

## LOCAL REMOVABLE SINGULARITY THEOREMS FOR MINIMAL LAMINATIONS

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### Abstract

In this paper we prove a *local removable singularity theorem* for certain minimal laminations with isolated singularities in a Riemannian three-manifold. This removable singularity theorem is the key result used in our proof that a complete, embedded minimal surface in  $\mathbb{R}^3$  with quadratic decay of curvature has finite total curvature.

### 1. Introduction.

There is a vast literature concerning the (local) analysis of mathematical objects around an isolated singularity. In this paper we shall be concerned with the local behavior of an embedded minimal surface, or more generally, of a minimal lamination  $\mathcal{L}$  of a punctured ball in a Riemannian three-manifold, with the central question of when such an  $\mathcal{L}$  extends as lamination across the puncture. We will characterize the removability of the possible singularity of the closure of  $\mathcal{L}$  at the puncture in terms of the growth of the norm of the second fundamental form of the leaves of  $\mathcal{L}$  when approaching the puncture; see the Local Removable Singularity Theorem (Theorem 1.1) below for this characterization.

Before stating the Local Removable Singularity Theorem, we set some specific notation to be used throughout the paper. Given a three-manifold  $N$  and a point  $p \in N$ , we denote by  $d_N$  the distance function in  $N$  to  $p$  and by  $B_N(p, r)$  the open metric ball of center  $p$  and radius  $r > 0$ . For a lamination  $\mathcal{L}$  of  $N$  and a leaf  $L$  of  $\mathcal{L}$ , we denote by  $|\sigma_L|$  the norm of the second fundamental form of  $L$ . Since leaves of  $\mathcal{L}$  do not intersect, it makes sense to consider the norm of the second fundamental form as a function defined on the union of the leaves of  $\mathcal{L}$ , which we denote by  $|\sigma_{\mathcal{L}}|$ . In the case  $N = \mathbb{R}^3$ , we use the notation  $\mathbb{B}(p, r) = B_{\mathbb{R}^3}(p, r)$  and  $\mathbb{B}(r) = \mathbb{B}(\vec{0}, r)$ . The boundary and closure of  $\mathbb{B}(r)$  will be respectively denoted by  $\partial\mathbb{B}(r) = \mathbb{S}^2(r)$  and  $\overline{\mathbb{B}}(r)$ .  $\mathbb{S}^1(r)$  represents the circle  $\{(x_1, x_2) \mid x_1^2 + x_2^2 = r^2\} \subset \mathbb{R}^2$ . Furthermore,  $R: \mathbb{R}^3 \rightarrow \mathbb{R}$  will stand for the distance function to the origin  $\vec{0} \in \mathbb{R}^3$ . Finally, for a surface  $M \subset \mathbb{R}^3$ ,  $K_M$  denotes its Gaussian curvature function.

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**Theorem 1.1** (Local Removable Singularity Theorem). *A minimal lamination  $\mathcal{L}$  of a punctured ball  $B_N(p, r) - \{p\}$  in a Riemannian three-manifold  $N$  extends to a minimal lamination of  $B_N(p, r)$  if and only if there exists a positive constant  $C$  such that  $|\sigma_{\mathcal{L}}| d_N(p, \cdot) \leq C$  in some subball. In particular under this hypothesis,*

- 1) *The curvature of  $\mathcal{L}$  is bounded in a neighborhood of  $p$ .*
- 2) *If  $\mathcal{L}$  consists of a single leaf  $M \subset B_N(p, r) - \{p\}$  which is a properly embedded minimal surface, then  $M$  extends smoothly through  $p$ .*

We remark that the natural generalization of the above local removable singularity theorem fails badly for codimension-one minimal laminations of  $\mathbb{R}^n$ , for  $n = 2$  and for  $n > 3$ . In the case  $n = 2$ , consider the cone  $\mathcal{C}$  over any two non-antipodal points on the unit circle;  $\mathcal{C}$  consists of two infinite rays making an acute angle at the origin. The punctured cone  $\mathcal{C} - \{\vec{0}\}$  is totally geodesic and so, the norm of the second fundamental form of  $\mathcal{C} - \{\vec{0}\}$  is zero but  $\mathcal{C}$  is not a smooth lamination at the origin. In the case  $n \geq 4$ , let  $\mathcal{C}$  denote the cone over any embedded, compact minimal hypersurface  $\Sigma$  in  $\mathbb{S}^{n-1}$  which is not an equator. Since the norm of the second fundamental form of  $\Sigma$  is bounded, then the norm of the second fundamental form of  $\mathcal{C} - \{\vec{0}\}$  multiplied by the distance function to the origin is also a bounded function on  $\mathcal{C} - \{\vec{0}\}$ . These examples demonstrate that Theorem 1.1 is precisely an ambiently three-dimensional result.

Theorem 1.1 is related to previous results by Colding and Minicozzi, where they obtain quadratic estimates for the area of a compact embedded minimal surface  $\Sigma \subset \mathbb{B}(R)$  with connected boundary  $\partial\Sigma \subset \mathbb{S}^2(R)$ , assuming a quadratic estimate of its Gaussian curvature and a concentration of the genus of  $\Sigma$  in a smaller ball; see Theorem 0.5 and Corollary 0.7 in [7].

An important application of Theorem 1.1 to the classical theory of minimal surfaces is to characterize complete embedded minimal surfaces with quadratic curvature decay. For the statement of the next theorem, we first recall that a complete Riemannian surface  $M$  has *intrinsic quadratic curvature decay constant*  $C > 0$  with respect to a point  $p \in M$ , if the absolute Gaussian curvature function  $|K_M|$  of  $M$  satisfies

$$|K_M(q)| \leq \frac{C}{d_M(p, q)^2} \quad \text{for all } q \in M,$$

where  $d_M$  denotes the Riemannian distance function. Since the intrinsic distance  $d_M$  dominates the ambient extrinsic distance in  $\mathbb{R}^3$ , we deduce that if a complete Riemannian surface  $M$  in  $\mathbb{R}^3$  with  $p = \vec{0} \in M$  has intrinsic quadratic curvature decay constant  $C$  with respect to  $\vec{0}$ , then it also has extrinsic quadratic decay constant  $C$  with respect to the radial distance  $R$  to  $\vec{0}$ , in the sense that  $|K_M|R^2 \leq C$  on  $M$ . For this reason, when we say that a minimal surface in  $\mathbb{R}^3$  has *quadratic*

*decay of curvature*, we will always refer to curvature decay with respect to the extrinsic distance  $R$  to  $\vec{0}$ , independently of whether or not  $M$  passes through  $\vec{0}$ . Note that the property of having quadratic decay of curvature is scale-invariant, a fact that will be crucial throughout this paper.

**Theorem 1.2** (Quadratic Curvature Decay Theorem). *A complete, embedded minimal surface in  $\mathbb{R}^3$  with compact boundary (possibly empty) has quadratic decay of curvature if and only if it has finite total curvature. In particular, a complete, connected embedded minimal surface  $M \subset \mathbb{R}^3$  with compact boundary and quadratic decay of curvature is properly embedded in  $\mathbb{R}^3$ . Furthermore, if  $C$  is the maximum of the logarithmic growths of the ends of  $M$ , then*

$$\lim_{R \rightarrow \infty} \sup_{M - \mathbb{B}(R)} |K_M| R^4 = C^2.$$

The Local Removable Singularity Theorem in this paper will be applied in forthcoming papers to obtain the following results.

- 1) A dynamics type result for the space of all limits of a given non-flat, properly embedded minimal surface in  $\mathbb{R}^3$  under divergent sequences of dilations in [22].
- 2) A blow-up technique on the scale of non-trivial topology for describing the local structure of a complete embedded minimal surface with injectivity radius zero in a homogeneously regular Riemannian three-manifold in [24].
- 3) Global structure theorems for certain possibly singular minimal laminations of  $\mathbb{R}^3$  in [25].
- 4) Bounds for the number of ends and for the index of stability of all complete, embedded minimal surfaces in  $\mathbb{R}^3$  with finite topology, more than one end and having fixed genus in [20].
- 5) Calabi-Yau type results. For example, a complete embedded minimal surface in  $\mathbb{R}^3$  of finite genus is properly embedded if and only if it has a countable number of ends, see [23]; this result generalizes a theorem by Colding and Minicozzi [9], who proved the sufficient implication in the case that the number of ends is finite.
- 6) We will extend in [21] the Local Removable Singularity Theorem from the minimal case (that is the leaves of  $\mathcal{L}$  in Theorem 1.1 are minimal surfaces) to the case of an  $H$ -lamination (all the leaves of  $\mathcal{L}$  have the same constant mean curvature  $H \in \mathbb{R}$ ). In this generalization, we will even allow the leaves of  $\mathcal{L}$  to intersect tangentially. This extended Local Removable Singularity Theorem is the key tool for the classification of all CMC (constant mean curvature) foliations of  $\mathbb{R}^3$  and  $\mathbb{S}^3$  with a closed countable number of singularities, where by a CMC foliation we mean that the leaves of the foliation have constant mean curvature, possibly varying from

leaf to leaf. We point out that the statement of this classification was announced in Theorem 6.8 of [27]; there we only gave a proof in the particular case of  $N = \mathbb{R}^3$  and  $\mathcal{S}$  is finite, and referred the reader to an earlier version of the present manuscript for the proof in the general case. Nevertheless, for the sake of simplicity we will only deal here with minimal surfaces and laminations, and postpone both the generalization of Theorem 1.1 to the case of  $H$ -laminations and the proof of the classification of CMC foliations of  $\mathbb{R}^3 - \mathcal{S}$  and  $\mathbb{S}^3 - \mathcal{S}$ ,  $\mathcal{S}$  being a closed countable set, to the paper [21].

The paper is organized as follows. In Section 2 we present the basic definitions and examples of minimal laminations in  $\mathbb{R}^3$  and in Riemannian three-manifolds. Section 3 is devoted to prove a key lemma on stable minimal surfaces which are complete outside a point in  $\mathbb{R}^3$ . In Section 4 we apply results by Colding and Minicozzi [8] to demonstrate a key technical result about minimal laminations of  $\mathbb{R}^3 - \{\bar{0}\}$  with quadratic decay of curvature, which will be used in Section 5 when proving the Local Removable Singularity Theorem. In Section 6 we apply the Local Removable Singularity Theorem to characterize complete embedded minimal surfaces with quadratic decay of curvature in  $\mathbb{R}^3$  as being surfaces with finite total curvature. We will give some applications to the case of minimal surfaces or minimal laminations with countably many singularities in Section 7. We finish the paper with an appendix that contains a second proof of the main technical result in Section 4, which does not depend on the results of Colding and Minicozzi in [8].

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## 2. Basic definitions and some examples.

**Definition 2.1.** Let  $M$  be a complete, embedded surface in a three-manifold  $N$ . A point  $p \in N$  is a *limit point* of  $M$  if there exists a sequence  $\{p_n\}_n \subset M$  which diverges to infinity in  $M$  with respect to the intrinsic Riemannian topology on  $M$  but converges in  $N$  to  $p$  as  $n \rightarrow \infty$ . Let  $\lim(M)$  denote the set of all limit points of  $M$  in  $N$ ; we

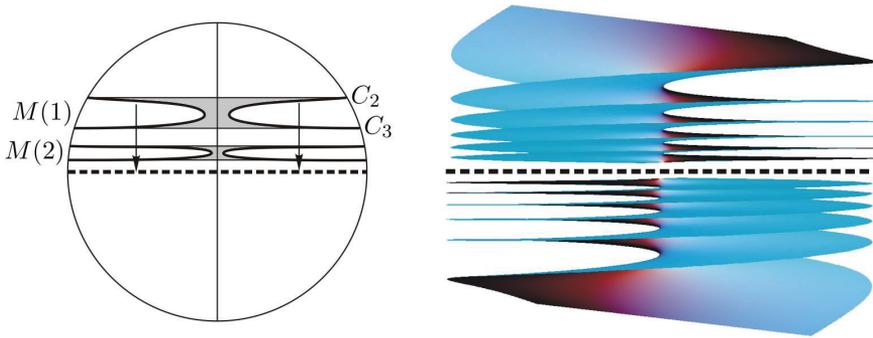
call this set the *limit set of  $M$* . In particular,  $\text{lim}(M)$  is a closed subset of  $N$  and  $\overline{M} - M \subset \text{lim}(M)$ , where  $\overline{M}$  denotes the closure of  $M$ .

**Definition 2.2.** A *codimension one lamination* of a Riemannian three-manifold  $N$  is the union of a collection of pairwise disjoint, connected, injectively immersed surfaces, with a certain local product structure. More precisely, it is a pair  $(\mathcal{L}, \mathcal{A})$  satisfying:

- 1)  $\mathcal{L}$  is a closed subset of  $N$ ;
- 2)  $\mathcal{A} = \{\varphi_\beta: \mathbb{D} \times (0, 1) \rightarrow U_\beta\}_\beta$  is an atlas of coordinate charts of  $N$  (here  $\mathbb{D}$  is the open unit disk in  $\mathbb{R}^2$ ,  $(0, 1)$  is the open unit interval and  $U_\beta$  is an open subset of  $N$ ); note that although  $N$  is assumed to be smooth, we only require that the regularity of the atlas (i.e. that of its change of coordinates) is of class  $C^0$ , i.e.  $\mathcal{A}$  is an atlas with respect to the topological structure of  $N$ .
- 3) For each  $\beta$ , there exists a closed subset  $C_\beta$  of  $(0, 1)$  such that  $\varphi_\beta^{-1}(U_\beta \cap \mathcal{L}) = \mathbb{D} \times C_\beta$ .

We will simply denote laminations by  $\mathcal{L}$ , omitting the charts  $\varphi_\beta$  in  $\mathcal{A}$ . A lamination  $\mathcal{L}$  is said to be a *foliation of  $N$*  if  $\mathcal{L} = N$ . Every lamination  $\mathcal{L}$  naturally decomposes into a collection of disjoint connected topological surfaces (locally given by  $\varphi_\beta(\mathbb{D} \times \{t\})$ ,  $t \in C_\beta$ , with the notation above), called the *leaves* of  $\mathcal{L}$ . Note that if  $\Delta \subset \mathcal{L}$  is any collection of leaves of  $\mathcal{L}$ , then the closure of the union of these leaves has the structure of a lamination within  $\mathcal{L}$ , which we will call a *sublamination*. The notion of limit point of a complete embedded surface (Definition 2.1) can be extended to the case of a lamination  $\mathcal{L}$  of a three-manifold  $N$  as follows. A point  $p \in \mathcal{L}$  is a *limit point* if there exists a coordinate chart  $\varphi_\beta: \mathbb{D} \times (0, 1) \rightarrow U_\beta$  as in Definition 2.2 such that  $p \in U_\beta$  and  $\varphi_\beta^{-1}(p) = (x, t)$  with  $t$  belonging to the accumulation set of  $C_\beta$ . It is easy to show that if  $p$  is a limit point of a lamination  $\mathcal{L}$ , then the leaf  $L$  of  $\mathcal{L}$  passing through  $p$  consists entirely of limit points of  $\mathcal{L}$ ; see Footnote 4. In this case,  $L$  is called a *limit leaf* of  $\mathcal{L}$ .

A lamination  $\mathcal{L}$  of  $N$  is said to be a *minimal lamination* if each of its leaves is a smooth surface with zero mean curvature. In this case, the function  $|\sigma_{\mathcal{L}}|$  that associates to each point  $p$  of  $\mathcal{L}$  the norm of the second fundamental form of the unique leaf of  $\mathcal{L}$  passing through  $p$ , makes sense on  $\mathcal{L}$ . A natural question to ask is whether or not the function  $|\sigma_{\mathcal{L}}|$  is locally bounded for any minimal lamination  $\mathcal{L}$  in a Riemannian three-manifold  $N$ . Concerning this question, we observe that the 1-sided curvature estimates for minimal disks by Colding and Minicozzi [8, 9] imply that  $|\sigma_{\mathcal{L}}|$  is locally bounded (to prove this, one only has to deal with limit leaves, where the 1-sided curvature estimates apply). Another important observation is that given a sequence of minimal laminations



**Figure 1.** Left: A catenoid type lamination. Right: A Colding-Minicozzi type lamination in a cylinder.

$\mathcal{L}_n$  of  $N$  with uniformly bounded second fundamental forms on compact subdomains of  $N$ , a subsequence of the  $\mathcal{L}_n$  converges to a minimal lamination of  $N$ ; see Proposition B1 in [8].

**2.1. Minimal laminations with isolated singularities.** We first construct examples in the closed unit ball of  $\mathbb{R}^3$  centered the origin, with the origin as the unique non-removable singularity. We then show how these examples lead to related singular minimal laminations in the hyperbolic space  $\mathbb{H}^3$ .

**EXAMPLE I: Catenoid type laminations.** Consider the sequence of horizontal circles  $C_n = \mathbb{S}^2(1) \cap \{x_3 = \frac{1}{n}\}$ ,  $n \geq 2$ . Note that each pair  $C_{2k}, C_{2k+1}$  bounds a compact unstable catenoid  $M(k) \subset \bar{\mathbb{B}}(1)$ . Clearly,  $M(k) \cap M(k') = \emptyset$  if  $k \neq k'$ . The sequence  $\{M(k)\}_k$  converges with multiplicity two outside of the origin  $\vec{0}$  to the closed horizontal disk  $\bar{\mathbb{D}}$  of radius 1 centered at  $\vec{0}$ . Thus,  $\{M(k)\}_k \cup \{\bar{\mathbb{D}} - \{\vec{0}\}\}$  is a minimal lamination of  $\bar{\mathbb{B}}(1) - \{\vec{0}\}$  which does not extend through the origin; see Figure 1 left.

**EXAMPLE II: Colding-Minicozzi examples.** In their paper [6], Colding and Minicozzi constructed a sequence of compact, embedded minimal disks  $D_n \subset \bar{\mathbb{B}}(1)$  with boundaries in  $\mathbb{S}^2(1)$ , that converges to a singular minimal lamination  $\bar{\mathcal{L}}$  of  $\bar{\mathbb{B}}(1)$  with an isolated singularity at  $\vec{0}$ . The related lamination  $\mathcal{L}$  of  $\bar{\mathbb{B}}(1) - \{\vec{0}\}$  consists of a unique limit leaf which is the punctured closed disk  $\bar{\mathbb{D}} - \{\vec{0}\}$ , together with two non-proper leaves that spiral into  $\bar{\mathbb{D}} - \{\vec{0}\}$  from opposite sides; see Figure 1 right.

Consider the exhaustion of  $\mathbb{H}^3$  (identified with  $\mathbb{B}(1)$  through the Poincaré model) by hyperbolic geodesic balls of hyperbolic radius  $n \in \mathbb{N}$  centered at the origin, together with compact minimal disks with boundaries on the boundaries of these balls, similar to the compact Colding-Minicozzi disks. We conjecture that these

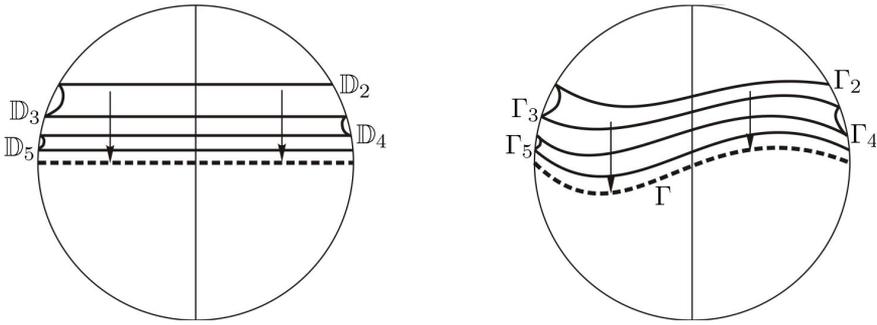
examples produce a similar limit lamination of  $\mathbb{H}^3 - \{\vec{0}\}$  with three leaves, one which is totally geodesic and the other two which are not proper and that spiral into the first one. We remark that one of the main results of Colding-Minicozzi theory (Theorem 0.1 in [8]) insures that such an example cannot be constructed in  $\mathbb{R}^3$ .

EXAMPLE III: *Catenoid type examples in  $\mathbb{H}^3$  and in  $\mathbb{H}^2 \times \mathbb{R}$ .* As in example I, consider the circles  $C_n = \mathbb{S}^n(1) \cap \{x_3 = \frac{1}{n}\}$ , where  $\mathbb{S}^2(1)$  is now viewed as the boundary at infinity of  $\mathbb{H}^3$ . Then each pair of circles  $C_{2k}, C_{2k+1}$  is the asymptotic boundary of a properly embedded annular minimal unstable surface  $M(k)$ , which is a surface of revolution called a catenoid (see e.g., Lemma 3.5 in Gomes [14]). The sequence  $\{M(k)\}_k$  converges with multiplicity two outside of  $\vec{0}$  to the horizontal totally geodesic subspace  $\mathbb{D}$  at height zero. Thus,  $\{M(k)\}_k \cup \{\mathbb{D} - \{\vec{0}\}\}$  is a minimal lamination of  $\mathbb{H}^3 - \{\vec{0}\}$ , which does not extend through the origin. A similar catenoidal construction can be done in  $\mathbb{H}^2 \times \mathbb{R}$ , where we consider  $\mathbb{H}^2$  in the disk model of the hyperbolic plane, using the minimal catenoids constructed in Theorem 1 of [32]. Note that the Half-space Theorem [19] excludes this type of singular minimal lamination in  $\mathbb{R}^3$ .

**2.2. Minimal laminations with limit leaves.**

EXAMPLE IV: *Simply-connected bridged examples.* Coming back to the Euclidean closed unit ball  $\mathbb{B}(1)$ , consider the sequence of horizontal closed disks  $\mathbb{D}_n = \mathbb{B}(1) \cap \{x_3 = \frac{1}{n}\}$ ,  $n \geq 2$ . Connect each pair  $\mathbb{D}_n, \mathbb{D}_{n+1}$  by a thin, almost vertical minimal bridge (in opposite sides for consecutive disks, as in Figure 2 left), and perturb slightly this non-minimal surface to obtain an embedded, stable minimal surface with boundary in  $\mathbb{B}(1)$  (this is possible by the *bridge principle* [31, 37, 38]). We denote by  $M$  the intersection of this surface with  $\mathbb{B}(1)$ . Then, the closure of  $M$  in  $\mathbb{B}(1)$  is a minimal lamination of  $\mathbb{B}(1)$  with two leaves, both being stable, one of which is  $\mathbb{D}$  (this is a limit leaf) and the other one is not flat and not proper.

A similar example with a non-flat limit leaf can be constructed by exchanging the horizontal circles by suitable curves in  $\mathbb{S}^2(1)$ . Consider a non-planar smooth Jordan curve  $\Gamma \subset \mathbb{S}^2(1)$  which admits a one-to-one projection onto a convex planar curve in a plane  $\Pi$ . Let  $\Gamma_n$  be a sequence of smooth Jordan curves in  $\mathbb{S}^2(1)$  converging to  $\Gamma$ , so that each  $\Gamma_n$  also projects injectively onto a convex planar curve in  $\Pi$  and  $\{\Gamma_n\}_n \cup \{\Gamma\}$  is a lamination on  $\mathbb{S}^2(1)$ . An elementary application of the maximum principle implies that each of the  $\Gamma_n$  is the boundary of a unique compact minimal surface  $\overline{M_n}$ , which is a graph over its projection to  $\Pi$ . Now join slight



**Figure 2.** Left: Almost flat minimal disks joined by small bridges. Right: A similar example with a non-flat limit leaf.

perturbations of the  $\overline{M}_n$  by thin bridges as in the preceding paragraph, to obtain a simply-connected minimal surface in the closed unit ball. Let  $M$  be the intersection of this surface with  $\mathbb{B}(1)$ . Then, the closure of  $M$  in  $\mathbb{B}(1)$  is a minimal lamination of  $\mathbb{B}(1)$  with two leaves, both being non-flat and stable, and exactly one of them is properly embedded in  $\mathbb{B}(1)$  and is a limit leaf; see Figure 2 right.

EXAMPLE V: *Simply-connected bridged examples in  $\mathbb{H}^3$ .* As in the previous subsection, the minimal laminations in example IV give rise to minimal laminations of  $\mathbb{H}^3$  consisting of two stable, complete, simply connected minimal surfaces, one of which is proper and the other one which is not proper in the space, and either one is not totally geodesic or both of them are not totally geodesic, depending on the choice of the Euclidean model surface in Figure 2 (for this existence, one can use Anderson [1] to create the corresponding minimal disks  $\overline{M}_n$  as in Example IV, and then use the bridge principle at infinity as described in Coskunuzer [10]). In this case, the proper leaf is the unique limit leaf of the minimal lamination. More generally, Theorem 13 in [30] implies that the closure of any complete, embedded minimal surface of finite topology in  $\mathbb{H}^3$  has the structure of a minimal lamination.

**3. Stable minimal surfaces which are complete outside of a point.**

**Definition 3.1.** A surface  $M \subset \mathbb{R}^3 - \{\vec{0}\}$  is *complete outside the origin*, if every divergent path in  $M$  of finite length has as limit point the origin.

If  $M$  is a complete, stable, orientable minimal surface in  $\mathbb{R}^3$ , then  $M$  is a plane [11, 13, 33]. Our goal in this section is to extend this result to the case where  $M$  is complete outside the origin.

- Remark 3.2.** 1) In Sections 4, 5 and 6 we shall study complete, embedded minimal surfaces  $M \subset \mathbb{R}^3$  with quadratic decay of curvature. Our approach is to produce from  $M$ , via a sequence of homothetic rescalings, a minimal lamination  $\mathcal{L}$  of  $\mathbb{R}^3 - \{\vec{0}\}$  with a limit leaf  $L$ . Since  $L$  is a leaf of a minimal lamination of  $\mathbb{R}^3 - \{\vec{0}\}$ , then  $L$  is complete outside the origin. After possibly passing to its orientable two-sheeted cover and applying the main theorem of [28], we can assume that  $L$  is stable, orientable and complete outside the origin. The following lemma will then be used to show that the closure of  $L$  is a plane. This planar leaf  $L$  will play a key role in proving that  $M$  must have finite total curvature.
- 2) The line of arguments in item 1 of this remark is inspired by ideas in our previous paper [26], where we proved that a properly embedded minimal surface of finite genus in  $\mathbb{R}^3$  cannot have one limit end. A key lemma in the proof of this result states that if such a surface  $M$  exists, then some sequence of homothetic shrinkings of  $M$  converges to a minimal lamination of  $\mathbb{R}^3 - \{\vec{0}\}$ . Furthermore, this lamination is contained in a closed half-space and contains a limit leaf  $L$ , which is different from the boundary of the half-space. Since  $L$  is a leaf of a minimal lamination of  $\mathbb{R}^3 - \{\vec{0}\}$ , then it is complete outside  $\vec{0}$  and as it is a limit leaf, it is stable. We then proved that the closure  $\bar{L}$  of  $L$  must be a plane. Using the plane  $\bar{L}$  as a guide for understanding the lamination, we obtained a contradiction.
- 3) The minimal case of Lemma 3.3 below was found independently by Colding and Minicozzi [4].

**Lemma 3.3** (Stability Lemma). *Let  $L \subset \mathbb{R}^3 - \{\vec{0}\}$  be a stable, immersed constant mean curvature (orientable if minimal) surface, which is complete outside the origin. Then,  $\bar{L}$  is a plane.*

*Proof.* We will present a detailed proof in the minimal case, since this is the version needed in this paper; see Lemma 6.4 of [27] for a proof in the general case of constant mean curvature. If  $\vec{0} \notin \bar{L}$ , then  $L$  is complete and so, it is a plane. Assume now that  $\vec{0} \in \bar{L}$ . Consider the metric  $\tilde{g} = \frac{1}{R^2}g$  on  $L$ , where  $g$  is the metric on  $L$  induced by the usual inner product  $\langle, \rangle$  of  $\mathbb{R}^3$ . Note that if  $L$  were a plane through  $\vec{0}$ , then  $\tilde{g}$  would be the metric on  $L$  of an infinite cylinder of radius 1 with ends at  $\vec{0}$  and at infinity. Since  $(\mathbb{R}^3 - \{\vec{0}\}, \hat{g})$  with  $\hat{g} = \frac{1}{R^2}\langle, \rangle$ , is isometric to  $\mathbb{S}^2(1) \times \mathbb{R}$ , then  $(L, \tilde{g}) \subset (\mathbb{R}^3 - \{\vec{0}\}, \hat{g})$  is complete.

We next show how the assumption of stability can be used to prove that  $(L, g)$  is flat. The laplacians and Gauss curvatures of  $g, \tilde{g}$  are related

by the equations  $\tilde{\Delta} = R^2\Delta$  and  $\tilde{K} = R^2(K_L + \Delta \log R)$ . Since  $\Delta \log R = \frac{2(1-\|\nabla R\|^2)}{R^2} \geq 0$ , then

$$-\tilde{\Delta} + \tilde{K} = R^2(-\Delta + K_L + \Delta \log R) \geq R^2(-\Delta + K_L).$$

Since  $K_L \leq 0$  and  $(L, g)$  is stable,  $-\Delta + K_L \geq -\Delta + 2K_L \geq 0$ , and so,  $-\tilde{\Delta} + \tilde{K} \geq 0$  on  $(L, \tilde{g})$ . As  $\tilde{g}$  is complete, the universal covering of  $L$  is conformally  $\mathbb{C}$  (Fischer-Colbrie and Schoen [13]). Since  $(L, g)$  is stable, there exists a positive Jacobi function  $u$  on  $L$ . Passing to the universal covering  $\hat{L}$ , we have  $\Delta \hat{u} = 2K_{\hat{L}}\hat{u} \leq 0$ . Thus  $\hat{u}$  is a positive superharmonic on  $\mathbb{C}$ , and hence constant. Therefore,  $0 = \Delta u - 2K_L u = -2K_L u$  on  $L$ , which means that  $K_L = 0$ . q.e.d.

We will need the following two corollaries. The first one follows directly from Lemma 3.3 and the fact that the two-sided cover of every limit leaf of a minimal lamination is stable, see [28]. For the second corollary, we refer the reader to Lemma 4.2 in [22] which implies that (with the notation of Corollary 3.5 below) the two-sided cover  $\hat{L}$  of  $L$  is stable; hence, Lemma 3.3 applies to give that  $\hat{L}$  (and so  $L$ ) is flat.

**Corollary 3.4.** *If  $L$  is a limit leaf of a minimal lamination of  $\mathbb{R}^3 - \{\vec{0}\}$ , then  $\bar{L}$  is a plane.*

**Corollary 3.5.** *Let  $\mathcal{L}$  be a minimal lamination of  $\mathbb{R}^3$  (resp. of  $\mathbb{R}^3 - \{0\}$ ) which is a limit of embedded minimal surfaces  $M_n$  with uniformly bounded second fundamental form on compact sets in  $\mathbb{R}^3$  (resp. of  $\mathbb{R}^3 - \{0\}$ ). Let  $L$  be a leaf of  $\mathcal{L}$  which is not a limit leaf of  $\mathcal{L}$ , such that the multiplicity of the limit  $\{M_n\}_n \rightarrow L$  is greater than one. Then,  $L$  (resp.  $\bar{L}$ ) is a plane.*

#### 4. Minimal laminations with quadratic decay of curvature.

In this section we will obtain a preliminary description of any non-flat minimal lamination  $\mathcal{L}$  of  $\mathbb{R}^3 - \{\vec{0}\}$  with quadratic decay of curvature, see Definition 4.2 below. We first consider the simpler case where  $\mathcal{L}$  consists of a properly embedded minimal surface in  $\mathbb{R}^3$ . When the decay constant for its curvature is small, then the topology and geometry of the surface is simple, as shown in the next lemma.

**Lemma 4.1.** *There exists  $C \in (0, 1)$  such that if  $M \subset \mathbb{R}^3 - \mathbb{B}(1)$  is a properly embedded, connected minimal surface with non-empty boundary  $\partial M \subset \mathbb{S}^2(1)$  and  $|K_M|R^2 \leq C$  on  $M$ , then  $M$  is an annulus which has a planar or catenoidal end.*

*Proof.* First let  $C$  be any positive number less than 1. Let  $f = R^2$  on  $M$ . Its critical points occur at those  $p \in M$  where  $M$  is tangent to  $\mathbb{S}^2(|p|)$ . The hessian  $\nabla^2 f$  at a critical point  $p$  is  $(\nabla^2 f)_p(v, v) = 2(|v|^2 - \sigma_p(v, v)\langle p, \vec{n} \rangle)$ ,  $v \in T_p M$ , where  $\sigma$  is the second fundamental

form of  $M$  and  $\vec{n}$  its Gauss map. Taking  $|v| = 1$ , we have  $\sigma_p(v, v) \leq |\sigma_p(e_i, e_i)| = \sqrt{|K_M|}(p)$ , where  $e_1, e_2$  is an orthonormal basis of principal directions at  $p$ . Since  $\langle p, \vec{n} \rangle \leq |p|$ , we have

$$(1) \quad (\nabla^2 f)_p(v, v) \geq 2 \left[ 1 - (|K_M|R^2)^{1/2} \right] \geq 2 \left( 1 - \sqrt{C} \right) > 0.$$

Hence, all critical points of  $f$  in the interior of  $M$  are non-degenerate local minima on  $M$ , and if  $p \in \text{Int}(M)$  is a local minimum of  $f$ , then  $M$  lies outside  $\mathbb{B}(|p|)$  locally around  $p$ , touching  $\mathbb{S}^2(|p|)$  only at  $p$ . Suppose  $f$  admits an interior critical point  $p$ . Since  $M$  is connected, we can choose a regular value  $R_1 > 1$  of  $f$  large enough so that  $p$  lies in the same component  $M_1$  of  $M \cap \overline{\mathbb{B}}(R_1)$  as  $\partial M$ . Since  $p$  is a non-degenerate local minimum and  $f$  has only non-degenerate critical points, then  $f$  is a Morse function on  $M$  and Morse theory implies that  $f|_{M_1}$  must have an index-one critical point, which is impossible. Therefore,  $f$  has no local minima on  $M$  except along  $\partial M$  where it attains its global minimum value. Hence,  $M$  intersects every sphere  $\mathbb{S}^2(r)$ ,  $r \geq 1$ , transversely in a connected simple closed curve, which implies that  $M$  is an annulus.

If  $M$  has finite total curvature, then it must be asymptotic to an end of a plane or of a catenoid, thus either the lemma is proved or  $M$  has infinite total curvature.

A general technique which we will use in later sections to obtain compactness of sequences of minimal surfaces is the following (see e.g., Meeks and Rosenberg [29]): If  $\{M_n\}_n$  is a sequence of minimal surfaces properly embedded in an open set  $B \subset \mathbb{R}^3$ , with their curvature functions  $K_{M_n}$  uniformly bounded on compact subsets of  $B$ , then a subsequence converges uniformly on compact subsets of  $B$  to a minimal lamination of  $B$  with leaves that have the same bound on the curvature as the surfaces  $M_n$ .

Suppose that the lemma fails. In this case, there exists a sequence of positive numbers  $C_n \rightarrow 0$  and minimal annuli  $M_n$  satisfying the conditions of the lemma, such that  $M_n$  has infinite total curvature and  $|K_{M_n}|R^2 \leq C_n$ . Since the  $M_n$  are annuli with infinite total curvature, the Gauss-Bonnet formula implies that there exists a sequence of numbers  $R_n \rightarrow \infty$  such that the total geodesic curvature of the outer boundary of  $M_n \cap \mathbb{B}(R_n)$  is greater than  $n$ . After extracting a subsequence, the surfaces  $\widetilde{M}_n = \frac{1}{R_n}M_n$  converge to a minimal lamination  $\mathcal{L}$  of  $\mathbb{R}^3 - \{\vec{0}\}$  that extends across  $\vec{0}$  to a lamination of  $\mathbb{R}^3$  by parallel planes (since  $|K_{\widetilde{M}_n}|R^2 \leq C_n$  and  $C_n \rightarrow 0$  as  $n \rightarrow \infty$ ). Furthermore,  $\mathcal{L}$  contains a plane  $\Pi$  passing through  $\vec{0}$ . Consider the great circle  $\Gamma = \Pi \cap \mathbb{S}^2(1)$  and let  $\Gamma(\varepsilon)$  be the  $\varepsilon$ -neighborhood of  $\Gamma$  in  $\mathbb{S}^2(1)$ , for a small number  $\varepsilon > 0$ . Each  $\widetilde{M}_n$  transversely intersects  $\mathbb{S}^2(1)$  in a simple closed curve  $\alpha_n$  and the Gauss map of  $\widetilde{M}_n$  along  $\alpha_n$  is almost constant and parallel to the unit normal vector to  $\Pi$ . Clearly, for  $n$  sufficiently large,

either  $\widetilde{M}_n \cap \Gamma(\varepsilon)$  contains long spiraling curves that join points in the two components of  $\partial\Gamma(\varepsilon)$  or it consists of a single closed curve which is  $C^2$ -close to  $\Gamma$ . This last case contradicts the assumption that the total geodesic curvature of  $M_n \cap \mathbb{S}^2(R_n)$  is unbounded. Hence, we must have spiraling curves in  $\widetilde{M}_n \cap \Gamma(\varepsilon)$ . In this case, there are planes  $\Pi_+, \Pi_-$  in  $\mathcal{L}$ , parallel to  $\Pi$ , such that  $\partial\Gamma(\varepsilon) = (\Pi_+ \cup \Pi_-) \cap \mathbb{S}^2(1)$ . In a small neighborhood  $U$  of  $(\Pi_+ \cup \Pi_-) \cap \mathbb{B}(2)$  which is disjoint from  $\Pi$ , the surfaces  $\widetilde{M}_n \cap U$  converge smoothly to  $\mathcal{L} \cap U$ . Since  $(\Pi_+ \cup \Pi_-) \cap \mathbb{B}(2)$  is simply connected, then a standard monodromy lifting argument implies that for  $n$  large,  $\widetilde{M}_n \cap \overline{\mathbb{B}}(1)$  contains two compact disks in  $U$  which are close to  $(\Pi_+ \cup \Pi_-) \cap \overline{\mathbb{B}}(1)$ . This contradicts the fact that each  $\widetilde{M}_n$  intersects  $\mathbb{S}^2(1)$  transversely in just one simple closed curve (see the first paragraph of this proof). This contradiction completes the proof of the lemma. q.e.d.

**Definition 4.2.** We denote by  $K_{\mathcal{L}}: \mathcal{L} \rightarrow \mathbb{R}$  the Gaussian curvature function of a lamination  $\mathcal{L}$ . A lamination  $\mathcal{L}$  of  $\mathbb{R}^3 - \{\vec{0}\}$  is said to have *quadratic decay of curvature* if  $|K_{\mathcal{L}}|R^2 \leq C$  on  $\mathcal{L}$  for some  $C > 0$ .

Our next goal is to show properness in  $\mathbb{R}^3 - \{\vec{0}\}$  for every leaf of a non-flat minimal lamination  $\mathcal{L}$  of  $\mathbb{R}^3 - \{\vec{0}\}$  with quadratic decay of curvature (Proposition 4.11 below). The proof of this property is a delicate technical argument, which we will break into separate statements.

**Lemma 4.3.** *Let  $\mathcal{L}$  be a non-flat minimal lamination of  $\mathbb{R}^3 - \{\vec{0}\}$  with quadratic decay of curvature. Suppose that  $L$  is a leaf of  $\mathcal{L}$  which is not proper in  $\mathbb{R}^3 - \{\vec{0}\}$ . Then after a rotation in  $\mathbb{R}^3$ ,  $L$  is contained in  $H^+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\}$ , and*

$$\lim(L) = \left\{ \text{limit points of } L \text{ in } \mathbb{R}^3 - \{\vec{0}\} \right\} = \{x_3 = 0\} - \{\vec{0}\}.$$

*In particular,  $L$  is proper in  $H^+$ .*

*Proof.* As  $L$  is not proper in  $\mathbb{R}^3 - \{\vec{0}\}$ , then  $L$  is not flat and there exists  $p \in \lim(L) \subset \mathbb{R}^3 - \{\vec{0}\}$ . Let  $L'$  be the leaf of  $\mathcal{L}$  that contains  $p$ . Since  $L' \cap \lim(L)$  is closed and open (Openness follows since a neighborhood of every point of  $L' \cap \lim(L)$  can be written as the limit of a sequence of graphs, all of them contained in  $L$ . Note that the same argument shows that if  $p$  is a limit point of a leaf  $L$  of a lamination, then the leaf passing through  $p$  consists entirely of limit points of  $L$ .) in  $L'$ , then  $L' \subset \lim(L)$ . In particular, by Corollary 3.4,  $L'$  is either a plane or a plane punctured at the origin, and  $L$  is contained in one of the half-spaces determined by  $L'$ . If  $L'$  does not pass through  $\vec{0}$ , then there exists  $\varepsilon > 0$  such that the  $\varepsilon$ -neighborhood  $L'(\varepsilon)$  of  $L'$  is at positive distance from  $\vec{0}$ . Since  $|K_L|R^2 \leq C$  for some  $C > 0$ , then  $L \cap L'(\varepsilon)$  has bounded curvature, which is impossible by the statement and proof of Lemma

1.3 in [29]; for the sake of completeness we now sketch this argument. Taking  $\varepsilon$  small, each component  $\Omega$  of  $L \cap L'(\varepsilon)$  is a multigraph over its orthogonal projection to  $L'$ . Actually  $\Omega$  is a graph over its projection on  $L'$  by a separation argument. Thus,  $\Omega$  is proper in  $L'(\varepsilon)$ , and the proof of the Half-space Theorem [19] gives a contradiction. Hence, the plane  $\overline{L'}$  passes through  $\vec{0}$ . This argument also shows that  $L'$  equals  $\lim(L)$  (otherwise we obtain a second punctured plane  $L'' \subset \lim(L)$  which is a leaf of  $\mathcal{L}$  and which passes through  $\vec{0}$ , which contradicts that  $L' \cap L''$  is empty). Now the lemma is proved. q.e.d.

Given  $\delta > 0$ , let  $C_\delta = \{(x_1, x_2, x_3) \mid x_3^2 = \delta^2(x_1^2 + x_2^2)\} \cap H^+$  (positive half-cone) and  $C_\delta^-$  the region of  $H^+$  below  $C_\delta$ .

**Lemma 4.4.** *Let  $L \subset H^+$  be a connected minimal surface which is complete outside the origin and whose Gaussian curvature  $K_L$  satisfies  $|K_L|R^2 \leq C$  for some  $C > 0$ . Then for any  $\varepsilon > 0$  small, there exists a  $\delta > 0$  such that in  $L \cap C_\delta^-$ , the inequality  $|\nabla_L x_3| \leq \varepsilon \frac{x_3}{R}$  holds.*

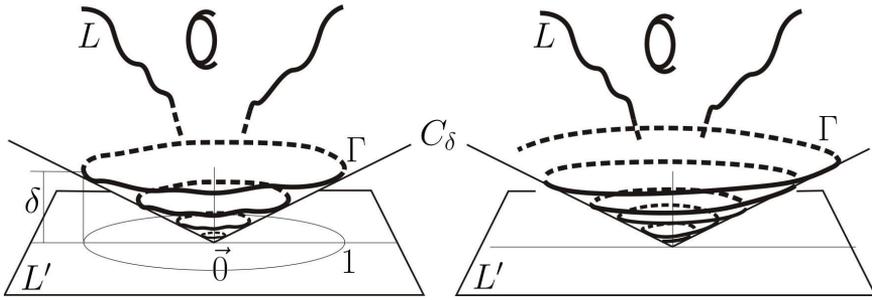
*Proof.* A consequence of  $|K_L|R^2 \leq C$  is that for all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for all  $p \in L \cap C_\delta^-$ , then the angle that the tangent space to  $L$  at  $p$  makes with the horizontal is less than  $\varepsilon$ .

Consider the conformal change of metric  $\tilde{g} = \frac{1}{R^2}g$  that we used in the proof of Lemma 3.3, where  $g$  is the induced metric on  $L$  by the usual inner product of  $\mathbb{R}^3$ . Recall that  $\tilde{g}$  was proven to be complete on  $L$ . Then, we can apply Theorem 6 in Cheng-Yau [2] to the harmonic function  $x_3$  on  $(L, \tilde{g})$  to obtain

$$(2) \quad \tilde{g}(\tilde{\nabla}x_3, \tilde{\nabla}x_3)(x)^{1/2} \leq \alpha x_3(x) \left( |\tilde{K}_\infty| + \frac{1}{a} \right),$$

where  $\tilde{\nabla}x_3$  is the gradient of  $x_3$  with respect to  $\tilde{g}$ ,  $x \in L \cap C_\delta^-$ ,  $\alpha > 0$  is a universal constant and  $\tilde{K}_\infty \in \mathbb{R}$  is a lower bound for the Gaussian curvature of the geodesic disc  $\tilde{D}_L(x, a)$  in  $(L, \tilde{g})$  centered at  $x$  with radius  $a > 0$ . As  $\tilde{g}$  is complete on  $L$ , we can take  $a \geq \frac{2\alpha}{\varepsilon}$ . For this value of  $a$ , we can choose  $\delta > 0$  small enough so that  $|\tilde{K}_\infty| < \frac{\varepsilon}{2\alpha}$  in  $\tilde{D}_L(x, a)$  (this follows since for fixed  $a > 0$  and  $\delta > 0$  arbitrarily small, the geodesic disks  $\tilde{D}_L(x, a)$  centered at  $x \in L \cap C_\delta^-$  can be considered to be domains in  $(L, g)$  which are locally approximated arbitrarily well in the  $C^2$ -norm as small  $x_3$ -graphs over a domain in the plane  $\{x_3 = 0\} - \{\vec{0}\}$ , and the restriction of  $\tilde{g}$  to  $\{x_3 = 0\} - \{\vec{0}\}$  is a flat metric). As  $\tilde{\nabla}x_3 = R^2 \nabla x_3$ , then (2) transforms into  $R|\nabla x_3| \leq \alpha x_3(x) \left( |\tilde{K}_\infty| + \frac{1}{a} \right) < \varepsilon x_3(x)$ , which proves the lemma. q.e.d.

Next we analyze the geometry of each component of  $L \cap C_\delta$  for  $\delta > 0$  sufficiently small, where  $L$  satisfies the hypotheses and conclusions of Lemma 4.3. Note that Lemma 4.4 applies in this setting and guarantees



**Figure 3.** Curves  $\Gamma \in \Lambda$  of type I (left) and one curve of type II (right).

that for  $\varepsilon > 0$  fixed and small, and  $\delta > 0$  sufficiently small,  $L$  intersects  $C_\delta$  transversely in a small angle that is uniformly bounded away from zero and each component of  $L \cap C_\delta$  is locally a radial graph over the circle  $C_\delta \cap \{x_3 = 1\}$ . Furthermore, in the natural polar coordinates in  $C_\delta$ , the radial lines intersect the collection of curves  $L \cap C_\delta$  almost orthogonally. Let  $\Lambda$  be the set of components of  $L \cap C_\delta$ . Then any  $\Gamma \in \Lambda$  is of one of the following two types, see Figure 3:

TYPE I.:  $\Gamma$  is a closed almost horizontal curve. In this case, any other  $\Gamma' \in \Lambda$  is also of type I, and there are an infinite number of these curves, converging to  $\{\vec{0}\}$ .

TYPE II.:  $\Gamma$  is a spiraling curve (with almost horizontal tangent vector) limiting down to  $\{\vec{0}\}$ .  $\Gamma$  rotates infinitely many times around  $C_\delta$ , and any other  $\Gamma' \in \Lambda$  is of type II. Note that in this case,  $\Lambda = \{\Gamma_1, \dots, \Gamma_k\}$  has a finite number of these spiraling components.

Observe that  $L \cap C_{\delta'}$  has the same pattern as  $\Lambda$ , for each  $\delta' \in (0, \delta)$ . **Our next goal is to find a contradiction in each of the two cases listed above.** These contradictions will lead to the conclusion that a leaf component  $L$  satisfying the hypotheses and conclusions of Lemma 4.3 cannot exist. This will be the content of Proposition 4.11 at the end of this section.

**4.1. Suppose the curves in  $\Lambda$  are of type I.** Let  $\Gamma \in \Lambda$ . Denote by  $E(\Gamma)$  the component of  $L \cap C_\delta^-$  whose boundary contains  $\Gamma$ . Lemma 4.4 implies that the third component of the unit normal vector to  $E(\Gamma)$  has a fixed non-zero sign on  $E(\Gamma)$ , and so,  $E(\Gamma)$  is locally a graph over its vertical projection to  $\{x_3 = 0\}$ . Since  $E(\Gamma)$  separates  $C_\delta^-$  (because  $E(\Gamma)$  is properly embedded in the closure of  $C_\delta^-$  and  $\Gamma \subset E(\Gamma)$  generates the first homology group of  $C_\delta^-(1)$ ), then  $E(\Gamma)$  is a global graph over its projection to  $\{x_3 = 0\}$ .

**Assertion 4.5.** *For  $\delta > 0$  sufficiently small, the boundary of  $E(\Gamma)$  equals  $\Gamma$  for each  $\Gamma \in \Lambda$ .*

*Proof.* Observe that otherwise there exists a point  $p = p(\delta) \in E(\Gamma) \cap C_\delta$  where the tangent plane  $T_p L$  is steeper than the tangent plane to  $C_\delta$ , i.e.  $\delta < |\nabla_L x_3|$ . Using again Lemma 4.4,

$$|\nabla_L x_3|(p) \leq \varepsilon \frac{x_3(p)}{R(p)} = \varepsilon \frac{\delta}{\sqrt{1 + \delta^2}},$$

where we have used that  $p \in C_\delta$  in the last equality. Therefore,  $\delta < \frac{\varepsilon \delta}{\sqrt{1 + \delta^2}}$ , or equivalently,  $\sqrt{1 + \delta^2} < \varepsilon$ , which is impossible for  $\varepsilon > 0$  small enough. This proves the assertion. q.e.d.

**Assertion 4.6.** *There exists a sequence of points  $\{q_n\}_n \subset L$  converging to  $\vec{0}$  such that for all  $n \in \mathbb{N}$ , we have  $(|K_L|R^2)(q_n) \geq 1$ .*

*Proof.* Reasoning by contradiction, suppose that there exists  $r > 0$  small such that  $|K_L|R^2 < 1$  in  $L \cap \overline{\mathbb{B}}(r)$ . By the arguments in the proof of Lemma 4.1,  $f = R^2$  is a Morse function with only local minima in  $L \cap \overline{\mathbb{B}}(r)$ , and so,  $L \cap \overline{\mathbb{B}}(r)$  consists of a non-empty family of compact disks and non-compact annuli with boundary on  $\mathbb{S}^2(r)$  and which are proper in  $\overline{\mathbb{B}}(r) - \{\vec{0}\}$ . Let  $\Omega$  be one of these components and suppose  $\Omega$  is an annulus. If  $\Omega$  is conformally  $\mathbb{D}^*$ , then  $\Omega$  extends smoothly across  $\vec{0}$ , which contradicts the maximum principle since  $L$  is contained in  $\{x_3 > 0\}$ . If  $\Omega$  is conformally  $\{\varepsilon < |z| \leq 1\}$  for some  $\varepsilon > 0$ , then each coordinate function of  $\Omega$  can be reflected in  $\{|z| = \varepsilon\}$  by Schwarz’s reflection principle, defining a branched conformal harmonic map on a larger annulus that maps the entire curve  $\{|z| = \varepsilon\}$  to a single point, which is impossible. This means that every component in  $L \cap \overline{\mathbb{B}}(r)$  is a compact disk. As the points of  $\mathbb{S}^1(r) \times \{0\}$  are limit points of  $L$ , we conclude that there exists a sequence of boundary curves  $\gamma_n$  of these disks components of  $L \cap \overline{\mathbb{B}}(r)$  that converges to  $\mathbb{S}^1(r) \times \{0\}$ , and such that for  $n$  large,  $\gamma_n$  is the boundary of an exterior non-compact minimal graph over its projection to  $\{x_3 = 0\}$ . This clearly contradicts that  $L$  is connected and finishes the proof of the assertion. q.e.d.

Since  $|K|R^2 \leq C$  on the minimal laminations  $\frac{1}{|q_n|}[L \cup (\{x_3 = 0\} - \{\vec{0}\})]$ , a subsequence of these laminations converges to a minimal lamination  $\mathcal{L}_1$  of  $x_3^{-1}([0, \infty)) - \{\vec{0}\}$  that contains  $\{x_3 = 0\} - \{\vec{0}\}$ . By Assertion 4.6,  $\mathcal{L}_1$  also contains a non-flat leaf  $L_1$  passing through a point in  $\mathbb{S}^2(1)$ . Since  $\mathcal{L}_1$  is a lamination outside the origin, then  $L_1$  is complete outside the origin. Furthermore,  $\vec{0}$  is in the closure of  $L_1$  in  $\mathbb{R}^3$  (otherwise the inequality  $|K_{L_1}|R^2 \leq C$  implies that  $L_1$  is a complete minimal surface with bounded Gaussian curvature, which contradicts that  $L_1$  is contained in a halfspace, see Xavier [39]).

**Assertion 4.7.** *In the above situation,  $L_1$  is proper in  $H^+$  and  $\lim(L_1) = \{x_3 = 0\} - \{\vec{0}\}$ .*

*Proof.* By Lemma 4.3 and since  $L_1 \subset H^+$ , it suffices to prove that  $L_1$  is not proper in  $\mathbb{R}^3 - \{\vec{0}\}$ . Arguing by contradiction, suppose that  $L_1$  is proper in  $\mathbb{R}^3 - \{\vec{0}\}$ . Since  $L_1$  is disjoint from the compact set  $\mathbb{S}^1(1) \times \{0\}$ , then the distance from  $\mathbb{S}^1(1) \times \{0\}$  to  $L_1$  is greater than some  $d > 0$ . In particular, the cylinder  $\mathbb{S}^1(1) \times [0, d]$  is disjoint from  $L_1$ . Fix  $n \in \mathbb{N}$ . Since the curves  $\mathbb{S}^1(1) \times \{d\}$  and  $\mathbb{S}^1(\frac{1}{n}) \times \{0\}$  are homotopic in a component of  $\{x_3 \geq 0\} - L_1$ , then using  $L_1$  as a barrier we obtain a compact least-area annulus  $A(n)$  disjoint from  $L_1$  with these curves as boundary. The annulus  $A(n)$  is a catenoid but no such catenoid exists for  $n$  large enough, since  $d > 0$  is fixed. This contradiction proves the assertion. q.e.d.

By Assertion 4.7, Lemma 4.4 and the arguments in the paragraph just after its proof, the intersection  $\Lambda_1$  of  $L_1$  with  $C_\delta$  consists of curves of type I or II. If the curves in  $\Lambda_1$  are of type II, then the corresponding spirals produce after shrinking back to  $L$  spiraling curves of type II on  $\Lambda$ , which is contrary to the hypothesis. Thus,  $\Lambda_1$  consists of curves of type I.

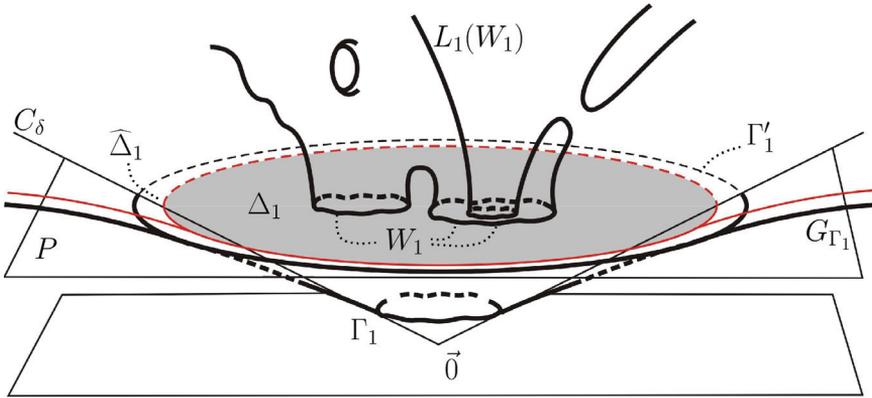
By our previous description of type I curves, the components in  $\Lambda_1$ , close to the origin are closed, almost horizontal curves that are naturally ordered by their distances to  $\vec{0}$  and have  $\{\vec{0}\}$  as limit set. Furthermore, Assertion 4.5 gives that for  $\delta > 0$  sufficiently small, every  $\Gamma_1 \in \Lambda_1$  bounds an annular end  $G_{\Gamma_1}$  of  $L_1 \cap C_\delta^-$  which is a graph over the exterior of the vertical projection of  $\Gamma_1$  in  $\{x_3 = 0\}$ .

**Assertion 4.8.** *There exists a compact horizontal disk  $\Delta_1 \subset H^+$  such that the following properties hold.*

- 1)  $\Delta_1 \cap L_1$  consists of a (non-zero) finite number of simple closed curves in  $\Delta_1 - \partial\Delta_1$ .
- 2)  $\Delta_1$  can be extended to a topological plane  $P(\Delta_1)$  which is a global graph over  $\{x_3 = 0\}$  and with  $P(\Delta_1) \cap L_1 = \Delta_1 \cap L_1$ .

*Proof.* Recall that given  $\Gamma_1 \in \Lambda_1$ , the end  $G_{\Gamma_1}$  of  $L_1 \cap C_\delta^-$  is annular and graphical, hence it has finite total curvature and it is planar or catenoidal (with positive logarithmic growth because  $L_1 \subset H^+$ ).

If there exists a curve  $\Gamma_1 \in \Lambda_1$  such that its corresponding graphical annular end  $G_{\Gamma_1}$  of  $L_1 \cap C_\delta^-$  is planar, then any curve  $\Gamma'_1 \in \Lambda_1 - \{\Gamma_1\}$  at smaller distance from  $\vec{0}$  than  $\Gamma_1$  also bounds a planar graphical annular end  $G_{\Gamma'_1}$  of  $L_1 \cap C_\delta^-$ . In this case, since between consecutive planar ends of  $L_1$  we can always find a horizontal plane  $P$  that intersects  $L_1$  transversally in a compact set, our claim holds by taking an appropriate disk  $\Delta_1$  in  $P$  and by letting  $P(\Delta_1) = P$ .



**Figure 4.** The shaded area is the compact horizontal disk  $\Delta_1$ , in the case that the end  $G_{\Gamma_1}$  is catenoidal.

Suppose now that a curve  $\Gamma_1 \in \Lambda_1$  bounds a catenoidal end  $G_{\Gamma_1} \subset L_1$ . Take a horizontal plane  $P \subset H^+$  whose height is large enough so that  $G_{\Gamma_1}$  intersects  $P$  transversely at an almost-circle  $\Gamma'_1$ . Consider the closed horizontal disk  $\widehat{\Delta}_1 \subset P$  bounded by  $\Gamma'_1$ . Now define  $\Delta_1 \subset P$  to be a slightly smaller closed disk in  $P$  which is contained in the interior of  $\widehat{\Delta}_1$  and such that

$$W_1 := \Delta_1 \cap L_1 = (\widehat{\Delta}_1 \cap L_1) - \Gamma'_1$$

is contained in the interior of  $\Delta_1$  (we can assume  $W_1 \neq \emptyset$  by taking  $\Gamma_1$  close enough to  $\vec{0}$ ). In this case, the topological plane  $P(\Delta_1)$  can be taken to be the union of  $\Delta_1$  with a minimal annular graph lying strictly above  $G_{\Gamma_1}$  and close to it. Observe that the set of curves  $W_1$  is the boundary of a proper, possibly disconnected subdomain  $L_1(W_1)$  of  $L_1$  which lies above  $P$ ; see Figure 4. This completes the proof of the assertion. q.e.d.

To find the desired contradiction in this case of type I curves, we will use a flux argument. Recall that for a curve  $\gamma \subset L$ , the *flux of  $x_3$*  along  $\gamma$  is defined as

$$\text{Flux}(\nabla x_3, \gamma) = \int_{\gamma} \frac{\partial x_3}{\partial \eta},$$

where  $\eta$  denotes a unit conormal to  $L$  along  $\gamma$ .

In the next flux argument, we will just consider the case where the annular graphical ends  $G_{\Gamma_1} \subset L_1 \cap C_{\delta}^-$  for  $\Gamma_1 \in \Lambda_1$  are catenoidal (the planar end case is similar and we leave the details to the reader). For  $n$

large, the shrunk disks  $D(n) = |q_n|\Delta_1$  intersect  $L$  transversally in a set  $W(n)$  which consists of a finite number of closed curves. Furthermore,  $W(n)$  bounds a proper, possibly disconnected subdomain  $L(W(n))$  of  $L$  and  $L(W(n))$  lies above the horizontal plane that contains  $D(n)$ . By construction,  $L(W(n))$  is the portion of  $L$  above a topological plane  $P(n)$  which is a global graph over  $\{x_3 = 0\}$  with  $P(n) \cap L = W(n)$ . Note that  $L(W(n)) \subset L(W(n+m))$  for every  $n, m \in \mathbb{N}$  with  $n$  large. Since  $x_3$  is proper on  $L(W(n+m))$ , the absolute value of the flux of  $\nabla x_3$  across  $\partial L(W(n+m))$  is not less than the absolute value of flux of  $\nabla x_3$  across  $\partial L(W(n))$ , which is positive. This is a contradiction, since the length of  $\partial L(W(n+m))$  converges to 0 as  $m \rightarrow \infty$ . This contradiction finishes the analysis when the curves in  $\Lambda$  are of type I.

**4.2. Suppose the curves in  $\Lambda$  are of type II.** Take a component  $\Gamma$  in  $\Lambda$ . By embeddedness, all of the curves in  $\Lambda - \{\Gamma\}$  have disjoint arcs trapped between one complete turn of  $\Gamma$ . Since  $L$  is proper in  $H^+$ , we have the number of curves in  $\Lambda$  is finite, say  $\Lambda = \{\Gamma(1), \dots, \Gamma(k)\}$ .

Consider the horizontal plane at height  $\delta$ , which intersects  $C_\delta$  in a circle of radius 1. After a small perturbation of  $\delta$ , we may assume that  $\{x_3 = \delta\}$  intersects  $L$  transversely and intersects each  $\Gamma(i)$  transversely,  $i = 1, \dots, k$ . Using natural cylindrical coordinates  $(r, \theta, x_3)$  in  $\{x_3 \geq 0\} - \{x_3\text{-axis}\}$ , each spiraling curve  $\Gamma(i)$  can be parameterized by the angle as  $(r_i(\theta), \theta, \delta r_i(\theta))$  for  $\theta \in (-\infty, \infty)$ , where  $r_i: \mathbb{R} \rightarrow (0, \infty)$  is a smooth function satisfying

$$\lim_{\theta \rightarrow -\infty} r_i(\theta) = \infty, \quad \lim_{\theta \rightarrow +\infty} r_i(\theta) = 0.$$

Furthermore, we may assume that the largest values of  $\theta$  such that the spiraling curves  $\Gamma(i)$  intersect  $\{x_3 = \delta\}$  are given by  $0 \leq \theta_1 < \theta_2 < \dots < \theta_k < 2\pi$ . This means that  $r_i(\theta_i) = 1$ ,  $r_i(\theta) < 1$  for  $\theta > \theta_i$ ,  $i = 1, \dots, k$ , see Figure 5 left.

As in the case of type I curves, we will use a flux argument to find a contradiction. Given  $t > 0$ , we define the function

$$(3) \quad F(t) = \sum_{i=1}^k \text{Flux}(\nabla x_3, \Gamma(i)|_{[\theta_i, \theta_i+t]}) \in \mathbb{R},$$

where the unit conormal vector with respect to which the above flux is computed is pointing inwards  $L \cap C_\delta^-$ .

**Lemma 4.9.** *The function  $F: (0, \infty) \rightarrow \mathbb{R}$  is bounded.*

*Proof.* Arguing by contradiction, we may assume that there exists a sequence  $t_n \rightarrow \infty$  such that  $|F(t_n)| \rightarrow \infty$ . Consider the sequence  $\lambda_n = |\Gamma(1)(\theta_1 + t_n)| > 0$ , which converges to zero as  $n \rightarrow \infty$ . After passing to a subsequence, the laminations  $\frac{1}{\lambda_n}[L \cup (\{x_3 = 0\} - \{\vec{0}\})]$  converge to a minimal lamination  $\mathcal{L}_1$  of  $\{x_3 \geq 0\} - \{\vec{0}\}$  with quadratic

decay of curvature. Note that  $\mathcal{L}_1$  contains a leaf  $L_1$  which passes through the limit point  $q_\infty$  of the sequence  $\frac{1}{\lambda_n}\Gamma(1)(\theta_1+t_n)$ , which lies in  $C_\delta \cap \mathbb{S}^2(1)$  (in particular,  $x_3(q_\infty) = \frac{\delta}{\sqrt{1+\delta^2}} > 0$ ).

We claim that  $\mathcal{L}_1$  is not flat. Otherwise,  $\mathcal{L}_1$  contains the plane  $\{x_3 = \frac{\delta}{\sqrt{1+\delta^2}}\}$  as a leaf. Therefore, for  $n$  large  $\frac{1}{\lambda_n}L$  contains an almost horizontal compact disk arbitrarily close to the disk  $\{(x_1, x_2, \frac{\delta}{\sqrt{1+\delta^2}}) \mid x_1^2 + x_2^2 \leq 1 + \frac{1}{1+\delta^2}\}$ . Since  $L$  is embedded, this contradicts the existence of the proper spiraling curves  $\Gamma(i)$ ,  $i = 1, \dots, k$ . Therefore,  $\mathcal{L}_1$  is not flat. The same argument proves that the leaf  $L_1$  of  $\mathcal{L}_1$  passing through  $q_\infty$  is not flat.

Since  $L_1$  is not flat, then  $L_1$  contains  $\vec{0}$  in its closure (argue as in the last sentence before the statement of Assertion 4.7). The proof of Assertion 4.7 applies without changes here and gives that  $\lim(L_1) = \{x_3 = 0\} - \{\vec{0}\}$  and  $L_1$  is proper in  $H^+$ , and thus Lemma 4.4 insures that the intersection  $\Delta_1$  of  $L_1$  with  $C_\delta$  consists of curves of type I or II. Type I curves of  $\Delta_1$  cannot occur, since we have explained that each of the types I and II persists after changing scale and taking limits.

The same analysis in the last paragraph shows that every leaf of  $\mathcal{L}_1$  different from  $\{x_3 = 0\} - \{\vec{0}\}$  contains  $\vec{0}$  in its closure, has  $\{x_3 = 0\} - \{\vec{0}\}$  as limit set in  $\mathbb{R}^3 - \{\vec{0}\}$ , is proper in  $H^+$  and intersects  $C_\delta$  in finitely many curves of type II. The non-existence of limit leaves of  $\mathcal{L}_1$  in  $H^+$  (by Corollary 3.4) implies that  $\mathcal{L}_1$  consists of a finite number of leaves. The same argument (by Corollary 3.5) shows that the multiplicity of the limit  $\frac{1}{\lambda_n}[L \cup (\{x_3 = 0\} - \{\vec{0}\})] \rightarrow \mathcal{L}_1$  is one; consequently,  $\mathcal{L}_1 \cap C_\delta$  consists of precisely  $k$  spiraling curves, each of which is a limit of a rescaled spiraling curve in  $L$ . This correspondence allows us to label the curves in  $\mathcal{L}_1 \cap C_\delta$  with the same indices as the related curves in  $L \cap C_\delta$ .

Consider an auxiliary smooth compact disk  $D \subset H^+ - C_\delta^-$  with  $\partial D \subset C_\delta$ , satisfying the following properties:

- (P1)  $D$  intersects transversely the union of the leaves in  $\mathcal{L}_1$ .
- (P2)  $\partial D$  intersects transversely each of the spirals in  $L_1 \cap C_\delta$  in a single point.

After shrinking back to the original scale, we find a sequence of compact disks  $\lambda_n D \subset H^+ - C_\delta^-$  which for  $n$  sufficiently large, satisfy the properties

- (P1)'  $\lambda_n D$  intersects transversely  $L$ .
- (P2)'  $\lambda_n \partial D$  intersects transversely each of the spirals  $\Gamma(1), \dots, \Gamma(k) \subset L_1$  in points of the form  $\Gamma(i)(\theta_i+t_{i,n})$ . We can assume without loss of generality that  $q_\infty$  is the limit of rescaled points in  $\Gamma(1)(\theta_1 + t_{1,n})$ .

(P3)'  $|t_n - t_{1,n}| \rightarrow 0$  as  $n \rightarrow \infty$  (since the angle coordinate is invariant under change of scale and the points  $\Gamma(1)(\theta_1 + t_n)$  converge after rescaling by  $1/\lambda_n$  to  $q_\infty$  as  $n \rightarrow \infty$ ).

(P4)'  $|t_n - t_{i,n}|$  is bounded independently of  $i, n$  for each  $i = 2, \dots, k$  (this holds since  $|\theta_i - \theta_1| \leq 2\pi$  and by property (P3)').

As  $|F(t_n)| \rightarrow \infty$  by hypothesis and the sum of lengths of the  $\Gamma(i)$  from  $\theta_i + t_n$  to  $\theta_i + t_{i,n}$  tends to zero as  $n \rightarrow \infty$  (by properties (P3)', (P4)'), then

$$(4) \quad \sum_{i=1}^k \text{Flux} \left( \nabla x_3, \Gamma(i)|_{[\theta_i, \theta_i + t_{i,n}]} \right) \rightarrow \infty \text{ as } n \rightarrow \infty.$$

We will next obtain the desired contradiction as an application of the fact that the total flux of a compact minimal surface along its boundary vanishes. To do this, consider the compact minimal surface with boundary  $\widehat{L}(n) \subset L$  bounded by  $\lambda_1 D$ ,  $\lambda_n D$  and  $C_\delta$ , where  $n$  is chosen large enough so that  $(\lambda_1 D) \cap (\lambda_n D) = \emptyset$ . Since the flux of  $\nabla x_3$  along  $\lambda_1 D$  is a fixed number and the flux of  $\nabla x_3$  along  $\lambda_n D$  tends to zero as  $n \rightarrow \infty$  (because the length of  $(\lambda_n D) \cap L$  tends to zero), then we conclude that the last displayed summation is bounded as  $n \rightarrow \infty$ , which is a contradiction. q.e.d.

By the description above, each of the components of  $L \cap C_\delta^-$  can be considered to be a graph of a function  $u_i = u_i(r, \theta)$  in polar coordinates defined over a region  $\Omega_i$  in the universal cover of the punctured plane  $\{x_3 = 0\} - \{\vec{0}\}$ , of the type

$$\Omega_i = \{(r, \theta) \mid r > r_i(\theta), \theta \in \mathbb{R}\},$$

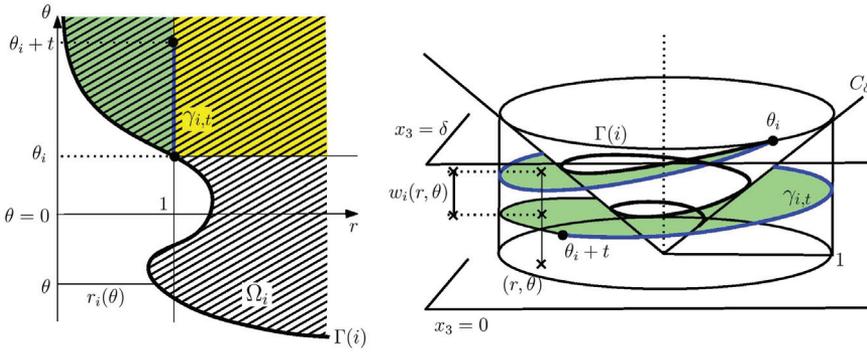
where the function  $r_i(\theta)$  was defined before the statement of Lemma 4.9. In particular, the restriction of  $u_i$  to the quadrant  $\{(r, \theta) \mid r \geq 1, \theta \geq \theta_i\}$  is an  $\infty$ -valued graph in the sense of Colding-Minicozzi, see Figure 5.

Consider the compact region  $R = \overline{C_\delta^-} \cap \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 \leq 1\}$ . By Lemma 4.4, each component of the intersection of  $L \cap R$  with any vertical halfplane  $\{\theta = \text{constant}\}$  has length not greater than 2 for  $\delta > 0$  sufficiently small. This property together with Lemma 4.9 and the Divergence Theorem implies that the following function is bounded:

$$(5) \quad \widehat{F}: (0, \infty) \rightarrow \mathbb{R}, \quad \widehat{F}(t) = \sum_{i=1}^k \text{Flux}(\nabla x_3, \gamma_{i,t}),$$

where  $\gamma_{i,t}$  is given in cylindrical coordinates by  $\gamma_{i,t}(\theta) = (1, \theta, u_i(1, \theta))$ ,  $\theta \in [\theta_i, \theta_i + t]$ , see Figure 5.

The desired contradiction in this case of type II curves will come from application of a slight adaptation of inequality (5.3) in the proof of Theorem 1.1 in Colding and Minicozzi [5]. The differences between our situation here and that of (5.3) in [5] are the following ones:



**Figure 5.** Schematic representation of one of the components of  $L \cap C_\delta^-$ . Left: In the  $(r, \theta)$ -plane (polar coordinates), the 'inner' spiraling curve  $\Gamma(i)$  corresponds to the global  $\theta$ -graph of the function  $r_i(\theta)$  defined before the statement of Lemma 4.9, and the 'outer' spiraling curve  $\gamma_{i,t}$  corresponds to the finite vertical segment  $\{1\} \times [\theta_i, \theta_i + t]$  (after a slight abuse of notation, we have also labeled these curves in the  $(r, \theta)$ -plane by  $\Gamma(i)$ ,  $\gamma_{i,t}$  respectively). The domain  $\Omega_i = \{(r, \theta) \mid r > r_i(t)\}$  is the domain of definition of the graphing function  $u_i$  that parameterizes the component of  $L \cap C_\delta^-$ . Right: The curve  $\Gamma(i)$  lies on the cone  $C_\delta$ , while  $\gamma_{i,t}$  is contained in the cylinder of radius 1. We have used the same green color in both the domain and the target to represent corresponding portions of the surface  $L \cap C_\delta^-$  under the graphing function  $u_i = u_i(r, \theta)$ , also note that the restriction of  $u_i$  to the yellow quadrant  $\{(r, \theta) \mid r \geq 1, \theta \geq \theta_i\}$  is an  $\infty$ -valued graph in the sense of [5]. The image of this quadrant under the graphing function lies outside the cylinder  $\{x_1^2 + x_2^2 = 1\}$  and it is not depicted on the right. Finally, note that the polar angle  $\theta$  increases to  $+\infty$  as the height of the spiraling curves approaches zero, and the separation  $w_i(r, \theta) = u_i(r, \theta) - u_i(r, \theta + 2\pi)$  is negative.

- We have  $k$   $\infty$ -valued minimal graphs  $u_1, \dots, u_k$  instead of two (see the statement of Theorem 1.1 in [5]); this is not a problem since inequality (5.3) is obtained in [5] by adding one inequality for each minimal graph (namely, (5.2) in [5]) and then using Stokes' theorem in Lemma 5.1 in [5]. Also note that Theorem 1.1 does not require that the minimal graphs are  $\infty$ -valued, but only that the number of sheets grow sufficiently fast in a certain sense.

- The range of the polar angle  $\theta$  for the  $\infty$ -valued minimal graph  $u_j(r, \theta)$  is  $[\theta_j, +\infty)$ , instead of the one in Theorem 1.1, that starts at  $\theta = 0$ ; this creates a translation by  $\theta_j$  of the polar angle variable in our application of inequality (5.3) of [5].

With this differences in mind, we can state the following inequality, that corresponds to inequality (5.3) in [5]:

$$\begin{aligned} & \sum_{j=1}^k \int_1^r \int_0^t \frac{|w_j(e^s, \theta_j)|}{t} ds dt - \sum_{j=1}^k \int_{\theta_j}^{\theta_j+2\pi} \int_{\partial \tilde{D}_1 \cap Q} u_j(e^{x+i(y+\tau)}) d\tau \\ & \qquad - 6\pi \sum_{j=1}^k u_j^3(1, \theta_j) \log r \\ & \leq - \sum_{j=1}^k \int_{\theta_j}^{\theta_j+2\pi} \int_1^r \left( \int_{(\Sigma_j)_{1,1}^{\tau, t+\tau}} \frac{\partial x_3}{\partial \eta} \right) \frac{1}{t} dt d\tau \leq 2C_1 \log r, \quad (*) \end{aligned}$$

where  $w_j(r, \theta) = u_j(r, \theta + 2\pi) - u_j(r, \theta) < 0$  is the separation between consecutive sheets of the multigraph  $u_j$ ,  $\tilde{D}_1 = \{(x, y) \mid x^2 + y^2 \leq 1\}$  and  $e^x = r$ ,  $y = \theta$ ,  $Q = \{(x, y) \mid x, y \geq 0\}$ , and  $(\Sigma_j)_{1,1}^{\tau, t+\tau}$  denotes the curve  $\theta \in [\tau, t + \tau] \mapsto (1, u_j(1, \theta))$  where as before, we are using polar coordinates  $(r, \theta)$  in the domain of  $u_j$  (hence  $(\Sigma_j)_{1,1}^{\tau, t+\tau}$  is a subarc of  $\gamma_{j, t+\tau}$ ). Note that the last inequality in  $(*)$  holds by Lemma 4.9. This is a contradiction for  $r$  large, since the left-hand-side of  $(*)$  has three terms, of which the third one grows as  $\log r$ , the second one is a fixed constant not depending on  $r$  and the first one can be bounded below by a positive constant times  $(\log r)^2$  by integrating the first inequality in Theorem 1.3 of [5] (which is valid in our situation since the  $\infty$ -valued minimal graphs are defined for  $r \in [1, \infty)$  and their gradient can be made arbitrarily small by choosing  $\delta > 0$  small enough). This contradiction finishes the analysis of type II curves.

**Remark 4.10.** In the above analysis of type I and II curves in  $\Lambda$  we have used a flux argument based on Colding-Minicozzi theory. In the appendix we provide a self-contained argument to get the same conclusions as above, by showing that a non-proper leaf  $L$  as in Lemma 4.3 is recurrent for Brownian motion, which by the Liouville theorem applied to the positive third coordinate function  $x_3|_L$ , gives the desired contradiction to the existence of such an  $L$ . See Proposition 8.2 for details.

We are now ready to prove the main result of this section, which will be improved in Corollary 6.3.

**Proposition 4.11.** *Let  $\mathcal{L}$  be a non-flat minimal lamination of  $\mathbb{R}^3 - \{\vec{0}\}$  with quadratic decay of curvature. Then, any leaf of  $\mathcal{L}$  is a properly*

embedded minimal surface in  $\mathbb{R}^3 - \{\vec{0}\}$ , and  $\mathcal{L}$  does not contain flat leaves.

*Proof.* Arguing by contradiction, suppose  $L$  is a leaf of  $\mathcal{L}$  which is not proper in  $\mathbb{R}^3 - \{\vec{0}\}$ . By Lemmas 4.3 and 4.4, after a rotation we can assume  $L \subset H^+$ ,  $\lim(L) = \{x_3 = 0\} - \{\vec{0}\}$  and for  $\delta > 0$  sufficiently small, we have the description of the set  $\Lambda$  of components of  $L \cap C_\delta$  given just after Lemma 4.4 in terms of curves of types I and II. By the above analysis, neither of these two cases can occur. This contradiction shows that every leaf  $L$  of  $\mathcal{L}$  is properly embedded in  $\mathbb{R}^3 - \{\vec{0}\}$ .

To finish the proof of the proposition, it remains to show that none of the leaves of  $\mathcal{L}$  are flat. Arguing again by contradiction, suppose  $\tilde{L} \in \mathcal{L}$  is a flat leaf and let  $\hat{L} \in \mathcal{L}$  be a non-flat leaf. If  $\hat{L}$  does not limit to  $\vec{0}$ , then  $\hat{L}$  has bounded curvature, and so, it cannot be contained in a halfspace [39]. This contradicts the Half-space Theorem as  $\hat{L}$  lies at one side of the closure of  $\tilde{L}$ , which is a plane. Hence,  $\vec{0}$  is a limit point of  $\hat{L}$ .

Now consider the sublamination  $\tilde{\mathcal{L}} = \{\hat{L}, \tilde{L}\}$ . Suppose that  $\tilde{L}$  has  $\vec{0}$  in its closure and we will obtain a contradiction. After a rotation, assume that  $\tilde{L}$  is the  $(x_1, x_2)$ -plane and the third coordinate of  $\hat{L}$  is positive. In this setting, the proof of Assertion 4.7 applies to  $L_1 = \hat{L}$  and gives the desired contradiction.

So we may assume that  $\tilde{L}$  is a plane which does not pass through  $\vec{0}$ . As  $\hat{L}$  is properly embedded in  $\mathbb{R}^3 - \{\vec{0}\}$ , the proof of the Half-space Theorem [19] gives that the distance between  $\hat{L}$  and  $\tilde{L}$  is positive. Consider the plane  $\Pi$  parallel to  $\tilde{L}$  at distance 0 from  $\hat{L}$ . Since  $\hat{L}$  is not a plane,  $\Pi$  must go through the origin, and we finish as before. Now the proof is complete. q.e.d.

### 5. The proof of Theorem 1.1.

We will divide the proof in five different cases. Cases 1, 2, 3 analyze the possibilities that can occur when the ambient space is  $\mathbb{R}^3$  (namely, in case 1 the lamination is assumed to reduce to a single properly embedded, connected minimal surface, in case 2 we drop the connectedness assumption but we still assume properness, while case 3 deals with the non-proper, possibly non-connected situation). Cases 4 and 5 deal with the general ambient setting of a Riemannian manifold (the hypotheses for the lamination in cases 4 and 5 are the same as in cases 2 and 3, respectively).

**CASE I: Suppose that  $N = \mathbb{R}^3$ ,  $p = \vec{0}$  (hence  $\overline{B}_N(p, r) = \overline{\mathbb{B}}(r)$ ) and  $\mathcal{L}$  consists of a single leaf  $M$  which is properly embedded in  $\overline{\mathbb{B}}(r) - \{\vec{0}\}$ .**

In this case it is known that the area of  $M$  is finite and  $M$  satisfies the monotonicity formula, see for instance Harvey and Lawson [18]. For the sake of completeness, we give a self-contained proof in our setting.

For  $0 < R_1 \leq R_2 \leq r$ , let  $A_M(R_1) = \text{Area}(M \cap \mathbb{B}(R_1))$ ,  $l_M(R_1) = \text{Length}(M \cap \mathbb{S}^2(R_1)) \in (0, \infty]$  and  $A_M(R_1, R_2) = \text{Area}(M \cap [\mathbb{B}(R_2) - \mathbb{B}(R_1)]) \in (0, \infty)$ . The Divergence Theorem applied to the vector field  $p^T = p - \langle p, \vec{n} \rangle \vec{n}$  (here  $\vec{n}$  is the Gauss map of  $M$ ) gives:

$$(6) \quad 2A_M(R_1, R_2) = \int_{M \cap [\mathbb{B}(R_2) - \mathbb{B}(R_1)]} \text{div}_M(p^t) = \int_{\partial_{R_1}} \langle p, \nu \rangle + \int_{\partial_{R_2}} \langle p, \nu \rangle,$$

where  $\partial_{R_i} = M \cap \mathbb{S}^2(R_i)$ ,  $i = 1, 2$ , and  $\nu$  is the unit exterior conormal vector to  $M \cap [\mathbb{B}(R_2) - \mathbb{B}(R_1)]$  along its boundary. The first integral in the right-hand-side is not positive, and Schwarz inequality applied to the second one gives  $2A_M(R_1, R_2) \leq R_2 l_M(R_2)$ . Taking  $R_1 \rightarrow 0$  and relabeling  $R_2$  as  $R$ , we have

$$(7) \quad 2A_M(R) \leq R l_M(R) \quad \text{for all } R \in [0, r].$$

In particular, the total area of  $M$  is finite. Next we observe that the monotonicity formula holds in our setting (i.e.  $R^{-2}A_M(R)$  is not decreasing for  $R \in [0, r]$ ). To see this, note that

$$(8) \quad R^3 \frac{d}{dR} \left( \frac{A_M(R)}{R^2} \right) = R A'_M(R) - 2A_M(R).$$

The coarea formula applied to the radial distance function  $R$  to  $\vec{0}$  gives

$$(9) \quad A'_M(R) = \int_{\partial_R} \frac{ds}{|\nabla R|} \geq l_M(R)$$

where  $\nabla R$  is the intrinsic gradient of  $R|_M$  and  $ds$  is the length element along  $\partial_R$ . Now (7), (8) and (9) imply the monotonicity formula.

As an important consequence of the finiteness of its area together with the monotonicity formula,  $M$  has limit tangent cones at the origin under expansions. To prove that  $M$  extends to a smooth minimal surface in  $\overline{\mathbb{B}(R_1)}$ , we discuss two situations separately. In the first one (paragraph I.1 below) we will deduce that  $M$  has finite topology, in which case the removability theorem is known (see [3], although we also provide a proof of the removability of the singularity in this situation), and to conclude the proof in Case I, we will show that the second situation (paragraph I.2 below) cannot hold.

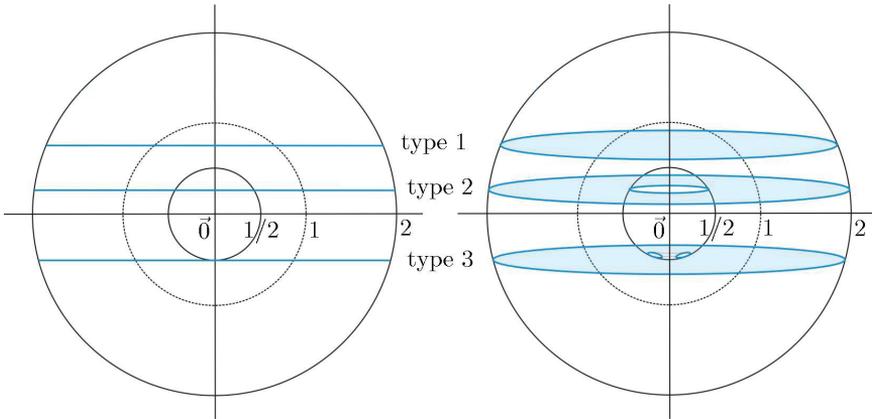
I.1. Suppose there exist constants  $C_1 < 1$  and  $r' \leq r$  such that  $|K_M|R^2 \leq C_1$  in  $M \cap \mathbb{B}(r')$ . Using the arguments in the proof of Lemma 4.1, we deduce that  $M \cap \mathbb{B}(r')$  consists of a finite number of annuli with compact boundary, transverse to the spheres centered at the origin such that each of these annuli has  $\vec{0}$  in its closure, together with a finite number of compact disks. Thus, for  $r'$  sufficiently small

we may assume that there are no such disk components. Let  $A$  be one of the annuli in  $M \cap \mathbb{B}(r')$ . If  $A$  is conformally  $\{\varepsilon < |z| \leq 1\}$  for some  $\varepsilon > 0$ , then each coordinate function of  $A$  can be reflected in  $\{|z| = \varepsilon\}$  (Schwarz's reflection principle), defining a conformal branched harmonic map which carries the entire curve  $\{|z| = \varepsilon\}$  to a single point, which is impossible. Thus,  $A$  is conformally  $\{0 < |z| \leq 1\}$  and so, its coordinate functions extend smoothly across  $\vec{0}$ , defining a possibly branched minimal surface  $A_0$  that passes through  $\vec{0}$ . If  $\vec{0}$  is a branch point of  $A_0$ , then  $A$  cannot be embedded in a punctured neighborhood of  $\vec{0}$ , which is a contradiction. Therefore,  $A_0$  is a smooth embedded minimal surface passing through  $\vec{0}$ . Since  $M$  is embedded, the usual maximum principle for minimal surfaces implies that there exists only one such surface  $A_0$ , and the theorem holds in this case.

I.2. Now assume that there exists a sequence  $\{p_n\}_n \subset M$  converging to  $\vec{0}$  such that  $1 \leq |K_M|R^2(p_n)$  for all  $n$ , and we will obtain a contradiction. The expanded surfaces  $\widetilde{M}_n = \frac{1}{|p_n|}M \subset \mathbb{R}^3 - \{\vec{0}\}$  also satisfy  $|K_{\widetilde{M}_n}|R^2 \leq C$ . After choosing a subsequence, the  $\widetilde{M}_n$  converge to a minimal lamination  $\mathcal{L}_1$  of  $\mathbb{R}^3 - \{\vec{0}\}$  with  $|K_{\mathcal{L}_1}|R^2 \leq C$ . Furthermore,  $\mathcal{L}_1$  contains a non-flat leaf  $L_1$  passing through a point in  $\mathbb{S}^2(1)$ , where it has absolute Gaussian curvature at least 1. By the monotonicity formula,  $R^{-2}A_M(R)$  is bounded as  $R \rightarrow 0$ . Geometric measure theory implies that any sequence of expansions of  $M$  converges (up to a subsequence) to a minimal cone over a configuration of geodesic arcs in  $\mathbb{S}^2(1)$ . Since any smooth point of such a minimal cone is flat, we contradict the existence of the non-flat minimal leaf  $L_1$ . This finishes the proof of Case I.

**CASE II: Suppose that  $N = \mathbb{R}^3$ ,  $p = \vec{0}$  and  $\mathcal{L}$  consists of a possibly disconnected, properly embedded minimal surface in  $\mathbb{B}(r) - \{\vec{0}\}$ .**

Consider the intersection of  $\mathcal{L}$  with the closed ball of radius  $r' \in (0, r)$ . We claim that every leaf  $L$  of  $\mathcal{L}$  having  $\vec{0}$  in its closure intersects  $\partial\mathbb{B}(r')$ ; otherwise, the supremum of the function  $R|_L$  is some  $r_1 \leq r'$  and there exists a leaf  $L_1$  in the closure of  $L$  that lies at the inner side of  $\mathbb{S}^2(r_1)$  and touches this sphere at some point, which contradicts the mean comparison principle for  $L_1$  and  $\mathbb{S}^2(r_1)$  and proves our claim. Since  $\partial\mathbb{B}(r')$  is compact, we conclude that there are at most a finite number of leaves of  $\mathcal{L}$  having  $\vec{0}$  in its closure. If there are two leaves of  $\mathcal{L} \cap \mathbb{B}(r')$  which have  $\vec{0}$  in their closure, then each of these leaves extends smoothly across the origin by the previous Case I, and we contradict the maximum principle for minimal surfaces. Therefore, at most one leaf of  $\mathcal{L}$  has the origin in its closure, and the other components, which are compact, do not intersect a certain ball  $\mathbb{B}(r')$  for some  $r' \in (0, r)$  small enough. Hence,  $\mathcal{L}$  extends in this case.



**Figure 6.** Type 1, 2, 3 connected components of  $\mathcal{L}_\varepsilon$ .

**CASE III: Suppose that  $N = \mathbb{R}^3$ ,  $p = \vec{0}$  and every punctured neighborhood of  $\vec{0}$  intersects a limit leaf of  $\mathcal{L}$**  (note that Cases I–III finish the  $\mathbb{R}^3$ -setting of Theorem 1.1).

Since the set of limit leaves of  $\mathcal{L}$  is closed, it follows that  $\mathcal{L}$  contains a limit leaf  $F$  with  $\vec{0}$  in the closure of  $F$ .

We claim that any blow-up limit of  $\mathcal{L}$  from  $\vec{0}$  converges outside  $\vec{0}$  to a flat lamination of  $\mathbb{R}^3$  by planes. Since  $|K_{\mathcal{L}}|R^2$  is scale-invariant, our claim follows by proving that for any  $\varepsilon > 0$ , there is  $r(\varepsilon) \in (0, r)$  such that  $|K_{\mathcal{L}}|R^2 < \varepsilon$  on  $\mathcal{L} \cap \mathbb{B}(r(\varepsilon))$ . Arguing by contradiction, suppose there exists a sequence of points  $q_n \in \mathcal{L}$  converging to  $\vec{0}$  such that  $|K_{\mathcal{L}}|(q_n)|q_n|^2$  is bounded away from zero. Then, after expansion of  $\mathcal{L}$  by  $\frac{1}{|q_n|}$  and taking a subsequence, we obtain a limit which is a non-flat minimal lamination  $\mathcal{L}_1$  of  $\mathbb{R}^3 - \{\vec{0}\}$  that satisfies the hypotheses in Proposition 4.11. In particular,  $\mathcal{L}_1$  does not contain flat leaves. The limit leaf  $F$  in  $\mathcal{L}$  produces under expansion a leaf  $F_1$  (which is stable since  $F$  is stable) in  $\mathcal{L}_1$ . Since  $F_1$  is complete outside the origin, the stability lemma implies that  $F_1$  is a plane, which is a contradiction. Now our claim is proved.

By the above claim, we know that any blow-up limit of  $\mathcal{L}$  is a minimal lamination of  $\mathbb{R}^3 - \{\vec{0}\}$  by parallel planes. It follows that for  $\varepsilon > 0$  sufficiently small, in the annular domain  $A = \{x \in \mathbb{R}^3 \mid \frac{1}{2} \leq |x| \leq 2\}$  the normal vectors to the leaves of  $\mathcal{L}_\varepsilon = (\frac{1}{\varepsilon}\mathcal{L}) \cap A$  are almost parallel, and after a rotation (which might depend on  $\varepsilon$ ), we will assume that the unit normal vector to the leaves of  $\mathcal{L}_\varepsilon$  lies in a small neighborhood of  $\{\pm(0, 0, 1)\}$ . Hence, for such a sufficiently small  $\varepsilon$ , each component  $C$  of  $\mathcal{L}_\varepsilon$  that intersects  $\mathbb{S}^2(1)$  is of one of the following four types, see Figure 6:

- 1) A compact disk with boundary in  $\mathbb{S}^2(2)$ ;
- 2) A compact annulus with one boundary curve in  $\mathbb{S}^2(\frac{1}{2})$  and the other boundary curve in  $\mathbb{S}^2(2)$ ;
- 3) A compact planar domain whose boundary consists of a single closed curve in  $\mathbb{S}^2(2)$  together with at least two closed curves in  $\mathbb{S}^2(\frac{1}{2})$ , and where the outer boundary curve bounds a compact disk in  $\frac{1}{\varepsilon}\mathcal{L}$ ;
- 4) An infinite multigraph  $\mathcal{M}$  whose limit set consists of two compact components of  $\mathcal{L}_\varepsilon$  of type 2. To see why this is the only possibility for the limit set  $\lim(\mathcal{M})$  of  $\mathcal{M}$ , note that  $\lim(\mathcal{M})$  cannot contain a component of type 1 by an elementary covering argument. Also,  $\lim(\mathcal{M})$  can be supposed not to contain components of type 3, by choosing  $\varepsilon$  smaller.

One consequence of the description above is that if a component  $\mathcal{M}$  of  $\mathcal{L}_\varepsilon$  of type 4 occurs, then  $\mathcal{M}$  cannot intersect the complement of the slab  $\{|x_3| \leq \frac{1}{2}\}$  in  $A$ . In fact, by taking  $\varepsilon > 0$  small enough, we conclude that spiraling multigraph components of  $\mathcal{L}_\varepsilon$  can only occur in the intersection of  $A$  with an open slab  $\Delta$  of small width around height 0. Note that the intersection of a spiraling component  $\mathcal{M}$  with  $\mathbb{S}^2(2)$  is an embedded spiraling curve  $\Gamma(2)$  contained in  $\Delta$  that limits to two closed, pairwise disjoint curves  $C_1(2), C_2(2) \subset \mathbb{S}^2(2) \cap \overline{\Delta}$  which are almost horizontal, and the same description holds for the intersection of  $\mathcal{M}$  with  $\mathbb{S}^2(\frac{1}{2})$ , defining an embedded spiraling curve  $\Gamma(\frac{1}{2})$  contained in  $\mathbb{S}^2(\frac{1}{2}) \cap \Delta$  that limits to two closed, pairwise disjoint, almost horizontal curves  $C_1(\frac{1}{2}), C_2(\frac{1}{2}) \subset \mathbb{S}^2(\frac{1}{2}) \cap \overline{\Delta}$ . In fact, this description also holds for the intersection of  $\mathcal{M}$  with every intermediate sphere  $\mathbb{S}^2(\tau)$ ,  $\frac{1}{2} \leq \tau \leq 2$ , and the union of the related closed curves  $\cup_{\tau \in [\frac{1}{2}, 2]} C_j(\tau)$ ,  $j = 1, 2$ , defines the two compact components of  $\mathcal{L}_\varepsilon$  of type 2 in the limit set of  $\mathcal{M}$ .

Next we check that type 4 components of  $\mathcal{L}_\varepsilon$  cannot occur for  $\varepsilon > 0$  sufficiently small: if for some sufficiently small  $\varepsilon_0$ ,  $\mathcal{L}_{\varepsilon_0}$  has a component of type 4, then this multigraph component persists for all  $\varepsilon \in (0, \varepsilon_0)$ , varying in a continuous manner in terms of  $\varepsilon$ , as well as the two annular components of type 2 which are the limit set of this multigraph component. Thus, the existence of a multigraph component in  $\mathcal{L}_{\varepsilon_0}$  implies that in the original scale,  $\mathcal{L} \cap \mathbb{B}(2\varepsilon_0)$  has two properly embedded annular leaves in  $\mathbb{B}(2\varepsilon_0) - \{\vec{0}\}$ . By our previously considered Case I in this proof, these two annular leaves extend smoothly to two minimal disks that only intersect at the origin, thereby contradicting the maximum principle for minimal surfaces. This contradiction shows that only components of types 1, 2, 3 can occur in  $\mathcal{L}_\varepsilon$  for  $\varepsilon$  small.

Recall that  $F$  is a limit leaf of  $\mathcal{L}$  with  $\vec{0}$  in its closure. We claim that there exists  $r_1 \in (0, r)$  such that  $F \cap [\mathbb{B}(r_1) - \{\vec{0}\}]$  is a proper annulus in  $\mathbb{B}(r_1) - \{\vec{0}\}$ . This follows from the fact that for any  $\varepsilon > 0$  small

enough,  $F$  produces a type 2 component of  $\mathcal{L}_\varepsilon$  (type 4 is discarded by the previous paragraph, and types 1, 3 can be discarded by taking  $\varepsilon$  small enough). Since  $F \cap [\overline{\mathbb{B}}(r_1) - \{\vec{0}\}]$  is a proper annulus, we can apply the already proven Case I to  $F \cap [\overline{\mathbb{B}}(r_1) - \{\vec{0}\}]$  and conclude that  $F \cap [\overline{\mathbb{B}}(r_1) - \{\vec{0}\}]$  extends smoothly across  $\vec{0}$  and  $\overline{F}$  is a compact minimal disk.

Next we claim that for  $\varepsilon > 0$  small enough and for every leaf component of  $\mathcal{L} \cap \overline{\mathbb{B}}(\varepsilon)$  the tangent planes to this leaf component at any of its points make an angle less than  $\pi/4$  with the tangent plane  $T_{\vec{0}}\overline{F}$ . Otherwise, we find a sequence of points  $p_n \in \mathcal{L}$  converging to  $\vec{0}$  as  $n \rightarrow \infty$ , such that the leaves  $L_n$  of  $\mathcal{L}$  passing through  $p_n$  have tangent planes  $T_{p_n}L_n$  making an angle larger than  $\pi/4$  with  $T_{\vec{0}}\overline{F}$ . After rescaling  $\mathcal{L}$  by the homothety with factor  $1/|p_n|$  centered at  $\vec{0}$ , we produce a sequence of laminations with the same quadratic decay constant. After passing to a subsequence, this sequence converges to a family of parallel planes, which is impossible since one of the planes is  $T_{\vec{0}}\overline{F}$  and another one is a plane passing through a point in  $\mathbb{S}^2(1)$  making an angle greater than  $\pi/4$  with  $T_{\vec{0}}\overline{F}$ . This contradiction proves our claim.

By the claim in the last paragraph, the inner product of the unit normal vector to any leaf component of  $\mathcal{L} \cap \overline{\mathbb{B}}(\varepsilon)$  with the normal direction to  $T_{\vec{0}}\overline{F}$  defines a Jacobi function with constant sign on such a leaf component. Hence,  $\mathcal{L} \cap \overline{\mathbb{B}}(\varepsilon)$  consists of a collection of stable surfaces. Finally, curvature estimates for stable minimal surfaces away from their boundaries (Schoen [36]) gives the desired lamination structure for  $\mathcal{L}$  around  $\vec{0}$ . This finishes the proof in this Case III.

**CASE IV: Suppose  $N$  is a Riemannian manifold,  $p \in N$  and for some  $r' \in (0, r)$   $\mathcal{L} \cap B_N(p, r')$  is a non-compact, possibly disconnected, properly embedded minimal surface  $M$  in  $B_N(p, r') - \{p\}$ .**

In this case,  $\exp_p$  yields  $\mathbb{R}^3$ -coordinates on  $B_N(p, r')$  centered at  $p \equiv \vec{0}$ , for  $r' > 0$  small enough. It follows from Theorems 3.1 and 4.1 in [18] that  $M$  is a locally rectifiable stationary current (relative to its boundary) with finite area. Hence, under homothetic expansions of coordinates,  $M$  has minimal limit tangent cones in  $\mathbb{R}^3$  at  $\vec{0}$ .

If there exists an  $\varepsilon > 0$  and a sequence  $\{p_n\}_n \subset M$  converging to  $p$  such that  $\varepsilon \leq |\sigma_M|(p_n)d_N(p_n, p)$  for all  $n$ , then a subsequence of the expanded surfaces  $\widetilde{M}_n = \frac{1}{d_N(p_n, p)}M$  in  $\frac{1}{d_N(p_n, p)}B_N(p, r')$  converges to a non-flat minimal lamination  $\mathcal{L}_\infty$  of  $\mathbb{R}^3 - \{\vec{0}\}$ . Since  $\mathcal{L}_\infty$  is not flat at some point of  $\mathbb{S}^2(1)$ , then  $\mathcal{L}_\infty$  has a leaf which is not a cone, which contradicts the conclusion of the previous paragraph.

Hence, any sequence of homothetic blow-ups of  $M$  has a subsequence which converges (possibly with finite multiplicity) to a minimal lamination of  $\mathbb{R}^3 - \{\vec{0}\}$  by parallel planes. Also, the arguments in the first

paragraph of the proof of Lemma 4.1 imply that  $M$  has a finite number of annular ends. Furthermore, the fact that under expansions of  $M$  we obtain minimal cones in  $\mathbb{R}^3 - \{\vec{0}\}$ , implies that  $M$  intersects almost orthogonally the geodesic spheres  $\partial B_N(p, r'')$  for all  $r'' > 0$  sufficiently small. Hence, each of the annular ends of  $M$  has linear area growth with respect to the complete metric  $\frac{1}{d_N(p, \cdot)^2} \langle, \rangle$  (here  $\langle, \rangle$  denotes the original metric on  $N$ ). Therefore, the ends of  $M$  are conformally punctured disks. Standard regularity theory implies that the conformal harmonic map from each of these annular ends of  $M$  into  $N$  extends smoothly across the punctured disks to a conformal harmonic map (see Grüter [16]), hence to a branched minimal immersion into  $B_N(p, r'')$  as well. Such a branched minimal immersion is free of branch points, since  $M$  is embedded. Finally, the maximum principle for minimal surfaces implies that  $M$  has only one annular end. This finishes the proof of Case IV.

To conclude the proof of the theorem, it remains to solve the following:

**CASE V: Suppose  $N$  is a Riemannian manifold,  $p \in N$  and for all  $r' \in (0, r)$ ,  $B_N(p, r') - \{p\}$  intersects a limit leaf of  $\mathcal{L}$ .**

In this case  $\mathcal{L}$  contains a limit leaf  $L$  with  $p \in \bar{L}$ . Since the proof of this case is very similar to the proof of Case III, we will only comment on the differences. The same arguments as in Case III prove that every blow-up limit of  $\mathcal{L}$  from  $p$  converges outside  $\vec{0} \in \mathbb{R}^3$  to a lamination of  $\mathbb{R}^3$  by planes (only exchange  $|K_{\mathcal{L}}|R^2$  by  $|\sigma_{\mathcal{L}}|d_N(p, \cdot)$ ). Then, for  $\varepsilon > 0$  sufficiently small the normal vectors to the leaves of  $\mathcal{L}_\varepsilon = \frac{1}{\varepsilon}(\mathcal{L} \cap \{x \in B_N(p, r') \mid \frac{\varepsilon}{2} \leq d_N(p, x) \leq 2\varepsilon\})$  at any of their points are almost parallel, where we are using  $\mathbb{R}^3$ -coordinates in  $B_N(p, r')$  via the exponential map  $\exp_p$  of  $N$  for  $r' \in (0, r)$  small enough. The components of  $\mathcal{L}_\varepsilon$  are of one of the types 1–4 in Case III (we only exchange  $\mathbb{S}^2(R)$  by  $\frac{1}{\varepsilon}\partial B_N(p, \varepsilon R)$ , where  $R = \frac{1}{2}$  or  $R = 2$ ). Type 4 components cannot occur if  $\varepsilon > 0$  is sufficiently small, by the same arguments as in Case III (such arguments rest on the validity of Case II, which we now substitute by Case IV). By the same arguments as in Case III, the limit leaf  $L$  is a proper annulus in  $B_N(p, r)$  that extends smoothly across  $p$ , and every blow-up limit of  $\mathcal{L}$  from  $p$  converges outside  $\vec{0} \in \mathbb{R}^3$  to a lamination of  $\mathbb{R}^3$  by planes parallel to the tangent plane  $T_p\bar{L}$  to the extended surface  $\bar{L} = L \cup \{p\}$ .

Consider a small geodesic disk  $D_{\bar{L}}(p, \delta)$  centered at  $p$  with radius  $\delta$  in  $\bar{L}$ , and let  $\eta$  be the unit normal vector field to  $D_{\bar{L}}(p, \delta)$  in  $N$ . Pick coordinates  $q = (x, y)$  in  $D_{\bar{L}}(p, \delta)$  and let  $t \in [-\tau, \tau] \mapsto \gamma_q(t)$  be the unit speed geodesic of  $N$  with initial conditions  $\gamma_q(0) = q, \gamma'_q(0) = \eta(q)$  (here  $\tau > 0$  can be taken independent of  $q \in D_{\bar{L}}(p, \delta)$ ). Then for some  $\tau > 0$  small,  $(x, y, t)$  produces “cylindrical” normal coordinates in a

neighborhood  $U$  of  $p$  in  $N$ , and we can consider the natural projection

$$\Pi: U \rightarrow D_{\bar{L}}(p, \delta), \quad \Pi(x, y, t) = (x, y).$$

Since every blow-up limit of  $\mathcal{L}$  from  $p$  converges outside  $\vec{0} \in \mathbb{R}^3$  to a lamination of  $\mathbb{R}^3$  by planes parallel to  $T_p\bar{L}$ , we conclude that for  $\delta$  and  $\tau$  sufficiently small, the angle of the intersection of any leaf component  $L_U$  of  $\mathcal{L} \cap U$  with any geodesic  $\gamma_q$  as above can be made arbitrarily close to  $\pi/2$ . Taking  $\delta$  much smaller than  $\tau$ , a monodromy argument implies any leaf component  $L_U$  of  $\mathcal{L} \cap U$  which contains a point at distance at most  $\delta/2$  from  $p$  is a graph over  $D_{\bar{L}}(p, \delta)$  (in other words,  $\Pi$  restricts to  $L_U$  as a diffeomorphism onto  $D_{\bar{L}}(p, \delta)$ ). We will finish this Case V by proving that this graphical property implies the desired uniform bound for the second fundamental form  $\sigma_{\mathcal{L}}$  of  $\mathcal{L}$  around  $p$ : otherwise there exists a sequence of points  $p_n$  in leaves  $L_n$  of  $\mathcal{L}$  converging to  $p$ , such that  $|\sigma_{L_n}|(p_n)$  diverges. Without loss of generality, we can assume that  $p_n \in U$  and  $p_n$  is a point where the following function attains its maximum:

$$f_n = |\sigma_{L_n}| d_{\bar{L}}(\Pi(\cdot), \partial D_{\bar{L}}(p, \delta)): L_n \cap U \rightarrow [0, \infty),$$

where  $d_{\bar{L}}$  denotes the intrinsic distance in  $\bar{L}$  to the boundary  $\partial D_{\bar{L}}(p, \delta)$ . Now expand the above coordinates  $(x, y, z)$  centered at  $p_n$  (so that  $p_n$  becomes the origin) by the factor  $|\sigma_{L_n}|(p_n) \rightarrow \infty$ . Under this expansion,  $U$  converges to  $\mathbb{R}^3$  with its usual flat metric and the geodesics  $\gamma_q$  converge to parallel lines (the canonical coordinates  $(x, y, z)$  in  $\mathbb{R}^3$  are not necessarily those coming from the  $(x, y, t)$ -coordinates in  $U$ ). The graphical property that  $\Pi$  restricts to  $L_U$  as a diffeomorphism onto  $D_{\bar{L}}(p, \delta)$  gives that after passing to a subsequence, the minimal graphs  $L_n \cap U$  converge to a minimal surface in  $\mathbb{R}^3$  which is a entire graph. By the Bernstein Theorem, such a limit is a flat plane. This contradicts that the homothetic expansion factors coincide with the norms of the second fundamental form of  $L_n \cap U$  at  $p_n$  for all  $n$ . This contradiction finishes the proof of Case V, and thus Theorem 1.1 is proved. q.e.d.

Theorem 1.1 supports the conjecture that a properly embedded minimal surface in a punctured ball extends smoothly through the puncture. An important partial result for this conjecture was obtained by Gulliver and Lawson [17], who proved it in the special case that the surface is stable (note that Theorem 1.1 generalizes the result by Gulliver and Lawson, by curvature estimates for stable minimal surfaces [36]). This isolated singularity conjecture is one of the fundamental open problems in minimal surface theory, and it is a special case of the following more general conjecture:

**Conjecture 5.1** (Fundamental Singularity Conjecture). *Suppose  $S \subset \mathbb{R}^3$  is a closed set whose one-dimensional Hausdorff measure is zero. If  $\mathcal{L}$  is a minimal lamination of  $\mathbb{R}^3 - S$ , then  $\mathcal{L}$  extends across  $S$  to a minimal lamination of  $\mathbb{R}^3$ .*

Since the union of a catenoid with a plane passing through its waist circle is a singular minimal lamination of  $\mathbb{R}^3$  whose singular set is the intersecting circle, the above conjecture represents the strongest possible conjecture. We point out to the reader that Conjecture 5.1 has a global nature, because there exist interesting minimal laminations of the open unit ball in  $\mathbb{R}^3$  punctured at the origin which do not extend across the origin, see Section 2. In hyperbolic three-space  $\mathbb{H}^3$ , there are rotationally invariant global minimal laminations which have a similar unique isolated singularity. The existence of these global singular minimal laminations of  $\mathbb{H}^3$  demonstrate that the validity of Conjecture 5.1 must depend on the metric properties of  $\mathbb{R}^3$ .

**6. The characterization of minimal surfaces with quadratic decay of curvature.**

In this section we will prove Theorem 1.2 stated in the introduction.

**Proposition 6.1.** *Let  $\mathcal{L}$  be a non-flat minimal lamination of  $\mathbb{R}^3 - \{\vec{0}\}$  with quadratic decay of curvature. Then,  $\mathcal{L}$  consists of a single leaf, which extends to a connected, properly embedded minimal surface in  $\mathbb{R}^3$ .*

*Proof.* By Proposition 4.11, each leaf  $L$  of  $\mathcal{L}$  is a minimal surface which is properly embedded in  $\mathbb{R}^3 - \{\vec{0}\}$ . Applying Theorem 1.1 to each  $L \in \mathcal{L}$ , we deduce that  $L$  extends to a properly embedded minimal surface in  $\mathbb{R}^3$ . Finally,  $\mathcal{L}$  consists of a single leaf by the maximum principle applied at the origin and the Strong Half-space Theorem [19].  
q.e.d.

**Theorem 6.2.** *Let  $M \subset \mathbb{R}^3$  be a complete, embedded, non-flat minimal surface with compact boundary (possibly empty). If  $M$  has quadratic decay of curvature, then  $M$  is properly embedded in  $\mathbb{R}^3$  with finite total curvature.*

*Proof.* We first check that  $M$  is proper when  $\partial M$  is empty. In this case where  $\partial M = \emptyset$ , the closure of  $M$  in  $\mathbb{R}^3 - \{\vec{0}\}$  is a minimal lamination of  $\mathbb{R}^3 - \{\vec{0}\}$  satisfying the conditions in Proposition 6.1. It follows that  $M$  is a properly embedded minimal surface in  $\mathbb{R}^3$  with bounded curvature.

We now prove that  $M$  is also proper when  $\partial M \neq \emptyset$ . Since  $\partial M$  is compact, we may assume  $\vec{0} \notin \partial M$  by removing a compact subset from  $M$ . Therefore, there exists an  $\varepsilon > 0$  such that  $\partial M \subset \mathbb{R}^3 - \mathbb{B}(\varepsilon)$ . Thus, Theorem 1.1 gives that  $\overline{M} \cap (\mathbb{B}(\varepsilon) - \{\vec{0}\})$  has bounded curvature, and so,  $M$  does as well (in order to apply Theorem 1.1 we need  $M \cap (\mathbb{B}(\varepsilon) - \{\vec{0}\})$  to be non-empty; but otherwise  $M$  would have bounded curvature so we would arrive to the same conclusion). If  $M$  were not proper in  $\mathbb{R}^3$ , then  $\overline{M} - \partial M$  has the structure of a minimal lamination of  $\mathbb{R}^3 - \partial M$  with a

limit leaf  $L$  which can be assumed to be disjoint from  $M$  (otherwise  $M$  is stable with compact boundary, hence  $M$  has finite total curvature by Fischer-Colbrie [12] and thus,  $M$  is proper). Since we may also assume, after possibly removing an intrinsic neighborhood of  $\partial M$ , that  $\bar{L} \cap \partial M = \emptyset$ , then  $L$  is complete and stable, and hence,  $L$  is a plane. Since  $M$  limits to  $L$  and  $M$  has bounded curvature, we obtain a contradiction by applying the arguments in the proof of Lemma 1.3 in Meeks and Rosenberg [29]. Hence,  $M$  is proper regardless of whether or not  $\partial M$  is empty.

From now on, we will assume that  $M$  is non-compact and properly embedded in  $\mathbb{R}^3$ . Since  $\partial M$  is compact (possibly empty), there exists an  $R_1 > 0$  such that  $\partial M \subset \mathbb{B}(R_1)$ . It remains to show that  $M$  has finite total curvature.

Let  $C_1 \in (0, 1)$  be the constant given by the statement of Lemma 4.1. Suppose first that there exists  $R_2 > R_1$  such that  $|K_M|R^2 \leq C_1$  in  $M - \mathbb{B}(R_2)$ . Applying Lemma 4.1 to each component of  $M - \mathbb{B}(R_2)$ , such components are annular ends with finite total curvature. Since  $M$  is proper, there are a finite number of such components as  $M \cap \mathbb{S}^2(R_2)$  is compact. Thus,  $M$  has finite total curvature, which proves the theorem in this case.

Now assume that there exists a sequence  $\{p_n\}_n \subset M$  diverging to  $\infty$  such that  $C_1 \leq |K_M|(p_n)|p_n|^2$  for all  $n$ , and we will find a contradiction. The homothetically shrunk surfaces  $\widetilde{M}_n = \frac{1}{|p_n|}M$  also have quadratic curvature decay and their boundaries collapse to  $\vec{0}$ . Thus, after choosing a subsequence, we may assume that the  $\widetilde{M}_n$  converge to a minimal lamination  $\mathcal{L}$  of  $\mathbb{R}^3 - \{\vec{0}\}$ , and  $|K_{\mathcal{L}}|R^2$  also decays quadratically. Since  $|K_{\widetilde{M}_n}|(\frac{1}{|p_n|}p_n) \geq C_1$  and we can assume  $\frac{1}{|p_n|}p_n \rightarrow \tilde{p}_\infty \in \mathbb{S}^2(1)$ , there exists a non-flat leaf  $L \in \mathcal{L}$  with  $\tilde{p}_\infty \in L$ . By Proposition 6.1,  $\mathcal{L} = \{L\}$  and  $\bar{L}$  is properly embedded in  $\mathbb{R}^3$ . If the convergence of the  $\widetilde{M}_n$  to  $\mathcal{L}$  had multiplicity greater than one, then  $L$  would be flat (see Corollary 3.5), but it is not. Also note that  $\bar{L}$  is connected, and so, it must pass through the origin. Since  $\bar{L}$  is properly embedded, the multiplicity of the limit  $\widetilde{M}_n \rightarrow L$  is one and  $\vec{0} \in \bar{L}$ , then we have  $\lim_{r \rightarrow 0} r^{-2} \text{Area}(\bar{L} \cap \mathbb{B}(r)) = \pi$  (i.e. the density of  $\bar{L}$  at the origin is 1) and there exists  $\varepsilon > 0$  such that for all  $\varepsilon' \in (0, \varepsilon]$ ,  $\bar{L} \cap \mathbb{B}(\varepsilon')$  consists of a non-flat disk passing through the origin and transverse to  $\mathbb{S}^2(\varepsilon')$  along its boundary.

We will study the function

$$R \in [R_1, \infty) \mapsto f(R) = R^{-2} \text{Area}(M \cap [\mathbb{B}(R) - \mathbb{B}(R_1)]),$$

which is non-decreasing by the monotonicity formula. By equation (6), we have

$$(10) \quad f(\varepsilon|p_n|) = \frac{1}{2\varepsilon^2|p_n|^2} \int_{\partial M} \langle p, \nu \rangle + \frac{1}{2\varepsilon^2|p_n|^2} \int_{M \cap \mathbb{S}^2(\varepsilon|p_n|)} \langle p, \nu \rangle,$$

where  $\nu$  is the unit exterior conormal vector to  $M \cap \mathbb{B}(\varepsilon|p_n|)$  along its boundary. Changing variables in the second integral in (10) we have

$$f(\varepsilon|p_n|) = \frac{1}{2\varepsilon^2|p_n|^2} \int_{\partial M} \langle p, \nu \rangle + \frac{1}{2\varepsilon^2} \int_{(\frac{1}{|p_n|}M) \cap \mathbb{S}^2(\varepsilon)} \langle q, \nu \rangle.$$

Since the sequence  $\{\frac{1}{|p_n|}M\}_n$  converges to  $L$  with multiplicity one on compact subsets of  $\mathbb{R}^3 - \{\vec{0}\}$ , then

$$(11) \quad \lim_{n \rightarrow \infty} f(\varepsilon|p_n|) = \frac{1}{2\varepsilon^2} \int_{L \cap \mathbb{S}^2(\varepsilon)} \langle q, \nu \rangle = \frac{1}{\varepsilon^2} \text{Area}[L \cap \mathbb{B}(\varepsilon)].$$

Since  $f$  is monotonically non-decreasing, we conclude from (11) that  $\lim_{R \rightarrow \infty} f(R)$  exists, and equals  $l(\varepsilon) = \frac{1}{\varepsilon^2} \text{Area}[L \cap \mathbb{B}(\varepsilon)]$ . This is a contradiction, since  $l(\varepsilon)$  is strictly increasing as a function of  $\varepsilon$  small as  $L$  is not flat. This contradiction proves the theorem. q.e.d.

**Corollary 6.3.** *Let  $\mathcal{L}$  be a non-flat minimal lamination of  $\mathbb{R}^3 - \{\vec{0}\}$ . If  $\mathcal{L}$  has quadratic decay of curvature, then  $\mathcal{L}$  consists of a single leaf, which extends to a properly embedded minimal surface with finite total curvature in  $\mathbb{R}^3$ .*

*Proof.* This follows easily from Proposition 6.1 and Theorem 6.2. q.e.d.

Theorem 1.2 follows immediately from Theorem 6.2. We just remark that the last statement in Theorem 1.2 follows from the finite total curvature property, since a non-flat, complete, embedded, non-compact minimal surface of finite total curvature has a positive number of catenoidal ends and possibly finitely many planar ends. A simple calculation shows that the growth constant  $C^2$  in Theorem 1.2 depends on the maximum logarithmic growth  $C$  of the catenoidal ends of  $M$ .

**Remark 6.4.** Given  $C > 0$ , let  $\mathcal{F}_C$  denote the family of all complete, embedded, connected minimal surfaces  $M \subset \mathbb{R}^3$  with quadratic curvature decay constant  $C$ , normalized so that the maximum of the function  $|K_M|R^2$  occurs at a point of  $M \cap \mathbb{S}^2(1)$ . In [27] we applied Theorem 1.2 to prove that  $\mathcal{F}_C$  is naturally a compact metric space and that for  $C$  fixed, there is a bound on the genus and number of ends of all surfaces in  $\mathcal{F}_C$  and that the subsets of  $\mathcal{F}_C$  with fixed topology are compact.

### 7. Minimal surfaces and minimal laminations with countably many singularities.

In Theorem 1 of [28] (see also [27]) we proved that the sublamination  $\text{Lim}(\mathcal{L})$  of limit leaves of a minimal lamination  $\mathcal{L}$  of a three-manifold  $N$  consists of stable minimal surfaces, and more strongly, their two-sided covers are stable. An immediate consequence of this result is that the set  $\text{Stab}(\mathcal{L})$  of stable leaves of  $\mathcal{L}$  is a sublamination of  $\mathcal{L}$  with

$\text{Lim}(\mathcal{L}) \subset \text{Stab}(\mathcal{L})$ . Using these observations together with standard curvature estimates [27, 35, 36] for stable two-sided minimal surfaces away from their boundaries, we will demonstrate the following consequence of Theorem 1.1.

**Corollary 7.1.** *Suppose that  $N$  is a not necessarily complete Riemannian three-manifold. If  $W \subset N$  is a closed countable subset and  $\mathcal{L}$  is a minimal lamination of  $N - W$ , then the closure of any collection of its stable leaves extends across  $W$  to a minimal lamination of  $N$  consisting of stable minimal surfaces. In particular:*

- 1) *The closure  $\overline{\text{Stab}(\mathcal{L})}$  in  $N$  of the collection of stable leaves of  $\mathcal{L}$  is a minimal lamination of  $N$  whose leaves are stable minimal surfaces.*
- 2) *The closure  $\overline{\text{Lim}(\mathcal{L})}$  in  $N$  of the collection of limit leaves of  $\mathcal{L}$  is a minimal lamination of  $N$ .*
- 3) *If  $\mathcal{L}$  is a minimal foliation of  $N - W$ , then  $\mathcal{L}$  extends across  $W$  to a minimal foliation of  $N$ .*

*Proof.* We start by proving the first statement in Corollary 7.1. Since the extension of the closure  $\mathcal{L}_1$  of any given collection of stable leaves in  $\mathcal{L}$  is a local question, it suffices to prove the corollary in small, open extrinsic balls in  $N$ . As  $W$  is countable, we can take these balls so that each of their boundaries are disjoint from  $W$ , and their closures in  $N$  are compact. It follows that for every such ball  $B_N$ , the set  $W \cap B_N$  is a complete countable metric space. By Baire's theorem, the set of isolated points in  $W \cap B_N$  is dense in  $W \cap B_N$ . Curvature estimates for stable minimal surfaces [34, 35, 36] together with Theorem 1.1 imply that  $\mathcal{L}_1 \cap B_N$  extends across every isolated point of  $W \cap B_N$  to a minimal lamination of  $B_N - W'$ , where  $W'$  is the subset of non-isolated points in  $W \cap B_N$ . Consider the minimal closed subset  $W''$  of  $W \cap B_N$  (under inclusion) such that  $\mathcal{L}_1$  does not extend across any point  $p \in W''$  to a minimal lamination. We want to prove that  $W'' = \emptyset$ . Arguing by contradiction, suppose  $W'' \neq \emptyset$ . As before,  $W''$  is a countable complete metric space and so, Baire's theorem insures that the set of its isolated points is dense in  $W''$ . But the lamination  $\mathcal{L}_2$  obtained by extension of  $\mathcal{L}_1$  across  $W - W''$ , extends through every isolated point of  $W''$  by the above arguments; hence we contradict the minimality of  $W''$  and the first statement in Corollary 7.1 is proved.

Note that items 1, 2, 3 of this corollary follow directly from the already proven first statement of the corollary and from the following facts:

- Limit leaves of a minimal lamination are stable [27].
- A smooth limit of limit leaves of a lamination is also a limit leaf.
- Every leaf of a foliation is a limit leaf.

Hence, the proof is complete.

q.e.d.

As an application of Corollary 7.1, we have the following generalization of the stability lemma (Lemma 3.3) in the minimal case. The proof of the next result follows immediately from the fact that every embedded, stable minimal surface in a Riemannian three-manifold has local curvature estimates, and so, its closure has the structure of a minimal lamination all whose leaves are stable.

**Corollary 7.2.** *Let  $M$  be a connected, embedded, stable minimal surface in a Riemannian three-manifold  $N$ . Suppose that  $M$  is complete outside a countable closed set  $S$  of  $N$ . Then, the closure of  $M$  has the structure of a minimal lamination of  $N$ , and the intrinsic metric completion of  $M$  is a leaf of this lamination. In particular, if  $N = \mathbb{R}^3$ , then the closure of  $M$  is a plane.*

**8. Appendix: An alternative proof of Proposition 4.11.**

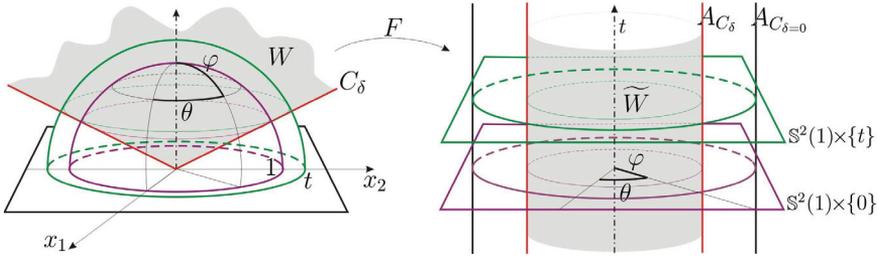
The main technical result of this paper, Proposition 4.11, was proved as a consequence of Lemmas 4.3 and 4.4, together with an analysis of type I and type II curves. In this analysis we used flux arguments, one of which was based in Colding-Minicozzi theory. We next explain how to rule out both cases with a different argument, based on conformal properties for minimal surfaces. The arguments that follow will use the same notation and results in Section 4, that will be assumed to hold up to Section 4.1.

Consider the metric  $\widehat{g} = \frac{1}{R^2} \langle, \rangle$  on  $\mathbb{R}^3 - \{\vec{0}\}$ . As we noticed in the proof of Lemma 3.3,  $(\mathbb{R}^3 - \{\vec{0}\}, \widehat{g})$  is isometric to  $\mathbb{S}^2(1) \times \mathbb{R}$  endowed with its product metric  $\widetilde{g}$ . In fact, the map

$$(12) \quad F: (\mathbb{R}^3 - \{\vec{0}\}, \widehat{g}) \rightarrow (\mathbb{S}^2(1) \times \mathbb{R}, \widetilde{g}), \quad F(p) = \left( \frac{p}{R(p)}, \log R(p) \right),$$

is an isometry. If  $C$  is any non-negative vertical half-cone  $C$  minus its vertex at  $\vec{0}$ , then  $A_C := F(C)$  is the flat cylinder  $(C \cap \mathbb{S}^2(1)) \times \mathbb{R}$ . In the particular case where  $C$  is the  $(x_1, x_2)$ -plane, then  $A_C = \mathbb{S}^1(1) \times \mathbb{R}$  is totally geodesic. We endow  $\mathbb{S}^2(1) \times \mathbb{R}$  with global coordinates  $(\varphi, \theta, t)$  so that  $(\varphi, \theta)$  are the natural spherical coordinates on  $\mathbb{S}^2(1)$  and  $t$  denotes the vertical linear coordinate in  $\mathbb{S}^2(1) \times \mathbb{R}$  (we will consider  $t$  to be the vertical height in  $\mathbb{S}^2(1) \times \mathbb{R}$ ); recall that  $\varphi \in [0, \pi]$  measures the angle with the positive vertical axis in  $\mathbb{R}^3$  and  $\theta \in [0, 2\pi)$ . Let  $W = H^+ - C_\delta^- \subset \mathbb{R}^3 - \{\vec{0}\}$  be the closure in  $H^+$  of the convex region above  $C_\delta$ , and let  $\widetilde{W} = F(W)$  be the closed solid cylinder bounded by  $A_{C_\delta}$ ; see Figure 7.

Consider a leaf  $L$  of a minimal lamination  $\mathcal{L}$  of  $\mathbb{R}^3 - \{\vec{0}\}$  satisfying the hypotheses and conclusions of Lemmas 4.3 and 4.4. Therefore,  $\widehat{L} = (L, \widehat{g}|_L) \subset (\mathbb{R}^3 - \{\vec{0}\}, \widehat{g})$  is complete. Given a surface  $\Sigma \subset \mathbb{R}^3 - \{\vec{0}\}$ , we will denote by  $\widetilde{\Sigma}$  the image  $F(\Sigma)$  under  $F$ .



**Figure 7.**  $F(p) = (\frac{p}{R(p)}, \log R(p))$  is an isometry between  $(\mathbb{R}^3 - \{\vec{0}\}, \hat{g})$  and  $(\mathbb{S}^2(1) \times \mathbb{R}, \tilde{g})$ .

**Lemma 8.1.** *In the above situation, the  $\tilde{g}$ -area of  $(\tilde{L} \cap \tilde{W}) \cap \{T < t < T+1\}$  is bounded independently of  $T \in \mathbb{R}$ . In particular, there exists  $c > 0$  such that for any  $a > 1$ , the  $\tilde{g}$ -area of  $(\tilde{L} \cap \tilde{W}) \cap \{T < t < T+a\}$  is not greater than  $ca$ .*

*Proof.* Suppose first that there exists a sequence  $r_n > 0$  such that the area of  $L \cap W \cap \{r_n < R < e \cdot r_n\}$  is greater than any constant times  $r_n^2$  for  $n$  large enough. Then the areas of the surfaces  $\frac{1}{r_n}(L \cap W \cap \{r_n < R < e \cdot r_n\})$  become unbounded in a compact region of  $H^+$ . As  $L$  is assumed to have quadratic curvature decay, then it follows that a subsequence of  $\frac{1}{r_n}L$  converges to a lamination of  $\mathbb{R}^3 - \{\vec{0}\}$ , with a leaf  $L_1$  which is either a limit leaf or a leaf for which the multiplicity of the convergence of the sequence  $\{\frac{1}{r_n}L\}_n$  to  $L_1$  is infinite. In either of these two cases,  $L_1 \subset H^+$  which must be a horizontal plane; see Corollaries 3.4 and 3.5. The existence of such a plane implies that the curves of the set  $\Lambda$  (defined in the paragraph just before Section 4.1) are of type I, and that below  $C_\delta$ ,  $L$  consists of annular graphs. This property implies that the lamination  $\mathcal{L}$  of which  $L$  is a leaf contains an entire graph, which must be a horizontal plane of positive height. Applying the arguments in the proof of Lemma 4.3 we can rule out the existence of such a plane, which then proves that the area of  $L \cap W \cap \{r < R < e \cdot r\}$  divided by  $r^2$  is bounded from above independently of  $r > 0$ .

The last property implies that there exists some constant  $c_1 > 0$  such that for any  $\lambda > 0$ ,

$$\begin{aligned} & r^{-2} \text{Area} [\lambda(L \cap W) \cap \{r < R < e \cdot r\}] \\ &= (r/\lambda)^{-2} \text{Area} \left[ (L \cap W) \cap \left\{ \frac{r}{\lambda} < R < e \cdot \frac{r}{\lambda} \right\} \right] \leq c_1. \end{aligned}$$

Since vertical translations  $\psi_T(q, t) = (q, t+T)$  in  $\mathbb{S}^2(1) \times \mathbb{R}$  are isometries of the product metric  $\tilde{g}$  on  $\mathbb{S}^2(1) \times \mathbb{R}$ , then homotheties centered at the origin are isometries of  $(\mathbb{R}^3 - \{0\}, \hat{g})$ . Thus,

$$\text{Area}_{\tilde{g}} \left[ (\tilde{L} \cap \tilde{W}) \cap \{T < t < T+1\} \right]$$

$$\begin{aligned}
 &= \text{Area}_{\tilde{g}} \left[ \psi_{-T}(\tilde{L} \cap \tilde{W}) \cap \{0 < t < 1\} \right] \\
 &= \text{Area}_{\tilde{g}} \left[ ((e^{-T}L) \cap W) \cap \{1 < R < e\} \right] \\
 &\stackrel{(A)}{\leq} \text{Area}_g \left[ (e^{-T}L) \cap W \cap \{1 < R < e\} \right] \leq c_1,
 \end{aligned}$$

where in (A) we have used that  $\hat{g} = \frac{1}{R^2}g$  and  $R \geq 1$  in the range where we are computing areas. This finishes the proof of the lemma. q.e.d.

**Proposition 8.2.** *The complete surface  $(\tilde{L} = F(L), \tilde{g}|_{\tilde{L}})$  has at most quadratic area growth.*

*Proof.* We will divide the proof in two cases, according to the type of curves in  $\Lambda$  (with the notation just after Section 4.1).

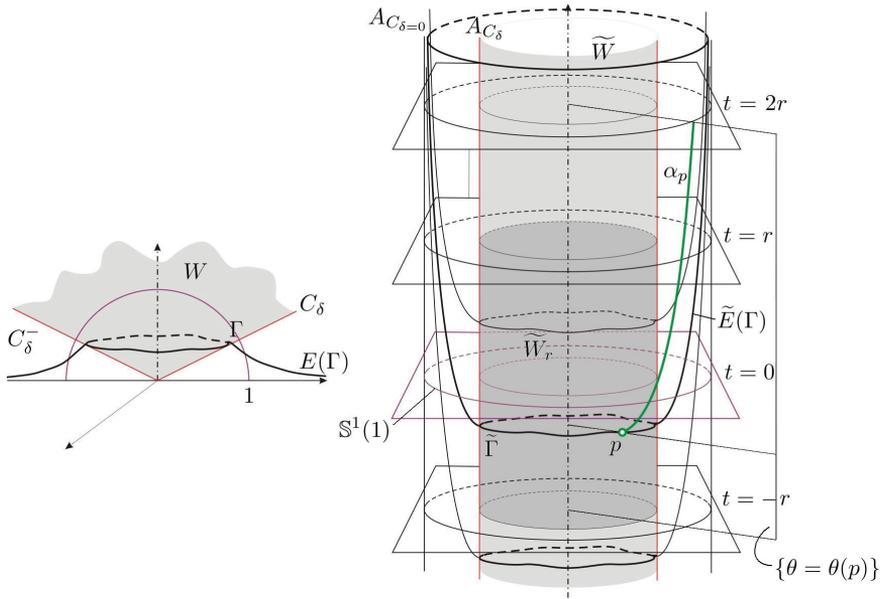
SUPPOSE THAT THE CURVES IN  $\Lambda$  ARE OF TYPE I.

Given  $\Gamma \in \Lambda$ ,  $\tilde{\Gamma} := F(\Gamma)$  is an almost horizontal circle in  $A_{C_\delta} = F(C_\delta) = \partial\tilde{W}$ . Let  $\tilde{\Lambda}$  denote the collection of these curves  $\tilde{\Gamma}$ . Let  $E(\Gamma)$  be the component of  $L \cap C_{\delta'}^-$  with  $\partial E(\Gamma) = \Gamma$ . Since each of the components in  $L \cap C_{\delta'}$  is of the same type as  $\Gamma$  for any  $\delta' \in (0, \delta)$ , then  $E(\Gamma)$  is an annulus. Let  $\tilde{E}(\Gamma) = F(E(\Gamma)) \subset \mathbb{S}^2(1) \times \mathbb{R}$  be the related surface. Note that  $\tilde{E}(\Gamma)$  is asymptotic to the end of  $\mathbb{S}^1(1) \times [0, \infty)$  and that  $\tilde{E}(\Gamma)$  is a small  $\varphi$ -graph over its projection to  $\mathbb{S}^1(1) \times \mathbb{R}$ ; see Figure 8. In fact, this  $\varphi$ -graph has small  $\tilde{g}$ -gradient as such a gradient measures the  $\tilde{g}$ -angle between the tangent space to  $\tilde{E}(\Gamma)$  with the vertical cylinders in  $\mathbb{S}^2(1) \times \mathbb{R}$ , but this  $\tilde{g}$ -angle coincides with the angle between the tangent space to  $E(\Gamma)$  with the cones  $C_{\delta'}$ ,  $\delta' \in (0, \delta)$  (since the metric  $\hat{g}$  is conformal to usual inner product in  $\mathbb{R}^3$ ), which can be made arbitrarily small by Lemma 4.4.

For  $d > 0$ , let  $[\tilde{E}(\Gamma)](d)$  be the subset of  $\tilde{E}(\Gamma)$  consisting of those points at intrinsic distance at most  $d$  from  $\tilde{\Gamma} = \partial E(\Gamma)$ . Then for  $\delta > 0$  sufficiently small and  $d \geq 1$ , the  $\tilde{g}$ -area of  $[\tilde{E}(\Gamma)](d)$  is less than  $3\pi d$  and the limit as  $d \rightarrow \infty$  of such an area divided by  $d$  is  $2\pi$ .

By Lemma 8.1, for  $T_0 > 1$  fixed and for  $T \in \mathbb{R}$ , the  $\tilde{g}$ -area of  $(\tilde{L} \cap \tilde{W}) \cap \{T - 1 < R < T + T_0 + 1\}$  is bounded independently of  $T$ . Since the angle between  $\partial\tilde{W}$  and  $\tilde{L}$  is small but bounded away from zero and the components of  $\tilde{\Lambda}$  that intersect the region  $\mathbb{S}^2(1) \times [T, T + T_0]$  are contained in  $\mathbb{S}^2(1) \times [T - \frac{1}{2}, T + T_0 + \frac{1}{2}]$ , then the total  $\tilde{g}$ -length of components of  $\tilde{\Lambda}$  that intersect the region  $\mathbb{S}^2(1) \times [T, T + T_0]$  is bounded from above independently of  $T$  as is the number of these components.

For  $r > 1$ , let  $\tilde{W}_r = \tilde{W} \cap (\mathbb{S}^2(1) \times [-r, r])$  and let  $\tilde{\Lambda}(r)$  denote the collection of those elements of  $\tilde{\Lambda}$  which intersect  $\partial\tilde{W}_r$ . For  $r$  large, let



**Figure 8.** Quadratic area growth in the case of curves in  $\Lambda$  of type I.

$\Delta(r) \subset \tilde{L}$  be union of  $\tilde{L} \cap \tilde{W}_r$  with the region

$$V_r = \bigcup_{\tilde{\Gamma} \in \tilde{\Lambda}(r)} \left( \bigcup_{p \in \tilde{\Gamma}} \alpha_p \right)$$

where for each  $p \in \tilde{\Gamma}$  with  $\tilde{\Gamma} \in \tilde{\Lambda}(r)$ , (with related component  $\Gamma = F^{-1}(\tilde{\Gamma}) \in \Lambda$ ),  $\alpha_p$  is the component of  $\tilde{E}(\Gamma) \cap \{\theta = \theta(p)\}$  whose end points are  $p$  and a point at  $\{t = 2r\}$ , see Figure 8 right. Given  $\tilde{\Gamma} \in \tilde{\Lambda}(r)$ , the geometric meaning of  $\bigcup_{p \in \tilde{\Gamma}} \alpha_p$  is the compact annular piece of the end  $\tilde{E}(\Gamma)$  of  $\tilde{L}$  bounded by the almost horizontal circle  $\tilde{\Gamma}$  and by a horizontal Jordan curve at constant height  $2r$  in  $\mathbb{S}^2(1) \times \mathbb{R}$  which is arbitrarily close to the circle  $\mathbb{S}^1(1) \times \{2r\}$  if  $r$  is taken large enough; see Figure 8 right. Note that  $V_r$  is just a finite union of these compact annuli, indexed by the curves  $\tilde{\Gamma} \in \tilde{\Lambda}(r)$ . By the last paragraph, there exists a universal constant  $c > 0$  such that for  $r > 1$ , the number of  $\tilde{\Gamma}$ -curves such that  $\partial \tilde{E}(\Gamma)$  intersects  $\mathbb{S}^2(1) \times [-r, r]$ , divided by  $r$ , is less than  $c$ . Also, note that for  $r$  large, the  $\tilde{g}$ -length of any of these  $\tilde{\Gamma}$ -curves is not greater than  $3\pi$  and the  $\tilde{g}$ -length of any of the considered  $\alpha_p$ -curves is less than or equal to  $4r$ . Therefore, the  $\tilde{g}$ -area of  $V_r$  is certainly less than  $3\pi \cdot 4r \cdot cr = 12\pi cr^2$  for  $r$  large. Since the  $\tilde{g}$ -area of  $\tilde{L} \cap \tilde{W}_r$  grows linearly in  $r$ , then the  $\tilde{g}$ -area of  $\Delta(r)$  is at most  $13\pi cr^2$  for  $r$  large.

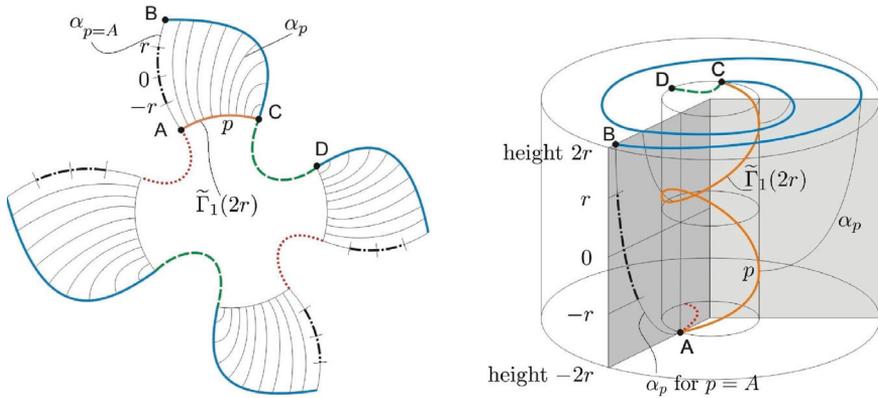
Fix  $p_0 \in \tilde{L} \cap W \cap (\mathbb{S}^2(1) \times \{0\})$  and let  $B_{\tilde{L}}(p_0, r)$  be the intrinsic open ball of radius  $r > 0$  centered at  $p_0$ . We will show that  $B_{\tilde{L}}(p_0, r) \subset \Delta(r)$ . First note that  $B_{\tilde{L}}(p_0, r)$  is contained in the region  $\mathbb{S}^2(1) \times (-r, r)$ . Let  $p$  be a point in  $B_{\tilde{L}}(p_0, r)$ . If  $p \in \tilde{W}_r$ , then by definition  $p \in \Delta(r)$ . Suppose  $p \notin \tilde{W}_r$  and let  $\gamma \subset \tilde{L}$  be a curve of  $\tilde{g}$ -length less than  $r$  joining  $p$  with  $p_0$ . Since  $\partial\Delta(r)$  does not intersect the region  $\mathbb{S}^2(1) \times (-r, r)$ , then  $\gamma$  does not intersect the boundary of  $\Delta(r)$ . As  $p_0 \in \Delta(r)$ , we conclude  $\gamma \subset \Delta(r)$ . Hence,  $p \in \Delta(r)$ . Finally, since  $B_{\tilde{L}}(p_0, r) \subset \Delta(r)$ , we deduce from the last paragraph that the intrinsic area growth of  $\tilde{L}$  is at most quadratic. This completes the proof of Proposition 8.2 provided that the curves in  $\Lambda$  are of type I.

NEXT ASSUME THAT THE CURVES IN  $\Lambda$  ARE OF TYPE II.

As in the previous case of type I curves, we consider the similar objects:

- $\Gamma \in \Lambda$ , which is now a spiraling curve in  $C_\delta$  limiting down to  $\bar{0}$  and rotating infinitely many times around  $C_\delta$ . We have  $k$  of these curve components in  $\Lambda = \{\Gamma_1, \dots, \Gamma_k\}$ .
- $E(\Gamma)$ , the component of  $L \cap C_\delta^-$  with  $\partial E(\Gamma) = \Gamma$ . Note that  $E(\Gamma)$  is topologically a half-plane.
- $\tilde{E}(\Gamma) = F(E(\Gamma))$ , which is also topologically a half-plane, and it is a small  $\varphi$ -multigraph with small gradient over the cylinder  $\mathbb{S}^1(1) \times \mathbb{R}$  (see the previous type I curve case for this argument). Away from its boundary,  $\tilde{E}(\Gamma)$  is asymptotic as a set to  $\mathbb{S}^1(1) \times \mathbb{R}$ .
- $\tilde{W} = F(W)$ , an infinite open solid vertical cylinder in  $\mathbb{S}^2(1) \times \mathbb{R}$  bounded by  $A_{C_\delta} = F(C_\delta)$ . Given  $r > 0$ ,  $\tilde{W}_r = \tilde{W} \cap (\mathbb{S}^2(1) \times [-r, r])$  is the portion of  $\tilde{W}$  between heights  $-r$  and  $r$ .
- $\tilde{\Gamma} = \partial\tilde{E}(\Gamma) = F(\Gamma)$ , an infinite spiral in  $A_{C_\delta}$  whose height function is proper. We have  $k$  of these helix-type curves, and  $\tilde{\Lambda} = F(\Lambda) = \{\tilde{\Gamma}_1, \dots, \tilde{\Gamma}_k\}$ .

For  $r > 1$ , define  $\tilde{\Lambda}(r)$  to be the set of components in  $\bigcup_{i=1}^k [\tilde{\Gamma}_i \cap W_r]$  which contain an end point at height  $r$  and the other end point at height  $-r$ . Since the curves in  $\tilde{\Lambda}$  are very horizontal, for any  $r > 1$  the set  $\tilde{\Lambda}(r)$  consists of  $k$  spiraling arcs  $\tilde{\Gamma}_1(r), \dots, \tilde{\Gamma}_k(r)$ . Given  $i \in \{1, \dots, k\}$  and a point  $p$  in a spiraling arc  $\tilde{\Gamma}_i(2r) \in \tilde{\Lambda}(2r)$ , we call  $\alpha_p$  to the component of  $\tilde{E}(\Gamma) \cap \{\theta = \theta(p)\}$  whose end points are  $p$  and a point at  $\{t = 2r\}$ . Thus,  $\alpha_p$  is a planar Jordan arc and as we move the point  $p$  along  $\tilde{\Gamma}_i(2r)$  (with  $i$  fixed), the arcs  $\{\alpha_p \mid p \in \tilde{\Gamma}_i(2r)\}$  define a foliation of the compact disk



**Figure 9.** Left: The cross-shaped region is an intrinsic representation of  $\Delta(r)$  if the curves in  $\Lambda$  are of type II, here  $k = 4$ . The  $k$  disks in the “wings” of  $\Delta(r)$  form the set  $V_r$ , which is foliated by the Jordan arcs  $\alpha_p$ . The remaining “central” portion  $\tilde{L} \cap \tilde{W}_{2r}$  of  $\Delta(r)$  might have positive genus. The boundary  $\partial(\tilde{L} \cap \tilde{W}_{2r})$  of this central region is a cyclic union of spiraling arcs  $\tilde{\Gamma}_i(2r)$  joined by arcs at constant heights  $2r$  (dashed in the figure) and  $-2r$  (dotted in the figure). Right: a schematic representation of  $\bigcup_{p \in \tilde{\Gamma}_1(2r)} \alpha_p \subset V_r \subset \Delta(r)$  in  $S^2(1) \times \mathbb{R}$  (the spiraling arc  $\tilde{\Gamma}_1(2r)$  should be almost horizontal, although we have represented it reasonably steep in order to clarify the picture). The (blue) curve joining  $B$  to  $C$  lies entirely at height  $2r$  in  $S^2(1) \times \mathbb{R}$ , and spirals an arbitrarily large number of times (taking  $r > 0$  sufficiently large) towards  $S^1(1) \times \{r\}$  from its convex side.

$\bigcup_{p \in \tilde{\Gamma}_i(2r)} \alpha_p \subset \tilde{E}(\Gamma_i)$ . Let

$$(13) \quad V_r = \bigcup_{i=1}^k \left( \bigcup_{p \in \tilde{\Gamma}_i(2r)} \alpha_p \right)$$

be the union of these  $k$  disks, and let  $\Delta(r) \subset \tilde{L}$  be union of  $\tilde{L} \cap \tilde{W}_{2r}$  with  $V_r$ ; see Figure 9.

The bound of  $\tilde{g}$ -area density given by Lemma 8.1 implies that there exists a universal constant  $c > 0$  such that for  $r > 1$  large, the sum of the lengths of the curves in  $\tilde{\Lambda}(2r)$ , divided by  $r$ , is less than  $c$ . This last property together with the fact that the  $\tilde{g}$ -length of the  $\alpha_p$ -curves is not greater than  $5r$  for  $\delta > 0$  sufficiently small, implies that the  $\tilde{g}$ -area of  $V_r$  is less than or equal to  $cr \cdot 5r = 5cr^2$ . Since the  $\tilde{g}$ -area of  $\tilde{L} \cap \tilde{W}_{2r}$

grows linearly in  $r$ , then we conclude that the  $\tilde{g}$ -area of  $\Delta(r)$  is at most  $6cr^2$  for  $r$  large.

Let  $p_0 \in \tilde{L} \cap \tilde{W}$  be a point at height 0. It remains to check that  $B_{\tilde{L}}(p_0, r) \subset \Delta(r)$  in order to conclude that the intrinsic  $\tilde{g}$ -area growth of  $\tilde{L}$  is at most quadratic in case of type II curves in  $\Lambda$ . Observe that the boundary of  $\Delta(r)$  intersects the region  $\text{Ext}(\tilde{W}_r)$  defined as the intersection of the exterior of  $\tilde{W}$  with  $\mathbb{S}^2(1) \times [-r, r]$  along a finite collection  $\Sigma = \{\alpha_1, \dots, \alpha_k\}$  of almost vertical arcs  $\alpha_i \subset \alpha_{p_i}$ , where  $p_i$  is the extremum of  $\tilde{\Gamma}_i(2r)$  at height  $t = -2r$  (these  $\alpha_i$ -curves are represented by dotted-dashed lines  $-\cdot-\cdot-\cdot-$  in Figure 9).

By the previous argument for type I curves, in order to prove

$$B_{\tilde{L}}(p_0, r) \subset \Delta(r)$$

it suffices to show that if  $\gamma \subset \tilde{L}$  is an arc of  $\tilde{g}$ -length less than  $r$  starting at  $p_0$ , then  $\gamma \cap \partial\Delta(r) = \emptyset$ . Observe that  $\gamma$  cannot intersect the portion of  $\partial\Delta(r)$  at height  $2r$  or  $-2r$  (since  $p_0$  is at height zero and the  $\tilde{g}$ -length of  $\gamma$  is less than  $r$ ), so it suffices to discard any intersection of  $\gamma$  with a portion of  $\partial\Delta(r)$  in  $V_r$ . So assume that the arc  $\gamma$  contains a point  $q \in V_r \cap \partial\Delta(r)$ . As  $\gamma$  has  $\tilde{g}$ -length less than  $r$  and starts at height zero, then  $\gamma$  lies entirely in  $\mathbb{S}^2(1) \times [-r, r]$ . This implies that  $q$  must lie in one of the almost vertical arcs  $\alpha_i, i = 1, \dots, k$ , and by continuity  $\gamma$  must intersect  $\tilde{\Gamma}_i \cap (\mathbb{S}^2(1) \times [-r, r])$  at some point  $q'$  so that the subarc  $\gamma_{q,q'}$  of  $\gamma$  between  $q$  and  $q'$  lies entirely in the topological half plane  $\tilde{E}(\Gamma_i)$ . Note that we can assign to the spiraling arc  $\tilde{\Gamma}_i$  a well-defined, real angle valued function  $\hat{\theta}$  which coincides with  $\theta \bmod 2\pi$ . Furthermore,  $\hat{\theta}$  extends to a continuous function (denoted in the same way) on  $\tilde{E}(\Gamma)$ . Since the curve  $\tilde{\Gamma}_i$  is almost horizontal and  $\hat{\theta} \circ \gamma_{q,q'}$  is continuous, the absolute difference between the  $\hat{\theta}$ -values of the end points of  $\gamma_{q,q'}$  is much larger than  $r$ . This implies that the  $\tilde{g}$ -length of  $\gamma_{q,q'}$  is also larger than  $r$ , which is a contradiction. This contradiction demonstrates that  $B_{\tilde{L}}(p_0, r) \subset \Delta(r)$ , and thus, the intrinsic area growth of  $\tilde{L}$  is at most quadratic provided that the curves in  $\Lambda$  are of type II. Now the proof of Proposition 8.2 is complete. q.e.d.

We are now ready to reprove Proposition 4.11. Arguing by contradiction, suppose  $L$  is a leaf of a minimal lamination  $\mathcal{L}$  of  $\mathbb{R}^3 - \{\vec{0}\}$  with quadratic decay of curvature, and suppose that  $L$  is not proper in  $\mathbb{R}^3 - \{\vec{0}\}$ . By Lemmas 4.3 and 4.4, after a rotation we can assume  $L \subset H^+, \lim(L) = \{x_3 = 0\} - \{\vec{0}\}$ . Consider the conformal change of metric  $\hat{g} = \frac{1}{R^2} \langle, \rangle$  on  $\mathbb{R}^3 - \{\vec{0}\}$ . Since the map  $F$  defined in (12) is an isometry, Proposition 8.2 ensures that the complete surface  $(L, \hat{g}|_L)$  has at most quadratic area growth. This property implies recurrence for Brownian motion, see for instance Grigor'yan [15]. Recurrence is a

property that only depends on the conformal class of  $\widehat{g}|_L$ , hence  $L$  (with its original metric induced by the usual inner product of  $\mathbb{R}^3$ ) is also recurrent. The classical Liouville theorem for recurrent manifolds applies to the positive harmonic coordinate function  $x_3$  on  $L$  and gives that  $x_3$  is constant on  $L$ , which is clearly a contradiction. This contradiction shows that every leaf  $L$  of  $\mathcal{L}$  is properly embedded in  $\mathbb{R}^3 - \{\vec{0}\}$ . The rest of the proof of the proposition is the same as the one given at the end of Section 4.

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