

# AN ALMOST FLAT MANIFOLD WITH A CYCLIC OR QUATERNIONIC HOLONOMY GROUP BOUNDS

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## Abstract

A long-standing conjecture of Farrell and Zdravkovska and independently S. T. Yau states that every almost flat manifold is the boundary of a compact manifold. This paper gives a simple proof of this conjecture when the holonomy group is cyclic or quaternionic. The proof is based on the interaction between flat bundles and involutions.

## 1. Introduction

A closed manifold  $M$  is *almost flat* if there is a sequence of metrics  $g_i$  on  $M$  so that  $|K_{g_i}| \text{diam}(M, g_i)^2 \rightarrow 0$  when  $i \rightarrow \infty$ , where  $K_{g_i}$  is the sectional curvature and  $\text{diam}(M, g_i)$  is the diameter of  $M$  with respect to the metric  $g_i$ . In his celebrated paper [5], Gromov generalized the classical Bieberbach theorem for flat manifolds and proved that every almost flat manifold is finitely covered by a *nilmanifold*, that is, the quotient of a simply connected nilpotent Lie group by a uniform lattice. (Conversely, work of Farrell and Hsiang [2] showed that every manifold finitely covered by a nilmanifold is homeomorphic to an almost flat manifold.) Ruh [9] strengthened Gromov's theorem and proved that an almost flat manifold is diffeomorphic to an *infranilmanifold*, that is, a double coset space  $\Gamma \backslash L \rtimes \text{Aut}(L) / \text{Aut}(L)$  where  $L$  is a simply connected nilpotent Lie group and  $\Gamma$  is a torsion-free subgroup of the affine group  $L \rtimes \text{Aut}(L)$  so that the kernel of  $\Gamma \rightarrow \text{Aut}(L)$  has finite index in  $\Gamma$  and is discrete and cocompact in  $L$ . In fact, Ruh produced a flat connection with parallel torsion on the tangent bundle of an almost flat manifold. The map  $\Gamma \rightarrow \text{Aut}(L)$  is the holonomy of this connection. Conversely, it is not difficult to see that every infranilmanifold is almost flat. The class of almost flat manifolds is much larger than flat manifolds; there are infinitely many almost flat manifolds in every dimension greater than two.

Almost flat manifolds play a fundamental role in Riemannian geometry. By the profound Cheeger-Fukaya-Gromov theorem [1], almost flat

manifolds are the fibers in a collapsing sequence of Riemannian manifolds with bounded curvature and diameter. All cuspidal ends of a complete Riemannian manifold with finite volume and negative pinched sectional curvature are almost flat manifolds, generalizing the fact that the cuspidal ends of a finite volume hyperbolic manifold are flat manifolds. Pedro Ontaneda recently proved an amazing converse (see Theorem A of [8]): If an almost flat manifold  $M$  is the boundary of a compact manifold, then there is a compact manifold  $X$  with boundary  $M$  so that  $X - M$  admits a complete, finite volume Riemannian metric with negative pinched sectional curvature.

By rescaling, one sees that for an almost flat manifold  $M$ , there is a sequence of Riemannian metrics  $g_i$  on  $M$  so that  $\text{diam}(M, g_i) = 1$  for all  $i$  and  $|K_{g_i}| \rightarrow 0$  as  $i \rightarrow \infty$ . By Chern-Weil theory, the Pontryagin numbers of an oriented closed manifold are integrals of the Pfaffian of the curvature forms, and for an almost flat manifold these integrals must converge to zero as  $i \rightarrow \infty$ , since by the volume comparison theorem the sequence  $\text{vol}(M, g_i)$  is bounded above. Therefore the Pontryagin numbers of an oriented almost flat manifold  $M$  all vanish. It follows that the disjoint union of  $M$  with itself is an oriented boundary. Furthermore, if  $M$  has an almost complex structure then, by the same reasoning, all the Chern numbers of  $M$  vanish. In particular, this implies that  $M$  bounds. (Throughout this paper when we say  $M$  bounds we mean that  $M$  is diffeomorphic to the boundary of a compact manifold.) This clearly suggests a natural and very interesting conjecture, *almost flat manifolds are boundaries*, due to Farrell and Zdravkovska [3], which is posed independently in the famous problem list of S.T.Yau [12]. It is a well-known theorem of Thom that a closed manifold bounds if and only if all its Stiefel-Whitney numbers vanish. Wall showed that a closed oriented manifold bounds an orientable manifold if and only if all Stiefel-Whitney numbers and all Pontryagin numbers are zero. The above discussion implies that if an oriented almost flat manifold bounds, then it bounds orientably.

A remarkable theorem of Hamrick and Royster [6] shows that every flat manifold bounds. But the corresponding statement for almost flat manifolds remains a conjecture. In some special cases it has been proven. The *holonomy group* of an infranilmanifold is the finite group  $G$  given by the image of the fundamental group  $\Gamma$  in  $\text{Aut}(L)$ . Farrell and Zdravkovska [3] proved that almost flat manifolds bound provided either that the holonomy group  $G$  has order two or that the holonomy group  $G$  acts effectively on the center of  $L$ . Upadhyay [11] proved that an almost flat manifold bounds if all of the following conditions hold:  $G$  is cyclic,  $G$  acts trivially on the center of  $L$ , and  $L$  is 2-step nilpotent.

This paper contains a new and quite simple proof of the above results and proves the more general statement:

**Theorem 1.1.** *Let  $M$  be an almost flat manifold and let  $\text{Syl}_2 G$  be the 2-sylow subgroup of its holonomy group. If  $\text{Syl}_2 G$  is cyclic or generalized quaternionic, then  $M$  is the boundary of a compact manifold.*

Since all rational Pontryagin numbers vanish, from the above theorem and cobordism theory it follows that every oriented almost flat manifold with such a holonomy group bounds an oriented manifold. However, it remains difficult to answer:

**Problem** (a). Does every almost flat Spin manifold (up to changing Spin structures) bound a Spin manifold? (b). Is the bordism class  $[M, h] \in \Omega_*(BG)$  given by the holonomy map of an almost flat manifold trivial?

For an almost flat Spin manifold  $M$  with holonomy map  $h : M \rightarrow BG$ , perhaps the  $\eta$ -invariant can be used to detect information on the bordism class of  $[M, h]$  (cf. [4].)

A strong conjecture of Farrell and Zdravkovska [3] asked whether an almost flat (or flat) manifold bounds a compact manifold whose interior admits a complete finite volume metric with negative (or constant negative) sectional curvature. Long and Reid [7] give counterexamples to the flat version of the conjecture using  $\eta$ -invariants. As a corollary of the recent result of Ontaneda mentioned above, we see that this strong conjecture is true in the case of cyclic or quaternionic holonomy:

**Corollary 1.2.** *An almost flat manifold whose 2-sylow subgroup of the holonomy group is cyclic or quaternionic bounds a compact manifold whose interior admits a complete, finite volume Riemannian metric with negative pinched sectional curvature.*

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## 2. Almost flat manifolds with cyclic or quaternionic holonomy bound

Recall that a *nilmanifold* is a quotient  $N \backslash L$  of a simply connected nilpotent Lie group  $L$  by a discrete cocompact subgroup  $N$ . A nilmanifold is parallelizable; indeed one projects a basis of left invariant vector fields on  $L$  to the nilmanifold. Since  $N \subset L$  are both nilpotent, their centers  $Z(N) \subset Z(L)$  are nontrivial. Translation by an element of order two in  $Z(N) \backslash Z(L)$  gives a fixed-point free involution on  $N \backslash L$ .

We now have two proofs that a nilmanifold bounds. The first proof is that a nilmanifold is parallelizable, so its Stiefel-Whitney numbers are zero, hence by Thom's theorem it bounds. The second proof is

that a nilmanifold admits a fixed-point free involution, and any closed manifold with a fixed-point free involution  $\tau : M \rightarrow M$  is a boundary:

$$\partial \left( \frac{M \times [0, 1]}{(m, t) \sim (\tau(m), 1 - t)} \right) = M.$$

The proof of our main result involves a combination of these two ideas. Lemma 2.5 is key.

An *infranilmanifold* is a double coset space  $M = \Gamma \backslash L \rtimes G/G$  where  $L$  is a simply connected nilpotent Lie group,  $G$  is a finite subgroup of  $\text{Aut}(L)$  and  $\Gamma$  is a discrete torsion-free cocompact subgroup of  $L \rtimes G$  which maps epimorphically to  $G$  under the projection  $L \rtimes G \rightarrow G$ . We require an infranilmanifold to have positive dimension.

Let  $N = \Gamma \cap L$ . Then  $N$  is a normal subgroup of  $\Gamma$  and the sequence

$$1 \rightarrow N \rightarrow \Gamma \rightarrow G \rightarrow 1$$

is short exact. Furthermore  $N$  is a discrete cocompact subgroup of  $L$  and hence is a finitely generated, torsion-free, nilpotent group. The group  $N$  is called the *nillattice* and  $G$  is called the *holonomy group* of the infranilmanifold.

We will replace the tautological regular covers

$$L \rtimes G/G \rightarrow N \backslash L \rtimes G/G \rightarrow \Gamma \backslash L \rtimes G/G,$$

using the diffeomorphisms  $L \cong L \rtimes G/G$  and  $N \backslash L \cong N \backslash L \rtimes G/G$ . We instead consider the covers

$$L \xrightarrow{p} N \backslash L \xrightarrow{\pi} \Gamma \backslash L \rtimes G/G$$

with  $p(l) = Nl$  and  $\pi(Nl) = \Gamma(l, e)G$ . The affine group  $\text{Aff}(L) = L \rtimes \text{Aut } L$  acts on  $L$  via  $(l', g)l = l'g(l)$ . Thus  $\Gamma < \text{Aff}(L)$  acts freely on  $L$  and, in fact,  $\pi \circ p$  is a regular  $\Gamma$ -cover. Likewise  $G$  acts freely on  $N \backslash L$  via  $g(Nl) = N\gamma l$  where  $\gamma \in \Gamma$  maps to  $g \in G$  and  $\pi$  is a regular  $G$ -cover. We call this  $G$ -action on  $N \backslash L$  the *affine action*.

The  $G$ -action on  $L$  given by  $G < \text{Aut } L$  leaves the center  $Z(L)$  invariant. The  $G$ -action on  $L$  can be reinterpreted as conjugation in the affine group by using the short exact sequence  $1 \rightarrow L \rightarrow L \rtimes G \rightarrow G \rightarrow 1$ . By comparing with the short exact sequence  $1 \rightarrow N \rightarrow \Gamma \rightarrow G \rightarrow 1$ , we see that the  $G$ -action also leaves  $Z(N)$  invariant but need not leave  $N$  invariant. We call the  $G$ -actions on  $L$ ,  $Z(L)$ ,  $Z(N)$ , and  $Z(N) \backslash Z(L)$  *conjugation actions*. They are all actions via group automorphisms.

**Definition 2.1.** A *central involution*  $\tau$  of an infranilmanifold  $M = \Gamma \backslash L \rtimes G/G$  is an element  $\tau \in Z(N) \backslash Z(L)$  of order 2 which is invariant under the conjugation action of  $G$ . This determines maps  $\tau : M \rightarrow M$ ,  $\Gamma(l, g)G \mapsto \Gamma(Tl, g)G$  and  $\tau : N \backslash L \rightarrow N \backslash L$ ,  $N(l, g)G \mapsto N(Tl, g)G$  where  $T \in Z(L)$  is a representative for  $\tau$ .

Central involutions were key in all previous work on this problem [6], [3], [11] and will be for us too.

- Lemma 2.2.** (i) *Let  $M$  be a closed manifold with an epimorphism of its fundamental group to a finite group  $G$ . Then  $M$  bounds if and only if  $H \backslash \widetilde{M}$  bounds where  $\widetilde{M}$  is the  $G$ -cover of  $M$  and  $H$  is a 2-Sylow subgroup of  $G$ .*
- (ii) *Any infranilmanifold  $M = \Gamma \backslash L \rtimes G/G$  with  $G$  a 2-group has a central involution.*

*Proof.* (i) Note that  $H \backslash \widetilde{M} \rightarrow M$  is a odd-degree cover, hence the domain and target have the same Stiefel-Whitney numbers; thus one bounds if and only if the other does.

(ii) Let  $\Sigma$  be the subgroup of  $Z(N) \backslash Z(L)$  generated by the elements of order 2. Since the 2-group  $G$  acts as group automorphisms of the abelian 2-group  $\Sigma$ , there must be at least two orbits of cardinality one, hence there must be a non-trivial element fixed by  $G$ . q.e.d.

To analyze the fixed point set of a central involution, we need a group theoretic remark.

**Remark 2.3.** Let  $X$  be an  $(H, K)$ -biset where both  $H$  and  $K$  act freely on  $X$ . Let  $q : X \rightarrow H \backslash X$  be the quotient map, let  $F$  be the fixed set of  $K$  acting on  $H \backslash X$ , and let  $\widetilde{F} = q^{-1}F$ . For  $x \in \widetilde{F}$ , there is a function  $\varphi_x : K \rightarrow H$  so that  $xk = \varphi_x(k)x$ . Since  $H$  acts freely, this function is uniquely defined, since  $X$  is a biset, this function is a homomorphism, and since  $K$  acts freely, it is a monomorphism. For a monomorphism  $\varphi : K \rightarrow H$ , let

$$\widetilde{F}_\varphi = \{x \in X \mid \forall k \in K, xk = \varphi(k)x\}.$$

Then  $\widetilde{F} = \coprod \widetilde{F}_\varphi$ . It is easy to see that  $h\widetilde{F}_\varphi = \widetilde{F}_{c_h \circ \varphi}$  where  $c_h : H \rightarrow H$  is conjugation. Thus  $q(\widetilde{F}_\varphi) = q(\widetilde{F}_{c_h \circ \varphi})$ . The group  $H$  acts on the set of monomorphisms  $K \rightarrow H$  by conjugation, let  $[\varphi]$  denote an orbit. Let  $F_{[\varphi]} = q(\widetilde{F}_\varphi)$ . Then  $F = \coprod F_{[\varphi]}$ .

Note that there is a bijection between  $(H, K)$ -bisets and left  $(H \times K)$ -sets, where  $h x k$  corresponds to  $h k^{-1} x$ .

Later we will apply Remark 2.3 to analyze the fixed point set of a central involution  $\tau : M \rightarrow M$  by setting  $X = N \backslash L$ ,  $H = G$  and  $K = \langle \tau \rangle$ .

A vector bundle  $E \rightarrow B$  is *flat* if it has finite structure group, that is,  $E \cong \widetilde{B} \times_G V$  for some finite, regular  $G$ -cover  $\widetilde{B} \rightarrow B$  and some  $\mathbb{R}G$ -module  $V$ . We call such a flat structure a  $(G, V)$ -*structure*. Such a bundle (over a CW-complex) is the pullback of the flat bundle  $EG \times_G V \rightarrow BG$  along a map  $B \rightarrow BG$ . The regular  $G$ -cover  $\widetilde{B} \rightarrow B$  can also be specified by a homotopy class of map  $B \rightarrow BG$  or by a  $G$ -conjugacy class of homomorphism  $\pi_1 B \rightarrow G$ .

Tangent bundles of infranilmanifolds are flat:

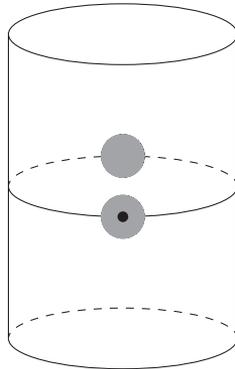
**Lemma 2.4.** *Consider an infranilmanifold  $M = \Gamma \backslash L \rtimes G/G$ . Note that  $T_e L$  is an  $\mathbb{R}G$ -module. The tangent bundle of  $M$  is flat:*

$$TM = (N \backslash L) \times_G T_e L$$

*Proof.* Note that  $L$ , like all Lie groups, is parallelizable. Indeed there is an isomorphism of vector bundles over  $L$  with  $\Phi : L \times T_e L \xrightarrow{\cong} TL$  given by  $\Phi(l, v) = d(L_l)_e v$ , where  $L_l : L \rightarrow L$  is left translation, defined by  $L_l(l') = ll'$ . Note  $L \rtimes G$  acts on  $L$  via  $(l', g)l = l'g(l)$ , on  $TL$  via the differential of this action, and on  $T_e L$  via  $v \mapsto (dg)_e v$ . It is straightforward to check that  $\Phi$  is  $L \rtimes G$ -equivariant with respect to the diagonal  $L \rtimes G$ -action on the domain and the action given by the differential on the target. Verify the  $L$ -equivariance and  $G$ -equivariance separately and use the identity  $g \circ L_l = L_{g(l)} \circ g : L \rightarrow L$ . Note  $\Gamma \subset L \rtimes G$  acts freely as deck transformations on  $L$  and the subgroup  $N$  acts trivially on  $T_e L$ , so the map  $\Phi$  descends to the desired isomorphism  $N \backslash L \times_G T_e L \cong TM$ . q.e.d.

**Lemma 2.5.** *Let  $\tau : TM \rightarrow TM$  be a nontrivial bundle involution on the tangent bundle of a closed connected manifold  $M$ . Let  $F \subset M$  be the fixed set of the involution restricted to  $M$ . If  $\tau$  restricted to  $TM|_F$  is the identity, then  $M$  bounds.*

*Proof.* Let  $I = [0, 1]$ . Extend the involution on  $M$  to an involution on  $M \times I$  by setting  $(m, t) \mapsto (\tau(m), 1 - t)$ . Choose a closed involution invariant tubular neighborhood  $N$  of the fixed set, thus  $F \times \{1/2\} \subset N \subset M \times I$  (see Figure 1).



**Figure 1.**  $M \times I$  with  $M = S^1$  and  $\tau(z) = \bar{z}$ . The disks are the tubular neighborhood  $N$ .

Note that  $(N, \partial N)$  is equivariantly diffeomorphic to the (disk bundle, sphere bundle) pair  $(D(\nu \oplus 1), S(\nu \oplus 1))$  where  $\nu = \nu(F \hookrightarrow M)$  is the normal bundle and where the involution on the bundle pair is given by fiberwise multiplication by  $-1$ . Let  $P(\nu \oplus 1)$  be the projective bundle

$S(\nu \oplus 1)/(v \sim -v)$ . Then  $W = (M \times I - \text{int } N) / \tau$  gives a cobordism of manifolds from  $M$  to  $P(\nu \oplus 1)$ .

We now upgrade to a cobordism of bundles. Note that  $TM \times I$  is a bundle over  $M \times I$ . The bundle involution on  $TM$  extends to a bundle involution  $TM \times I$  by setting  $(v, t) \mapsto (\tau(v), 1 - t)$ . This descends to a bundle  $\xi$  over  $W$  which restricts to  $TM$  over  $M$ . We wish to identify  $\xi|_{P(\nu \oplus 1)}$ . By homotopy invariance of pullbacks of vector bundles (see Proposition 1.3 of [10]), there is an equivariant isomorphism of vector bundles  $TM \times I|_{D(\nu \oplus 1)} \cong \pi_D^* TM|_F$  where  $\pi_D : D(\nu \oplus 1) \rightarrow F$  is the bundle projection. Since the involution on  $TM|_F$  is trivial, there is an induced isomorphism  $\xi|_{P(\nu \oplus 1)} \cong \pi_P^* TM|_F$  where  $\pi_P : P(\nu \oplus 1) \rightarrow F$  is the bundle projection.

Thus  $\xi$  gives a cobordism of bundles from  $TM$  to  $\pi_P^* TM|_F$ . Since Stiefel-Whitney numbers are cobordism invariants, for all partitions  $J$  of  $\dim M$ ,

$$\begin{aligned} w_J(TM)[M] &= w_J(\pi_P^* TM|_F)[P(\nu \oplus 1)] \\ &= (\pi_P^* w_J(TM|_F))[P(\nu \oplus 1)] \\ &= 0 \end{aligned}$$

since  $w_J(TM|_F) = 0$  because  $|J| = \dim M > \dim F$ . Thus all Stiefel-Whitney numbers of  $M$  vanish, so by Thom’s theorem  $M$  bounds.

q.e.d.

**Remark 2.6.** One wonders if there could be a direct proof of Lemma 2.5 which avoids the use of Thom’s Theorem.

Note that any involution  $\tau : M \rightarrow M$  induces the involution  $d\tau : TM \rightarrow TM$ , but this involution does not restrict to the identity on  $TM|_F$ .

**Theorem 2.7.** *An almost flat manifold bounds provided that the 2-sylow subgroup of the holonomy group is cyclic or generalized quaternionic.*

*Proof.* Let  $M = \Gamma \backslash L \rtimes G/G$  be an infranilmanifold with the 2-sylow subgroup of  $G$  cyclic or generalized quaternionic. By Lemma 2.2(i), we can pass to odd degree cover and assume that  $G$  is a 2-group. Then there is a unique element  $g \in G$  of order 2. (In fact, according to a theorem of Burnside, a 2-group has a unique element of order 2 if and only if it is cyclic or generalized quaternionic.) Since the center of a  $p$ -group is nontrivial,  $g$  is central.

By Lemma 2.2(ii), there is a central involution  $\tau \in Z(N) \backslash Z(L)$ . Since it is  $G$ -invariant, it induces a nontrivial involution  $\tau : M \rightarrow M$ . By Remark 2.3, applied with  $X = N \backslash L$ ,  $H = G$ , and  $K = \langle \tau \rangle$ , the fixed set of  $\tau$  is

$$F = \{[x] \in M \mid x \in N \backslash L, \tau x = gx\}.$$

The involution  $\tau$  on  $M$  extends to the flat tangent bundle  $TM = (N \setminus L) \times_G T_e L$  via  $\tau[x, v] = [\tau x, gv]$ . Then  $\tau$  restricted to  $TM|_F$  is the identity since for  $[x] \in F$ ,  $\tau[x, v] = [\tau x, gv] = [gx, gv] = [x, v]$ . Thus by Lemma 2.5  $M$  is a boundary. q.e.d.

The question of whether an infranilmanifold with holonomy group the Klein 4-group bounds remains open.

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