_i) & \sin(\alpha_i) \\ \sin(\alpha_i) & -\cos(\alpha_i) \end{pmatrix}.$$

We obtain

$$\tilde{S}_i \mathcal{S}_{i,1}(s) \tilde{S}_i = \tilde{S}_{i,2}(s).$$

Now (3.26) can be written in the following way:

(3.30)
$$\widehat{\mathcal{B}}_i^{(\omega)}(s) = \begin{pmatrix} I_{2\times 2} & g_i(s) I_{2\times 2} + \omega h_i(s) \mathcal{S}_{i,1}(s) \\ \tilde{S}_i (g_i(s) I_{2\times 2} + \omega h_i(s) \mathcal{S}_{i,1}(s)) \tilde{S}_i & I_{2\times 2} \end{pmatrix}$$

As a next step we will prove the Fredholm property of $\mathcal{B}_i^{(\omega)}$. We will closely follow Lewis [**13**] and we first recall Lemma 6.2 from that paper.

Lemma 3.4. *We consider the equation*

$$\frac{\sin(\gamma z)}{\gamma z} - \omega \frac{\sin(\gamma)}{\gamma} = 0, \quad \gamma \in (0, 2\pi). \tag{$*$}$$

(i) *Let $\omega = 1$. For $0 < \gamma \le \gamma_{\mathrm{crit}}$, equation $(*)$ has no solution in $\Gamma_{0,1}$, $\Gamma_{0,1} := \{z \in \mathbf{C} \mid 0 \le Re(z) < 1\}$, for $\gamma_{\mathrm{crit}} < \gamma < 2\pi$ there is exactly one solution $z_0(1,\gamma) \in \Gamma_{0,1}$. This solution is real and decreases monotonically from 1 to $1/2$ if γ varies between γ_{crit} and 2π.*

(ii) *Let $-1 \le \omega < 1$. For $0 < \gamma \le \pi$, equation $(*)$ has no solution in $\Gamma_{0,1}$, for $\pi < \gamma < 2\pi$ there is exactly one solution $z_0(\omega,\gamma) \in \Gamma_{0,1}$, which decreases monotonically from 1 to $1/2$ if γ runs from π to 2π.*

Remark. We define

$$\begin{aligned} \bar{z}(\omega) &:= \bar{z}(\omega, \alpha_1, \dots, \alpha_n) \\ &:= \min_{i=1}^{n} \{z_0(\omega, \pi + |\alpha_i|), z_0(-\omega, \pi + |\alpha_i|)\} \\ &= \min_{i=1}^{n} \{z_0(-\omega, \pi + |\alpha_i|)\}, \quad \omega \in [0,1], \end{aligned} \tag{3.31}$$

because $z_0(\omega, \alpha)$ is monotonically decreasing as a function of ω. The lemma of Lewis shows

$$\frac{1}{2} < \bar{z}(1) \le \bar{z}(\omega) < 1, \tag{3.32}$$

and

$$\frac{\sin((\pi \pm \alpha_i)z)}{(\pi \pm \alpha_i)z} \pm \omega \frac{\sin(\pi \pm \alpha_i)}{\pi \pm \alpha_i} \neq 0, \quad i \in \{1, \dots, n\}, \tag{3.33}$$

$z \in \{z \in \mathbf{C} \setminus \{0\} \mid 0 \le \mathrm{Re}\,(z) < \bar{z}(\omega)\}$.

The following technical lemma is proved in Section 5.

Lemma 3.5. *Let $\omega \in [0,1]$. Then the following operators are Fredholm*:

(i) $\mathcal{B}_i^{(\omega)} : (L^p(0,1))^4 \to (L^p(0,1))^4$, $p \in (1/\bar{z}(\omega), \infty]$, $i \in \{1, \dots, n\}$,

(ii) $\mathcal{B}_i^{(\omega)} : (X_1^{p,0}(0,1))^4 \to (X_1^{p,0}(0,1))^4$, $p \in [1, 1/(1-\bar{z}(\omega)))$, $i \in \{1, \dots, n\}$,

(iii) $\mathcal{B}_i^{(\omega)} : (X_\rho^{2,0}(0,1))^4 \to (X_\rho^{2,0}(0,1))^4$, $\rho \in [0, 1/2 + \bar{z}(\omega))$, $i \in \{1, \dots, n\}$.

Now we can determine the index of these Fredholm operators.

Lemma 3.6. *Let $\omega \in [0,1]$ and $i \in \{1, \dots, n\}$. We have that*

(i) $\mathcal{B}_i^{(\omega)} : (L^p(0,1))^4 \to (L^p(0,1))^4$, $p \in (1/\bar{z}(\omega), \infty]$,

(ii) $\mathcal{B}_i^{(\omega)} : (X_1^{p,0}(0,1))^4 \to (X_1^{p,0}(0,1))^4$, $p \in [1, 1/(1-\bar{z}(\omega)))$,

(iii) $\mathcal{B}_i^{(\omega)} : (X_\rho^{2,0}(0,1))^4 \to (X_\rho^{2,0}(0,1))^4$, $\rho \in [0, 1/2 + \bar{z}(\omega))$,

are Fredholm operators with index 0.

Proof. $\mathcal{B}_i^{(\omega)}$, $\omega \in [0,1]$, is a homotopy between $\mathcal{B}_i^{(0)}$ and $\mathcal{B}_i^{(1)}$ in case (i) and (ii) and for p fixed and in case (iii) for fixed ρ. It remains in the set of Fredholm operators by Lemma 3.5 if p and ρ are restricted to the range which is given in Lemma 3.5 So it is sufficient to prove that the index of $\mathcal{B}_i^{(0)}$ is 0.

We define

$$\begin{aligned}\widetilde{B}_i(s) &:= \begin{pmatrix} I_{2\times 2} & -I_{2\times 2} \\ I_{2\times 2} & I_{2\times 2} \end{pmatrix}^{-1} \hat{\mathcal{B}}_1^{(0)}(s) \begin{pmatrix} I_{2\times 2} & -I_{2\times 2} \\ I_{2\times 2} & I_{2\times 2} \end{pmatrix} \\ &= \begin{pmatrix} (1+g_i(s))I_{2\times 2} & 0 \\ 0 & (1-g_i(s))I_{2\times 2} \end{pmatrix}.\end{aligned}$$

So we see that $\mathcal{B}_i^{(0)}$ can be diagonalized by a transformation not dependent on s. This means that we can apply the theory for scalar equations. By the proof of Lemma 3.5 we already know that $1 \pm g_i(s) \neq 0$, for all s, $|Re(s)| < \overline{z}(0)$.

But in this special case we can prove even more:

$$\begin{aligned} |g_i(s)| &= \left| \int_0^\infty k_D(t) t^{s-1}\, dt \right| \\ &\le \int_0^\infty |k_D(t)| t^{\mathrm{Re}\,(s)-1}\, dt \\ &= \frac{\sin(|\alpha_i| \mathrm{Re}\,(s))}{\sin(\pi \mathrm{Re}\,(s))} \\ &< 1 \end{aligned}$$

for all s with $\mathrm{Re}\,(s) \in (-\bar{z}(0), \bar{z}(0))$, because $\sin(|\alpha_i| \bar{z}(0)) \le \sin(\pi \bar{z}(0))$ and $\bar{z}(0) = \min_{i=1}^n \{\pi/(\pi + |\alpha_i|)\}$. If $x \in \mathbf{R}$, $|x| < \bar{z}(0)$, we get for the index of the functions $1 \pm g_i(x+iy)$, $y \in \mathbf{R}$,

$$\mathrm{Index}_{y=-\infty}^{\infty} (1 \pm g_i(x+iy)) = 0.$$

Now we obtain with the correspondence between $\mathcal{B}_i^{(0)}$ and Wiener-Hopf operators on $\mathbf{R}^+$, see the proof of Lemma 3.5, and by [**10**, Theorem I.8.1]: (a) $\mathcal{B}_i^{(0)}$ is invertible on $(L^p(0,1))^4$, $p \in (1/\bar{z}(0), \infty]$,

(b) $\mathcal{B}_i^{(0)}$ is invertible on $(X_1^{p,0}(0,1))^4$, $p \in [1, 1/(1-\bar{z}(0))\,)$,

(c) $\mathcal{B}_i^{(0)}$ is invertible on $(X_\rho^{2,0}(0,1))^4$, $\rho \in [0, 1/2 + \bar{z}(0))$.

This shows (a), (b) and (c) for $\omega = 0$. Thus the remarks at the beginning of the proof and the inequality $\bar{z}(\omega) \le \bar{z}(0)$ show the statement of the lemma. □

The transformation which we used in the proof of the last lemma is now applied again. With the help of the matrices $\tilde{S}_i$ defined in (3.29) and $\hat{C}_i^{(\omega)}(s)$, cf. (5.3), we construct a matrix $\tilde{B}_i^{(\omega)}(s)$ which is similar to $\widehat{\mathcal{B}}_i^{(\omega)}(s)$ but has a simpler structure:

$$\begin{aligned} \tilde{B}_i^{(\omega)}(s) &:= \begin{pmatrix} I_{2\times 2} & -I_{2\times 2} \\ I_{2\times 2} & I_{2\times 2} \end{pmatrix}^{-1} \begin{pmatrix} I_{2\times 2} & 0 \\ 0 & \tilde{S}_i \end{pmatrix} \hat{\mathcal{B}}_1^{(\omega)}(s) \\ &\quad \cdot \begin{pmatrix} I_{2\times 2} & 0 \\ 0 & \tilde{S}_i \end{pmatrix} \begin{pmatrix} I_{2\times 2} & -I_{2\times 2} \\ I_{2\times 2} & I_{2\times 2} \end{pmatrix} \\ &= \begin{pmatrix} I_{2\times 2} + \hat{C}_i^{(\omega)}(s) & 0 \\ 0 & I_{2\times 2} - \hat{C}_i^{(\omega)}(s) \end{pmatrix} \end{aligned} \tag{3.34}$$

So $\tilde{B}_i^{(\omega)}(s)$ and $\hat{\mathcal{B}}_i^{(\omega)}(s)$ are similar with a transformation independent of s. In the next lemma we study the eigenvalues of $\tilde{B}_i^{(\omega)}(s)$; the lemma is proved in Section 5.

Lemma 3.7. *There is a $q > 0$ such that all eigenvalues of $Re\,(\tilde{B}_i^{(\omega)}(s))$, $Re\,(s) = 0$, are greater than q for $\omega \in [0,1]$, $i \in \{1,\dots,n\}$.*

Remark. The statement of Lemma 3.7 is wrong for $\mathrm{Re}\,(s) = 1/2$. A numerical calculation shows that $\mathrm{Re}\,(\hat{C}_i^{(1)}(1/2))$ has an eigenvalue greater than 1.05 if $\alpha_i = 0.6 * \pi$.

Theorem 3.8. *Let $\omega \in [0,1]$, $i \in \{1,\dots,n\}$. Then*

(i) $\mathcal{B}_i^{(\omega)} : (L^p(0,1))^4 \to (L^p(0,1))^4$, $p \in (1/\bar{z}(\omega), \infty]$,

(ii) $\mathcal{B}_i^{(\omega)} : (X_1^{p,0}(0,1))^4 \to (X_1^{p,0}(0,1))^4$, $p \in [1, 1/(1-\bar{z}(\omega))\,)$,

(iii) $\mathcal{B}_i^{(\omega)} : (X_\rho^{2,0}(0,1))^4 \to (X_\rho^{2,0}(0,1))^4$, $\rho \in [1, 1/2 + \bar{z}(\omega))$,

are invertible, see Lemma 3.4 *and* (3.31) *for the definition of $\bar{z}(\omega)$.*

Proof. Lemma 3.7 and the transformation used in (3.34) show that $\hat{\mathcal{B}}_i^{(\omega)}(s)$ is strongly elliptic for $\mathrm{Re}\,(s) = 0$. By the correspondence to Wiener-Hopf operators, cf. Section 5, we get that

(i) $\mathcal{B}_i^{(\omega)} : (L^\infty(0,1))^4 \to (L^\infty(0,1))^4$ is invertible and

(ii) $\mathcal{B}_i^{(\omega)} : (X_1^{1,0}(0,1))^4 \to (X_1^{1,0}(0,1))^4$ is invertible.

$\mathcal{B}_i^{(\omega)} : (L^p(0,1))^4 \to (L^p(0,1))^4$, $p \in (1/\bar{z}(0), \infty]$, is a Fredholm operator of index 0 by Lemma 3.6 and $L^\infty(0,1) \subset L^p(0,1)$ is dense. Then it follows by a standard argument for Fredholm operators, see [**18**], that

$$N(\mathcal{B}_i^{(\omega)}|_{(L^p(0,1))^4}) \subset N(\mathcal{B}_i^{(\omega)}|_{(L^\infty(0,1))^4}) = \{0\},$$

where $N(L)$ denotes the kernel of the linear mapping L. This implies $\mathcal{B}_i^{(\omega)}$ is invertible and this proves (i).

Lemma 3.6 also shows that $\mathcal{B}_i^{(\omega)} : (X_1^{p,0}(0,1))^4 \to (X_1^{p,0}(0,1))^4$, $p \in [1, 1/(1-\bar{z}(0)))$, is a Fredholm operator with index 0. But we have $(X_1^{p,0}(0,1))^4 \subset (X_1^{1,0}(0,1))^4$ and so we get

$$N(\mathcal{B}_i^{(\omega)}|_{(X_1^{p,0}(0,1))^4}) \subset N(\mathcal{B}_i^{(\omega)}|_{(X_1^{1,0}(0,1))^4}) = \{0\}.$$

This proves (ii), and the inclusion

$$N(\mathcal{B}_i^{(\omega)}|_{(X_\rho^{2,0}(0,1))^4}) \subset N(\mathcal{B}_i^{(\omega)}|_{(X_1^{2,0}(0,1))^4}) = \{0\}, \quad \rho \in [1, 1/2 + \bar{z}(0)).$$

shows (iii) in a similar way. □

Corollary 3.9. *The operator* $I + \mathcal{K}^{(\omega,1)} : (L^2(0,T))^2 \to (L^2(0,T))^2$, *see* (3.7), *has an inverse and* $\mathcal{B}^{(\omega)} : (L^2(0,T))^2 \to (L^2(0,T))^2$, *see* (3.2), *is a Fredholm operator with index 0.*

Proof. Note that $\chi_i(I + \mathcal{K}^{(\omega,1)})\chi_i$ can be identified with $\mathcal{B}_i^{(\omega)}$, see (3.6), (3.7), (3.12) and (3.16). Now by Theorem 3.8 each operator $\mathcal{B}_i^{(\omega)}$ has an inverse so that $I + \mathcal{K}^{(\omega,1)} = I + \sum_{i=1}^n \chi_i \mathcal{K}^{(\omega,1)} \chi_i$ is invertible. Lemma 3.1 and (3.9) prove the second statement of the corollary. □

Lemma 3.10. *Let* $\vec{u} \in \left(L^2(0,1)\right)^4$, $\omega \in [0,1]$, $\rho \in [1, 1/2 + \bar{z}(\omega))$ *and* $l \in \mathbf{N}$. *Then*

(i) $\mathcal{B}_i^{(\omega)} \vec{u} \in (X_\rho^{2,l}(0,1))^4$ implies $\vec{u} \in (X_\rho^{2,l}(0,1))^4$,

(ii) $\mathcal{B}_i^{(\omega)} \vec{u} \in (X_\rho^{2,l}(0,1))^4 \dot{+} \mathbf{R}^4$ *implies* $\vec{u} \in (X_\rho^{2,l}(0,1))^4 \dot{+} \mathbf{R}^4$.

Proof. We follow closely [**5**, Theorem 1.10] and define

$$T_i^{(\omega)}(z) := \begin{pmatrix} 0 & k_{i,1}^{(\omega)}(z) \\ k_{i,2}^{(\omega)}(z) & 0 \end{pmatrix},$$

see (3.19). Equations (3.20)–(3.22) and (3.17)–(3.18) show that all entries of $T_i^{(\omega)}$ are in $X_1^{2,m}$, for all $m \in \mathbf{N}$. A simple calculation shows

$$[zD\mathcal{B}_i^{(\omega)}\vec{u}](z) = [\mathcal{B}_i^{(\omega)}(zD\vec{u})](z) - T_i^{(\omega)}(z)\vec{u}(1),$$

for $\vec{u} \in (X_0^{2,1}(0,1))^4$. By an induction we get

$$[(zD+2)^m \mathcal{B}_i^{(\omega)} \vec{u}](z) = [\mathcal{B}_i^{(\omega)}(zD+2)^m \vec{u})](z) + (T_{i,m}^{(\omega)}(z)\vec{u})(z),$$

$\vec{u} \in (X_0^{2,m}(0,1))^4$, $m \in \mathbf{N}$. Here $T_{i,m}^{(\omega)}$ is a finite dimensional operator, which consists of linear combinations of $T_i^{(\omega)}(z)$ and its derivatives. In [**6**] it is shown that

$$\phi_m := (zD+2)^m : (X_\rho^{p,m}(0,1))^4 \longrightarrow (X_\rho^{p,0}(0,1))^4, \quad 1 \le p \le \infty,$$

is an isomorphism for all $m \in \mathbf{N}$.

This implies

$$\mathcal{B}_i^{(\omega)} = \phi_m^{-1} \circ \mathcal{B}_i^{(\omega)} \circ \phi_m + \phi_m^{-1} \circ T_{i,m}^{(\omega)},$$

where $\mathcal{B}_i^{(\omega)}$ is invertible on $(X_\rho^{2,m}(0,1))^4$ by Theorem 3.8 and $\phi_m^{-1} \circ T_{i,m}^{(\omega)}$ is a finite dimensional operator and hence compact. Because of this we have that $\mathcal{B}_i^{(\omega)}$ is a Fredholm operator and its index is 0. But we also have

$$N(\mathcal{B}_i^{(\omega)}|_{(X_\rho^{2,m}(0,1))^4}) \subset N(\mathcal{B}_i^{(\omega)}|_{(X_\rho^{2,0}(0,1))^4}) = \{0\}.$$

This shows that $\mathcal{B}_i^{(\omega)}$ is an isomorphism on $(X_\rho^{2,m}(0,1))^4$ and proves (i).

We recall the formula

$$[zD\mathcal{B}_i^{(\omega)}\vec{u}](z) = [\mathcal{B}_i^{(\omega)}(zD\vec{u})](z) - T_i^{(\omega)}(z)\vec{u}(1)$$

from above. By [**6**] it follows that

$$zD : (X_\rho^{p,m}(0,1))^4 \dotplus \mathbf{R}^4 \longrightarrow (X_\rho^{p,m-1}(0,1))^4$$

is surjective with kernel $\mathbf{R}^4$. This implies

$$\mathcal{B}_i^{(\omega)}(\mathbf{R}^4) \subset (X_\rho^{p,m}(0,1))^4 \dotplus \mathbf{R}^4,$$

and $\mathcal{B}_i^{(\omega)}$ is a Fredholm operator with index 0 on $(X_\rho^{p,m}(0,1))^4 \dotplus \mathbf{R}^4$. But the kernel of $\mathcal{B}_i^{(\omega)}$ in $\left(L^2(0,1)\right)^4$ is trivial by Theorem 3.8 and so (ii) follows. $\square$

Lemma 3.11. *Let $\vec{u} \in (L^2(0,T))^2$ be a solution of the equation* (3.2), *with $\omega \in [0,1]$ and $\vec{f}|_{[s_i,s_{i+1}]} \in (C^l[s_i,s_{i+1}])^2$, $i = 0(1)n-1$, $l \in \mathbf{N}$. Then we have*

(i) $\vec{u}|_{(s_i,s_{i+1})} \in (C^l(s_i,s_{i+1}))^2$, $i = 0(1)n-1$.

(ii)

$$\vec{v}_+(t) := \vec{u}(s_i + t) \in (X_\rho^{2,l}(0,1))^4 \dotplus \mathbf{R}^4, \quad i \in \{0,\dots,n-1\},$$
$$\vec{v}_-(t) := \vec{u}(s_i - t) \in (X_\rho^{2,l}(0,1))^4 \dotplus \mathbf{R}^4, \quad i \in \{1,\dots,n\},$$

with $\rho \in [1, 1/2 + \bar{z}(\omega))$ and $\bar{z}(\omega)$ defined in (3.31).

Proof. (i) For $\vec{u}$ we have, see (3.2),

$$\vec{u} = \vec{f} - \mathcal{K}^{(\omega)}\vec{u}.$$

Lemma 3.1 and the assumption on $\vec{f}$ prove statement (i).

(ii) By (3.12) and (3.13) we see that the function $\vec{v} := (\vec{v}_-, \vec{v}_+)^T \in \left(L^2(0,1)\right)^4$ fulfills the equation

$$\mathcal{B}_i^{(\omega)}\vec{v} = \begin{pmatrix} \vec{g}_- \\ \vec{g}_+ \end{pmatrix} =: \vec{g},$$

where

$$\vec{g}_\pm(x) = \vec{f}(s_i \pm x) - (\mathcal{K}^{(\omega,2)}\vec{u})(s_i \pm x).$$

The assumptions on $\vec{f}$ and Lemma 3.1 imply

$$\vec{g} \in (C^l[0,1])^4.$$

For $l \geq 1$ and $\rho \in [1, 1/2 + \bar{z}(\omega))$ we have

$$\vec{g}(x) = (\vec{g}(x) - \vec{g}(0)) + \vec{g}(0) \in (X_\rho^{2,l}(0,1))^4 \dotplus \mathbf{R}^4.$$

By Lemma 3.10 we get

$$\vec{v} \in (X_\rho^{2,l}(0,1))^4 \dotplus \mathbf{R}^4,$$

and this proves (ii). $\square$

Corollary 3.12. *Let $\vec{u} \in (L^2(0,T))^2$ be the solution of (3.2), $\omega \in [0,1]$, $\vec{f}|_{[s_i,s_{i+1}]} \in (C^1[0,1])^2$, for all i. Then we have*

$$\vec{u}|_{[s_i,s_{i+1}]} \in (H^1[s_i,s_{i+1}])^2.$$

Proof. Because of $X_1^{2,1}(0,1) \dotplus \mathbf{R} \subset H^1[0,1]$ and by Lemma 3.11, it follows that $\vec{u}|_{J_j} \in (H^1(J_j))^2$, for all $j \in \{1,\ldots,3n\}$. The continuity of $\vec{u}$ on $J_{3i} \cup J_{3i+1} \cup J_{3i+2}$, $i \in \{1,\ldots,n\}$, follows from 3.11 (i). ◻

Lemma 3.13. *Let $\vec{u} \in (C[0,1])^4$. Then we have*

$$\lim_{x \searrow 0} (\mathcal{B}_i^{(\omega)} \vec{u})(x) = E_i^{(\omega)} \vec{u}(0),$$

where

$$E_i^{(\omega)} = I_{4\times 4} + \begin{pmatrix} 0 & C_{E_i}^{(\omega)} \\ C_{E_i}^{(\omega)} & 0 \end{pmatrix}, \tag{3.35}$$

and

$$C_{E_i}^{(\omega)} = \frac{\alpha_i}{\pi} I_{2\times 2} + \frac{\omega}{\pi} \sin(\alpha_i) \begin{pmatrix} \cos(\alpha_i) & \sin(\alpha_i) \\ \sin(\alpha_i) & -\cos(\alpha_i) \end{pmatrix}.$$

Proof. It is proved in [**2**] that

$$\lim_{x \searrow 0} (\mathcal{B}_i^{(\omega)} \vec{u})(x) = \hat{\mathcal{B}}_i^{(\omega)}(0) \vec{u}(0),$$

for continuous $\vec{u}$ (only in the scalar case, but this is sufficient). We get:

$$\begin{aligned}
\lim_{s\to 0} g_i(s) &= \frac{\alpha_i}{\pi} \\
\lim_{s\to 0} \omega h_i(s) &= \frac{\omega}{\pi} \sin(\alpha_i) \\
\lim_{s\to 0} \tilde{S}_{i,1}(s) &= \begin{pmatrix} \cos(\alpha_i) & \sin(\alpha_i) \\ \sin(\alpha_i) & -\cos(\alpha_i) \end{pmatrix} \\
\lim_{s\to 0} \tilde{S}_{i,2}(s) &= \begin{pmatrix} \cos(\alpha_i) & \sin(\alpha_i) \\ \sin(\alpha_i) & -\cos(\alpha_i) \end{pmatrix},
\end{aligned}$$

see (3.27) and (3.28) for the definitions. But then (3.26) gives the result. ◻

By a calculation we get:

Lemma 3.14. *Let $\omega \in [0,1]$, $\alpha_i \in (-\pi,\pi)$. The matrix $E_i^{(\omega)}$ has the factorization*

$$E_i^{(\omega)} = \frac{1}{\sqrt{2}} \begin{pmatrix} U_i & -U_i \\ U_i & U_i \end{pmatrix} \begin{pmatrix} D_{i,1}^{(\omega)} & 0 \\ 0 & D_{i,2}^{(\omega)} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} U_i^T & U_i^T \\ -U_i^T & U_i^T \end{pmatrix}, \tag{3.36}$$

where U_i is a unitary matrix

$$U_i = \begin{pmatrix} \cos(\alpha_i/2) & -\sin(\alpha_i/2) \\ \sin(\alpha_i/2) & \cos(\alpha_i/2) \end{pmatrix} \tag{3.37}$$

and $D_{i,1}^{(\omega)}$ and $D_{i,2}^{(\omega)}$ are diagonal matrices

$$\begin{aligned} D_{i,1}^{(\omega)} &= \begin{pmatrix} 1 + (\alpha_i + \omega\sin(\alpha_i))/\pi & 0 \\ 0 & 1 + (\alpha_i - \omega\sin(\alpha_i))/\pi \end{pmatrix} \\ D_{i,2}^{(\omega)} &= \begin{pmatrix} 1 - (\alpha_i + \omega\sin(\alpha_i))/\pi & 0 \\ 0 & 1 - (\alpha_i - \omega\sin(\alpha_i))/\pi \end{pmatrix} \end{aligned} \tag{3.38}$$

which are nonsingular.

The last two lemmas make it easy to prove.

Theorem 3.15. *Let $\omega \in [0,1]$ and $\vec{u} \in (L^2(0,T))^2$ be solutions of the equation* (3.2)*, $\vec{f} \in (C[0,T])^2$, $\vec{f}|_{[s_i,s_{i+1}]} \in (C^1[s_i,s_{i+1}])^2$. Then we have*

$$\vec{u} \in (C[0,T])^2.$$

Proof. Because of Lemma 3.11 we only have to show the continuity of $\vec{u}$ in s_i, $i \in \{1,\dots,n\}$. For fixed i we define

$$\begin{aligned} \vec{v}_\pm(x) &:= \vec{u}(s_i \pm x), \\ \vec{w}_\pm(x) &:= \vec{f}(s_i \pm x) - (\mathcal{K}^{(\omega,2)}\vec{u})(s_i \pm x), \quad x \in [0,1], \\ \vec{v} &:= \begin{pmatrix} \vec{v}_- \\ \vec{v}_+ \end{pmatrix}, \\ \vec{w} &:= \begin{pmatrix} \vec{w}_- \\ \vec{w}_+ \end{pmatrix}. \end{aligned}$$

We have by (3.9), (3.12) and (3.13)

$$\mathcal{B}_i^{(\omega)}\vec{v} = \vec{w}.$$

Our assumptions on $\vec{f}$ and Lemma 3.1 imply $\vec{w} \in (C^1[0,1])^4$ and $\vec{w}_-(0) = \vec{w}_+(0)$. By Corollary 3.12 we get $\vec{v} \in (H^1[0,1])^4 \subset (C[0,1])^4$. Lemma 3.13 gives

$$E_i^{(\omega)}\vec{v}(0) = \begin{pmatrix} \vec{w}_-(0) \\ \vec{w}_-(0) \end{pmatrix}.$$

Lemma 3.14 shows

$$\begin{aligned} \vec{v}(0) &= (E_i^{(\omega)})^{-1} \begin{pmatrix} \vec{w}_-(0) \\ \vec{w}_-(0) \end{pmatrix} \\ &= \begin{pmatrix} U_i(D_{i,1}^{(\omega)})^{-1}U_i^T\vec{w}_-(0) \\ U_i(D_{i,1}^{(\omega)})^{-1}U_i^T\vec{w}_-(0) \end{pmatrix}. \end{aligned}$$

This shows $\vec{v}_-(0) = \vec{v}_+(0) = U_i(D_{i,1}^{(\omega)})^{-1}U_i^T\vec{w}_-(0)$, and we have proved the continuity in s_i. □

Corollary 3.16. *Let $\omega \in [0,1]$, $\vec{u} \in (L^2(0,T))^2$ be solutions of the equation (3.2) and $\vec{f} \in (C[0,T])^2$, $\vec{f}|[s_i, s_{i+1}] \in (C^1[s_i, s_{i+1}])^2$. Then we have*

$$\vec{u} \in (H^1[0,T])^2.$$

Proof. The proof follows from Corollary 3.12 and Theorem 3.15 because $\vec{u}$ is a piecewise H^1–function, which is continuous on the whole interval. □

Let $\mathcal{B}$ be the operator defined in (1.1), or equivalently, the operator defined in (3.2) with $\omega = \bar{\omega}$, cf. (1.3).

Theorem 3.17. $\mathcal{B} : (L^2(0,T))^2 \to (L^2(0,T))^2$ *is an isomorphism.*

Proof. By Corollary 3.9, $\mathcal{B}$ is a Fredholm operator with index 0. If $\vec{u} \in (L^2(0,T))^2$ is a solution of

$$\mathcal{B}\vec{u} = 0,$$

then we know by Corollary 3.16 that $\vec{u} \in (H^1[0,T])^2$. We define the double layer potential $\vec{U}(x) := (K^{(\bar{\kappa})}u)(x)$, see (2.39). By Lemma 2.10 we get

$$P\vec{U}|_\Omega = 0, \quad \vec{U} \in V_P^1,$$

and the relation (2.43) implies $\gamma_0\vec{U}|_\Omega = 0$. By Lemma 2.2 we know $\vec{U}|_\Omega = 0$ and this implies $\gamma_1^{(\bar{\kappa})}\vec{U}|_\Omega = 0$. By Lemma 2.11 $\vec{U}$ is also a weak solution of

$$P\vec{U} = 0 \text{ in } \Omega^c.$$

Lemma 2.12 gives $\gamma_1^{(\bar{\kappa})}\vec{U}|_{\Omega^c} = 0$ and Lemma 2.13 implies $\vec{U}|_{\Omega^c} = 0$. This shows $\gamma_0\vec{U}|_{\Omega^c} = 0$. But by Lemma 2.12 we obtain

$$0 = [\gamma_0\vec{U}] = [\gamma_0 K_1^{(\omega)}\vec{u}] = -\vec{u}.$$

So $\mathcal{B}$ is injective and this proves the theorem. □

Remark. Theorem 3.8 shows in an analogous way that

$$\mathcal{B} : (L^p(0,T))^2 \longrightarrow (L^p(0,T))^2$$

is a Fredholm operator with index 0 for $p \in [1/\bar{z}(\bar{c}), \infty]$, see Lemma 3.4 and (3.31) for the definition of $\bar{z}(\bar{c})$. The inclusion $L^p(0,T) \subset L^2(0,T)$ proves that $\mathcal{B} : (L^p(0,T))^2 \to (L^p(0,T))^2$ is an isomorphism for $p \in [2, \infty]$.

Theorem 3.18. *Let* $\vec{f} \in (C[0,T])^2$, $\vec{f}|_{[s_i,s_{i+1}]} \in (C^l[s_i,s_{i+1}])^2$, $i = 0(1)n-1$, $l \in \mathbf{N}$, *and* $\vec{u} \in (L^2(0,T))^2$ *be the solution of equation* (1.1). *Then* $\vec{u}$ *has the following properties*:

(i) $\vec{u} \in (C[0,T])^2$,

(ii) $\vec{u}|_{(s_i,s_{i+1})} \in (C^l(s_i,s_{i+1}))^2$, $i \in \{0,\dots,n-1\}$,

(iii)

$$\vec{u}(s_i + t) \in \left(X_\rho^{2,l}(0,1)\right)^2 \dot{+} \mathbf{R}^2, \quad i \in \{0,\dots,n-1\},$$
$$\vec{u}(s_i - t) \in \left(X_\rho^{2,l}(0,1)\right)^2 \dot{+} \mathbf{R}^2, \quad i \in \{1,\dots,n\},$$

with $\rho \in [1, 1/2 + \bar{z}(\bar{\omega}))$, $\bar{z}(\bar{\omega})$ defined in (3.31).

Proof. Corollary 3.16 shows (i) and Lemma 3.11 implies (ii) and (iii). $\square$

4. On the numerical approximation of the solution of the Lamé equation. In this section we use a collocation method to approximate the solution of equation (1.1). We use piecewise polynomials on $[0,T]$, which are continuous and periodic on $[0,T]$. The meshes must be graded near the corners to get a good convergence rate and a cut off technique (i^*-trick, see [**7**] and [**3**]) has to be used to guarantee the stability of the method. The proof of stability here is not standard, because the operator is not strongly elliptic in L^2, see the remark after Lemma 3.7.

As a preliminary step, we first discuss the stability of certain finite section approximations to equation (1.1). We introduce projections P_h on $(L^2(0,T))^2$ and projections Q_h on the reference space $\left(L^2(0,1)\right)^4$ to construct these approximations.

Let $h \in (0,1)$. The projector $P_h : (L^2(0,T))^2 \to (L^2(0,T))^2$ is defined by

$$(P_h\vec{u})(x) = \begin{cases} \vec{u}(x) & |x - s_i| > h, \quad \forall\, i \in \{0,\dots,n\}, \\ 0, & \text{else.} \end{cases} \tag{4.1}$$

Recall that s_i are the preimages of the corner points under the parametrization $\gamma : [0,T] \to \Gamma$. The projectors P_h have the property:

$$\lim_{h\to 0} P_h = I_{(L^2(0,T))^2}, \quad \text{strongly.}$$

The finite section approximation for $\mathcal{B}^{(\omega)}$, see (3.2), is defined by

$$\mathcal{B}_h^{(\omega)} := P_h \mathcal{B}^{(\omega)} P_h. \tag{4.2}$$

Our first aim is to prove that $\mathcal{B}_h^{(\omega)}$ has a bounded inverse for $h < h_0$, where $h_0 > 0$ is some constant. To prove the stability, we have to study the corresponding finite section approximation for $\mathcal{B}_i^{(\omega)}$, cf. (3.12). The projector $Q_h : \left(L^2(0,1)\right)^4 \to \left(L^2(0,1)\right)^4$, $h \in (0,1)$, is given by

$$(Q_h\vec{u})(x) = \begin{cases} \vec{u}(x) & x \geq h \\ 0 & x \leq h \end{cases} \tag{4.3}$$

and the finite section approximation for $\mathcal{B}_i^{(\omega)}$ by

$$\mathcal{B}_{i,h}^{(\omega)} = Q_h \mathcal{B}_i^{(\omega)} Q_h. \tag{4.4}$$

Lemma 4.1. *Let $\omega \in [0,1]$. There exists $h_0 > 0$ such that $\mathcal{B}_{i,h}^{(\omega)}$ has an inverse in $Q_h \left(L^2(0,1)\right)^4$ for all $h < h_0$ and $i \in \{1, \dots, n\}$. There is a constant C for which*

$$\|(\mathcal{B}_{i,h}^{(\omega)})^{-1}\|_{Q_h(L^2(0,1))^4} \le C, \quad h < h_0, i \in \{1, \dots, n\}.$$

Proof. The proof is based on the stability results for the finite section approximation for Wiener-Hopf operators in [**10**]. The finite section approximation of a Wiener-Hopf operator W is stable if the symbol matrix $\hat{W}(s)$ has determinant different from zero, $s \in \mathbf{R}$, and if the left and right partial indices of the symbol matrix $\hat{W}(s)$ are all zero, [**10**, Theorem VIII.6.2]. The left partial indices of $\hat{W}(s)$ are the right partial indices of $\hat{W}(-s)$, [**10**, p. 222]. The vanishing of the right partial indices of $\hat{W}(s)$ is equivalent to the invertibility of the operator W, [**10**, Theorem VIII.6.1].

Now we denote by W the Wiener-Hopf operator which corresponds to $\mathcal{B}_i^{(\omega)}$. Then $\hat{W}(s) = \hat{\mathcal{B}}_i^{(\omega)}(1/2 + i\,s)$, $s \in \mathbf{R}$, see (3.26). Because $\mathcal{B}_i^{(\omega)}$ is invertible all right partial indices of $\hat{W}(s)$ vanish.

We denote by $C_i^{(\omega)}$ the operator on $\left(L^2(0,1)\right)^4$ which has the symbol matrix $\hat{\mathcal{B}}_i^{(\omega)}(1-s)$. Then the Wiener-Hopf operator W_1 which corresponds to $C_i^{(\omega)}$ has the symbol matrix $\hat{W}(-s)$. If we can show that $C_i^{(\omega)}$ is invertible then the right indices of $\hat{W}(-s)$ are zero and then the finite section approximation for W and so for $\mathcal{B}_i^{(\omega)}$ is stable.

It remains to show that the operator $C_i^{(\omega)} : \left(L^2(0,1)\right)^4 \to \left(L^2(0,1)\right)^4$ is invertible. If $\mathrm{Re}\,(s) = 1$, then $\hat{C}_i^{(\omega)}(s) = \hat{\mathcal{B}}_i^{(\omega)}(1-s)$ is strongly elliptic, Lemma 3.7, i.e., all eigenvalues of $\mathrm{Re}\,(\hat{C}_i^{(\omega)}(s))$, $\mathrm{Re}\,(s) = 1$, are greater than some positive constant. This implies

$$C_i^{(\omega)} : (L^1(0,1))^4 \longrightarrow (L^1(0,1))^4$$

is invertible. Now we remark that $\det(\hat{C}_i^{(\omega)}(s)) \neq 0$, for all s with $\mathrm{Re}\,(s) \in [1/2, 1]$, see (3.4) and the following text. Therefore the operator $C_i^{(\omega)}$ is a Fredholm operator on $(L^p(0,1))^4$ for all $p \in [1,2]$ and $\omega \in [0,1]$. For $\omega = 0$ we get the operator, which corresponds to the double layer potential, see (3.4). This operator is invertible in $(L^p(0,1))^4$ and so $C_i^{(\omega)}$ is a Fredholm operator with index 0 for all $p \in [1,2]$, $\omega \in [0,1]$. But

$$N(C_i^{(\omega)}|_{(L^p(0,1))^4}) \subset N(C_i^{(\omega)}|_{(L^1(0,1))^4}) = \{0\},$$

which proves the invertibility of $C_i^{(\omega)}$, especially for $p = 2$. $\square$

In the following we restrict ourselves to the case $\omega = \bar{\omega}$ and omit the upper index ω for $\mathcal{B}_h^{(\bar{\omega})}$, $\mathcal{K}^{(\bar{\omega},1)}$ and $\mathcal{K}^{(\bar{\omega},2)}$.

Theorem 4.2. *There exists an $h_0 > 0$ and a constant $C > 0$ such that for $h \in (0, h_0)$ the operator $\mathcal{B}_h$ is invertible in $P_h((L^2(0,T))^2)$ and*

$$\|\mathcal{B}_h^{-1}\|_{P_h((L^2(0,T))^2)} \leq C.$$

Proof. By (3.9), we get

$$\mathcal{B}_h = P_h(I + \mathcal{K}^{(1)})P_h + P_h\mathcal{K}^{(2)}P_h$$

with compact $\mathcal{K}^{(2)}$, see Lemma 3.1. Moreover $\mathcal{B}$ is invertible by Theorem 3.17, and [**10**, II.3.1] shows that we only have to prove the invertibility of $P_h(I + \mathcal{K}^{(1)})P_h$. The operator $P_h(I + \mathcal{K}^{(1)})P_h$, $h \leq 1$, takes the form

$$P_h + \sum_{i=0}^{n-1} \chi_i P_h \mathcal{K}^{(1)} P_h \chi_i$$

and the operator $\chi_i P_h(I + \mathcal{K}^{(1)})P_h\chi_i$ can be identified with $\mathcal{B}_{i,h}$, cf. the proof of Corollary 3.9. Therefore Lemma 4.1 proves the theorem. $\square$

Now we introduce the graded meshes and the corresponding spline spaces. Choose a grading exponent $q > 0$, let $\Theta_m := (x_j^{(m)})_{j=0}^m$, $m \in \mathbf{N}$, be the partition of the interval $[0,1]$ given by

$$x_j^{(m)} := \left(\frac{j}{m}\right)^q \tag{4.5}$$

and define

$$h_j^{(m)} := x_j^{(m)} - x_{j-1}^{(m)}, \quad j = 1(1)m. \tag{4.6}$$

Here and in the following we will not explicitly indicate the dependence of $x_j^{(m)}$ and $h_j^{(m)}$ on q.

Remark. We assume this special partition only for simplicity. All of the following statements are true for partitions $(x_j^{(m)})_{j=1}^m$, which fulfill

$$\frac{c_1}{j}\left(\frac{j}{m}\right)^q \le h_j^{(m)} \le \frac{c_2}{j}\left(\frac{j}{m}\right)^q, \tag{4.7}$$

with constants c_1, c_2 independent of j and m.

For a sequence Θ_m, we define by

$$\Pi_m^d := \Pi_m^d(\Theta_m) \tag{4.8}$$

the space of all continuous functions, which are piecewise polynomials with respect to the partition Θ_m and of degree smaller than or equal to d. If we choose $(\xi_k)_{k=0}^d \subset [0,1]$, $0 = \xi_0 < \ldots < \xi_d = 1$, then the projector $P_m^d : C[0,1] \to \Pi_m^d$, $P_m^d = P_m^d(\Theta_m, (\xi_k)_{k=0}^d)$ is defined by

$$(P_m^d u)(x_j^{(m)} + \xi_k h_{j+1}^{(m)}) = u(x_j^{(m)} + \xi_k h_{j+1}^{(m)}), \tag{4.9}$$

$j = 0(1)m-1$, $k = 0(1)d$. For $j \ge 1$ we define

$$\Pi_{m,j}^d := \{u \in \Pi_m^d \mid u|_{[0,x_j^{(m)}]} = 0\} \tag{4.10}$$

and $P_{m,j}^d : C[0,1] \to C[0,1]$ by

$$(P_{m,j}^d u)(x_l^{(m)} + \xi_k h_{l+1}^{(m)}) = \begin{cases} u(x_l^{(m)} + \xi_k h_{l+1}^{(m)}) & l = j+1(1)m-1, \\ & k = 0(1)d, \\ u(x_l^{(m)} + \xi_k h_{l+1}^{(m)}) & l = j,\ k = 1(1)d, \\ 0 & \text{else.} \end{cases} \tag{4.11}$$

As a next step we introduce partitions of $[0,T]$, which are refinements of the subdivision (3.6). For $q>1$ we define a sequence of partitions $\Delta_m := (s_j^{(m)})_{j=0}^{3mn+1}$, $m \in \mathbf{N}$, of $[0,T]$ with

$$0 = s_0^{(m)} < \cdots < s_{3mn+1}^{(m)} = T, \tag{4.12}$$

by the demand that the $3mn+1$ real numbers

$$\begin{gathered} s_i + \left(\frac{j}{m}\right)^q, \quad j = 0(1)m, i = 0(1)n-1, \\ s_i + 1 + \frac{j}{m}(s_{i+1} - s_i - 2), \quad j = 0(1)m, i = 0(1)n-1, \\ s_i - \left(\frac{j}{m}\right)^q, \quad j = 0(1)m, i = 1(1)n, \end{gathered}$$

are elements of $\{s_j^{(m)} \mid j = 0(1)3mn+1\}$. The stepwidth $\delta_j^{(m)}$ is defined by

$$\delta_j^{(m)} := s_j^{(m)} - s_{j-1}^{(m)}, \quad j = 1(1)3mn+1. \tag{4.13}$$

Remark. Here we also consider this special mesh only for simplicity. For a sequence of partitions $(s_j^{(m)})_{j=0}^{M(m)}$, $M(m) \sim m$, it is sufficient that the greatest stepwidth goes to zero like $1/m$ and that near the points s_i the mesh satisfies condition (4.7). All results in this section are valid if this is fulfilled.

We define by

$$\widetilde{\Pi}_m^d = \widetilde{\Pi}_m^d(\Delta_m), \quad d \in \mathbf{N}, \tag{4.14}$$

the space of all continuous functions on $C[0,T]$, which are piecewise polynomials with respect to Δ_m and of degree smaller than or equal to d. Given $0 = \xi_0 < \xi_1 < \cdots < \xi_d = 1$, we define the projector $\widetilde{P}_m^d : C[0,T] \to \widetilde{\Pi}_m^d$ by

$$\begin{gathered} (\widetilde{P}_m^d u)(s_j^{(m)} + \xi_k \delta_{j+1}^{(m)}) = u(s_j^{(m)} + \xi_k \delta_{j+1}^{(m)}), \\ j = 0(1)3mn, k = 0(1)d. \end{gathered} \tag{4.15}$$

For $j \in \mathbf{N}$, $j \le m$, we define

$$\Xi_j^{(m)} := \left[0, \left(\frac{j}{m}\right)^q\right] \cup \bigcup_{l=1}^{n-1} \left[s_l - \left(\frac{j}{m}\right)^q, s_l + \left(\frac{j}{m}\right)^q\right] \cup \left[T - \left(\frac{j}{m}\right)^q, T\right]$$

and a further projector R_j^m in $(L^2(0,T))^2$ by

$$(R_j^m \vec{u})(x) = \begin{cases} \vec{u}(x) & x \in [0,T] \setminus \Xi_j^{(m)}, \\ 0 & x \in \Xi_j^{(m)}. \end{cases} \tag{4.16}$$

Finally we define the modifications of the space $\widetilde{\Pi}_m^d$,

$$\widetilde{\Pi}_{m,j}^d := \{u \in \widetilde{\Pi}_m^d \mid u|_{\Xi_j^{(m)}} \equiv 0\}, \tag{4.17}$$

and its projector $\widetilde{P}_{m,j}^d : C[0,T] \to \widetilde{\Pi}_{m,j}^d$ by

$$(\widetilde{P}_{m,j}^d u)(s_l^{(m)} + \xi_k \delta_{l+1}^{(m)}) = \begin{cases} u(s_l^{(m)} + \xi_k \delta_{l+1}^{(m)}) & s_l^{(m)} + \xi_k \delta_{l+1}^{(m)} \in [0,T] \setminus \Xi_j^{(m)} \\ 0 & \text{else} \end{cases} \tag{4.18}$$

for all l, k. All of the above spaces and projectors can be defined for functions with values in $\mathbf{R}^l$, $l \in \mathbf{N}$.

For $\vec{f} \in (C[0,T])^2$ we denote by $\vec{u}_m \in \widetilde{\Pi}_{m,j}^d$ the solution of the collocation equation

$$\widetilde{P}_{m,j^*}^d (I + \mathcal{K}) \vec{u}_m = \widetilde{P}_{m,j^*}^d \vec{f}. \tag{4.19}$$

Remark. If we look at the proof of Theorem 4.2, then it is clear that for a fixed $j \ge 1$ we have

$$\|R_j^m (I + \mathcal{K}) \vec{u}\|_{(L^2([0,T] \setminus \Xi_j^{(m)})} \ge c \|\vec{u}\|_{(L^2[0,T] \setminus \Xi_j^{(m)})}, \tag{4.20}$$

for all $\vec{u} \in (L^2([0,T] \setminus \Xi_j^{(m)}))^2$, m sufficiently large.

The following lemmas allow us to obtain the stability of the collocation by small perturbations of the last estimate (4.20).

Lemma 4.3. *Let $q > 0$ and $0 = \xi_0 < \xi_1 < \cdots < \xi_d = 1$ be given. For every $\varepsilon > 0$ there is an $\tilde{i} \geq 1$ and $\tilde{m} \geq 1$ such that*

$$\|(I - P^d_{m,j})u\|_{L^2(x_j^{(m)},1)} \leq \varepsilon \|u\|_{X_0^{2,1}}, \quad u \in X_0^{2,1}(0,1),$$

$j \geq \tilde{i}$, $m \geq \tilde{m}$.

Proof. For $u \in X_0^{2,1}$ we have $u \in H^1[\alpha, 1]$, for all $\alpha > 0$. $P^d_{m,j}u$ is well defined for $j \geq 1$. If $m \in \mathbf{N}$, $j \geq 1$, we get

$$\begin{aligned}\|(I - P^d_{m,j})u\|^2_{L^2(x_l^{(m)},x_{l+1}^{(m)})} &\leq c(h_{l+1}^{(m)})^2 \int_{x_l^{(m)}}^{x_{l+1}^{(m)}} u'(x)^2\,dx \\ &\leq c\left(\frac{h_{l+1}^{(m)}}{x_l^{(m)}}\right)^2 \int_{x_l^{(m)}}^{x_{l+1}^{(m)}} (xu'(x))^2\,dx,\end{aligned}$$

where c depends only on $(\xi_k)_k$, see [**5**, Section 2]. For $\varepsilon > 0$ there exists an $i^*(\varepsilon) \geq 1$ and $m^*(\varepsilon)$ such that

$$\left(\frac{h_{l+1}^{(m)}}{x_l^{(m)}}\right)^2 \leq \frac{\varepsilon^2}{c}, \quad l \geq i^*,$$

and therefore

$$\int_{x_l^{(m)}}^{x_{l+1}^{(m)}} ((I - P^d_{m,j})u)^2\,dx \leq \varepsilon^2 \int_{x_l^{(m)}}^{x_{l+1}^{(m)}} (xu'(x))^2\,dx.$$

Summation over l gives

$$\int_{x_j^{(m)}}^1 ((I - P^d_{m,j})u)^2\,dx \leq \varepsilon^2 \int_{x_j^{(m)}}^1 (xu'(x))^2\,dx, \quad j \geq i^*,$$

and this implies

$$\begin{aligned}\|(I - P^d_{m,j})u\|_{L^2(x_j^{(m)},1)} &\leq \varepsilon \|xu'\|_{L^2(x_j^{(m)},1)} \\ &\leq \varepsilon \|u\|_{X_0^{2,1}(0,1)},\end{aligned}$$

for all $j \geq i^*$, $m \geq m^*$. This shows the lemma with $\tilde{i} = i^*$ and $\tilde{m} = m^*$. $\square$

Lemma 4.4. *Let $\omega \in \mathbf{R}$, $i \in \{1, \dots, n\}$. The operator*

$$xD\mathcal{K}_i^{(\omega)} : \big(L^2(0,1)\big)^4 \longrightarrow \big(L^2(0,1)\big)^4$$

is continuous.

Proof. The definition of $\mathcal{K}_i^{(\omega)}$ in (3.13) and the formulas (3.19)–(3.22) show that we only have to prove that

$$xDL_j : L^2(0,1) \longrightarrow L^2(0,1), \quad j = 1(1)4,$$

is continuous, where

$$(L_j u)(x) := \int_0^1 l_j\left(\frac{x}{\tau}\right) u(\tau)\,\frac{d\tau}{\tau},$$

see (3.17) for the definitions of the l_j. For $u \in C_0^\infty(0,1)$, we have

$$\begin{aligned}(\tilde{L}_j u)(x) &:= [xD(L_j u)](x)\\ &= \int_0^1 l_j'\left(\frac{x}{\tau}\right)\left(\frac{x}{\tau}\right) u(\tau)\frac{d\tau}{\tau}.\end{aligned}$$

This shows that $\tilde{L}_j$ is a Mellin convolution with kernel $l_j'(s)s$. But $l_j'(s)s$ fulfills $(H1^p)$, $p \in (-1,1)$. This shows [**5**]

$$\tilde{L}_j : L^2(0,1) \longrightarrow L^2(0,1)$$

is continuous and the lemma is proved. $\square$

Lemma 4.5. *Let $\omega \in \mathbf{R}$, $q > 0$, and $0 = \xi_0 < \xi_1 < \cdots < \xi_d = 1$. For every $\varepsilon > 0$ there are $\tilde{i}(\varepsilon) \geq 1$ and $\tilde{m}(\varepsilon) \geq 1$ such that*

$$\|(I - \widetilde{P}_{m,j}^d)\mathcal{K}^{(\omega)}\vec{u}\|_{(L^2([0,T]\setminus\Xi_j^{(m)}))^2} \leq \varepsilon\|\vec{u}\|_{(L^2(0,T))^2},$$

$j \geq \tilde{i}$, $m \geq \tilde{m}$.

Proof. First we write

$$(I - \widetilde{P}^d_{m,j})\mathcal{K}^{(\omega)}\vec{u} = (I - \widetilde{P}^d_{m,j})\mathcal{K}^{(\omega,1)}\vec{u} + (I - \widetilde{P}^d_{m,j})\mathcal{K}^{(\omega,2)}\vec{u},$$

see (3.7), (3.8). By Lemma 3.1 we know that

$$\mathcal{K}^{(\omega,2)} : (L^2(0,T))^2 \longrightarrow \prod_{i=0}^{3n-1} (C^1(J_i))^2$$

is continuous. By the definition of Δ_m we know that there is an $m_0^*(\varepsilon) \geq 1$, such that

$$\|(I - \widetilde{P}^d_{m,j})\mathcal{K}^{(\omega,2)}\vec{u}\|_{(L^2(0\pi))^2} \leq \frac{\varepsilon}{2}\|\vec{u}\|_{(L^2(0,T))^2}, \quad m \geq m_0^*(\varepsilon). \tag{4.21}$$

But $(I - \widetilde{P}^d_{m,j})(\mathcal{K}^{(\omega,1)}\vec{u})(x)$ is different from zero only for $x \in [s_i - 1, s_i + 1]$. We apply Lemma 4.3 in the neighborhood of each s_i and define $\vec{v} \in \big(L^2(0,1)\big)^4$ by

$$\begin{aligned} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}(x) &:= \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}(s_i - x), \\ \begin{pmatrix} v_3 \\ v_4 \end{pmatrix}(x) &:= \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}(s_i + x). \end{aligned}$$

Lemma 4.3 shows that for all $\eta > 0$ there is an $i_1^*(\eta) \geq 1$ and $m_1^*(\eta) \geq 1$ such that

$$\begin{aligned} \|(I - \widetilde{P}^d_{m,j})\mathcal{K}^{(\omega,1)}\vec{u}\|_{(L^2((s_i-1,s_i+1)\setminus\Xi_j^{(m)}))^2} &\leq \eta\|\mathcal{K}_i^{(\omega)}\vec{v}\|_{(X_0^{2,1}(0,1))^4} \\ &\leq \eta c\|\vec{v}\|_{(L^2(0,1))^4} \\ &\leq \eta\tilde{c}\|\vec{u}\|_{(L^2(s_i-1,s_i+1))^2}, \\ i \geq i_1^*, m \geq m_1^*. \end{aligned}$$

Here we have applied Lemma 4.4. So we can choose $i^*(\varepsilon) \geq 1$ and $m_2^*(\varepsilon) \geq 1$ with

$$\begin{aligned} \|(I - \widetilde{P}^d_{m,j})\mathcal{K}^{(\omega,1)}\vec{u}\|_{(L^2((s_i-1,s_i+1)\setminus\Xi_j^{(m)}))^2} &\leq \frac{\varepsilon}{2}\|\vec{u}\|_{(L^2(s_i-1,s_i+1))^2}, \\ i \geq i^*(\varepsilon), m \geq m_2^*(\varepsilon). \end{aligned} \tag{4.22}$$

Equations (4.21) and (4.22) together prove the theorem for $m^*(\varepsilon) := \max\{m_1^*(\varepsilon), m_2^*(\varepsilon)\}$. $\square$

Now we can prove the stability of our modified collocation method $\widetilde{P}_{m,j}^d(I+\mathcal{K})\widetilde{P}_{m,j}^d$ if j is sufficiently large.

Theorem 4.6. *Let $q > 0$ and $0 = \xi_0 < \ldots < \xi_d = 1$. There exist $i^*, m^* \in \mathbf{N}$, such that, for all $i \geq i^*$,*

$$\|\widetilde{P}_{m,i}^d(I+\mathcal{K})\vec{u}\|_{(L^2(0,T))^2} \geq c\|\vec{u}\|_{(L^2(0,T))^2},$$

$\vec{u} \in \widetilde{\Pi}_{m,i}^d$, $m \geq m^*$, where the constant $c > 0$ does not depend on $\vec{u}$ or m.

Proof. By Theorem 4.2 there is an $h_0 > 0$, such that

$$\|P_h(I+\mathcal{K})P_h\vec{u}\|_{(L^2(0,T))^2} \geq c\|P_h\vec{u}\|_{(L^2(0,T))^2}, \quad h \in (0,h_0), \tag{4.23}$$

where c is independent of $\vec{u}$ and h. By Lemma 4.5 there exist i^* and m^* with

$$\|(I-\widetilde{P}_{m,j}^d)\mathcal{K}\vec{u}\|_{(L^2([0,T]\setminus\Xi_j^{(m)}))^2} \leq \frac{c}{2}\|\vec{u}\|_{(L^2(0,T))^2}, \tag{4.24}$$

$m \geq m^*$, $j \geq i^*$. Now we fix j. For $m \geq m_1^*$ the projector R_j^m, see (4.16), also fulfills the inequality (4.23), see the remark before Lemma 4.4. We further have

$$R_j^m \circ \widetilde{P}_{m,j}^d = \widetilde{P}_{m,j}^d. \tag{4.25}$$

For $m \geq \max\{m^*, m_1^*\}$, we get for $\vec{u} \in (\widetilde{\Pi}_{m,j}^d)^2$

$$\begin{aligned}
\|\widetilde{P}_{m,j}^d(I+\mathcal{K})\vec{u}\|_{(L^2(0,T))^2} &\overset{(4.25)}{\geq} \|R_j^m(I+\mathcal{K})R_j^m\vec{u}\|_{(L^2(0,T))^2} \\
&\qquad - \|(\widetilde{P}_{m,j}^d - R_j^m)(I+\mathcal{K})R_j^m\vec{u}\|_{(L^2(0,T))^2} \\
&\overset{(4.23)}{\geq} c\|R_j^m\vec{u}\|_{(L^2(0,T))^2} \\
&\qquad - \|(\widetilde{P}_{m,j}^d - R_j^m)\mathcal{K}R_j^m\vec{u}\|_{(L^2(0,T))^2} \\
&= c\|\vec{u}\|_{(L^2(0,T))^2} \\
&\qquad - \|(\widetilde{P}_{m,j}^d - I)\mathcal{K}R_j^m\vec{u}\|_{(L^2([0,T]\setminus\Xi_j^{(m)}))^2} \\
&\overset{(4.24)}{\geq} \frac{c}{2}\|\vec{u}\|_{(L^2(0,T))^2}.
\end{aligned}$$

□

Lemma 4.7. *Let $\vec{u} \in (L^2(0,T))^2$, $\rho \in (1, 3/2)$. Assume that, for every i,*

$$\vec{u}|_{[s_i, s_{i+1}]} \in (C^l(s_i, s_{i+1}))^2,$$
$$\vec{v}_i^{\pm}(x) := \vec{u}(s_i \pm x) \in \left(X_\rho^{2,l}(0,1)\right)^2 \dot{+} \mathbf{R}^2,$$

and let $q \geq 2r$, $r := \min\{l, d+1\}$. For $j^* \in \mathbf{N}$ we have

$$\|(I - \widetilde{P}_{m,j^*}^d)\vec{u}\|_{(L^2(0,T))^2} \leq \frac{c(\vec{u}, j^*)}{m^r}.$$

Proof. By the triangle inequality we get

$$\begin{aligned}
\|(I - \widetilde{P}_{m,j^*}^d)\vec{u}\|_{(L^2(0,T))^2} \leq & \sum_{j=0}^{n-1} \|(I - \widetilde{P}_{m,j^*}^d)\vec{u}\|_{(L^2(s_j+1, s_{j+1}-1))^2} \\
& + \sum_{j=0}^{n-1} \|(I - \widetilde{P}_{m,j^*}^d)\vec{u}\|_{(L^2(s_j, s_j+1))^2} \\
& + \sum_{j=1}^{n} \|(I - \widetilde{P}_{m,j^*}^d)\vec{u}\|_{(L^2(s_j-1, s_j))^2}.
\end{aligned}$$

For the first summand we get

$$|(I - \widetilde{P}_{m,j^*}^d)\vec{u}(x)| \leq \frac{c_1(\vec{u})}{m^r}, \quad x \in [s_i + 1, s_{i+1} - 1], \tag{4.26}$$

because here $\vec{u}$ is a C^l-function, $l \geq r$.

The terms in the second and third sum can all be estimated by the approximation error for $\vec{v}_i^{\pm}(x)$ on $(0,1)$. We look at one term in the second summand and because of $2r \geq r/\rho$ we get by [**5**, Lemma 2.20]

$$\|(I - P_{m,j^*}^d)\vec{v}_i^{+}\|_{L^2(x_{j^*}^{(m)}, 1))^2} \leq \frac{c_2}{m^r} \|\vec{v}_i^{+}\|_{\left(X_\rho^{2,l}(0,1)\right)^2}, \tag{4.27}$$

where the constant c_2 depends only on $(\xi_k)_{k=0}^d$. Now we notice

$$\vec{v}_i^{+} = \vec{w}_i^{+} + v_0^{+}, \vec{w}_i^{+} \in \left(X_\rho^{2,l}(0,1)\right)^2, v_0^{+} \in \mathbf{R}^2.$$

We get

$$\begin{aligned}
\|(I-P^d_{m,j^*})\vec{v}_i^{\,+}\|_{(L^2(0,x^{(m)}_{j^*}))^2} &= \|\vec{v}_i^{\,+}\|_{(L^2(0,x^{(m)}_{j^*}))^2} \\
&\le \left(\int_0^{x^{(m)}_{j^*}} \|\vec{w}_i^{\,+}(x)\|^2\,dx\right)^{1/2} \\
&\quad + \left(\int_0^{x^{(m)}_{j^*}} \|\vec{v}_0^{\,+}\|^2\,dx\right)^{1/2} \\
&\le \left((x^{(m)}_{j^*})^{2\rho}\int_0^{x^{(m)}_{j^*}} (x^{-\rho}\|\vec{w}_i^{\,+}(x)\|)^2\,dx\right)^{1/2} \\
&\quad + (x^{(m)}_{j^*})^{1/2}\|\vec{v}_0^{\,+}\| \\
&\le c_3(\vec{v}_i^{\,+})(x^{(m)}_{j^*})^{1/2} \\
&\stackrel{q\ge 2r}{\le} c_4(\vec{v}_i^{\,+})\frac{1}{m^r}.
\end{aligned} \tag{4.28}$$

Now (4.26)–(4.28) prove the lemma. □

Remark. In the proof of Lemma 4.7 we see that the large grading exponent $2r$ is only necessary for the proof of (4.28). If it is possible to prove stability for a modified projector P^d_{m,j^*} where the functions are constant (but not necessarily zero) in a vicinity of zero, then a grading exponent r/ρ would be sufficient for the approximation result in Lemma 4.7.

The next theorem shows that for m large enough the solution $\vec{u}_m$ of the collocation equation (4.19) is well defined and we get an estimate for the error.

Theorem 4.8. *Let $\vec{f}\in(C[0,T])^2$, $\vec{f}|_{[s_i,s_{i+1}]}\in(C^l[s_i,s_{i+1}])^2$, $i=0(1)n-1$, $l\in\mathbf{N}$. We denote by $\vec{u}$ the solution of equation* (1.1)*, see Theorem* 3.17*. Let $0=\xi_0<\xi_1<\dots<\xi_d=1$, $d\in\mathbf{N}$, $q\ge 2r$, $r:=\min\{l,d+1\}$. There exists an $i^*\in\mathbf{N}$, such that for $j^*\ge i^*$ and all sufficiently large m the equation* (4.19) *has a solution $\vec{u}_m$ and we get*

$$\|\vec{u}-\vec{u}_m\|_{(L^2(0,T))^2}\le\frac{c}{m^r}.$$

Proof. By Theorem 4.6 there exists an $i^* \in \mathbf{N}$, such that for $j^* \geq i^*$

$$\|\widetilde{P}^d_{m,j^*}(I+\mathcal{K})\vec{v}\|_{(L^2(0,T))^2} \geq c\|\vec{v}\|_{(L^2(0,T))^2}, \quad \forall\, \vec{v} \in \widetilde{\Pi}^d_{m,j^*}, \tag{4.29}$$

$m \geq m^*$, $c > 0$. Because $\widetilde{\Pi}^d_{m,j^*}$ is finite-dimensional this shows the solvability of (4.19), and we get by the triangle inequality

$$\|\vec{u} - \vec{u}_m\|_{(L^2(0,T))^2} \leq \|\vec{u} - \widetilde{P}^d_{m,j^*}\vec{u}\|_{(L^2(0,T))^2} + \|\widetilde{P}^d_{m,j^*}\vec{u} - \vec{u}_m\|_{(L^2(0,T))^2}. \tag{4.30}$$

For the second summand we get by (4.29),

$$\begin{aligned}
\|\widetilde{P}^d_{m,j^*}\vec{u} - \vec{u}_m\|_{(L^2(0,T))^2} &\leq \frac{1}{c}\|\widetilde{P}^d_{m,j^*}(I+\mathcal{K})\widetilde{P}^d_{m,j^*}\vec{u} \\
&\qquad - \widetilde{P}^d_{m,j^*}(I+\mathcal{K})\vec{u}_m\|_{(L^2(0,T))^2} \\
&= \frac{1}{c}\|\widetilde{P}^d_{m,j^*}(I+\mathcal{K})\widetilde{P}^d_{m,j^*}\vec{u} \\
&\qquad - \widetilde{P}^d_{m,j^*}\vec{f}\|_{(L^2(0,T))^2} \\
&= \frac{1}{c}\|\widetilde{P}^d_{m,j^*}(I+\mathcal{K})\widetilde{P}^d_{m,j^*}\vec{u} \\
&\qquad - \widetilde{P}^d_{m,j^*}(I+\mathcal{K})\vec{u}\|_{(L^2(0,T))^2} \\
&= \frac{1}{c}\|\widetilde{P}^d_{m,j^*}\mathcal{K}(\widetilde{P}^d_{m,j^*}\vec{u} - \vec{u})\|_{(L^2(0,T))^2} \\
&\leq c_1\|\widetilde{P}^d_{m,j^*}\vec{u} - \vec{u}\|_{(L^2(0,T))^2}.
\end{aligned} \tag{4.31}$$

Here the continuity of $\widetilde{P}^d_{m,j^*}\mathcal{K}$, see Lemma 4.5, has been used. Equations (4.30) and (4.31) now give

$$\|\vec{u} - \vec{u}_m\|_{(L^2(0,T))^2} \leq (1+c_1)\|\vec{u} - \widetilde{P}^d_{m,j^*}\vec{u}\|_{(L^2(0,T))^2},$$

but $\vec{u}$ fulfills the assumptions of Lemma 4.7, with $\rho \in [1, 1/2 + \bar{z}(\bar{\omega}))$, by Lemma 3.11, and this proves Theorem 4.8. □

5. Proof of some auxiliary results.

Proof of Lemma 3.5. By the transformation $x \to e^{-x/p}$ we have the correspondence between the operator $\mathcal{B}_i^{(\omega)}$ on $(0,1)$ and a Wiener-Hopf operator $W_i^{(\omega)}$ on $[0,\infty)$.

The Wiener-Hopf operator is a Fredholm operator if the determinant of its symbol $\hat{W}_i^{(\omega)}(s)$ is different from zero on the real line [**10**, Theorem VIII, 6.1]. If we consider $\mathcal{B}_i^{(\omega)}$ as an operator on $(L^p(0,1))^4$ the corresponding Wiener-Hopf operator has the symbol

$$\widehat{\mathcal{B}}_i^{(\omega)}(s), \quad \mathrm{Re}\,(s) = 1/p. \tag{5.1}$$

If we consider $\widehat{\mathcal{B}}_i^{(\omega)}(s)$ as an operator on $(X_\rho^{p,0}(0,1))^4$, the corresponding Wiener-Hopf operator has the symbol

$$\widehat{\mathcal{B}}_i^{(\omega)}(s), \quad \mathrm{Re}\,(s) = 1/p - \rho, \tag{5.2}$$

see [**5**].

This implies that we have to study the zeros of the function

$$\det(\hat{\mathcal{B}}_i^{(\omega)}(s)), \quad \omega \in [0,1].$$

By formula (3.30) we have

$$\begin{aligned}
&\hat{\mathcal{B}}_i^{(\omega)}(s) = \begin{pmatrix} I_{2\times 2} & 0 \\ 0 & \tilde{S}_i \end{pmatrix} \tilde{\mathcal{B}}_i^{(\omega)}(s) \begin{pmatrix} I_{2\times 2} & 0 \\ 0 & \tilde{S}_i \end{pmatrix}, \text{ where} \\
&\tilde{\mathcal{B}}_i^{(\omega)}(s) := \begin{pmatrix} I_{2\times 2} & \hat{C}_i^{(\omega)}(s) \\ \widehat{C}_i^{(\omega)}(s) & I_{2\times 2} \end{pmatrix}, \text{ with} \\
&\hat{C}_i^{(\omega)}(s) = (g_i(s) I_{2\times 2} + \omega h_i(s) \mathcal{S}_{i,1}(s)) \widetilde{S}_i.
\end{aligned} \tag{5.3}$$

Because of $\det(\hat{S}_i) = -1$ we get

$$\det(\hat{\mathcal{B}}_i^{(\omega)}(s)) = \det(\tilde{\mathcal{B}}_i^{(\omega)}(s)).$$

Now we follow the proofs of Lewis in [**13**].

First we obtain

$$\det(\tilde{\mathcal{B}}_i^{(\omega)}(s)) = \det(I_{2\times 2} - \hat{C}_i^{(\omega)}(s)) \det(I_{2\times 2} + \hat{C}_i^{(\omega)}(s)).$$

Furthermore we have

$$\begin{aligned}
I_{2\times 2} - \hat{C}_i^{(\omega)}(s) = \frac{1}{\sin(\pi s)} \Big[&- \sin(\alpha_i s) \widetilde{S}_i \\
&\begin{pmatrix} \sin(\pi s) - \omega s \sin(\alpha_i)\cos(\alpha_i) & -\omega s \sin(\alpha_i)\sin(\alpha_i s) \\ \omega s \sin(\alpha_i)\sin(\alpha_i s) & \sin(\pi s) - \omega s \sin(\alpha_i)\cos(\alpha_i) \end{pmatrix} \Big] \\
&=: \frac{1}{\sin(\pi s)} (A_2 + A_1).
\end{aligned}$$

Here A_1 has the form

$$A_1 = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} \\ -a_{12}^{(1)} & a_{11}^{(1)} \end{pmatrix}$$

(which is called antireflective by Lewis) and A_2 has the form

$$A_2 = \begin{pmatrix} a_{11}^{(2)} & a_{12}^{(2)} \\ a_{12}^{(2)} & -a_{11}^{(2)} \end{pmatrix}$$

(called reflective by Lewis). This implies

$$\begin{aligned} \det(A_1 \pm A_2) &= \det(A_1) + \det(A_2) \\ &= ((a_{11}^{(1)})^2 + (a_{12}^{(1)})^2) - ((a_{11}^{(2)})^2 + (a_{12}^{(2)})^2); \end{aligned}$$

hence,

$$\begin{aligned} \det(I_{2\times 2} - \hat{C}_i^{(\omega)}(s)) &= \frac{1}{\sin(\pi s)^2}((\sin(\pi s)\cos(\alpha_i s) - \omega s \sin(\alpha_i))^2 \\ &\qquad - (\cos(\pi s)\sin(\alpha_i s))^2) \\ &=: \frac{1}{\sin(\pi s)^2}(\alpha^2 - \beta^2) \\ &= \frac{1}{\sin(\pi s)^2}(\alpha - \beta)(\alpha + \beta), \end{aligned}$$

where

$$\begin{aligned} \alpha &= \sin(\pi s)\cos(\alpha_i s) - \omega s \sin(\alpha_i), \\ \beta &= \cos(\pi s)\sin(\alpha_i s). \end{aligned}$$

For $s = 0$, α and β have a simple zero. Now we have

$$\begin{aligned} & \alpha - \beta && = 0 \\ \stackrel{s\neq 0}{\Longleftrightarrow}\ & \frac{\sin((\pi - \alpha_i)s)}{(\pi - \alpha_i)s} - \omega \frac{\sin(\pi - \alpha_i)}{\pi - \alpha_i} && = 0. \end{aligned}$$

By Lemma 3.4 of Lewis we get

$$\alpha - \beta \neq 0, \quad \text{for } s \in \mathbf{C}, \quad 0 \leq \mathrm{Re}\,(s) < \bar{z}(\omega), \quad s \neq 0.$$

On the other hand,

$$\begin{aligned} & \alpha+\beta && = 0 \\ \overset{s\neq 0}{\Longleftrightarrow}\ & \frac{\sin((\pi+\alpha_i)s)}{(\pi+\alpha_i)s} + \omega\frac{\sin(\pi+\alpha_i)}{\pi+\alpha_i} && = 0. \end{aligned}$$

Again by the lemma of Lewis we know

$$\alpha+\beta \neq 0, \quad \text{for } 0 \le Re(s) < \bar{z}(\omega), \quad s \neq 0.$$

This implies

$$\det(I_{2\times 2} - \hat{C}_i^{(\omega)}(s)) \neq 0, \quad 0 \le \mathrm{Re}\,(s) < \bar{z}(\omega). \tag{5.4}$$

Analogous to $I_{2\times 2} - \hat{C}_i^{(\omega)}(s)$ we now analyze $I_{2\times 2} + \hat{C}_i^{(\omega)}(s)$ and get

$$\det(I_{2\times 2} + \hat{C}_i^{(\omega)}(s)) \neq 0, \quad 0 \le \mathrm{Re}\,(s) < \bar{z}(\omega). \tag{5.5}$$

$\det(I_{2\times 2} \pm \hat{C}_i^{(\omega)}(s))$ are even functions, therefore (5.4) and (5.5) show

$$\det(I_{2\times 2} \pm \hat{C}_i^{(\omega)}(s)) \neq 0, \quad \text{if } |\mathrm{Re}\,(s)| < \bar{z}(\omega).$$

This gives

$$\det(\hat{\mathcal{B}}_i^{(\omega)}(s)) \neq 0, \quad \text{if } |\mathrm{Re}\,(s)| < \bar{z}(\omega).$$

Lemma 3.2 now shows (iii) for $1/2-\rho > -\bar{z}(\omega)$, and Lemma 3.2 and (5.1) imply

$$\mathcal{B}_i^{(\omega)} : (L^p(0,1))^4 \longrightarrow (L^p(0,1))^4$$

is a Fredholm operator for

$$0 \le \frac{1}{p} < \bar{z}(\omega) \Longleftrightarrow \frac{1}{\bar{z}(\omega)} < p \le \infty.$$

Lemma 3.2 and (5.2) show

$$\mathcal{B}_i^{(\omega)} : (X_1^{p,0}(0,1))^4 \longleftrightarrow (X_1^{p,0}(0,1))^4$$

is a Fredholm operator for

$$-\bar{z}(\omega) < \frac{1}{p} - 1 \leq 0 \Longleftrightarrow 1 \leq p < \frac{1}{1-\bar{z}(\omega)}.$$

□

Proof of Lemma 3.7. We have

$$\hat{C}_i^{(\omega)}(s) = g_i(s)\tilde{S}_i + \omega h_i(s)\mathcal{S}_{i,1}(s)\tilde{S}_i$$

For $\tilde{\mathcal{S}}_{i,1}(s) := \mathcal{S}_{i,1}(s)\tilde{S}_i$ we get

$$\tilde{\mathcal{S}}_{i,1}(s) = \begin{pmatrix} \cos(\alpha_i s) & \sin(\alpha_i s) \\ -\sin(\alpha_i s) & \cos(\alpha_i s) \end{pmatrix}.$$

$$\begin{aligned}
&\mathrm{Re}\,(\hat{C}_i^{(\omega)}(s)) \\
&\quad = \frac{1}{2}\Big(g_i(s)\tilde{S}_i + \overline{g_i(s)}\tilde{S}_i + \omega h_i(s)\tilde{\mathcal{S}}_{i,1}(s) + \omega\overline{h_i(s)}\tilde{\mathcal{S}}_{i,1}(s)^*\Big) \\
&\quad = \mathrm{Re}\,(g_i(s))\tilde{S}_i \\
&\qquad + \omega\begin{pmatrix} \mathrm{Re}\,(h_i(s)\cos(\alpha_i s)) & \sqrt{-1}\,\mathrm{Im}\,(h_i(s)\sin(\alpha_i s)) \\ -\sqrt{-1}\mathrm{Im}\,(h_i(s)\sin(\alpha_i s)) & \mathrm{Re}\,(h_i(s)\cos(\alpha_i s)) \end{pmatrix}
\end{aligned}$$

The first term is a reflective matrix and the second term is antireflective. This will be used in the calculation of the eigenvalues of $\mathrm{Re}\,(\hat{C}_i^{(\omega)}(s))$.

$$\begin{aligned}
&\det(\mathrm{Re}\,(\hat{C}_i^{(\omega)}(s)) - \lambda I_{2\times 2}) \\
&\ = \det(\begin{pmatrix} \omega\mathrm{Re}\,(h_i(s)\cos(\alpha_i s)) - \lambda & \sqrt{-1}\omega\,\mathrm{Im}\,(h_i(s)\sin(\alpha_i s)) \\ -\sqrt{-1}\omega\,\mathrm{Im}\,(h_i(s)\sin(\alpha_i s)) & \mathrm{Re}\,(h_i(s)\cos(\alpha_i s)) - \lambda \end{pmatrix} \\
&\qquad + \mathrm{Re}\,(g_i(s))\tilde{S}_i) \\
&\ = (\omega\mathrm{Re}\,(h_i(s)\cos(\alpha_i s)) - \lambda)^2 \\
&\qquad - \omega^2\,\mathrm{Im}\,((h_i(s)\sin(\alpha_i s)))^2 - \mathrm{Re}\,(g_i(s))^2 \\
&\ = \lambda^2 - 2\omega\,\mathrm{Re}\,(h_i(s)\cos(\alpha_i s))\lambda \\
&\qquad + \omega^2(\mathrm{Re}\,(h_i(s)\cos(\alpha_i s))^2 \\
&\qquad - \mathrm{Im}\,(h_i(s)\sin(\alpha_i s))^2) - \mathrm{Re}\,(g_i(s))^2.
\end{aligned}$$

The two solutions for λ are given by

$$
\begin{aligned}
\lambda_{1/2}(s) = {} & \omega \operatorname{Re}(h_i(s)\cos(\alpha_i s)) \\
& \pm (\operatorname{Re}(g_i(s))^2 + \omega^2 \operatorname{Im}(h_i(s)\sin(\alpha_i s))^2)^{1/2}
\end{aligned}
\tag{5.6}
$$

We recall that

$$
\begin{aligned}
h_i(s)\sin(\alpha_i s) &= \sin(\alpha_i)\frac{s\sin(\alpha_i s)}{\sin(\pi s)}, \\
h_i(s)\cos(\alpha_i s) &= \sin(\alpha_i)\frac{s\cos(\alpha_i s)}{\sin(\pi s)}, \\
g_i(s) &= \frac{\sin(\alpha_i s)}{\sin(\pi s)}.
\end{aligned}
$$

We substitute $s = \sqrt{-1}y$, $y \in \mathbf{R}$, and obtain

$$
\begin{aligned}
h_i(s)\sin(\alpha_i s) &= \sqrt{-1}\,\sin(\alpha_i)\frac{y\sinh(\alpha_i y)}{\sinh(\pi y)}, \\
h_i(s)\cos(\alpha_i s) &= \sin(\alpha_i)\frac{\sqrt{-1}\,y\cosh(\alpha_i y)}{\sqrt{-1}\,\sinh(\pi y)}, \\
&= \sin(\alpha_i)\frac{y\cosh(\alpha_i y)}{\sinh(\pi y)}, \\
g_i(s) &= \frac{\sinh(\alpha_i y)}{\sinh(\pi y)}.
\end{aligned}
$$

By (5.6) we get

$$
\begin{aligned}
\lambda_{1/2}(y) &= \omega\sin(\alpha_i)\frac{y\cosh(\alpha_i y)}{\sinh(\pi y)} \\
&\qquad \pm \left(\frac{\sinh(\alpha_i y)^2}{\sinh(\pi y)^2} + \omega^2\sin(\alpha_i)^2\frac{\sinh(\alpha_i y)^2}{\sinh(\pi y)^2}\right)^{1/2} \\
&= \omega\sin(\alpha_i)\frac{y\cosh(\alpha_i y)}{\sinh(\pi y)} \\
&\qquad \pm \left|\frac{\sinh(\alpha_i y)}{\sinh(\pi y)}\right|(1+\omega^2\sin(\alpha_i)^2 y^2)^{1/2}.
\end{aligned}
$$

Let $\alpha_i \geq 0$. We define

$$\begin{aligned} f_1(\alpha_i, y) &:= \lambda_1(y) \\ &= \omega \sin(\alpha_i) \frac{y \cosh(\alpha_i y)}{\sinh(\pi y)} + \frac{\sinh(\alpha_i y)}{\sinh(\pi y)} (1 + \omega^2 \sin(\alpha_i)^2 y^2)^{1/2} \\ f_2(\alpha_i, y) &:= \lambda_2(y) \\ &= \omega \sin(\alpha_i) \frac{y \cosh(\alpha_i y)}{\sinh(\pi y)} - \frac{\sinh(\alpha_i y)}{\sinh(\pi y)} (1 + \omega^2 \sin(\alpha_i)^2 y^2)^{1/2}. \end{aligned}$$

For $\alpha_i < 0$ we have

$$\begin{aligned} \lambda_1(y) &= -\omega \sin(-\alpha_i) \frac{y \cosh(\alpha_i y)}{\sinh(\pi y)} + \frac{\sinh(-\alpha_i y)}{\sinh(\pi y)} (1 + \omega^2 \sin(\alpha_i)^2 y^2)^{1/2} \\ &= -f_2(-\alpha_i, y) \\ \lambda_2(y) &= -\omega \sin(-\alpha_i) \frac{y \cosh(\alpha_i y)}{\sinh(\pi y)} - \frac{\sinh(-\alpha_i y)}{\sinh(\pi y)} (1 + \omega^2 \sin(\alpha_i)^2 y^2)^{1/2} \\ &= -f_1(-\alpha_i, y) \end{aligned}$$

We are only interested in the absolute values of $\lambda_{1/2}(y)$, so we only have to consider $f_{1/2}(\alpha_i, y)$, $\alpha_i \in [0, \pi)$, $y \in \mathbf{R}$. But we further have

(i) $f_{1/2}(\alpha_i, y) = f_{1/2}(\alpha_i, -y)$, $\forall\, \alpha_i$ and $y \in \mathbf{R}$.

This means we only have to look at the case $\alpha_i \in [0, \pi)$, $y \geq 0$. But then we observe that

$$|f_1(\alpha_i, y)| \geq |f_2(\alpha_i, y)| \text{ and } f_1(\alpha_i, y) \geq 0.$$

So it remains to show that there exists

$$q_i^* < 1 \text{ such that } f_1(\alpha_i, y) \leq q_i^*, \quad \forall\, y \geq 0. \tag{5.7}$$

The statement of the lemma then follows with

$$q := 1 - \max_{i=1}^{n} \{q_i^*\}.$$

Define

$$f(\alpha_i, y) := \sin(\alpha_i) \frac{y \cosh(\alpha_i y)}{\sinh(\pi y)} + \frac{\sinh(\alpha_i y)}{\sinh(\pi y)} (1 + \sin(\alpha_i)^2 y^2)^{1/2}$$

(here ω is equal to 1). We have the following properties

(ii) $f_1(\alpha_i, y) \le f(\alpha_i, y)$, $\forall\, \alpha_i \in [0,\pi), y \ge 0$.

(iii) $\lim_{|y|\to\infty} f(\alpha_i, y) = 0$.

(iv) $f(0,y) \equiv 0$.

(v) $f(\pi, y) \equiv 1$.

From (ii) it follows that we have to prove (5.7) only for $f(\alpha_i, y)$. (iii)–(v) show that it is sufficient to prove that the mapping

$$\alpha_i \longrightarrow f(\alpha_i, y) \quad \text{is monotonically increasing}, \quad y \ge 0. \tag{5.8}$$

Then (5.7) is proved. But we have

$$\begin{aligned}
\frac{\partial f}{\partial \alpha_i}(\alpha_i, y) = \cos(\alpha_i)\frac{y\cosh(\alpha_i y)}{\sinh(\pi y)} + \sin(\alpha_i)\frac{y^2\sinh(\alpha_i y)}{\sinh(\pi y)} \\
&+ \frac{y\cosh(\alpha_i y)}{\sinh(\pi y)}(1+\sin(\alpha_i)^2 y^2)^{1/2} \\
&+ \sin(\alpha_i)\cos(\alpha_i)\frac{y^2\sinh(\alpha_i y)}{\sinh(\pi y)}(1+\sin(\alpha_i)^2y^2)^{-1/2} \\
&= \frac{1}{\sinh(\pi y)}\Big(y\cosh(\alpha_i y)\underbrace{(\cos(\alpha_i) + (1+\sin(\alpha_i)^2y^2)^{1/2}}_{>0} \\
&+ y^2\underbrace{\sin(\alpha_i)\sinh(\alpha_i y)}_{\ge 0}\underbrace{(1+\cos(\alpha_i)(1+\sin(\alpha_i)^2y^2)^{-1/2})}_{\ge 0} \\
&\ge 0.
\end{aligned}$$

This proves (5.8) and the lemma. $\square$

Acknowledgments. The second author is grateful to Prof. Dr. H.N. Mülthei, Johannes Gutenberg–Universität Mainz, and to Prof. Dr. S. Prössdorf (†), Weierstraß–Institut für Angewandte Analysis und Stochastik, who made it possible that he could spend half a year at the Weierstraß–Institut für Angewandte Analysis und Stochastik.

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