

EIGENFUNCTIONS WITH PRESCRIBED NODAL SETS

ALBERTO ENCISO & DANIEL PERALTA-SALAS

Abstract

In this paper we consider the problem of prescribing the nodal set of low-energy eigenfunctions of the Laplacian. Our main result is that, given any separating closed hypersurface Σ in a compact n -manifold M , there is a Riemannian metric on M such that the nodal set of its first nontrivial eigenfunction is Σ . We present a number of variations on this result, which enable us to show, in particular, that the first nontrivial eigenfunction can have as many non-degenerate critical points as one wishes.

1. Introduction

The eigenfunctions of the Laplacian on a closed n -dimensional Riemannian manifold (M, g) satisfy the equation

$$\Delta u_k = -\lambda_k u_k,$$

where $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ are the eigenvalues of M . In this paper we will be concerned with the geometry of the nodal sets $u_k^{-1}(0)$ of the eigenfunctions of the Laplacian, which is a classic topic in geometric analysis with a number of important open problems [18, 19].

Since the first nontrivial eigenfunction is u_1 , we will be especially interested in the shape of the nodal set $u_1^{-1}(0)$. More generally, this paper focuses on the study of the nodal set of low-energy eigenfunctions; in particular, we will not consider the measure-theoretic properties of the nodal set $u_k^{-1}(0)$ as $k \rightarrow \infty$, which is an important topic that has been thoroughly studied e.g. in [6, 11, 12].

In fact, the central question that we will address in this paper is the following: given a hypersurface (i.e., a codimension-1 submanifold) Σ of M , is there a Riemannian metric g such that Σ is the nodal set $u_1^{-1}(0)$? In the case of the unit two-dimensional sphere, a detailed study of the possible configurations of the nodal sets of the eigenfunctions has been carried out by Eremenko, Jakobson and Nadirashvili [9]. In any closed surface, Komendarczyk [14] has shown that, given any homotopically trivial closed curve γ , there is a metric such that $u_1^{-1}(0)$ is diffeomorphic

to γ . This result has been extended to arbitrary curves on surfaces by Lisi [15] using techniques from contact topology.

Our main theorem asserts that, given an n -manifold M and any closed (i.e., compact boundaryless) hypersurface $\Sigma \subset M$, there is a metric g on M such that Σ is a connected component of the nodal set of the eigenfunction u_1 . Moreover, if Σ separates (that is, if the complement $M \setminus \Sigma$ is the union of two disjoint open sets), then one can show that the nodal set does not have any other connected components. More precisely, we have the following statement. Throughout, we will assume that the hypersurfaces are all connected, $n \geq 3$ and all objects are of class C^∞ .

Theorem 1.1. *Let Σ be a closed orientable hypersurface of M . Then there exists a Riemannian metric g on M such that Σ is a connected component of the nodal set $u_1^{-1}(0)$. If Σ separates, then the nodal set is exactly Σ .*

An analogous result for the first l eigenfunctions will be proved in Theorem 2.1 and Proposition 3.1; however, in this Introduction we have chosen to restrict our attention to the first nontrivial eigenfunction to keep the statements as simple as possible. Somewhat related results on level sets with prescribed topologies were derived, using completely different methods, for Green's functions in [7] and for harmonic functions in \mathbb{R}^n in [8]. It should also be noted that the results that we prove in this paper are robust in the sense that if g is the metric with eigenfunctions of prescribed nodal sets that we construct in this paper, then any other metric close enough to g in the C^2 norm possesses the same property. In particular, the metric can be taken analytic whenever the manifold is analytic.

The strategy of the proof of Theorem 1.1 is quite versatile and can be used to derive a number of related results. For instance, an easy application of the underlying philosophy enables us to prove that, given any n -manifold M , there is a metric such that the eigenfunction u_1 has as many isolated critical points as one wishes:

Theorem 1.2. *Given any positive integer N , there is a Riemannian metric on M whose eigenfunction u_1 has at least N non-degenerate critical points.*

An analog of this result for the first l nontrivial eigenfunctions is given in Proposition 3.2. It is worth recalling that, on surfaces, Cheng [4] gave a topological bound for the number of critical points of the k^{th} eigenfunction that lie on the nodal line.

The ideas of the proof of the main theorem remain valid in the case of manifolds with boundary. Specifically, let now (M, g) be a compact Riemannian n -manifold with boundary and consider the sequence of

its Dirichlet eigenfunctions, which with some abuse of notation we still denote by u_k and satisfy the equation

$$\Delta u_k = -\lambda_k u_k \quad \text{in } M, \quad u_k|_{\partial M} = 0.$$

Now $0 < \lambda_1 < \lambda_2 \leq \dots$ are the Dirichlet eigenvalues of M . The problem under consideration is related to Payne's classical conjecture, which asserts that when M is a bounded simply connected planar domain the nodal line of u_2 is an arc connecting two distinct points of the boundary. Payne's conjecture is known to hold for convex domains [16, 2], but the general case is still open. Yau [19, Problem 45] raised the question of the validity of Payne's conjecture in n -manifolds with boundary. In this direction, Freitas [10] showed that there is a metric on the two-dimensional ball for which Payne's conjecture does not hold, as the nodal set is a closed curve contained in the interior of the ball. The techniques used in this paper readily yield a powerful higher-dimensional analog of this result. For simplicity and in view of Payne's conjecture, we state it in the case of the n -ball, although a totally analogous statement holds true in any compact n -manifold with boundary (see Proposition 3.3):

Theorem 1.3. *Let Σ be a closed orientable hypersurface contained in the n -dimensional ball \mathbb{B}^n . Then there exists a Riemannian metric g on $\overline{\mathbb{B}^n}$ such that the nodal set of its second Dirichlet eigenfunction is Σ .*

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2. Proof of the main theorem

In this section we will prove Theorem 2.1 below, a consequence of which is Theorem 1.1. The gist of the proof, which uses ideas introduced by Colin de Verdière [5] to prescribe the multiplicity of the first nontrivial eigenvalue λ_1 , is to choose the metric so that the low-energy eigenvalues are simple and the corresponding eigenfunctions are close, in a suitable sense, to functions whose nodal set is known explicitly.

From now on and until Subsection 3.3, M will be a compact manifold without boundary of dimension $n \geq 3$.

Step 1: Definition of the metrics. Consider a small neighborhood $\Omega \subset M$ of the orientable hypersurface Σ , which we can identify with $(-1, 1) \times \Sigma$. Let us take a metric g_0 on M whose restriction to Ω is

$$g_0|_{\Omega} = dx^2 + g_{\Sigma},$$

where x is the natural coordinate in $(-1, 1)$ and g_{Σ} is a Riemannian metric on Σ . We can assume that the first nontrivial eigenvalue of the

Laplacian on Σ defined by g_Σ is larger than $l^2\pi^2/4$, where l is a positive integer.

It is then clear that the first $l + 1$ Neumann eigenfunctions of the domain Ω can be written as

$$v_k = |\Sigma|^{-\frac{1}{2}} \cos \frac{k\pi(x+1)}{2}, \quad 0 \leq k \leq l,$$

where $|\Sigma|$ stands for the area of Σ . Observe that, with this normalization,

$$\int_{\Omega} v_j v_k = \delta_{jk}.$$

Denoting by Δ_0 the Laplacian corresponding to the metric g_0 , these eigenfunctions satisfy the equation

$$\Delta_0 v_k = -\mu_k v_k \quad \text{in } \Omega, \quad \partial_\nu v_k|_{\partial\Omega} = 0$$

with $\mu_k := k^2\pi^2/4$.

For each $\epsilon > 0$, let us define a piecewise smooth metric g_ϵ on M by setting

$$g_\epsilon := \begin{cases} g_0 & \text{in } \overline{\Omega}, \\ \epsilon g_0 & \text{in } M \setminus \overline{\Omega}. \end{cases}$$

To define the spectrum of this discontinuous metric one resorts to the quadratic form

$$\begin{aligned} Q_\epsilon(\varphi) &:= \int_M |d\varphi|_\epsilon^2 dV_\epsilon \\ &= \int_{\Omega} |d\varphi|^2 + \epsilon^{\frac{n}{2}-1} \int_{\Omega^c} |d\varphi|^2 \end{aligned}$$

together with the natural L^2 norm corresponding to the metric g_ϵ :

$$\begin{aligned} \|\varphi\|_\epsilon^2 &:= \int_M \varphi^2 dV_\epsilon \\ &= \int_{\Omega} \varphi^2 + \epsilon^{\frac{n}{2}} \int_{\Omega^c} \varphi^2 \end{aligned}$$

Here the possibly disconnected set $\Omega^c := M \setminus \overline{\Omega}$ stands for the interior of the complement of Ω , the subscripts ϵ refer to quantities computed with respect to the metric g_ϵ and we are omitting the subscripts (and indeed the measure in the integrals) when the quantities correspond to the reference metric g_0 . As is well known, the domain of the quadratic form Q_ϵ can be taken to be the Sobolev space $H^1(M)$ (recall that, M being compact, this Sobolev space is independent of the smooth metric one uses to define it).

By the min-max principle, the k^{th} eigenvalue $\lambda_{k,\epsilon}$ of this quadratic form is

$$(2.1) \quad \lambda_{k,\epsilon} = \inf_{W \in \mathcal{W}_k} \max_{\varphi \in W \setminus \{0\}} q_\epsilon(\varphi),$$

where \mathcal{W}_k stands for the set of $(k + 1)$ -dimensional linear subspaces of $H^1(M)$ and

$$(2.2) \quad q_\epsilon(\varphi) := \frac{Q_\epsilon(\varphi)}{\|\varphi\|_\epsilon^2}$$

is the Rayleigh quotient. The k^{th} eigenfunction $u_{k,\epsilon}$ is then a minimizer of the above variational problem for $\lambda_{k,\epsilon}$, in the sense that any subspace that minimizes the variational problem can be written as $\text{span}\{u_{0,\epsilon}, \dots, u_{k,\epsilon}\}$.

Step 2: $\lambda_{k,\epsilon}$ is almost upper bounded by the corresponding Neumann eigenvalue. Let us now show that the k^{th} eigenvalue of the quadratic form Q_ϵ is upper bounded by the Neumann eigenvalues of Ω as

$$(2.3) \quad \limsup_{\epsilon \searrow 0} \lambda_{k,\epsilon} \leq \mu_k.$$

To see this, consider the function $\psi_k \in H^1(M)$ given by

$$\psi_k := \begin{cases} v_k & \text{in } \bar{\Omega}, \\ \hat{v}_k & \text{in } \Omega^c, \end{cases}$$

where \hat{v}_k is the harmonic extension to Ω^c of the Neumann eigenfunction v_k , defined as the solution to the boundary value problem

$$\Delta_0 \hat{v}_k = 0 \quad \text{in } \Omega^c, \quad \hat{v}_k|_{\partial\Omega} = v_k.$$

Standard elliptic estimates show that the Sobolev norms of \hat{v}_k are controlled in terms of those of $v_k|_{\partial\Omega}$; in particular

$$(2.4) \quad \int_{\Omega^c} |d\hat{v}_k|^2 + \int_{\Omega^c} \hat{v}_k^2 \leq C \|v_k\|_{H^{\frac{1}{2}}(\partial\Omega)}^2 \leq C \|v_k\|_{H^1(\Omega)}^2 = C(\mu_k + 1),$$

where C is a constant independent of ϵ and k , and we have used that the Neumann eigenfunctions are normalized.

We are now ready to derive the upper bound for $\lambda_{k,\epsilon}$. We will find it convenient to denote by $O(\epsilon^m)$ a quantity that is bounded (possibly not uniformly in k) by $C\epsilon^m$. From (2.4) it stems that for any linear

combination $\varphi, \varphi' \in \text{span}\{\psi_1, \dots, \psi_k\}$ we have

$$\begin{aligned}
 \int_M \varphi \varphi' dV_\epsilon &= \int_\Omega \varphi \varphi' + \epsilon^{\frac{n}{2}} \int_{\Omega^c} \varphi \varphi' \\
 &= \int_\Omega \varphi \varphi' + O(\epsilon^{\frac{n}{2}}) \|\varphi\|_{H^1(\Omega)} \|\varphi'\|_{H^1(\Omega)} \\
 &= \int_\Omega \varphi \varphi' + O(\epsilon^{\frac{n}{2}}) (1 + \mu_k) \|\varphi\|_{L^2(\Omega)} \|\varphi'\|_{L^2(\Omega)} \\
 (2.5) \qquad &= \int_\Omega \varphi \varphi' + O(\epsilon^{\frac{n}{2}}) \|\varphi\|_{L^2(\Omega)} \|\varphi'\|_{L^2(\Omega)}.
 \end{aligned}$$

To pass to the last line we have simply used that, for any finite k , we can absorb the constant $1 + \mu_k$ in the $O(\epsilon^{\frac{n}{2}})$ term. An immediate consequence of this inequality is that, for all $k \leq l$ and sufficiently small ϵ (depending on l), the linear space

$$\text{span}\{\psi_1, \dots, \psi_k\}$$

is a k -dimensional subspace of $H^1(M)$.

A similar argument using (2.4) allows us to estimate $Q_\epsilon(\varphi)$, with $\varphi \in \text{span}\{\psi_1, \dots, \psi_k\}$, as

$$\begin{aligned}
 Q_\epsilon(\varphi) &= \int_\Omega |d\varphi|^2 + \epsilon^{\frac{n}{2}-1} \int_{\Omega^c} |d\varphi|^2 \\
 &= \int_\Omega |d\varphi|^2 + O(\epsilon^{\frac{n}{2}-1}) (1 + \mu_k) \|\varphi\|_{L^2(\Omega)}^2 \\
 (2.6) \qquad &= \int_\Omega |d\varphi|^2 + O(\epsilon^{\frac{n}{2}-1}) \|\varphi\|_{L^2(\Omega)}^2.
 \end{aligned}$$

In view of the min-max formulation (2.1) and omitting the condition that $\varphi \neq 0$ for notational simplicity, we then obtain from (2.5) and (2.6) that

$$\begin{aligned}
 \lambda_{k,\epsilon} &\leq \max_{\varphi \in \text{span}\{\psi_1, \dots, \psi_k\}} q_\epsilon(\varphi) \\
 &= \max_{\varphi \in \text{span}\{\psi_1, \dots, \psi_k\}} \frac{\int_\Omega |d\varphi|^2 + O(\epsilon^{\frac{n}{2}-1}) \|\varphi\|_{L^2(\Omega)}^2}{[1 + O(\epsilon^{\frac{n}{2}})] \|\varphi\|_{L^2(\Omega)}^2} \\
 &= [1 + O(\epsilon^{\frac{n}{2}})] \max_{\chi \in \text{span}\{\psi_1, \dots, \psi_k\}} q_\Omega(\chi) + O(\epsilon^{\frac{n}{2}-1}) \\
 &= [1 + O(\epsilon^{\frac{n}{2}})] \mu_k + O(\epsilon^{\frac{n}{2}-1}),
 \end{aligned}$$

which proves (2.3). Here

$$q_\Omega(\chi) := \frac{\int_\Omega |d\chi|^2}{\int_\Omega \chi^2}$$

is the Rayleigh quotient in Ω and we recall that the bounds for the $O(\epsilon^m)$ terms are *not* uniform in k .

Step 3: Convergence to the Neumann eigenfunctions. Let us consider the linear space

$$\mathcal{H}_2 := \{ \varphi \in H^1(M) : \varphi|_{\Omega^c} \in H_0^1(\Omega^c), \varphi|_{\Omega} = 0 \}.$$

This is a closed subspace of $H^1(M)$, so it is standard that there is another closed subspace \mathcal{H}_1 of $H^1(M)$ such that

$$(2.7) \quad H^1(M) = \mathcal{H}_1 \oplus \mathcal{H}_2$$

and

$$(2.8) \quad \int_M g_0(\nabla\varphi_1, \nabla\varphi_2) = 0$$

for all $\varphi_j \in \mathcal{H}_j$. Here the gradient is computed using the reference metric g_0 . A short computation shows that in fact one has

$$\mathcal{H}_1 := \{ \varphi \in H^1(M) : \Delta_0\varphi = 0 \text{ in } \Omega^c \}.$$

We will denote by \mathcal{P}_j the projector associated with the subspace \mathcal{H}_j .

Our goal now is to analyze the first l eigenfunctions of Q_ϵ for small ϵ . Hence, let us fix some nonnegative integer $k \leq l$. By the upper bound (2.3), if ϵ is small enough we have that

$$\lambda_{k,\epsilon} < \mu_k + 1,$$

which implies that

$$(2.9) \quad \lambda_{k,\epsilon} = \inf_{W \in \mathcal{W}'_k} \max_{\varphi \in W \setminus \{0\}} q_\epsilon(\varphi).$$

The difference with (2.1) is that now \mathcal{W}'_k is the set of $(k+1)$ -dimensional subspaces of $H^1(M)$ such that

$$q_\epsilon(\varphi) < \mu_k + 1$$

for all nonzero $\varphi \in \mathcal{W}'_k$.

A further simplification is the following. Let us use the direct sum decomposition (2.7) to write φ as

$$\varphi = \varphi_1 + \varphi_2,$$

where we will henceforth use the notation $\varphi_j := \mathcal{P}_j\varphi$. The observation now is that if

$$\|\varphi_2\|_\epsilon^2 \geq c\|\varphi\|_\epsilon^2$$

for some $c > 0$, then

$$\begin{aligned}
 q_\epsilon(\varphi) &\geq \frac{\epsilon^{\frac{n}{2}-1} \int_{\Omega^c} |d\varphi_2|^2}{\|\varphi\|_\epsilon^2} \\
 &\geq \frac{c\epsilon^{\frac{n}{2}-1} \int_{\Omega^c} |d\varphi_2|^2}{\|\varphi_2\|_\epsilon^2} \\
 &\geq \frac{c \int_{\Omega^c} |d\varphi_2|^2}{\epsilon \int_{\Omega^c} \varphi_2^2} \\
 &\geq \frac{c_0 c}{\epsilon},
 \end{aligned}$$

where the positive, ϵ -independent constant c_0 is the first Dirichlet eigenvalue of Ω^c with the reference metric g_0 . Hence we easily infer that, if φ is in a subspace belonging to \mathcal{W}'_k with $k \leq l$, then necessarily

$$(2.10) \quad \|\varphi_2\|_\epsilon^2 \leq O(\epsilon) \|\varphi\|_\epsilon^2.$$

By mimicking the proof of the estimate (2.4), we readily find that if $q_\epsilon(\varphi) \leq \mu_k + 1$, one has that

$$(2.11) \quad \int_{\Omega^c} \varphi_1^2 + \int_{\Omega^c} |d\varphi_1|^2 \leq C \int_{\Omega} \varphi_1^2.$$

We can now use the orthogonality relation (2.8) to write, for any nonzero $\varphi \in W$ with $W \in \mathcal{W}'_k$,

$$\begin{aligned}
 q_\epsilon(\varphi) &= \frac{\int_M |d\varphi_1|_\epsilon^2 dV_\epsilon + \int_M |d\varphi_2|_\epsilon^2 dV_\epsilon}{\|\varphi\|_\epsilon^2} \\
 &= [1 + O(\epsilon^{\frac{1}{2}})] \frac{\int_M |d\varphi_1|_\epsilon^2 dV_\epsilon + \int_M |d\varphi_2|_\epsilon^2 dV_\epsilon}{\|\varphi_1\|_\epsilon^2} \\
 &\geq [1 + O(\epsilon^{\frac{1}{2}})] \frac{\int_M |d\varphi_1|_\epsilon^2 dV_\epsilon}{\|\varphi_1\|_\epsilon^2} \\
 &= [1 + O(\epsilon^{\frac{1}{2}})] \frac{\int_\Omega |d\varphi_1|^2 + \epsilon^{\frac{n}{2}-1} \int_{\Omega^c} |d\varphi_1|^2}{\int_\Omega \varphi_1^2 + \epsilon^{\frac{n}{2}} \int_{\Omega^c} \varphi_1^2} \\
 &= [1 + O(\epsilon^{\frac{1}{2}})] \frac{\int_\Omega |d\varphi_1|^2 + O(\epsilon^{\frac{n}{2}-1}) \int_\Omega \varphi_1^2}{\int_\Omega \varphi_1^2} \\
 &= [1 + O(\epsilon^{\frac{1}{2}})] q_\Omega(\varphi_1|\Omega) + O(\epsilon^{\frac{n}{2}-1}).
 \end{aligned}$$

Here we have used the inequalities (2.10)-(2.11), which in particular imply

$$\|\varphi\|_\epsilon = [1 + O(\epsilon^{\frac{1}{2}})] \|\varphi_1\|_\epsilon.$$

Therefore, by the min-max principle (2.9),

$$\begin{aligned} \lambda_{k,\epsilon} &\geq [1 + O(\epsilon^{\frac{1}{2}})] \inf_{W \in \mathcal{W}'_k} \max_{\varphi \in W \setminus \{0\}} q_\Omega(\varphi_1|\Omega) + O(\epsilon^{\frac{n}{2}-1}) \\ &\geq [1 + O(\epsilon^{\frac{1}{2}})] \mu_k + O(\epsilon^{\frac{n}{2}-1}). \end{aligned}$$

Together with the upper bound (2.3), this shows that

$$\lim_{\epsilon \searrow 0} \lambda_{k,\epsilon} = \mu_k.$$

Since the eigenvalues μ_k are simple for $k \leq l$, it is well-known that for $k \leq l$ this implies that there are nonzero constants β_k such that the restriction to Ω of the eigenfunctions $u_{k,\epsilon}$ converges to the Neumann eigenfunctions v_k , up to a multiplicative constant, in the H^1 norm associated with the quadratic form. Therefore there is some constant β_k such that

$$(2.12) \quad \lim_{\epsilon \searrow 0} \|u_{k,\epsilon} - \beta_k v_k\|_{H^1(\Omega)} = 0 \quad \text{for } k \leq l.$$

For simplicity of notation, we will redefine the normalization constants of the eigenfunctions $u_{k,\epsilon}$ if necessary to take $\beta_k = 1$. In view of (2.12), standard elliptic estimates then ensure that $u_{k,\epsilon} \rightarrow v_k$ as $\epsilon \rightarrow 0$ in $C^m(K)$, for any compact subset K of Ω and any integer m .

Step 4: Characterization of the nodal sets. Let us fix a small enough $\epsilon > 0$ and take a sequence of uniformly bounded smooth functions $\chi_{j,\epsilon} \in C^\infty(M)$ converging pointwise to:

$$\lim_{j \rightarrow \infty} \chi_{j,\epsilon}(x) = \begin{cases} 1 & \text{if } x \in \overline{\Omega}, \\ \epsilon & \text{if } x \in \Omega^c. \end{cases}$$

It is then well-known (see e.g. [3]) that the k^{th} eigenvalue of the Laplacian corresponding to the metric $g_{j,\epsilon} := \chi_{j,\epsilon} g_0$ then converges to $\lambda_{k,\epsilon}$ as $j \rightarrow \infty$, and that its k^{th} eigenfunction

$$u_{k,\epsilon}^j$$

converges in $H^1(M)$ (and in $C^m(K)$ for any compact subset of $M \setminus \partial\Omega$ and any m) to $u_{k,\epsilon}$, where we are assuming that $k \leq l$ to ensure that the eigenvalues are simple. The convergence result proved at the end of Step 3 then yields that, for ϵ small enough, j large and any fix compact set $K \subset \Omega$, the difference

$$\|u_{k,\epsilon}^j - v_k\|_{C^m(K)}$$

can be made as small as one wishes for $0 \leq k \leq l$.

It is clear that the nodal set of the Neumann eigenfunction v_k is empty for $k = 0$ and diffeomorphic to k copies of Σ for $1 \leq k \leq l$, namely,

$$(2.13) \quad v_k^{-1}(0) = \left\{ \frac{(1 + 2i - k)}{k} : i \in [0, k - 1] \cap \mathbb{Z} \right\} \times \Sigma.$$

Here we are identifying Ω with $(-1, 1) \times \Sigma$ as before. Since dv_k does not vanish on the nodal set and we have shown that $u_{k,\epsilon}^j$ converges to v_k in the C^m norm on compact subsets of Ω , Thom's isotopy theorem [1, Section 20.2] implies that for small enough ϵ , large j and $k \leq l$, the nodal set $(u_{k,\epsilon}^j)^{-1}(0)$ has at least k connected components, which are diffeomorphic to Σ . More precisely, there is a diffeomorphism $\Phi_{k,\epsilon}^j$ of M , arbitrarily close to the identity in any fixed $C^m(M)$ norm, such that $\Phi_{k,\epsilon}^j(v_k^{-1}(0))$ is a collection of connected components of the nodal set $(u_{k,\epsilon}^j)^{-1}(0)$.

Furthermore, if Σ separates, Courant's nodal domain theorem then guarantees that the nodal set cannot have any further components that separate. In this case, a component of the nodal set that does not separate must sit in a region where $u_{k,\epsilon}^j$ has a constant sign, which means that this component is a local extremum of the eigenfunction. A theorem of Uhlenbeck [17] ensures that critical nodal sets are non-generic, so that if necessary one can find a small perturbation $\tilde{g}_{j,\epsilon}$ of the metric $g_{j,\epsilon}$ so that the corresponding k^{th} eigenfunction $\tilde{u}_{k,\epsilon}^j$ does not have any components of the nodal set that are critical and, a fortiori, its critical set is exactly $\tilde{\Phi}_{k,\epsilon}^j(v_k^{-1}(0))$ for some diffeomorphism $\tilde{\Phi}_{k,\epsilon}^j$ that is close to the identity in $C^m(M)$. To use the same notation no matter whether Σ separates or not, we shall henceforth drop the tildes in our notation.

For large enough j and small ϵ , let us consider the pulled-back metric

$$g := (\Phi_{1,\epsilon}^j)^* g_{j,\epsilon}$$

and call u_k its k^{th} eigenfunction, which is given by

$$u_k = u_{k,\epsilon}^j \circ \Phi_{1,\epsilon}^j.$$

The nodal set of u_k has then a connected component given by Σ if $k = 1$ and a collection of components given by $\Phi_k(v_k^{-1}(0))$ if $2 \leq k \leq l$, with

$$(2.14) \quad \Phi_k := (\Phi_{1,\epsilon}^j)^{-1} \circ \Phi_{k,\epsilon}^j$$

a diffeomorphism that can be taken as close to the identity in any fixed C^m norm as one wishes. We have therefore proved the following

Theorem 2.1. *If ϵ is small enough and j is large enough, for $k \leq l$ the nodal sets of the eigenfunctions u_k corresponding to the metric g contain a collection of connected components diffeomorphic to the set (2.13), which is given by k copies of Σ . The diffeomorphism, which is given by (2.14), can be taken arbitrarily close to the identity in the $C^m(M)$ norm, and in fact is exactly the identity for $k = 1$. Furthermore, if Σ separates, the nodal set of u_k does not have any additional components.*

3. Applications

The strategy of the proof of Theorem 2.1 is quite versatile and can be employed to construct eigenfunctions with prescribed behavior in different situations. We shall now present three concrete applications of this philosophy, where we will consider eigenfunctions with prescribed nodal sets of different topologies, low-energy eigenfunctions with a large number of non-degenerate critical points and nodal sets of Dirichlet eigenfunctions in manifolds with boundary.

3.1. Eigenfunctions with nodal sets of different topologies. In Theorem 2.1 we showed how to construct metrics whose k^{th} eigenfunction has a nodal set diffeomorphic to k copies of a given hypersurface Σ (and possibly other connected components, if Σ does not separate). To complement this result, in the following proposition we shall show how one can use the same argument to construct a metric where the nodal set of the k^{th} eigenfunction has a prescribed connected component Σ_k , possibly not diffeomorphic for distinct values of k .

Proposition 3.1. *Let $\Sigma_1, \dots, \Sigma_l$ be pairwise disjoint closed orientable hypersurfaces of M . Then for all $1 \leq k \leq l$, there exists a Riemannian metric g on M such that Σ_k is a connected component of the nodal set $u_{k+l-1}^{-1}(0)$.*

Proof. The proof is an easy modification of that of the main theorem. Indeed, let us consider (pairwise disjoint) small neighborhoods Ω_k of the orientable hypersurfaces Σ_k . As before, we will identify Ω_k with $(-1, 1) \times \Sigma_k$. The starting point now is a smooth reference metric g_0 on M taking the form

$$g_0|_{\Omega_k} = \Gamma_k^2 dx^2 + g_{\Sigma_k},$$

in each domain Ω_k . Here g_{Σ_k} is a metric on Σ_k whose first eigenvalue is assumed larger than π^2 and Γ_k are constants such that

$$1 = \Gamma_1 > \Gamma_2 > \dots > \Gamma_l > \frac{1}{2}.$$

Hence the first nonzero Neumann eigenvalue of Ω_k is

$$\mu_k := \frac{\pi^2}{4\Gamma_k^2},$$

and the corresponding normalized eigenfunction is

$$v_k := \frac{1}{\sqrt{\Gamma_k|\Sigma_k|}} \cos \frac{\pi(x+1)}{2}.$$

Notice that here v_k and μ_k do *not* have the same meaning as in Section 2, even though they will play totally analogous roles in the proof.

We now set $\Omega := \bigcup_{k=1}^l \Omega_k$ and consider the piecewise smooth metric

$$g_\epsilon := \begin{cases} g_0 & \text{in } \bar{\Omega}, \\ \epsilon g_0 & \text{in } M \setminus \Omega. \end{cases}$$

Arguing exactly as in Steps 2 and 3 of the proof of the main theorem with the new definitions of μ_k and v_k and noticing that 0 is a Neumann eigenvalue of the domain Ω with multiplicity l , one concludes that the $(k + l - 1)^{\text{th}}$ eigenvalue $\lambda_{k+l-1,\epsilon}$ of the quadratic form defined by the metric g_ϵ tends to μ_k as $\epsilon \searrow 0$ for all $1 \leq k \leq l$, and that the corresponding eigenfunction $u_{k+l-1,\epsilon}$ converges to v_k in $C^m(K)$, for K any compact subset of Ω_k .

As in Step 4 of Section 2, we can smooth the metric g_ϵ by introducing a sequence of smooth metrics $g_{j,\epsilon}$. Arguing as in Step 4, we infer that for large j and small ϵ the nodal set of the eigenfunction $u_{k+l-1,\epsilon}^j$ then has a connected component diffeomorphic to Σ_k for $1 \leq k \leq l$, with the corresponding diffeomorphism $\Phi_{k,\epsilon}^j$ being arbitrarily close to the identity in any fixed C^m norm. The point now is that, as the hypersurfaces Σ_k are pairwise disjoint, the diffeomorphisms $\Phi_{k,\epsilon}^j$ can be assumed to differ from the identity only in a small neighborhood S_k of each hypersurface. This allows us to define a diffeomorphism Φ_ϵ^j of M by setting $\Phi_\epsilon^j := \Phi_{k,\epsilon}^j$ in each S_k and letting Φ_ϵ^j be the identity outside these sets.

The statement then follows by taking $g := (\Phi_\epsilon^j)^* g_{j,\epsilon}$, with large j and small ϵ , since in this case the eigenfunctions are $u_{k+l-1} = u_{k+l-1,\epsilon}^j \circ \Phi_\epsilon^j$.
q.e.d.

3.2. Critical points of low-energy eigenfunctions. The study of level sets of a function is deeply connected with that of its critical points. Hence here we will use Morse theory and our construction of eigenfunctions with prescribed nodal sets to show that there are metrics whose low-energy eigenfunctions have an arbitrarily high number of critical points. Notice that Theorem 1.2 is a corollary of Proposition 3.2 below. Regarding the behavior of the critical points of high-energy eigenfunctions, let us recall that Jakobson and Nadirashvili [13] proved that there are metrics (even on the two-dimensional torus) for which the number of critical points of the k^{th} eigenfunction does not tend to infinity as $k \rightarrow \infty$.

Proposition 3.2. *Given any positive integers N and l , there is a smooth metric on M such that the k^{th} eigenfunction u_k has at least N non-degenerate critical points for all $1 \leq k \leq l$.*

Proof. Let us fix some ball $B \subset M$ and take a domain D whose closure is contained in B . This ensures that $\Sigma := \partial D$ separates. Theorem 2.1 shows that there is a smooth metric g such that the nodal set of its k^{th}

eigenfunction u_k is diffeomorphic to k copies of Σ . Furthermore, the corresponding eigenvalues λ_k are simple for $0 \leq k \leq l$ and in Step 4 of Section 2 we showed that the differential of u_k does not vanish on its nodal set.

A theorem of Uhlenbeck [17] ensures that one can take a C^{m+1} -small perturbation \tilde{g} of the metric g so that the first l eigenfunctions are Morse, that is, all their critical points are non-degenerate. Standard results from perturbation theory show that the eigenfunctions \tilde{u}_k of the perturbed metric are close in the $C^m(M)$ norm to u_k , so in particular the nodal set of \tilde{u}_k has a connected component $\tilde{\Sigma}_k \subset B$ diffeomorphic to Σ for all $1 \leq k \leq l$. Here we are using the fact that the gradient of u_k does not vanish on its nodal set and Thom’s isotopy theorem.

Call D_k the domain contained in B that is bounded by $\tilde{\Sigma}_k$ and let us denote by $\tilde{\nabla}$ the covariant derivative associated with the metric \tilde{g} . Since $\tilde{\nabla}\tilde{u}_k$ is a nonzero normal vector on ∂D_k , which can be assumed to point outwards without loss of generality, we can resort to Morse theory for manifolds with boundary to show that the number of critical points of \tilde{u}_k of Morse index i is at least as large as the i^{th} Betti number of the domain \overline{D}_k , for $0 \leq i \leq n - 1$. Since \overline{D}_k is diffeomorphic to \overline{D} , the proposition then follows by choosing the domain \overline{D} so that the sum of its Betti numbers is at least N (this can be done, e.g., by taking ∂D diffeomorphic to a connected sum of N copies of any nontrivial product of spheres, such as $\mathbb{S}^1 \times \mathbb{S}^{n-2}$, since in this case the first Betti number is N).
 q.e.d.

3.3. Nodal sets of Dirichlet eigenfunctions. Motivated by Payne’s problem, in this subsection we will consider the nodal set of Dirichlet eigenfunctions on manifolds with boundary. Hence here M will be a compact manifold with boundary of dimension $n \geq 3$. Notice that Theorem 1.3 is a corollary of Proposition 3.3 below.

Proposition 3.3. *Let M be a compact n -manifold with boundary and let Σ be a closed orientable hypersurface contained in the interior of M . Then there exists a Riemannian metric g on M such that Σ is a connected component of the nodal set of its second Dirichlet eigenfunction. If Σ separates, then the nodal set is exactly Σ .*

Proof. The proof goes along the lines of that of the main theorem. Specifically, the construction of the metrics g_0 and g_ϵ is exactly as in Step 1 of Section 2. The variational formulation of the Dirichlet eigenvalues of the associated quadratic form Q_ϵ is analogous, the main difference being that its domain is now $H_0^1(M)$. A minor notational difference is that, since the first eigenvalue is customarily denoted in this context by λ_1 instead of λ_0 , \mathcal{W}_k (and \mathcal{W}'_k in Step 3) now consist of k -dimensional, instead of $(k + 1)$ -dimensional, subspaces.

Step 2 also carries over verbatim to the case of manifolds with boundary, with the proviso that the harmonic extension \hat{v}_k must also include a Dirichlet boundary condition at ∂M , i.e.,

$$\Delta_0 \hat{v}_k = 0 \quad \text{in } M \setminus \overline{\Omega}, \quad \hat{v}_k|_{\partial\Omega} = v_k, \quad \hat{v}_k|_{\partial M} = 0.$$

Steps 3 and 4 also remain valid in the case of manifolds with boundary, the only difference being that the space \mathcal{H}_1 must also include Dirichlet boundary conditions:

$$\mathcal{H}_1 := \{ \varphi \in H_0^1(M) : \Delta_0 \varphi = 0 \text{ in } \Omega^c \}.$$

The proposition is then proved using the same reasoning as in Section 2, noticing that the diffeomorphism $\Phi_{1,\epsilon}^j$ can be assumed to differ from the identity only in a small neighborhood of the hypersurface Σ . q.e.d.

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INSTITUTO DE CIENCIAS MATEMÁTICAS
CONSEJO SUPERIOR DE INVESTIGACIONES CIENTÍFICAS
28049 MADRID, SPAIN
E-mail address: aenciso@icmat.es

INSTITUTO DE CIENCIAS MATEMÁTICAS
CONSEJO SUPERIOR DE INVESTIGACIONES CIENTÍFICAS
28049 MADRID, SPAIN
E-mail address: dperalta@icmat.es