J. DIFFERENTIAL GEOMETRY
99 (2015) 313-357

ON RICCI CURVATURE AND VOLUME GROWTH IN DIMENSION THREE

Martin Reiris

Abstract

We prove that any complete metric on $\mathbb{R}^3$ minus an open ball, with non-negative Ricci curvature and quadratic Ricci-curvature decay, has cubic volume growth.

1. Introduction

In the study of non-compact manifolds, a simple and at the same time rich invariant worth investigating is the rate of volume growth of geodesic spheres. The importance of this invariant roots in the fact that it can provide global information under either local or global conditions on the curvature.

Let (M, g) be a non-compact and connected complete Riemannian manifold of dimension $n \geq 3$. If the Ricci curvature is non-negative, then the Bishop—Gromov volume-comparison tells that the volume growth is at most Euclidean,

$$\lim_{r \to \infty} \frac{Vol(B(o,r))}{r^n} = \rho \neq \infty, \tag{1}$$

and Yau showed [**30**] that it is at least linear,

$$\liminf_{r \to \infty} \frac{Vol(B(o,r))}{r} = \rho' > 0.$$

If $Ric \geq 0$ and furthermore M is three-dimensional, then G. Liu [**17**] proved recently that M is either diffeomorphic to $\mathbb{R}^3$ or the universal cover splits with a $\mathbb{R}$-factor. This extends an earlier result of Schoen and Yau [**27**] saying that M is diffeomorphic to $\mathbb{R}^3$ when $Ric > 0$. Assuming now $Ric \geq 0$, $n = 3$, and Euclidean volume growth (i.e., $\rho \neq 0$), then Zhu [**31**] showed that M is contractible. Combined with Liu's result this says that the only open three-manifold admitting a complete metric of non-negative Ricci curvature and Euclidean volume growth is $\mathbb{R}^3$.

In arbitrary dimensions but also when (M, g) has Euclidean volume growth and non-negative Ricci curvature, Perelman [**22**] proved that M is contractible when $\rho \geq \rho_n$. Also in this context, Cheeger and Colding [**6**] showed that the limit of scalings of (M, g) has always the structure of a metric cone, Perelman [**23**] realized that it may not be unique, and Tian and Cheeger [**8**] gave curvature conditions for uniqueness. Volume growth is also tied to the fundamental group as put forward by Milnor [**20**]. In manifolds of non-negative Ricci curvature, this fact was exploited by Li [**15**] and further by Anderson [**2**].

Playing somehow in the reverse direction, several authors have also studied volume growth under various curvature-decaying conditions. For instance, Cheeger, Gromov, and Taylor [**7**] proved less than Euclidean volume growth (i.e., $\rho = 0$) under the *lower quadratic curvature decay* $Ric \geq \Lambda/r^2$ ($r(p) = dist(p, o)$), and, by disregarding the assumption $Ric \geq 0$, Lott and Shen [**19**] gave interesting examples of complete metrics in $\mathbb{R}^n$ with slow volume growth and quadratic curvature decay, that is, with $|Ric| \leq \Lambda/r^2$.

In this article, we provide optimal hypothesis in dimension three to have cubic volume growth. The main result, which we state next, applies to complete metrics g on the manifold with boundary $\mathbb{R}^3 \setminus \mathbb{B}^3$, where $\mathbb{B}^3$ is the open unit ball of $\mathbb{R}^3$.

Theorem 1.1. *Let g be a complete metric on $\mathbb{R}^3 \setminus \mathbb{B}^3$ with non-negative Ricci curvature and quadratic curvature decay. Then g has cubic volume growth.*

Combined with the results of G. Liu and Zhu, we obtain the following Corollary.

Corollary 1.1. *Let (M, g) be a non-compact and boundaryless three-dimensional manifold of non-negative curvature and quadratic curvature decay. Then (M, g) has Euclidean volume growth iff M is diffeomorphic to $\mathbb{R}^3$.*

In view of this corollary, the importance of having a theorem like Theorem 1.1 stated for metrics on the manifold with boundary $\mathbb{R}^3 \setminus \mathbb{B}^3$, which gives much more applicability, is now apparent.

Theorem 1.1 is optimal in various ways. First, the Lott–Shen examples show that metrics in $\mathbb{R}^3$ with quadratic curvature decay can display dramatic volume distortions if one disregards entirely the signature of the Ricci tensor. Second, there are simple spherically symmetric metrics in $\mathbb{R}^3$ with $Ric \geq 0$ (outside a ball), $\rho = 0$, and $|Ric| \leq \Lambda/r^{2\alpha}$ for any $\alpha < 1$ that show that the hypothesis of quadratic curvature decay in Theorem 1.1 can be hardly weakened. Third, replacing in Theorem 1.1 $\mathbb{R}^3$ by $\mathbb{R}^n$ and "cubic" by "Euclidean" can also create a false statement. For instance, the flat product metric on $\mathbb{S}^1 \times \mathbb{R}^+$, which has linear volume growth, shows that it is false when $n = 2$, while the Tau-NUT Ricci flat instanton [**14**] in $\mathbb{R}^4$, which has cubic volume growth, shows that it is false when $n = 4$. We do not know at the moment if $n = 3$ is the only dimension where such statement holds.

This article was partly motivated by investigations on the asymptotic of isolated bodies in General Relativity, a subject very closely related to the Theory of Gravitational Instantons. Concretely, Theorem 1.1 found the following application in [**25**].

Theorem. *Any vacuum (strictly) stationary spacetime-end whose Killing field has its norm bounded away from zero is necessarily asymptotically flat with Schwarzschidian fall off.*

In this statement a *spacetime-end*, as was defined in [**25**], is diffeomorphic to $(\mathbb{R}^3 \setminus \mathbb{B}^3) \times \mathbb{R}$, and its Lorentzian metric is complete up to be the boundary. Strictly stationary means that the Killing field is time-like everywhere. In physical applications, one must understand spacetime-ends as a part of a bigger globally hyperbolic one far away from the "sources." In this sense, the theorem says that isolated systems in General Relativity are necessarily asymptotically flat with Schwarzschidian fall off. For this application, it is crucial that Theorem 1.1 is stated for metric in $\mathbb{R}^3 \setminus \mathbb{B}^3$. This is partly because, at least in vacuum, the strictly stationary Killing field on an end does not extend to a time-like Killing field in the bigger globally hyperbolic space unless the space is Minkowski [**3**]. We point out that by the proximity of the subjects it is likely that Theorem 1.1 can find further applications in the study of four-dimensional Instantons as well.

In the next part of the introduction we explain in great detail the arguments behind the proof of Theorem 1.1. We delineate the contents of the article afterward.

Let us start with some preliminary words on the statement of Theorem 1.1. By extending g from $\mathbb{R}^3 \setminus \mathbb{B}^3$ to $\mathbb{R}^3$ [1] we can assume without loss of generality that a metric g in $\mathbb{R}^3$ is given, having $Ric \geq 0$ outside $\mathbb{B}^3$ and quadratic curvature decay, namely $|Ric| \leq \Lambda/r^2$ with $r(p) = dist(o,p)$ and $o = (0,0,0)$. Now let $\mathcal{T}(\partial\mathbb{B}^3, r)$ be the tubular neighborhood of $\partial\mathbb{B}^3$ inside $\mathbb{R}^3 \setminus \mathbb{B}^3$ and of radius r. By the volume comparison, the quotient $Vol(\mathcal{T}(\partial\mathbb{B}^3, r))/r^3$ is monotonically deceasing in r and therefore has a limit that coincides with the limit (1) in the space $(\mathbb{R}^3, g)$. Thus we need to prove that $\rho > 0$.

The outline of the proof of Theorem 1.1 (which proceeds by contradiction) is somehow simple. In gross terms one proves that if the volume growth is non-cubic (i.e., $\rho = 0$), then (under the hypothesis of Theorem 1.1) one can partition $\mathbb{R}^3$ into a set of compact manifolds with a sufficient understanding of their topology to be able to prove that their union is topologically incompatible with $\mathbb{R}^3$. More precisely, one is able to write $\mathbb{R}^3$ as the union of an open set with compact closure and containing the origin o, and a set

$$\bigcup_{i=i_0}^{i=\infty} M(T^{2o}_{i+1}, T^{2o}_i), \tag{2}$$

[1] This can be done also to have every point in $\partial\mathbb{B}^3$ at a constant distance from o.

where each $M(T^{2o}_{i+1}, T^{2o}_i)$ is a three-manifold with the tori T^{2o}_{i+1} and T^{2o}_i as its only boundary components. Moreover, in this union every $M(T^{2o}_{i+1}, T^{2o}_i)$ is an *irreducible manifold with incompressible boundary* (hereafter IIB-manifold), with the interiors $M(T^{2o}_{i+1}, T^{2o}_i)^\circ$ pairwise disjoint. In this scenario, one reaches a contradiction as follows. Pick a coordinate sphere $\mathbb{S}^2_{\bar{r}} = \partial\mathbb{B}^3(o, \bar{r})$ in $\mathbb{R}^3$ of coordinate radius $\bar{r}$ large enough to ensure that $\mathbb{S}^2_{\bar{r}}$ lies inside the union (2) (indeed, inside a finite union of $M(T^{2o}_{i+1}, T^{2o}_i)$'s). Because the union is also irreducible (see Proposition 2.1), the sphere $\mathbb{S}^2_{\bar{r}}$ must bound a three-ball in it, which forcefully must be $\mathbb{B}^3(o, \bar{r})$. But then the origin o must belong to the union (2), which was not assumed.

We explain now the construction of the manifolds $M(T^{2o}_{i+1}, T^{2o}_i)$ and point out the hypotheses that are needed to show that they are IIB manifolds. This will help the reader to identify the important steps inside the technical discussion later.

For every $b > a > 0$, let $A_g(a, b) = B_g(o, b) \setminus \overline{B_g(o, a)}$ be the *(open) metric annulus* with radii a and b and center the origin o. Then, to the effect of constructing the manifolds $M(T^{2o}_{i+1}, T^{2o}_i)$, one first defines *annuli decompositions* as follows.

Definition 1. A set $\mathcal{U} = \{U_{k,l}; k = k_0 + 2j, j = 0, 1, 2, 3, ..., l = 1, 2, \ldots, l(k)\}$ of compact and connected three-submanifolds of $(\mathbb{R}^3, g)$ with smooth boundary is an annuli decomposition iff the following conditions are fulfilled:

1. $U_{k,l} \subset A_g(10^{k-1}, 10^{k+3})$,
2. $U_{k,l} \cap A_g(10^{k-1}, 10^k) \neq \emptyset$, and $U_{k,l} \cap A_g(10^{k+1}, 10^{k+2}) \neq \emptyset$,
3. $\partial U_{k,l} \subset \big(A_g(10^{k-1}, 10^k) \cup A_g(10^{k+1}, 10^{k+2})\big)$,
4. If $(k, l) \neq (k', l')$, then $U^\circ_{k,l}$ and $U^\circ_{k',l'}$ are disjoint and if $U_{k,l}$ and $U_{k',l'}$ intersect then they do in a set of boundary components (of both, $U_{k,l}$ and $U_{k',l'}$),
5. $U_{k_0-2} := \mathbb{R}^3 \setminus \big(\bigcup_{U_{k,l} \in \mathcal{U}} U_{k,l}\big)^\circ$ is compact.

In other words, an annuli decomposition is just a partition of $\mathbb{R}^3$ into a set of pieces $\{U_{k,l}\}$ adapted to the set of metric annuli $\big\{A_g(10^{k-1}, 10^{k+3}),\ k = k_0, k_0 + 2, \ldots\big\}$ and enjoying uniform "size" conditions (i.e., satisfying *items 1–3*). They are soft structures and exist independently of the metric g. An example of one is represented in Figure 1.

Then one proves that under the hypothesis of Theorem 1.1 there is an annuli decomposition of $(\mathbb{R}^3, g)$ where every piece $U_{k,l}$ posses a well-understood geometry and topology. This is the content of the following proposition, which we prove in Section 2.5 and which guarantees the existence of an annuli decomposition whose pieces $U_{k,l}$, when endowed with the scaled metrics $g_k := g/10^{2k}$, are close in the Gromov–Hausdorff (GH) distance to an interval or a two-orbifold $X_{k,l}$ and for which, in addition, there is a fibration $f_{k,l} : U_{k,l} \to X_{k,l}$ encoding precisely the relation between the geometry and topology of $(U_{k,l}, g_k)$ and the spaces $X_{k,l}$.

Proposition 2.4. *Let g be a complete metric in $\mathbb{R}^3$ with $Ric \geq 0$ outside a compact set, quadratic curvature decay, and less-than-cubic-volume growth. Then there is an annuli decomposition $\mathcal{U}$ with the following properties:*

For every $\epsilon > 0$, there is $k(\epsilon)$ such that for any $k \geq k(\epsilon)$ every piece $(U_{k,l}, g_k)$ is ϵ-close in the GH-metric to a space $X_{k,l}$ of one of the following two forms:

$\tilde{\mathbf{D}}\mathbf{1}$. *An interval, in which case $U_{k,l}$ is either diffeomorphic to $\mathbb{T}^2 \times \mathbb{I}$ or a solid torus $\mathbb{B}^2 \times \mathbb{S}^1$.*

$\tilde{\mathbf{D}}\mathbf{2}$. *A two-orbifold, in which case $U_{k,l}$ is diffeomorphic to a Seifert manifold with at least one boundary component.*

Moreover, there are fibrations $f_{k,l} : U_{k,l} \to X_{k,l}$, such that for any $k \geq k(\epsilon)$ the fibers $f_{k,l}^{-1}(x)$, which are diffeomorphic either to $\mathbb{T}^2$ or $\mathbb{S}^1$, are ϵ-collapsed, and:

$\tilde{\mathbf{I}}\mathbf{1}$. *In case $\tilde{\mathbf{D}}\mathbf{1}$, either $Sing(X_{k,l})$ is empty or is one of the extreme points of the interval. In addition, for any non-singular point x, the fiber $f_{k,l}^{-1}(x)$ is diffeomorphic to $\mathbb{T}^2$, and if x is a singular point, then $f_{k,l}^{-1}(x)$ is diffeomorphic to $\mathbb{S}^1$.*

$\tilde{\mathbf{I}}\mathbf{2}$. *In case $\tilde{\mathbf{D}}\mathbf{2}$, the fibers $f_{k,l}^{-1}(x)$, which are all diffeomorphic to $\mathbb{S}^1$, are the fibers of the Seifert-fibration.*

The proposition is an immediate consequence of the Cheeger–Gromov–Fukaya theory of volume collapse with bounded diameter and curvature applied to the sequence of annuli $\{(A_g(10^{k-1}, 10^{k+4}), g_k)\}$ and in this sense there is little novel in it. Despite the stringent constraints on the nature of the pieces $U_{k,l}$ of this annuli decomposition, the topology of $\mathbb{R}^3$ is, at this point, not contradicted in any way. Proposition 2.4 requires only the non-negativity of Ricci outside a compact set in a mild form (and may not even be necessary).

The manifolds $M(T^{2o}_{i+1}, T^{2o}_i)$ are defined from the annuli decomposition $\mathcal{U}$ in Proposition 2.4 as follows. Denote by $\mathcal{N}$ to the set of boundary components of the manifolds $U_{k,l}$ forming $\mathcal{U}$. These manifolds are tori and denoted by T^2. For every $T^2 \in \mathcal{N}$, let $M(T^2)$ be the closure of the bounded region enclosed by T^2 in $\mathbb{R}^3$. Then consider the set $\mathcal{N}^o = \{T^{2o}_i, i = 1, 2, 3, \ldots\}$ of all the tori in $\mathcal{N}$ such that $o \in M(T^{2o}_i)$ and ordered in such a way that $M(T^{2o}_i) \subset M(T^{2o}_{i'})$ whenever $i' > i$. Finally, define

$$M(T^{2o}_{i+1}, T^{2o}_i) := M(T^{2o}_{i+1}) \setminus M(T^{2o}_i)^\circ.$$

As said, to prove Theorem 1.1 we need to show that for any $i \geq i_0$, with i_0 big enough, the manifolds $M(T^{2o}_{i+1}, T^{2o}_i)$ are IIB manifolds. This is done by ruling out the presence of certain crucial pieces $U_{k,j}$ inside the $M(T^{2o}_{i+1}, T^{2o}_i)$'s for $i \geq i_0$. It is at this stage when one must rely heavily upon all the hypotheses of Theorem 1.1. Let us discuss this more concretely in the next paragraph. Full details, however, must be found inside the proof of Theorem 1.1.

First, some terminology. Given $\epsilon > 0$ let, $i(\epsilon) > 0$ be large enough that for any piece $U_{k,l}$ in $\mathbb{R}^3 \setminus M(T^{2o}_{i(\epsilon)})^\circ$ we have $k \geq k(\epsilon)$, where $k(\epsilon)$ is the one provided by Proposition 2.4. Then we say that a piece $U_{k,l}$ in $\mathbb{R}^3 \setminus M(T^{2o}_{i(\epsilon)})^\circ$ is of type $I(\epsilon)$ if $(U_{k,l}, g_k)$ is ϵ-close in the GH-distance to an interval. If not, then $(U_{k,l}, g_k)$ is ϵ-close to a two-orbifold and we say that it is of type $II(\epsilon)$. From now on we study the manifolds $M(T^{2o}_{i+1}, T^{2o}_i)$ with $i \geq i(\epsilon)$. First, as shown easily in Section 2.3, the two consecutive tori T^{2o}_i and T^{2o}_{i+1} are boundary components of one single piece $U_{k,l}$. If such piece is of type $I(\epsilon)$, then it has to be diffeomorphic to $\mathbb{T}^2 \times \mathbb{I}$ and is thus a IIB manifold. If not, then we have two options: (i) all the $U_{k,l}$ pieces forming $M(T^{2o}_{i+1}, T^{2o}_i)$ are of type $II(\epsilon)$ or (ii) at least one of the pieces $U_{k,l}$ forming $M(T^{2o}_{i+1}, T^{2o}_i)$ is of type $I(\epsilon)$. In case (i), $M(T^{2o}_{i+1}, T^{2o}_i)$ is then a union of Seifert manifolds (with the Seifert fibrations coinciding at any intersection) and therefore a Seifert manifold itself. Thus, in this case $M(T^{2o}_{i+1}, T^{2o}_i)$ is Seifert, with two boundary components (T^{2o}_{i+1} and T^{2o}_i) and hence a IIB manifold (see Section 2.2). We treat now the more involved case (ii). Denote by $\mathscr{U}_{i+1,i}$ to the union of all the pieces $U_{k,l}$ in $M(T^{2o}_{i+1}, T^{2o}_i)$ of type $II(\epsilon)$ and by $\widehat{\mathscr{U}_{i+1,i}}$ to the only connected component of $\mathscr{U}_{i+1,i}$ containing T^{2o}_{i+1} and T^{2o}_i. Let $\widehat{\mathscr{N}}_{i+1,i}$ be the set of of boundary components of $\widehat{\mathscr{U}_{i+1,i}}$ other than T^{2o}_{i+1} and T^{2o}_i. Thus every torus T^2 in $\widehat{\mathscr{N}}_{i+1,i}$ is the boundary of a piece of type $I(\epsilon)$ and a piece of type $II(\epsilon)$. In particular, such a torus inherits from the piece of type $II(\epsilon)$ the Seifert fibration $\{\mathscr{C}\}$ by short loops. Also, every T^2 in $\widehat{\mathscr{N}}_{i+1,i}$ either bounds, as an embedded torus in $\mathbb{R}^3$, a solid torus (i.e., $M(T^2)$ is a solid torus) or not. Tori $T^2 \in \widehat{\mathscr{N}}_{i+1,i}$ in the former case are denoted by $T^{2\blacklozenge}$, while tori in the last case are denoted by $T^{2\lozenge}$. If the mentioned loops $\{\mathscr{C}\}$ in a torus $T^{2\blacklozenge} \in \widehat{\mathscr{N}}_{i+1,i}$ are non-contractible inside $M(T^{2\blacklozenge})$, then, as shown easily in Section 2.2, one can extend the Seifert fibration $\{\mathscr{C}\}$ to the whole $M(T^{2\blacklozenge})$. On the other hand, a decisive step in this article consists in proving that if ϵ is chosen small enough (as we assume below), then there are no tori $T^{2\blacklozenge} \in \widehat{\mathscr{N}}_{i+1,i}$ for which the loops $\{\mathscr{C}\}$ are contractible inside $M(T^{2\blacklozenge})$. This is a consequence of the lengthy Proposition 3.1, which relies upon Proposition 2.3. The non-negativity of Ricci is here fundamentally used. After this type of boundary component $T^{2\blacklozenge} \in \widehat{\mathscr{N}}_{i+1,i}$ is ruled out, one can argue as follows. Write $M(T^{2o}_{i+1}, T^{2o}_i)$ as

$$\Big[\Big(\bigcup_{U_{k,j}\subset\widehat{\mathscr{U}}_{i+1,i}} U_{k,j}\Big)\bigcup\Big(\bigcup_{T^{2\blacklozenge}\in\widehat{\mathcal{N}}_{i+1,i}} M(T^{2\blacklozenge})\Big)\Big]\bigcup\Big[\bigcup_{T^{2\lozenge}\in\widehat{\mathcal{N}}_{i+1,i}} M(T^{2\lozenge})\Big].$$

Then, as we just explained, the manifold in the first square bracket is naturally a Seifert manifold with at least two boundary components (T^{2o}_{i+1} and T^{2o}_i), and hence a IIB manifold. On the other hand, a simple argument in three-dimensional topology (see Section 2.2) shows that every manifold $M(T^{2\lozenge})$ inside the second square bracket is a IIB manifold too. Proposition 2.1 then shows that $M(T^{2o}_{i+1}, T^{2o}_i)$ is also a IIB manifold when $i \geq i_0 := i(\epsilon)$, as desired.

The article is organized as follows. In Section 2.1, we introduce some basic notation. In Section 2.2, we discuss elementary facts about two-manifolds embedded in $\mathbb{R}^3$ and IIB-manifolds. This section is important and is used several times. Then, in Section 2.3, we reintroduce annuli decomposition and prove their basic properties. The whole Section 2.4 develops the main elements of the Cheeger–Gromov–Fukaya theory on three-manifolds with boundary. To our knowledge this has not been discussed previously in the literature with the necessary detail. This justifies our exhaustive presentation, which, incidentally, could be of use in other investigations. The section ends with Lemma 2.3, which is the first simple but relevant application. Lemma 2.3 will be used in the proof of the fundamental Proposition 3.1. In Section 2.5, we prove the commented Proposition 2.4. Section 3 is the crucial section of the article. It starts proving the fundamental Proposition 3.1 and ends with the proof of Theorem 1.1 along the lines mentioned above. We explain in the appendix a couple of technical propositions whose inclusion inside the text would cause much disruption. The article has a good amount of background material, examples, and illustrations.

Acknowledgment. We would like to thank the referee for useful expository suggestions.

2. Preliminaries

2.1. Basic notation. $\mathbb{S}^n$, $n \geq 1$ will be the unit sphere in $\mathbb{R}^{n+1}$ and $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ the two-dimensional torus. $\mathbb{S}^1$ and $\mathbb{T}^2$ will be thought both as manifolds and as Lie groups. Furthermore, $\mathbb{B}^n(o, r) = \{\bar{x} \in \mathbb{R}^n, |\bar{x}| < r\}$ will be the open ball of center the origin $o = (0, 0, 0)$ and radius r ($|\bar{x}|$ is the Euclidean norm of a point $\bar{x}$ of $\mathbb{R}^n$). $\mathbb{B}^n = \mathbb{B}^n(o, 1)$. $\mathbb{I} = \mathbb{B}^1$.

Let (M, g) be a compact connected Riemannian manifold with boundary. The Riemannian metric g induces a metric $d_g = dist_g$ in M (as usual) by

$$d_g(p,q) = \inf\{length_g(\gamma(p,q)), \gamma(p,q)\ C^1\text{-curve from } p \text{ to } q\}. \tag{3}$$

However, if $(\Omega, g) \subset (M, g)$, then on Ω one can consider two different distances, the distance (3) of (Ω, g) and the distance (3) of (M, g) restricted to Ω. This situation will appear often, and for this reason and to avoid confusion we will denote them by d_g^{Ω} and d_g^{M}, respectively.

In this article, the Riemannian space (Ω, g) will also denote the metric space (Ω, d_g^{Ω}).

We will always use the following definitions of *diameter* $diam_g(\Omega)$ and *radius (to the boundary)* $rad_g(\Omega)$, even when $(\Omega, g) \subset (M, g)$:

$$diam_g(\Omega) = \sup\{d_g^{\Omega}(p,q); p,q \in \Omega\},\ \ rad_g(\Omega) = \sup\{d_g^{\Omega}(p, \partial\Omega); p \in \Omega\}.$$

Manifold interiors $\Omega \setminus \partial\Omega$ are denoted by Ω°. To us, a metric ball of center $p \in \Omega^{\circ}$ and radius r is a *geodesic ball* if $r < d_g^{\Omega}(p, \partial\Omega)$.

The ends of theorems, lemmas, and propositions are marked with "q.e.d," while the end of claims, steps, or examples, are marked with a ◀.

2.2. Surfaces in $\mathbb{R}^3$ and IIB three-manifolds. From now on, we let S be a smoothly embedded compact, orientable, and boundaryless surface in $\mathbb{R}^3$. Any S divides $\mathbb{R}^3$ into two open connected components. We will denote by $M(S)$ the closure of the bounded component. For instance, if $S \sim \mathbb{S}^2$, then S bounds a three-ball [**1**]. If $S \cap S' = \emptyset$, then either

$$M(S) \cap M(S') = \emptyset,\ \ M(S) \subset M(S')^{\circ},\ \ \text{or}\ \ M(S') \subset M(S)^{\circ}. \tag{4}$$

Moreover, if $S' \subset M(S)^{\circ}$, then S belongs to $\mathbb{R}^3 \setminus M(S')$ and therefore $M(S') \subset M(S)^{\circ}$. In particular, if $S' \sim \mathbb{S}^2$ and $S' \subset M(S)^{\circ}$, then S' bounds a three-ball inside $M(S)^{\circ}$. Recall that a three-manifold is *irreducible* if every embedded two-sphere bounds a three-ball. Thus for any S, $M(S)$ is an irreducible manifold.

We claim that if $S \sim \mathbb{T}^2$, then either $M(S)$ is a solid torus—i.e., $\sim \bar{\mathbb{B}}^2 \times \mathbb{S}^1$—or $S = \partial M(S)$ is incompressible in $M(S)$, where, recall, N is an *incompressible* boundary component of a manifold M if $i_* : \pi_1(N) \to \pi_1(M)$ is injective (here $i : N \to M$ is the inclusion). To see this, think of S as a surface in $\mathbb{S}^3$ via $S \subset \mathbb{R}^3 \subset \big(\mathbb{R}^3 \cup \{\infty\}\big) \sim \mathbb{S}^3$. If $M(S)$ is a solid torus, we are done. If not, then $\mathbb{S}^3 \setminus M(S)^{\circ}$ is a solid torus (this is due to Alexander [**1**]). If $\mathbb{S}^3 \setminus M(S)^{\circ}$ represents the unknot, then $M(S)$ is a solid torus but we are assuming that it is not. Then $\mathbb{S}^3 \setminus M(S)^{\circ}$ is not the unknot. Theorem 11.2 in [**16**] shows that in this case S is incompressible in $M(S)$, as claimed. Summarizing, for any $S \sim \mathbb{T}^2$, $M(S)$ is either a solid torus or an irreducible manifold with incompressible boundary.

Other examples of irreducible manifolds with incompressible boundary components (in short, "IIB" manifolds) are compact *Seifert* manifolds with at least two boundary components [**28**, pp. 431–432 and Corollary 3.3]. Recall that a Seifert manifold is one admitting a foliation $\mathcal{C}$ by circles $\mathscr{C}$ around any of which there is a fibered neighborhood isomorphic to a *fibered solid torus or Klein bottle* (see [**28**, pg 428]. The class of IIB manifolds is closed under sums along boundary components. Precisely, we have (Lemma 1.1.4 in [**29**]) the following.

Proposition 2.1.

I. *Let M_1 and M_2 be two* IIB *manifolds. and let $f : N_1 \to N_2$ be a diffeomorphism between a boundary component N_1 of M_1 and a boundary component N_2 of M_2. Then the manifold that results from identifying through f the boundary N_1 of M_1 to the boundary N_2 of M_2 is a* IIB *manifold.*

II. *Let M_1 be a* IIB *manifold, and let $f : N_1 \to N_2$ be a diffeomorphism between the boundary components $N_1 \neq N_2$ of M_1. Then the manifold that results from identifying through f the boundary N_1 to the boundary N_2 of the manifold M_1 is a* IIB *manifold.*

Therefore, any sum of IIB *manifolds along any number of boundary components is a* IIB *manifold.*

However, there is a simple but important situation when the sum of a IIB manifold and a non-IIB manifold results in a IIB manifold. The case is when M_1 is a Seifert manifold with Seifert structure $\mathcal{C}$ and at least three-boundary components, M_2 is a solid torus, and the gluing function $f : N_1(\subset \partial M_1) \to N_2(= \partial M_2)$ send circles $\mathscr{C}$ in $\mathcal{C}$ into non-contractible circles $f(\mathscr{C})$ as circles in M_2. The reason is that in this situation the $\mathbb{S}^1$-foliation $f(\mathcal{C})$ of $N_2 = \partial M_2$ can always be extended to a Seifert structure in M_2 and thus making the sum a Seifert manifold with at least two boundary components and therefore a IIB manifold. To construct the extension of $f(\mathcal{C})$, proceed as follows. On $M_2 \sim \mathbb{B}^2\times\mathbb{S}^1$, denote points by $(\bar{x}, s)$, $\bar{x} \in \mathbb{B}^2$, and $s \in \mathbb{S}^1$. Then, for any $r \in [0, 1]$ define $F_r : \mathbb{B}^2 \times \mathbb{S}^1 \to \mathbb{B}^2 \times \mathbb{S}^1$ by $F_r(\bar{x}, s) = (r\bar{x}, s)$. The desired extension of $f(\mathcal{C})$ is $\{F_r(\mathscr{C}); \mathscr{C} \in \mathcal{C}, r \in [0, 1]\}$.

2.3. Annuli decompositions. Let g be a complete metric in $\mathbb{R}^3$. For every $b > a > 0$, we let $A_g(a, b) = B_g(o, b) \setminus \overline{B_g(o, a)}$ (resp. $A_g[a, b] = \overline{B_g(o, b)} \setminus B_g(o, a)$) be the *open (resp. closed) annulus* with radii a and b and center the origin o.

Definition 1. A set $\mathcal{U} = \{U_{k,l}; k = k_0 + 2j, j = 0, 1, 2, 3, \ldots, l = 1, 2, \ldots, l(k)\}$ of compact and connected three-submanifolds of $\mathbb{R}^3$ with smooth boundary is an annuli decomposition iff the following conditions are fulfilled:

1) $U_{k,l} \subset A_g(10^{k-1}, 10^{k+3})$,

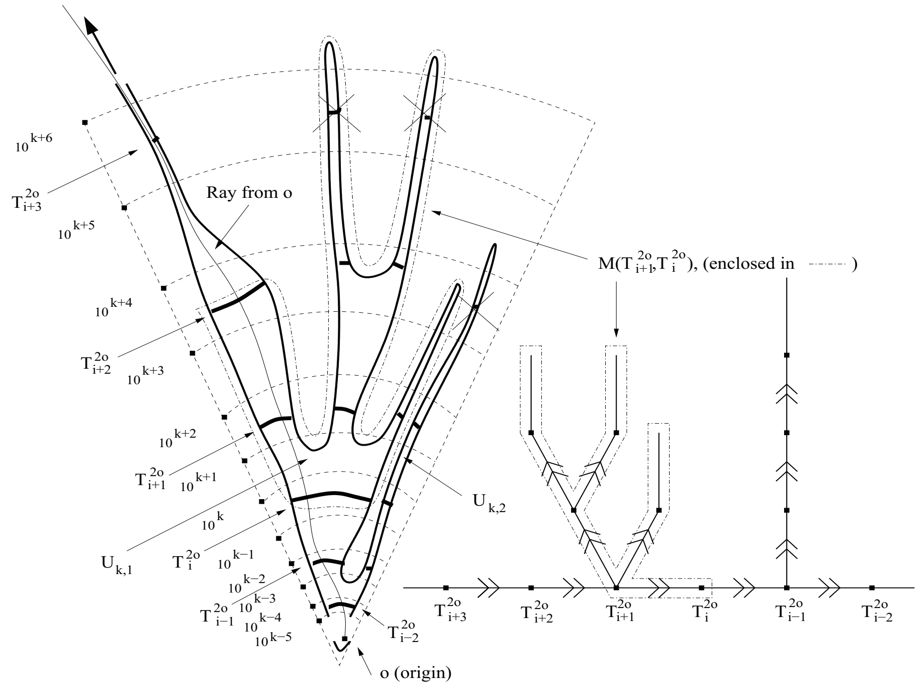

Figure 1. On the left side, the figure schematizes a part of an annuli decomposition. We have indicated only the pieces $U_{k,1}$ and $U_{k,2}$ but every region enclosed by thick lines is a piece $U_{k,l}$. We have also explicitly indicated the surfaces S^o_i, however, with a T^{2o}_i, as this is the notation to be used in Section 3 where the proof of Theorem 1.1 is carried out. On the right side is represented the corresponding part of the tree induced by the order $\ll$. On both sides we have enclosed in a dash/point line $(-\cdot-\cdot-)$ the manifold $M(T^{2o}_{i+1}, T^{2o}_i)$. On the left it can also be seen crossed thick lines. This is to exemplify the construction of the special annuli decomposition in Proposition 2.4 inside Section 2.5 (to be used inside the proof of Theorem 1.1). The cross indicates that such "cuts," as we refer them there, are to be discarded.

2) $U_{k,l} \cap A_g(10^{k-1}, 10^k) \neq \emptyset$, and $U_{k,l} \cap A_g(10^{k+1}, 10^{k+2}) \neq \emptyset$,
3) $\partial U_{k,l} \subset \big(A_g(10^{k-1}, 10^k) \cup A_g(10^{k+1}, 10^{k+2})\big)$,
4) If $(k,l) \neq (k',l')$, then $U^\circ_{k,l}$ and $U^\circ_{k',l'}$ are disjoint, and if $U_{k,l}$ and $U_{k',l'}$ intersect, then they do in a set of boundary components (of both, $U_{k,l}$ and $U_{k',l'}$),
5) $U_{k_0-2} := \mathbb{R}^3 \setminus \big(\bigcup_{U_{k,l}\in\mathcal{U}} U_{k,l}\big)^\circ$ is compact.

Let $\mathcal{N}$ be the set of boundary components of the manifolds $U_{k,l}$ in an annuli decomposition $\mathcal{U}$. Elements of $\mathcal{N}$ are pairwise disjoint compact, orientable, and embedded surfaces. We can order them as follows: $S \ll S'$ iff $M(S) \subset M(S')$. The order is not necessarily a linear order, as there can be two elements not related. However, there is an important subset that is linearly ordered—this is the set $\mathcal{N}^o = \{S \in \mathcal{N}; o \in M(S)\}$ (use (4)). Thus $\mathcal{N}^o = \{S_1^o, S_2^o, S_3^o, \dots\}$ with $S_1^o \ll S_2^o \ll S_3^o \ll \dots$. We will be using this notation (the upper-index o is from "origin"). We will also write $M(S_i^o, S_{i'}^o) := M(S_i^o) \setminus M(S_{i'}^o)^\circ$ for the region enclosed by S_i^o and $S_{i'}^o$, $i > i'$.

We note that any consecutive S_i^o and S_{i+1}^o are necessarily boundary components of a single piece $U_{k,l} \subset M(S_{i+1}^o, S_i^o)$. This is seen as follows. Let $U_{k,l}(S_{i+1}^o)$ be the only piece in $M(S_{i+1}^o)$ having S_{i+1}^o as a boundary component, and assume that S_i^o is not a boundary component of $U_{k,l}(S_{i+1}^o)$. Then $U_{k,l}(S_{i+1}^o)$ is disjoint from $M(S_i^o)$ (otherwise $U_{k,l}(S_{k,l}^o) \subset M(S_i^o)$ and therefore $S_{i+1}^o \ll S_i^o$), and because $o \in M(S_i^o)$ we conclude that $o \notin U_{k,l}(S_{i+1}^o)$. Now, if a boundary component S of $U_{k,l}(S_{k,l}^o)$ other than S_{i+1}^o is in $\mathcal{N}^o$, then it must be $S_i^o \ll S \ll S_{i+1}^o$, which is impossible. Thus $o \notin M(S)$ for any boundary component S of $U_{k,l}(S_{i+1}^o)$ other than S_{i+1}^o. Hence we can write

$$M(S_{i+1}^o) = U_{k,l}(S_{i+1}^o) \bigcup \Bigg[\bigcup_{S \in (\{\partial U_{k,l}(S_{i+1}^o)\} \setminus S_{i+1}^o)} M(S) \Bigg],$$

to conclude that $o \notin M(S_{i+1}^o)$, which is a contradiction.

A representation of an annuli decomposition can be seen in Figure 1. The figure shows also the tree induced by the order $\ll$.

2.4. Collapse with bounded diameter and curvature.

2.4.1. The Gromov–Hausdorff distance and a relevant example. The *Gromov–Hausdorff distance* (GH-distance) [**24**] between two compact metric spaces (X, d_X) and (Y, d_Y) is defined as the infimum of the $\delta > 0$ such that there exists, on the disjoint union $X \sqcup Y$, a metric $d_{X \sqcup Y}$ extending d_X and d_Y such that

$$Y \subset \mathcal{T}_{d_{X\sqcup Y}}(X, \delta) \text{ and } X \subset \mathcal{T}_{d_{X\sqcup Y}}(Y, \delta), \tag{5}$$

where $\mathcal{T}_{d_{X\sqcup Y}}(X, \delta)$ and $\mathcal{T}_{d_{X\sqcup Y}}(Y, \delta)$ are the $d_{X\sqcup Y}$-metric neighborhoods of X and Y and radius δ, respectively.

We introduce now some terminology to be used during the rest of the article. We will say that a sequence of compact manifolds (M_i, g_i) *metrically collapses* to a space (X, d) if it converges in the GH-topology to (X, d) and the Hausdorff dimension of X is less than that of M_i (which we assume is constant). If the GH-distance between (M, g) and (X, d) is less than or equal to ϵ, then we say that (M, g) is *ϵ-close to* (X, d). If the GH-distance between (M, g) and a point is less than or equal to ϵ, we say that (M,g) is ϵ-*collapsed* (for the distance of (M,g) to a point, see [**24**]).

We present below an example where we estimate the distance between two metric spaces that will be relevant to us in the proof of the Step **C** inside the proof of the Proposition 3.1.

Example of a Gromov–Hausdorff distance estimation. Let I be a compact interval in $\mathbb{R}$ of length $|I| \geq 1$. Let h be a flat metric in $\mathbb{T}^2$ of diameter Γ. Provide $X = \mathbb{T}^2 \times I$ with the metric d_X induced from the Riemannian flat product-metric $g = dx^2 + h$. Intuitively, if Γ is small, then (X, d_X) should be close metrically to the interval I. More precisely, it should be close to the metric space $(Y, d_Y) = (I, |\ |)$, where $d_Y(x_1, x_2) = |x_1 - x_2|$. We show now the following upper and lower bounds for the GH-distance between (X, d_X) and (Y, d_Y) when $\Gamma \leq 1$:

$$\frac{\Gamma}{5} \leq dist_{GH}(X,Y) \leq \frac{\Gamma}{2}. \tag{6}$$

• *The upper bound.* Points in $\mathbb{T}^2$ are denoted by t, points in I by x, and thus points in $X = \mathbb{T}^2 \times I$ by (t,x). Let t_0 be a point in $\mathbb{T}^2$ such that $\overline{B_h(t_0, \Gamma/2)} = \mathbb{T}^2$ (such point always exists). f $\epsilon > 0$ define the distance $d^{\epsilon}_{X \sqcup Y}$ as equal to d_X and d_Y when restricted to X and Y, respectively, and as $d^{\epsilon}_{X\sqcup Y}((t,x), x') = d_X((t,x),(t_0,x')) + \epsilon$ for the distance between $(t,x) \in \mathbb{T}^2 \times I$ and $x' \in I$. Now (5) holds for $\delta(\epsilon) = \Gamma/2 + 2\epsilon$ and for any $\epsilon > 0$. Therefore, $dist_{GH}(X,Y) \leq \frac{\Gamma}{2}$.

• *The lower bound.* Make $dist_{GH}\big(X,Y)\big) = \Gamma/\mu$ for a μ that we will estimate as $\mu < 5$. Let t_1 and t_2 be two points in $\mathbb{T}^2$ such that $dist_h(t_1,t_2) = \Gamma$. Let also $p_1 = (t_1, 0),\ p_2 = (t_2, 0),\ p_3 = (t_1, \Gamma),\ p_4 = (t_2, \Gamma)$, forming an "square" in X; i.e., $d_X(p_1,p_2) = d_X(p_2,p_4) = d_X(p_4,p_3) = d_X(p_3,p_1) = \Gamma$ and $d_X(p_1,p_4) = d_X(p_2,p_3) = \sqrt{2}\Gamma$. By the definition of the GH-distance, for every $\epsilon > 0$ there is $d^{\epsilon}_{X\sqcup Y}$ extending d_X and d_Y and satisfying (5) with $\delta(\epsilon) = \Gamma/\mu + \epsilon$. Therefore, there are points x_1, x_2, x_3, and x_4 in I such that for every $j = 1,2,3,4$ we have $d^{\epsilon}_{X\sqcup Y}(p_j, x_j) \leq \frac{\Gamma}{\mu} + \epsilon$. From this and the triangle inequalities

$$d_Y(x_j,x_k) \leq d^{\epsilon}_{X\sqcup Y}(x_j,p_j) + d_X(p_j,p_k) + d^{\epsilon}_{X\sqcup Y}(p_k,x_k),$$
$$d_X(p_j,p_k) \leq d^{\epsilon}_{X\sqcup Y}(x_j,p_j) + d_Y(x_j,x_k) + d^{\epsilon}_{X\sqcup Y}(p_k,x_k),$$

we get, when (j,k) is not $(1,4)$ or $(2,3)$,

$$|x_j - x_k| \leq 2\frac{\Gamma}{\mu} + \Gamma + 2\epsilon, \text{ and } \Gamma \leq 2\frac{\Gamma}{\mu} + |x_j - x_k| + 2\epsilon, \tag{7}$$

while when (j,k) is $(1,4)$ or $(2,3)$ we get

$$\sqrt{2}|x_j - x_k| \leq 2\frac{\Gamma}{\mu} + \sqrt{2}\Gamma + 2\epsilon, \text{ and } \sqrt{2}\Gamma \leq 2\frac{\Gamma}{\mu} + |x_j - x_k| + 2\epsilon. \tag{8}$$

We will use inequalities (7) and (8) in what follows. Suppose that $x_1 \leq x_3$ (the case $x_1 \geq x_3$ is symmetric). Then:

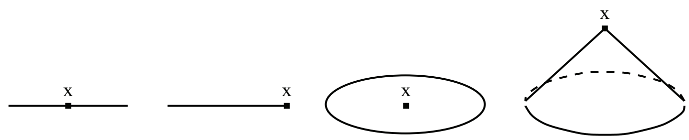

Figure 2. The local models of collapse. From left to right, models: **I.a**, **I.b**, **II.a**, **II.b**.

• If $x_4 \leq x_3$ we have $|x_1 - x_4| = ||x_1 - x_3| - |x_3 - x_4||$, which using (7) is less than or equal to $4\Gamma/\mu + 4\epsilon$, i.e. $|x_1 - x_4| \leq 4\Gamma/\mu + 4\epsilon$. On the other hand, from this and (8) we obtain $\sqrt{2}\Gamma \leq 6\Gamma/\mu + 6\epsilon$, for every $\epsilon > 0$ and therefore $\mu < 5$.

• If $x_4 \geq x_3$, then we have two possibilities (i) $x_2 \geq x_1$ or (ii) $x_2 \leq x_1$. (i) is symmetric to the case we have considered before under the change $x_1 \to x_3$, $x_3 \to x_1$ and $x_4 \to x_2$. We consider then (ii). In this case we have $|x_2 - x_4| = |x_2 - x_1| + |x_1 - x_3| + |x_3 - x_4|$, which by (7) is greater than or equal to $3\Gamma - 6\Gamma/\mu - 6\epsilon$, i.e. $|x_2 - x_4| \geq 3\Gamma - 6\Gamma/\mu - 6\epsilon$. From this and (7), again we obtain $4\Gamma/\mu \geq \Gamma - 4\epsilon$, for every $\epsilon > 0$ and therefore $\mu < 5$. ◀

2.4.2. The local models of collapse and examples. Locally there are only five types of models describing the metric limit of boundaryless compact three-manifolds collapsing in volume with curvature and diameter bounds. If (X, d) is a limit metric space and $x \in X$, then either x is the only point of X or there is a neighborhood of x locally isometric to one of the following four possibilities:

- **I.a** an interval $I = (-a, a)$, with $-a < x = 0 < a$, provided with the standard metric $d(x_1, x_2) = |x_1 - x_2|$,
- **I.b** an interval $I = [0, a)$, with $x = 0 < a$, provided with the standard metric $d(x_1, x_2) = |x_1 - x_2|$,
- **II.a** a disc $D = \mathbb{B}^2(o, a)$, $x = o$, provided with a metric d induced from a $C^{1,\beta}$-Riemannian metric,
- **II.b** a disc $D = \mathbb{B}^2(o, a)$, $x = o$, provided with a metric d induced from the quotient of a $C^{1,\beta}$-Riemannian metric by the action of $\mathbb{Z}_q$, $q \geq 1$ by isometries leaving the origin o fixed.

The point $x = 0$ in case **I.a** and the point $x = o$ in case **II.b**. will be here called *singular points* and denoted by $Sing(X)$. A manifold locally of the form **II.a** or **II.b** will be called a $C^{1,\beta}$ orbifold.

That **I.a, I.b, II.a,** and **II.b** are the only possible models is an important consequence of the Cheeger–Gromov–Fukaya theory of collapse under curvature bounds [**10**]. We comment on this in what follows. First, the limit space is always of integer dimension and therefore if it not a point it must be of dimension 1 or 2 as stated in Theorem 0.6 (and the paragraph below it) of [**10**, p. 2]. That when the dimension is 2 the models are of the forms **II.a** and **II.b** is the content of Proposition 11.5

of [**12**, p. 186] (Proposition 11.5 is a Corollary to Theorem 11.1 [**10**, p. 184], which is a restatement of Theorem 0.6 in [**10**]). That when the dimension is 1 the models are of the forms **I.a** and **I.b** follows from Theorem 0.5 of [**10**, p.2] after Definition 0.4. Indeed by Theorem 0.5 and Definition 0.4 there is a neighborhood of x homeomorphic to the quotient of $\mathbb{B}^m$ (with $x = o$ and for some m) by the action of a Lie subgroup of $O(3)$. Thus, if the Hausdorff dimension is 1, then the space must be of the type **I.a** or **I.b**, as these are the only possible quotients of dimension 1. Note that it is excluded for the instance of the union of three or more segments in a point (if we remove $x = o$ from the quotient the space must be still connected).

Below we are going to give four examples showing how each of the four cases above can be realized. They are illustrative and do not play any other role in the article. For this reason, the presentation is rather synthetic. The examples give sequences of Riemannian manifolds (M_n, g_n) converging to a (X, d) as in **I.a**, **I.b**, **II.a**, and **II.b** (in this order). We define first the sequence (M_n, g_n) and give what is going to be the limit space (X, d). After the definitions for every one of the cases **I.a, I.b, II.a**, and **II.b** are made, we list the geometric properties of the convergence process applying to each. The justifications are just computational and, because they play no role in the article, are left to the reader. Finally, let us mention that the examples show essentially all that can occur locally in volume collapse with curvature and diameter bounds besides collapse to a point (see Lemma 2.1).

We will use the following notation. The rotational group of $\mathbb{R}^2 \sim \mathbb{C}$ will be denoted by $\mathcal{R}$. Obviously, $\mathbb{U}(1) \sim \mathcal{R}$ under the homomorphism $u \in \mathbb{U}(1) \to R(u) \in \mathcal{R}$, with $R(u)z = uz$ for any $z \in \mathbb{C}$. Also, for any natural number q let $\mathcal{R}^q \sim \mathbb{Z}_q$ be the subgroup of rotations generated by $R(e^{2\pi i/q})$. Finally, the group of rotations on the first factor $\mathbb{R}^2$ in $\mathbb{R}^2 \times \mathbb{R}^2$ will be denoted by $\mathcal{R}_1$, and the group of rotations on the second factor will be denoted by $\mathcal{R}_2$. Note that the set $\mathbb{B}^2 \times \mathbb{S}^1 \subset \mathbb{R}^2 \times \mathbb{R}^2$ and the set $\mathbb{T}^2 \subset \mathbb{R}^2 \times \mathbb{R}^2$ are invariant under $\mathcal{R}_1 \times \mathcal{R}_2$. In particular, $\mathbb{T}^2 \times \mathbb{I} \subset \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}$ is invariant under $\mathcal{R}_1 \times \mathcal{R}_2$.

Example I.a.

• (M_n, g_n)—Let $\tilde{M} = \mathbb{T}^2 \times \mathbb{I}$ and provided with a smooth and $\mathcal{R}_1 \times \mathcal{R}_2$-invariant Riemannian metric $\tilde{g}$. Let $G_n \sim \mathbb{Z}_n \times \mathbb{Z}_n$ be the group generated by the rotations $\mathcal{R}_1(e^{2\pi i/n})$, $\mathcal{R}_2(e^{2\pi i/n})$. Let $M_n = \tilde{M}/G_n$ be the quotient of $\tilde{M}$ by G_n, $\pi_n : \tilde{M} \to M_n$ the covering map, and g_n the projected metric on M_n, namely, $\pi_n^*(g_n) = \tilde{g}$.

• (X, d)—Let $X = \mathbb{T}^2 \times \mathbb{I}/(\mathcal{R}_1 \times \mathcal{R}_2)$ with the induced quotient metric d, and let $f_n : M_n \to X$ be the projection.

Example I.b.

• (M_n, g_n)—Let $\tilde{M} = \mathbb{B}^2 \times \mathbb{S}^1$ and provided with a smooth and $\mathcal{R}_1 \times \mathcal{R}_2$-invariant Riemannian metric $\tilde{g}$. Let $G_n \sim \mathbb{Z}_{n^2}$ be the group generated by the rotations $\mathcal{R}_1(e^{2\pi i/n}) \times \mathcal{R}_2(e^{2\pi i/n^2})$. Let $M_n = \tilde{M}/G_n$ be the quotient of $\tilde{M}$ by G_n, $\pi_n : \tilde{M} \to M_n$ the covering map, and g_n the projected metric on M_n, namely, $\pi_n^*(g_n) = \tilde{g}$.

• (X, d)—Let $X = \mathbb{B}^2/(\mathcal{R}_1 \times \mathcal{R}_2)$ with the induced quotient metric d, and let $f_n : M_n \to X$ be the projection.

Example II.a.

• (M_n, g_n)—Let $\tilde{M} = \mathbb{B}^2 \times \mathbb{S}^1$ and provided with a smooth and $\mathcal{R}_2$-invariant Riemannian metric $\tilde{g}$. Let $G_n \sim \mathbb{Z}_n$ be the subgroup of $\mathcal{R}_2$ generated by the rotations $\mathcal{R}_2(e^{2\pi i/n})$. Let $M_n = \tilde{M}/G_n$ be the quotient of $\tilde{M}$ by G_n, $\pi_n : \tilde{M} \to M_n$ the covering map, and g_n the projected metric on M_n, namely, $\pi_n^*(g_n) = \tilde{g}$.

• (X, d)—Let $X = \mathbb{B}^2$, with the induced quotient metric d. Let $f_n : M_n \to X$ be the projection.

Example II.b.

• (M_n, g_n)—Let $\tilde{M} = \mathbb{B}^2 \times \mathbb{S}^1$ provided with a smooth and $\mathcal{R}_1 \times \mathcal{R}_2$-invariant Riemannian metric $\tilde{g}$. Let $0 < p < q$ be two relatively prime natural numbers, and let $G_n \sim \mathbb{Z}_{qn}$ be the subgroup of $\mathcal{R}_1 \times \mathcal{R}_2$ generated by the rotations $\mathcal{R}_1(e^{2\pi pi/q}) \times \mathcal{R}_2(e^{2\pi i/qn})$. Let $M_n = \tilde{M}/G_n$ be the quotient of $\tilde{M}$ by G_n, $\pi_n : \tilde{M} \to M_n$ the covering map, and g_n the projected metric on M_n, namely, $\pi_n^*(g_n) = \tilde{g}$.

• (X, d)—Let $X = \mathbb{B}^2/\mathcal{R}^q$, with the induced quotient metric d. Let $f_n : M_n \to X$ be the projection.

With these definitions for the examples **I.a**, **I.b**, **II.a**, and **II.b**, it is straightforward to check that:

1) $Sing(X) = \emptyset$ in cases **I.a**, **II.a** and $Sing(X) = \{o\}$ in cases **I.b** and **II.b**.
2) In every example, the sequence (M_n, g_n) converges in the GH-topology to (X, d). The group G_n of Deck transformations on $\tilde{M}$ converges to $G = \mathcal{R}_1 \times \mathcal{R}_2 \sim \mathbb{T}^2$ in cases **I.a** and **I.b**, to $G = \mathcal{R}_2 \sim \mathbb{S}^1$ in case **II.a**, and to $G := \mathcal{R}_1^q \times \mathcal{R}_2 \sim \mathbb{Z}_q \times \mathbb{S}^(1)$ in case **II.b**. Moreover, $X = \tilde{M}/G$. Let $\pi : \tilde{M} \to X$ be the projection. Then $Centr_G(\pi^{-1}(Sing(X))) = \mathcal{R}_1^q$, where $Centr$ is the centralizer.
3) In every example, $f_n : M_n \to X$ is a fibration and $length_{g_n}(f_n^{-1}(x)) \to 0$. Moreover, $f_n : M_n \setminus f_n^{-1}(Sing(X)) \to X \setminus Sing(X)$ is a $\mathbb{T}^2$-fiber bundle in cases **I.a** and **I.b** and a $\mathbb{S}^1$-fiber bundle in cases **II.a** and **II.b**. $Centr(Sing(X))$ acts freely on $f_n^{-1}(x)$ for any $x \in$

$X \setminus Sing(X)$ and $f_n^{-1}(Sing(X)) \sim f_n^{-1}(x)/Centr(\pi^{-1}(Sing(X)))$. ◀

2.4.3. Volume collapse of three-manifolds with boundary and with curvature and diameter bounds—an statement. We discuss now briefly what we will mean by *three-manifolds with non-necessarily smooth boundary*. The reader should keep in mind that the notion is just for the purpose of working later with some necessary generality, with no intention whatsoever in developing a new concept, which, as a matter of fact, would be here purposeless. Let M be a compact set on an open manifold P. Then we say that M is a *compact manifold with non-necessarily smooth boundary* (in short, *manifold with NNSB*) if M is equal to the closure (in P) of its interior (in P). In this sense, the *boundary* ∂M of M is defined as M minus the topological interior of M (in P), and the *manifold's interior* $M^\circ := M \setminus \partial M$ therefore coincides with the topological interior (in P). Note that we do not assume that M° is connected. A subset of M is *a submanifold with NNSB* if it is a manifold with NNSB as a subset of P. Of course any compact manifold with smooth boundary is a manifold with NNSB. If P carries a Riemannian metric g, then we say that (M, g) is a Riemannian manifold with NNSB. In this case, the Riemannian metric g induces a metric $d = d_g^M$ in every connected component of M°. For the discussion below, we do not need to extend d to a metric on M°. The distance from a point $p \in M^\circ$ to ∂M can be defined in several equivalent and natural ways—for instance, $d(p, \partial M)$ as the supremum of the radius of the geodesic balls of center p, lying entirely in M°. Then $d(p, \partial M)$ is realized by the g-length of a geodesic starting at p and ending at ∂M and whose interior lies in M°. Define the tubular neighborhoods $\mathcal{T}_d(\partial M, \epsilon) := \partial M \cup \{p \in M^\circ, d(p, \partial M) < \epsilon\}$.

Definition 2. Let $\mathfrak{N}_0 : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be a non-necessarily continuous function. Then define $\mathcal{M}(\mathfrak{N}_0)$ as the set of compact Riemannian manifolds with NNSB (M, g), such that for any $1 > \epsilon_0 > 2\epsilon_1 > 0$, the minimum number of geodesic balls of radius ϵ_1 covering $M \setminus \mathcal{T}_d(\partial M, \epsilon_0)$ is bounded above by $\mathfrak{N}_0(\epsilon_0, \epsilon_1)$.

Remark 2.1. The values of $\mathfrak{N}_0$ outside the set $\{(\epsilon_0, \epsilon_1), 1 > \epsilon_0 > 2\epsilon_1 > 0\}$ are of no relevance.

We would like to comment briefly about the reason for this definition. Recall that given $\Lambda_0 > 0$, $D_0 > 0$ there is $\mathfrak{N}_0 : \mathbb{R}^+ \to \mathbb{R}^+$, depending on them, such that for any compact boundaryless Riemannian three-manifold with $|Ric| \leq \Lambda_0$, $diam_g(M) \leq D_0$ the minimum number of balls of radius ϵ covering M is bounded above by $\mathfrak{N}_0(\epsilon)$ (this is due to Gromov; see [**24**, p. 281]). Moreover, the existence of such $\mathfrak{N}_0$ is equivalent to the precompactness of the family of compact and boundaryless Riemannian three-manifolds with $|Ric| \leq \Lambda_0$ and $diam_g(M) \leq D_0$, as a set inside the family of compact metric spaces provided with the GH-topology ([**24**, p. 280]). However, in the family of compact manifolds with NNSB, and even those with smooth boundary, and with $|Ric| \leq \Lambda_0$ and $diam_g(M) \leq D_0$, one cannot guarantee the existence of $\mathfrak{N}_0 : \mathbb{R}^+ \to \mathbb{R}^+$ nor the precompactness of such family. Consider, for instance, the following example. For any $n \geq 2$, let $V_n = [1/n, 1] \times \mathbb{S}^1$ be endowed with the flat metric $dx^2 + n^2x^2d\varphi^2$, where φ is the coordinate in the $\mathbb{S}^1$ factor (and recall that $\mathbb{S}^1$ has total length 2π). For any n, the diameter of V_n is less than or equal to $2\pi + 2$. On $M_n = V_n \times \mathbb{S}^1$, consider the flat product metric $g_n = dx^2 + n^2x^2d\varphi^2 + (1/n)^2d\theta^2$, where θ is the coordinate in the $\mathbb{S}^1$ factor defining M_n. Also, for any n, $diam_{g_n}(M_n, g_n) \leq 2\pi + 2 + 1/n < 10$. Despite of this and despite that the manifolds (M_n, g_n) are flat, they do not collapse to a compact metric space (as $n \to \infty$). Even more, we have that for any $1/2 > \epsilon > 0$ no pointed sequence (Ω_n, g_n, p_n) of compact connected regions of M_n with smooth boundary $\partial\Omega_n$, $\partial\Omega_n \subset \mathcal{T}_{d^{M_n}_{g_n}}(\partial M_n, \epsilon)$ collapses to a compact metric space. This occurs even when $d^{M_n}_{g_n}(p_n, \partial\Omega_n) \geq 1/4$ (for instance).

But any family $\mathcal{M}(\mathfrak{N}_0)$ satisfies the following kind of precompactness.

Proposition 2.2. *Let (M_i, g_i) be a sequence in a family $\mathcal{M}(\mathfrak{N}_0)$, and on every connected component of M_i° let $d_i = d^{M_i}_{g_i}$. Then for every $1 > \epsilon_0 > 0$ we have the following:*

1. *There are at most $\mathfrak{N}_0(\epsilon_0, \epsilon_0/3)$ connected components $\breve{M}_i^\circ$ of M_i° intersecting $M_i \setminus \mathcal{T}_{d_i}(\partial M_i, \epsilon_0)$.*
2. *For every sequence $\breve{M}_i^\circ$ of connected components of M_i° intersecting $M_i \setminus \mathcal{T}_{d_i}(\partial M_i, \epsilon_0)$, there is a subsequence (index again by "i") such that $(\breve{M}_i^\circ \setminus \mathcal{T}_{d_i}(\partial M_i, \epsilon_0), d_i)$ converges in the GH-topology to a compact metric space (X, d).*

Remark 2.2. Note that distances in $\breve{M}_i^\circ \setminus \mathcal{T}_{d_i}(\partial M_i, \epsilon)$, which can be a connected set or not, are with respect to $d_i = d^{M_i}_{g_i}$.

Proof. *Item 1.* By definition, $\mathfrak{N}_0(\epsilon_0, \epsilon_0/3)$ bounds from above the minimum number of balls of radius $\epsilon_0/3$ covering $M_i \setminus \mathcal{T}_{d_i}(\partial M_i, \epsilon_0)$. But given one such cover there must be at least one ball for every connected component $\breve{M}_i^\circ$ intersecting $M_i \setminus \mathcal{T}_{d_i}(\partial M_i, \epsilon_0)$. *Item 2.* From Definition 2, the function $\mathfrak{N}_0(\epsilon_0, \epsilon_1)$ as a function of ϵ_1 and with ϵ_0 fixed as in the hypothesis bounds from above the minimum number of d_i-balls of radius ϵ_1 covering $\breve{M}_i^\circ \setminus \mathcal{T}_{d_i}(\partial M_i, \epsilon_0)$. The proposition follows from Lemma 1.9 in [**24**, p. 280]. q.e.d.

In the example below, we describe a nontrivial family of manifolds with boundary that are of great interest to us and lie in a class $\mathcal{M}(\mathfrak{N}_0)$.

Example I. Let g be a fixed complete Riemannian metric in $\mathbb{R}^3$. Suppose that $Ric_g \geq 0$ outside $B_g(o, r_0)$ and that $|Ric_g| \leq \Lambda_0/r^2$. Suppose, too, that $0 < c_0 < c_1$ are given and fixed. We claim that there are $\mathfrak{N}_0$ and r_1, both depending only on Λ_0, r_0, c_0, c_1, such that for any $\bar{r} \geq r_1$, the Riemannian annuli (with NNSB) $(M_{\bar{r}}, g_{\bar{r}})$,

$$M_{\bar{r}} := A_g[c_0\bar{r}, c_1\bar{r}] = \overline{B_{g_{\bar{r}}}(o, c_1)} \setminus B_{g_{\bar{r}}}(o, c_0), \quad g_{\bar{r}} := \frac{1}{\bar{r}^2} g,$$

lie in $\mathcal{M}(\mathfrak{N}_0)$. We prove the claim in the following.

(a) From Z-d Liu's ball-covering property, more precisely, Remark 2 in [**18**, pg. 215]; take there $S = A_g[c_0\bar{r}, c_1\bar{r}]$, $C_0 = c_1/c_0$, and $\mu = c_0/3$, we know that there are r_1 and n_0, both depending only on c_0, c_1, r_0, Λ_0, such that for any $\bar{r} > r_1$, n_0 bounds from above the minimum number of $g_{\bar{r}}$-balls (inside $\mathbb{R}^3$) with centers in $M_{\bar{r}}$, of radii $c_0/8$, and covering $M_{\bar{r}}$.

(b) Assume from now on and without loss of generality that $r_1 \geq 2r_0/c_0$, and let $\bar{r} > r_1$. In this situation, a standard argument due to Gromov [**13**] using the volume comparison shows that, for any $\epsilon_1 < c_0/16$ and for any $g_{\bar{r}}$-ball with center in $M_{\bar{r}}$ and of $g_{\bar{r}}$-radius $c_0/8$, the minimum number of $g_{\bar{r}}$-balls of $g_{\bar{r}}$-radii ϵ_1 covering it is bounded above by $8c_0^3/\epsilon_1^3$.[2]

From (a) and (b), we deduce that for any $\epsilon_1 < c_0/16$ and $\bar{r} > r_1$ the minimum number of $g_{\bar{r}}$-balls of $g_{\bar{r}}$-radii ϵ_1 covering $M_{\bar{r}}$ is bounded by $8n_0c_0^3/\epsilon_1^3$. Now let ϵ_0 and ϵ_1 with $2\epsilon_1 < \epsilon_0 < 1$ and $\epsilon_1 < c_0/16$. Then as $\big(M_{\bar{r}} \setminus \mathcal{T}_{d_{\bar{r}}}(\partial M_{\bar{r}}, \epsilon_0)\big) \subset M_{\bar{r}}$ (here $d_{\bar{r}} = d_{g_r}^{M_{\bar{r}}}$), it follows that the minimum number of $g_{\bar{r}}$-balls of $g_{\bar{r}}$-radii ϵ_1 covering $M_{\bar{r}} \setminus \mathcal{T}_{d_{\bar{r}}}(\partial M_{\bar{r}}, \epsilon_0)$ is bounded by $8n_0c_0^3/\epsilon_1^3$. This shows the following. If $\bar{r} > r_1$, then $(M_{\bar{r}}, g_{\bar{r}})$ belongs to $\mathcal{M}(\mathfrak{N}_0)$, where, for $1 > \epsilon_0 > 2\epsilon_1 > 0$ (cf. Remark 2.1), $\mathfrak{N}_0(\epsilon_0, \epsilon_1)$ is defined as $\mathfrak{N}_0(\epsilon_0, \epsilon_1) = 8n_0c_0^3/\epsilon_1^3$ if $\epsilon_1 < \min\{1/2, c_0/8\}$ and as $\mathfrak{N}_0(\epsilon_0, \epsilon_1) = \mathfrak{N}_0(\epsilon_0, c_0/8) = 8^4 n_0$ if $\epsilon_1 \in [\min\{1/2, c_0/8\}, 1/2)$. ◀

Example II. For any D_0, Λ_0, and δ_0, there is $\mathfrak{N}_0(D_0, \Lambda_0, \delta_0)$ such that for any (M, g) Riemannian-manifold, with $|Ric_g| \leq \Lambda_0$, and connected

[2]This is seen as follows. Let $B(p, b)$ be a ball inside a manifold with $Ric \geq 0$. Let n_a be the maximum number of disjoint balls of radii $a/3$ with centers in $B(o, p)$. Let $\{B(q_i, a/3), i = 1, \ldots, n_a\}$ be a set of disjoint balls with centers in $B(p, b)$. Then $\{B(q_i, a), i = 1, \ldots, n_a\}$ covers $B(p, b)$, and thus n_a bounds the minimum number of balls with centers in $B(p, b)$ and of radii a necessary to cover $B(p, b)$. By the volume comparison, we have $Vol(B(q_i, a/3)) \geq (a/6(b + a))^3 Vol(B(q_i, 2(b + a)))$ for all i. Also for all i, we have $B(q_i, a/3) \subset B(p, b + a)$ and $B(p, b + a) \subset B(q_i, 2(b + a))$. Therefore, $Vol(B(p, b+a)) \geq \sum Vol(B(q_i, a)) \geq n_a(a/6(b+a))^3 Vol(B(p, b+a))$ and thus $n_a \leq (6(b+a)/a)^3$. Now in our situation we have $p \in M_{\bar{r}}$, with $\bar{r} > r_1 \geq 2r_0/c_0$. From the discussion earlier we deduce that the minimum number of balls of $g_{\bar{r}}$ radii $\epsilon_1 \leq c_0/16$ covering $B_{g_{\bar{r}}}(p, c_0/8)$ is less than or equal to $(6(c_0/8 + \epsilon_1)/\epsilon_1)^3 \leq 8c_0^3/\epsilon_1^3$. The condition $r_1 \geq 2r_0/c_0$ is used to make sure that the construction is inside a region with $Ric \geq 0$.

compact region $\Omega \subset M$ with smooth boundary having

$$diam_g(\Omega) \leq D_0 \text{ and } d_g^M(\partial\Omega, \partial M) > \delta_0,$$

the connected manifold (with NNSB) $\overline{\mathcal{T}_{d_g^M}(\Omega, \delta_0)}$ lies is $\mathcal{M}(\mathfrak{N}_0, \Lambda_0)$. The proof is not difficult and is left to the reader. ◀

It is instructive to go back and recall the discussion before Proposition 2.2. There we presented a sequence (M_n, g_n) that, in light of Proposition 2.2, did not belong to a single $\mathcal{M}(\mathfrak{N}_0)$. Now, in the light of Example II, the manifolds (M_n, g_n) (for all n) cannot be extended beyond their boundary to manifolds $(\bar{M}_n, g_n)$ with $|Ric_{g_n}| \leq \Lambda_1$ and $d_{g_n}^{\bar{M}_n}(M_n, \partial\bar{M}_n) \geq \delta_0 > 0$.

As a consequence of Example II we have:

Example III. Let $R_0 > 0$ and $\Lambda_0 > 0$ be given. Then there is $\mathfrak{N}_0(R_0, \Lambda_0)$ such that any closure of a geodesic ball (see Section 2.1) of radius $r_0 \leq R_0$ inside a manifold (M, g) with $|Ric| \leq \Lambda_0$ lies in $\mathcal{M}(\mathfrak{N}_0)$. To see this, note that $B_g(p, r_0) = \mathcal{T}_{d_g^M}(B_g(p, r_0/2), r_0/2)$ and then use Example II. ◀

We will denote by $\mathcal{M}(\mathfrak{N}_0, \Lambda_0)$ the set of Riemannian three-manifolds (with NNSB) in the class $\mathcal{M}(\mathfrak{N}_0)$ and with $|Ric| \leq \Lambda_0$. In Example I, the manifolds $(M_{\bar{r}}, g_{\bar{r}})$ lie in $\mathcal{M}(\mathfrak{N}_0, c_0^{-2}\Lambda_0)$, where $\mathfrak{N}_0, c_0$, and Λ_0 are as in the example.

Definition 3. Let (M, g) be a compact manifold (with NNSB). Let $0 < \underline{\epsilon} < \overline{\epsilon} < 1$. Then a compact connected region Ω (with NNSB) is said to be an $(\underline{\epsilon}, \overline{\epsilon})$-connected component of M if $\partial\Omega \subset \mathcal{T}_{d_g^M}(\partial M, \overline{\epsilon}) \setminus \overline{\mathcal{T}_{d_g^M}(\partial M, \underline{\epsilon})}$. The set of $(\underline{\epsilon}, \overline{\epsilon})$-components of three-manifolds in a class $\mathcal{M}(\mathfrak{N}_0, \Lambda_0)$ will be denoted by $\mathcal{M}_{\underline{\epsilon}}^{\overline{\epsilon}}(\mathfrak{N}_0, \Lambda_0)$.

Thus when we write $(\Omega, g) \in \mathcal{M}_{\underline{\epsilon}}^{\overline{\epsilon}}(\mathfrak{N}_0, \Lambda_0)$, we imply that (Ω, g) is the $(\overline{\epsilon}, \underline{\epsilon})$-connected component of an $(\bar{M}, g) \in \mathcal{M}(\mathfrak{N}_0, \Lambda_0)$.

A sequence (M_i, g_i) is *volume collapsing* if $Vol_{g_i}(M_i) \to 0$. The following important lemma is essentially Proposition 1.5 in [**4**] (up to some modifications[3]) and with some additional information from [**10**].

[3]Unfortunately, Proposition 1.5 in [**4**] is stated without proof. An argumentative proof can be found in [**3**] (for the Lemma 1.4 p. 982, which is the equivalent to Proposition 1.5 in [**4**]) but we were not able to check every claim in there, especially concerning the existence of U_i (in the terminology of [**3**]) with $\epsilon/2 \leq dist(\partial U_i, \partial\Omega_i) \leq \epsilon$. The problems have to do with the fact that a priori the sequence (D_i, g_i, x_i) (in the terminology of [**4**]) do not belong to any family $\mathcal{M}(\mathfrak{N}_0)$, and this may cause some inconveniences as indicated in the discussion before Proposition 2.2. It is essentially to avoid these inconveniences that we included the hypothesis that the sequence (M_i, g_i) belongs a priori to some fixed family $\mathcal{M}(\mathfrak{N}_0)$. We would like to thank Michael Anderson for conversations on Propositions 1.4 and 1.5 in [**4**].

Lemma 2.1. *Let (M_i, g_i) be a volume-collapsing sequence in a given $\mathcal{M}(\mathfrak{N}_0, \Lambda_0)$ and such that for some $p_i \in M_i$ we have $d_{g_i}^{M_i}(p_i, \partial M_i) \geq \Gamma_0 > 0$. Then for every $0 < \underline{\epsilon} < \overline{\epsilon} < \min\{1, \Gamma_0/2\}$ there is a sequence (Ω_i, g_i) of $(\underline{\epsilon}, \overline{\epsilon})$-connected components of M_i, with $p_i \in \Omega_i$, and a subsequence of it (indexed again by "i") converging in the GH-topology to a space (X, d) of one of the following two forms:*

D1. *an interval $([0, \bar{x}], |\ |)$, with $Sing(X) = \emptyset$ or $Sing(X) = \{\bar{x}\}$, or,*
D2. *a $C^{1,\beta}$-two-orbifold, with either $Sing(X) = \emptyset$ or $Sing(X) = \{\bar{x}_1, \ldots, \bar{x}_n\} \subset X^\circ$.*

Moreover (for $i \geq i_0$), **I1** *and* **I2**, *below, hold.*

I1. *There are fibrations $f_i : \Omega_i \to X$, with asymptotically collapsing fibers $f_i^{-1}(x)$, such that,*
- *For* **D1**: *$f_i : \Omega_i \setminus f_i^{-1}(Sing(X)) \to X \setminus Sing(X)$ is a $\mathbb{T}^2$-fibre-bundle and if $Sing(X) \neq \emptyset$ then $f_i^{-1}(\bar{x}) \sim \mathbb{T}^2/(\mathbb{S}^1 \times \mathbb{Z}_q)$, where the quotient is by a free action.*
- *For* **D2**: *$f_i : \Omega_i \setminus f_i^{-1}(Sing(X)) \to X \setminus Sing(X)$ is a $\mathbb{S}^1$-fibre-bundle and if $Sing(X) \neq \emptyset$ then $f_i^{-1}(\bar{x}_j) \sim \mathbb{S}^1/\mathbb{Z}_{q_j}$, where the quotient is by a free action.*

I2. *There are finite coverings $\pi_i : \tilde{\Omega}_i \to \Omega_i$, such that:*
- *For* **D1**: *$(\tilde{\Omega}_i, \tilde{g}_i)$ converges in $C^{1,\beta}$ to a $\mathbb{T}^2$-symmetric Riemannian manifold.*
- *For* **D2**: *$(\tilde{\Omega}_i, \tilde{g}_i)$ converges in $C^{1,\beta}$ to a $\mathbb{S}^1$-symmetric Riemannian manifold.*

In either case, for any $x \in X \setminus Sing(X)$, $\pi_i^{-1}(f_i^{-1}(x))$ converges in C^1 to the $\mathbb{T}^2$ or $\mathbb{S}^1$ orbits.

The fibrations f_i have one more property [**9**]: for any neighborhood W of $Sing(X)$, the map $f_i : f_i^{-1}(X \setminus W) \to X \setminus W$ is an almost-Riemannian submersion; more precisely, we have

$$e^{-o(i)} \leq |f_{i*}(V)| \leq e^{o(i)}, \text{ where } o(i) \xrightarrow{i \to \infty} 0,$$

and for any unit-norm vector V perpendicular to the fibers.

Remark 2.3. Note that the space (Ω_i, g_i) represents $(\Omega_i, d_{g_i}^{\Omega_i})$ (see Sec. 2.1) rather than $(\Omega_i, d_{g_i}^{M_i})$. Compare this with *item 2* in Proposition 2.2.

Once one assumes that the sequence (M_i, g_i) is in $\mathcal{M}(\mathfrak{N}_0, \Lambda_0)$, the proof of Lemma 2.1 reduces to pointing to the appropriate reference in Fukaya's work. Here we give an overview of why this is so. The proof itself is postponed to the appendix.

We introduce first the following terminology. We say that two metric spaces (Y, d_Y) and (Z, d_Z) are *locally isometric under a homeomorphism* $\phi : Y \to Z$ if for all $y \in Y$ and $\phi(y) = z$ there are $\delta(y)$ and $\delta(z)$ such that $\phi : (B_{d_Y}(y, \delta(y)), d_Y) \to (B_{d_Z}(z, \delta(z)), d_Z)$ is an isometry. Of course, there are non-isometric metric spaces that are locally isometric.[4] As a matter of fact if $(\Omega, g) \subset (M, g)$ then $(\Omega^\circ, d_g^\Omega)$ is locally isometric under the identity homeomorphism to (Ω°, d_g^M), but they are not globally isometric in general.

Suppose now that a sequence of compact boundaryless manifolds (M_i, g_i) with uniformly bounded curvature and diameter collapses to a metric space (X, d), and suppose that $p_i \to x$. Let $exp : T_{p_i}M_i \to M_i$ be the exponential map, and let $g_i(p_i)$ be the metric g_i on $T_{p_i}M_i$. Finally, let $BT_{g_i(p_i)}(p_i, R_0)$ be the $g_i(p_i)$-ball of radius R_0 in $T_{p_i}M_i$. There is $R_0(\Lambda_0)$ small enough, for which the map

$$exp : BT_{g_i(p_i)}(p_i, R_0) \to B_{g_i}(p_i, R_0)$$

is of maximal rank. Let g_i^* be the pull-back metric. Then Fukaya's technique to describe the space around x [**10**, Ch. 3], consists in working with $(BT_{g_i^*}(p_i, R_1), g_i^*)$, with $R_1 \le R_0$ small enough, and making the following observations[5]

1) One can find a subsequence of it converging to a Riemannian manifold (BT, g^*) [**10**, p. 9].
2) For every i, the space $(B_{g_i}(p_i, R_1/2), g_i)$ is isometric to the quotient of the space $(BT_{g_i^*}(p_i, R_1/2), g_i^*)$ by an appropriate *local group*[6] of isometries G_i and that G_i converges to a local group G [**10**, p. 9], that is locally isomorphic to a Lie group [**10**, Lemma 3.1].
3) $(B_d(x, R_1/2), d)$ is locally isometric to $(BT(R_1/2), g^*)/G$, where $BT(R_1/2)$ is the g^*-ball of radius $R_1/2$ in BT (i.e., the limit of $BT_{g_i^*}(p_i, R_1/2)$) [7] .

Thus by *item 3*, to study locally the space (X, d) around x it is enough to study the limit spaces $(BT(R_1/2), g^*)/G$, and this is what is done in [**10**]. What is important to us about this conclusion is that one can study the collapse of manifolds with boundary as long as one works on a finite number of balls at a definite distance away from the boundary. This is essentially what is done in the proof of Lemma 2.1 in the appendix and where the condition $(M_i, g_i) \in \mathcal{M}(\mathfrak{N}_0, \Lambda_0)$ is used.

2.4.4. A relevant application of Lemma 2.1. We describe now a relevant application of Lemma 2.1 that will be of use to us in Proposition 3.1. We describe it first in rough terms and then in a precise statement. Consider any solid torus with curvature bounded above by Λ_0 (fixed) and that is metrically close to an interval I of length between $\infty \ge L_0 > |I| \ge 1 > 0$ (with L_0 fixed) and with boundary metrically close to a point. Then any curve $\mathscr{C}$ in its boundary that is not a

[4]For instance, compare the set $\{\varphi \in \mathbb{S}^1, 0 < \varphi < 3\pi/4\}$ with the restriction of the standard metric in $\mathbb{S}^1$ and $((0, 3\pi/4), |\ |)$.

[5]We do not comment here about some technical issues on smoothing.

[6]See [**10**] and ref. therein.

[7]This is easy to check and is left to the reader.

contractible to a point (from now on simply "contractible") as a curve in the boundary but that is contractible as a curve in the solid torus must have length greater or equal to some $l_0(\Lambda_0, L_0) > 0$. A proof of this phenomenon can be given along the following lines. Suppose that a curve $\mathscr{C}$ in the boundary of the solid torus Ω that is not a contractible curve as a curve in $\partial\Omega$ but is contractible as a curve in Ω has very small length. Then one can "unwrap" Ω—namely, take a non-collapsed cover $\tilde{\Omega}$, which is also a solid torus. In particular, $\partial\Omega$ is covered by a non-collapsed two-torus $\partial\tilde{\Omega}$. But then the closed curve $\mathscr{C}$, which is contractible in Ω, lifts to a closed, equal length and non-contractible curve $\tilde{\mathscr{C}}$ in $\partial\tilde{\Omega}$. But there are no non-contractible curves $\tilde{\mathscr{C}}$ in $\partial\tilde{\Omega}$ of very small length. This idea is made rigorous in the proof of Proposition 2.3. This behavior is explicit in Example **I.b**, as we explain in what follows. In that example the Riemannian solid tori (M_n, g_n) are collapsing to a segment of length 1. No matter the value of n, consider the $\mathscr{C}_0 = \pi_n(\tilde{\mathscr{C}_0})$ where $\tilde{\mathscr{C}_0} = \mathbb{S}^1 \times \{1\} \subset \mathbb{B}^2 \times \mathbb{S}^1$. The g_n-length of $\mathscr{C}_0$ is equal to the length of $\tilde{\mathscr{C}_0}$ and therefore equal to 2π. Moreover, any curve $\mathscr{C}$ in ∂M_n that is non-contractible as a curve in ∂M_n but that is contractible as a curve in M_n has length greater than that of $\mathscr{C}_0$—i.e., 2π. In other words, no matter the value of n, there are no such curves having a small length.

We give a statement of what we described above in Proposition 2.3. The statement is a bit more general than what was explained before, as we do not make hypothesis on the boundary of the solid tori. For this reason, too, it is more general than what we will need in this article, but it can be useful in other investigations. The proof is given in all detail partly to exemplify how the techniques apply.

Proposition 2.3. *For any Λ_0, $\delta_0 < 1/2$, and L_0 there is $\ell_0 > 0$ such that for any sequence (Ω_i, g_i) of solid tori inside a volume-collapsing sequence of Riemannian manifolds (M_i, g_i) with $|Ric_{g_i}| \leq \Lambda_0$ and having*

Q1. *$rad_{g_i}(\Omega_i) \geq 1$, $d^{M_i}_{g_i}(\partial\Omega_i, \partial M_i) \geq \delta_0 > 0$, and which is*

Q2. *metrically collapsing to an interval $(I, |\ |)$, and*

Q3. *posesses a sequence of closed curves $\mathscr{C}_i \subset \partial\Omega_i$ non-contractible in $\partial\Omega_i$ but contractible in Ω_i with $length_{g_i}(\mathscr{C}_i) \leq \ell_0$,*

we have $|I| \geq L_0$.

Remark 2.4. The hypothesis that the sequence (M_i, g_i) is volume collapsing can be seen to be unnecessary.

Proof. For the proof, it is worth to keep reference to Figure 3. We will argue by contradiction. Suppose that there is Λ_0, $\delta_0 < 1/2$, and L_0 such that for every $m = 1, 2, 3, \ldots$ there are sequences (in "i") $(\Omega_{m,i}, g_{m,i}) \subset (M_{m,i}, g_{m,i})$, where for every m, $(M_{m,i}, g_{m,i})$ is a volume-0collapsing sequence of Riemannian manifolds with $|Ric_{g_{m,i}}| \leq \Lambda_0$, such that

$\bar{\mathbf{Q}}$1. $rad_{g_{m,i}}(\Omega_{m,i}) \geq 1$, $d^{M_{m,i}}_{g_{m,i}}(\partial\Omega_{m,i}, \partial M_{m,i}) \geq \delta_0 > 0$, and which is

$\bar{\mathbf{Q}}\mathbf{2}$. metrically collapsing to an interval $(I_m, |\ |)$, with $L_0 \geq |I_m|$, and which

$\bar{\mathbf{Q}}\mathbf{3}$. posesses a sequence of closed curves $\mathscr{C}_{m,i} \subset \partial\Omega_{m,i}$ non-contractible in $\partial\Omega_{m,i}$ but contractible in $\Omega_{m,i}$ and of $length_{g_{m,i}}(\mathscr{C}_{m,i}) \leq 1/m$.

Using that every sequence (in "i") $(M_{m,i}, g_{m,i})$ is volume collapsing and using $\bar{\mathbf{Q}}\mathbf{2}$, one can select for every m an $i(m)$ such that

$$Vol_{g_{m,i(m)}}(M_{m,i(m)}) \leq 1/m, \text{ and } dist_{GH}(\Omega_{m,i(m)}, I_m) \leq 1/m.$$

In particular, the sequence (in "m") $(M_{m,i(m)}, g_{m,i(m)})$ is volume collapsing. Also, as we have $rad_{g_{m,i(m)}}(\Omega_{m,i(m)}) \geq 1$ and because of $\bar{\mathbf{Q}}\mathbf{2}$, then we have $L_0 \geq |I_m| \geq 1$ for every m.[8] Therefore, there is a subsequence of $(\Omega_{m,i(m)}, g_{m,i(m)})$ (indexed again by "m") metrically collapsing to an interval I' with $L_0 \geq |I'| \geq 1$. We continue working with this subsequence in what follows. This implies, in particular, that $diam_{g_{m,i(m)}}(\Omega_{m,i(m)}) \leq D_0$ for some D_0 and for all m.[9] Finally, note (to be used below) that from $\bar{\mathbf{Q}}\mathbf{3}$ there is, for every m, a curve $\mathscr{C}_{m,i(m)} \subset \partial\Omega_{m,i(m)}$ non-contractible in $\partial\Omega_{m,i(m)}$ but contractible in $\Omega_{m,i(m)}$ and of $length_{g_{i(m)}}(\mathscr{C}_{m,i(m)}) \leq 1/m$.

By Example II, if we let $M'_m = \mathcal{T}_{d_m}(\Omega_{m,i(m)}, \delta_0)$ with $d_m = d_{g_{m,i(i)}}^{M_{m,i(m)}}$, then $(M'_m, g_{m,i(m)})$ lies in $\mathcal{M}(\mathfrak{N}_0, \Lambda_0)$ for some $\mathfrak{N}_0(D_0, \Lambda_0, \delta_0)$. On the other hand, as $M'_m \subset M_{m,i(m)}$, then $(M'_m, g_{m,i(m)})$ is also a volume-collapsing sequence. Hence, by Lemma 2.1, one can find a sequence of $(\delta_0/4, \delta_0/2)$-connected components of M'_m containing $\Omega_{m,i(m)}$, to be denoted by $\hat{\Omega}_m$, and having a subsequence (indexed again by "m") metrically collapsing to an interval $\hat{I}$ containing I'. We continue using this subsequence in what follows. For the sake of concreteness, assume that $\hat{I}$ is the interval $[0, |\hat{I}|]$.

Consider the fibrations $f_m : \hat{\Omega}_m \to \hat{I}$ as is explained in Lemma 2.1. As $m \to \infty$, the fibers $f_m^{-1}(x)$ collapse to a point and so does $\partial\hat{\Omega}_m = f_m^{-1}(0)$ to the point 0 in $\hat{I}$. Observe that the right point of I' must be the right point of $\hat{I}$—that is, $|\hat{I}|$—and therefore it is a singular point, namely, $Sing(\hat{I}) = \{|\hat{I}|\}$. We observe, too, that from the very definition of M'_m we have, for every $q \in \partial\Omega_{m,i(m)}$, $d_{g_{m,i(m)}}^{\hat{\Omega}_m}(q, \partial\hat{\Omega}_m) < \delta_0 < 1/2$.[10] It follows from this that for $m \geq m_0$ with m_0 big enough

[8] In general, if $(X_m, d_{X_m} \xrightarrow{GH} (X, d_X)$ and $d_{X_m}(x_m, x'_m) \geq \Gamma$ for all m, then there are x and x' in X with $d_X(x, x') \geq \Gamma$ (use the definition of GH-convergence). On the other hand, if $rad_{g_{m,i(m)}}(\Omega_{m,i(m)}) \geq 1$, then there are x_m and x'_m in $\Omega_{m,i(m)}$ such that $d_{g_{m,i(m)}}^{\Omega_{m,i(m)}}(x_m, x'_m) \geq 1$).

[9] In general, if $(X_m, d_m) \xrightarrow{GH} (X, d)$, then there is D_0 such that $diam_{d_{X_m}}(X_m) \leq D_0$ for all m (use the definition of GH-convergence).

[10] Note for this that for any $q \in \partial\Omega_{m,i(m)}$ we must have $B_{g_{m,i(m)}}(q, \delta_0) \cap \partial\hat{\Omega}_m \neq \emptyset$, because $\hat{\Omega}_m$ is a $(\delta_0/4.\delta_0/2)$-c.c.

(i) $\partial\Omega_{m,i(m)} \subset f_m^{-1}([0,1/2])$, (ii) $f_m^{-1}(1/2)$ lies in the interior of $\Omega_{m,i(m)}$, and (iii) $f_m^{-1}(0)$ lies in the exterior of $\Omega_{m,i(m)}$. In this way, $\partial\Omega_{m,i(m)}$ separates $f_m^{-1}([0,1/2])$, which is diffeomorphic to $\mathbb{T}^2 \times [0,1/2]$, into two connected components. This implies[11] that $\partial\Omega_{m,i(m)}$ is isotopic to $f_m^{-1}(x)$ for any $x \in [0,1/2]$. In particular, if $\mathscr{C}_{m,i(m)}$ is non-contractible in $\partial\Omega_{m,i(m)}$, then it is also non-contractible in $f_m^{-1}([0,1/2])$. Moreover, by Lemma 2.1 there is a subsequence (indexed again by "m") and coverings $\pi_m : \tilde{\hat{\Omega}}_m \to \hat{\Omega}_m$ such that $(\tilde{\hat{\Omega}}_m, \tilde{g}_{m,i(m)})$ converges in $C^{1,\beta}$ to a $\mathbb{T}^2$-symmetric metric on $\mathbb{B}^2 \times \mathbb{S}^1$ and $(\pi_m^{-1}(f_m^{-1}([0,1/2])), \tilde{g}_{m,i(m)})$ converges in $C^{1,\beta}$ to a $\mathbb{T}^2$-symmetric metric on $\mathbb{T}^2 \times \mathbb{I}$. For this reason, there are m_1 and ℓ_1, such that for any $m \geq m_1$ any non-contractible closed curve in $(\pi_m^{-1}(f_m^{-1}([0,1/2])), \tilde{g}_{m,i(m)})$ has length greater or equal to ℓ_1. But for every m, the curve $\mathscr{C}_{m,i(m)}$ is closed and contractible in $\Omega_{m,i(m)}$ and thus contractible also in $\tilde{\hat{\Omega}}_m$. Therefore, its lift $\tilde{\mathscr{C}}_{m,i(m)}$ to $\pi_m^{-1}(f_m^{-1}[0,1/2])) \subset \tilde{\hat{\Omega}}_m$ is also closed and has the same length, which, as was observed above, is less than or equal to $1/m$. If $m \geq \max\{m_1, 2/\ell_1\}$, then $length_{g_{i(m)}}(\mathscr{C}_{m,i(m)}) \leq \ell_1/2$, which is not possible. q.e.d.

2.5. A special annuli decomposition. The results of the previous section allow us to show the existence of an annuli decomposition with special properties. Again for every k we define the scaled metric $g_k = \frac{1}{10^{2k}} g$. Therefore, $A_g(10^{n_1+k}, 10^{n_2+k}) = A_{g_k}(10^{n_1}, 10^{n_2})$.

Proposition 2.4. *Let g be a complete metric in $\mathbb{R}^3$ with $Ric \geq 0$ outside a compact set, quadratic curvature decay, and less than cubic-volume growth. Then there is an annuli decomposition $\mathcal{U}$ with the following properties:*

For every $\epsilon > 0$ there is $k(\epsilon)$ such that for any $k \geq k(\epsilon)$ every piece $(U_{k,l}, g_k)$ is ϵ-close in the GH-metric to a space $X_{k,l}$ of one of the following two forms:

- $\tilde{\mathbf{D}}\mathbf{1}$. *an interval, in which case $U_{k,l}$ is either diffeomorphic to $\mathbb{T}^2 \times \mathbb{I}$ or a solid torus $\mathbb{B}^2 \times \mathbb{S}^1$, or*
- $\tilde{\mathbf{D}}\mathbf{2}$. *a two-orbifold, in which case $U_{k,l}$ is diffeomorphic to a Seifert manifold with at least one boundary component.*

There are fibrations $f_{k,l} : U_{k,l} \to X_{k,l}$, such that for any $k \geq k(\epsilon)$ the fibers $f_{k,l}^{-1}(x)$, which are diffeomorphic either to $\mathbb{T}^2$ or $\mathbb{S}^1$, are ϵ-collapsed. Moreover

- $\tilde{\mathbf{I}}\mathbf{1}$. *In case $\tilde{\mathbf{D}}\mathbf{1}$, either $Sing(X_{k,l})$ is empty or is one of the extreme points of the interval. In addition, for any non-singular point x,*

[11] This is a simple exercise in topology; use Alexander's theorem in [**1**] for two-tori in $\mathbb{S}^3$.

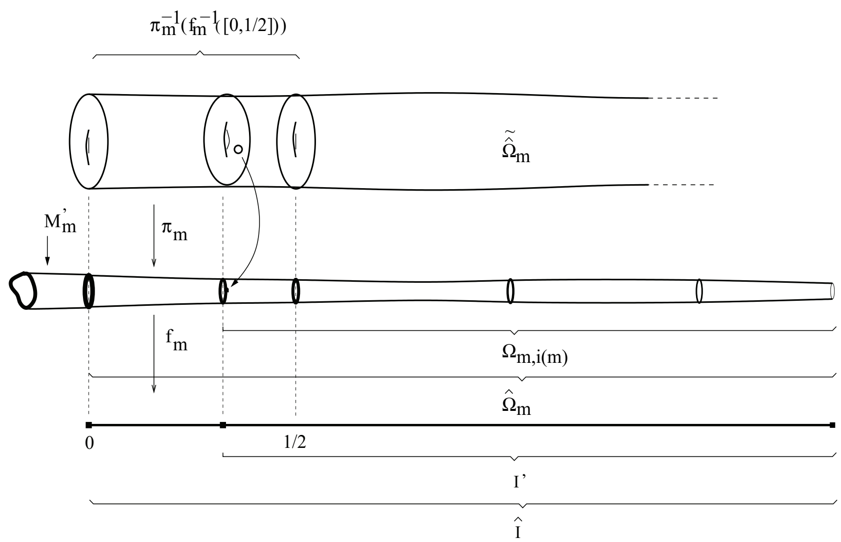

Figure 3. A representation of the argument given in the proof of Proposition 2.3. The little curve in the cover manifold represents the lift $\tilde{\mathscr{C}}_{m,i(m)}$ of $\mathscr{C}_{m,i(m)}$. If $m \geq \max\{m_1, 2\ell_1\}$, the length of $\tilde{\mathscr{C}}_{m,i(m)}$ would be too small to be non-contractible in $\pi_m^{-1}(f_m^{-1}([0,1/2]))$.

the fiber $f_{k,l}^{-1}(x)$ is diffeomorphic to $\mathbb{T}^2$ and if x is a singular point then $f_{k,l}^{-1}(x)$ is diffeomorphic to $\mathbb{S}^1$.

Ĩ2. *In case* **D̃2**, *the fibers $f_{k,l}^{-1}(x)$, which are all diffeomorphic to $\mathbb{S}^1$, are the fibers of the Seifert-fibration.*

For the proof, we will need the notion of "a cut of $(\mathbb{R}^3, g)$ along the annulus $A_{g_k}(10^{-1}, 1)$," which we now explain. We say that (given k) a set $\{S_{k,j}, j = 1, \dots, j(k)\}$ of embedded two-manifolds is a cut along $A_{g_k}(10^{-1}, 1)$ iff

(Cut1) $S_{k,j} \subset A_{g_k}(10^{-1}, 1)$ for all $j = 1, \dots, j(k)$; and

(Cut2) every curve $\alpha : [0,1] \to \mathbb{R}^3$ with $\alpha(0) \in B_{g_k}(o, 10^{-1})$ and $\alpha(1) \in \left(\mathbb{R}^3 \setminus \overline{B_{g_k}(o,1)}\right)$ intersects at least one of the $S_{k,j}$'s; and

(Cut3) the property (Cut2) does not hold if one deletes one of the $S_{k,j}$'s from the set.

Observe that if a set of manifolds $\{S_{k,j}, j = 1, \dots, j(k)\}$ satisfy (Cut1) and (Cut2), then one can remove (if necessary) some elements of the set to satisfy also (Cut3). Note, too, that if $\{S_{k,j}, j = 1, \dots, j(k)\}$ is a cut, then the connected component $\mathscr{S}_k$ of $\mathbb{R}^3 \setminus (\cup_j S_{k,j})$ containing the origin o has exactly the manifolds $S_{k,j}, j = 1, \dots, j(k)$ as its boundary components. Moreover,

$$\overline{B_{g_k}(o,10^{-1})} \subset \mathscr{S}_k^\circ \quad \text{and} \quad \mathscr{S}_k \subset B_{g_k}(o,1).$$

We move now to the proof of Proposition 2.4.

Proof of Proposition 2.4. For the proof, it may be worth to keep in mind Figure 1. As explained in the Example I in Section 2.4.3, the spaces $(A_{g_k}[10^{-2},10^4],g_k)$ lie in $\mathcal{M}(\mathfrak{N}_0,10^2\Lambda_0)$ for some k-independent $\mathfrak{N}_0$. Moreover, for any $p\in A_{g_k}[1,10]$ we have that $d^{\mathbb{R}^3}_{g_k}(p,\partial A_{g_k}[10^{-2},10^4])$ $>1/2$. Granted these two facts, we can use then Lemma 2.1 to obtain with no difficulty the following:

There is $k_0>0$ and a set $\{\tilde{U}_{k,l}, l=1,\ldots,l(k); k=k_0,k_0+2,\ldots\}$ of (for each k) $(10^{-2}/2,10^{-2})$-connected components of $(A_{g_k}[10^{-2},10^4],g_k)$ with the following properties:

1) *$\{\tilde{U}_{k,l}, l=1,\ldots,l(k)\}$ covers $A_{g_k}[10^{-1},10^3]$ for every $k=k_0,k_0+2,\ldots$.*
2) *There are intervals or two-orbifolds, to be denoted by $\tilde{X}_{k,l}$, and for every $m=1,2,3,\ldots$ there is k_m, such that if $k\geq k_m$, then $(\tilde{U}_{k,l},g_k)$ is $1/m$-close in the GH-metric to $\tilde{X}_{k,l}$.*
3) *There are fibrations $\tilde{f}_{k,l}:\tilde{U}_{k,l}\to\tilde{X}_{k,l}$, with the properties* $\tilde{\mathbf{I}}\mathbf{1}$ *and* $\tilde{\mathbf{I}}\mathbf{2}$*, such that if $k\geq k_m$, their fibers are $1/m$-collapsed.*

It is important that the fibrations $\tilde{f}_{k,l}:\tilde{U}_{k,l}\to\tilde{X}_{k,l}$ can be chosen in such a way that (i) if $\tilde{U}_{k,l}$ and $\tilde{U}_{k',l'}$ overlap and have fibers of the same dimension (namely, both have fibers of dimension 1 or both have fibers of dimension 2), then the foliations of fibers coincide on the overlap, and (ii) if 1 has fibers of dimension one and the other of dimension 2, then fibers of dimension 1 are included in fibers of dimension 2. For this the reader can consult the geometric construction of the fibrations in [**9**] and [**10**].

Now, the manifolds $U_{k,l}$ of the desired annuli decomposition will be defined below as appropriate submanifolds of the $\tilde{U}_{k,l}$. Once this is performed, the desired fibrations $f_{k,l}:U_{k,l}\to X_{k,l}$ are defined by the restrictions $f_{k,l}:=\tilde{f}_{k,l}|_{U_{k,l}}:U_{k,l}\to X_{k,l}:=\tilde{f}_{k,l}(U_{k,l})$, where here $\tilde{U}_{k,l}$ is the piece containing $U_{k,l}$ and $\tilde{f}_{k,l}:\tilde{U}_{k,l}\to\tilde{X}_{k,l}$ its fibration. We now explain how the regions $U_{k,l}$ are constructed.

Fix a value of k in $\{k_0,k_0+2,\ldots\}$. Then on those $\tilde{X}_{k,l}$ that are intervals select points $\tilde{x}_{k,l,j}$ and on those $\tilde{X}_{k,l}$ that are a two-orbifolds select disjoint closed curves $\tilde{\mathscr{C}}_{k,j,i}$, in such a way that the set of tori $\{S_{k,l,j}\}:=\{\tilde{f}^{-1}_{k,l}(\tilde{x}_{k,l,j}),\tilde{f}^{-1}_{k,l,j}(\tilde{\mathscr{C}}_{k,j,i}), \text{ all } l,j\}$ is a "cut of $\mathbb{R}^3$ along the annulus $A_{g_k}(10^{-1},1)$," that is, satisfying (Cut1)-(Cut3).

Thus for every k in $\{k_0,k_0+2,\ldots\}$ there is a cut $\{S_{k,l,j}\}$, and associated to it is the connected set $\mathscr{S}_k$ as defined before the start of the proof.

Then for every k in $\{k_0, k_0+2, \ldots\}$ let $\mathcal{U}_k$ be the set of connected components of $\mathscr{S}_{k+2} \setminus \mathscr{S}_k^{\circ}$. Every piece in $\mathcal{U}_{k+2}$ shares a boundary component with a piece in $\mathcal{U}_k$, but not necessarily vice versa.

For every $k = k_0, k_0+2, \ldots$ let $\mathcal{U}_k^-$ be the set formed by those pieces of $\mathcal{U}_k$ that do not share a boundary component with a piece of $\mathcal{U}_{k+2}$ and define $\mathcal{U}_k^+ = \mathcal{U}_k \setminus \mathcal{U}_k^-$. Having done this, define for every k the set of pieces $U_{k,l}$ (that we are looking for) as the connected components of the region

$$\Big(\bigcup_{U_{k,l}^+ \in \mathcal{U}_k^+} U_{k,l}^+\Big) \bigcup \Big(\bigcup_{U_{k+2,l}^- \in \mathcal{U}_{k+2}^-} U_{k+2,l}^-\Big).$$

The reader can check directly that, with this definition of the set $\{U_{k,l}\}$, *items 1–5* of the definition of annuli decomposition are readily satisfied. Every $U_{k,l}$ lies inside a unique $\tilde{U}_{k,l}$ and the fibrations $f_{k,l}$ are defined as was explained earlier. q.e.d.

3. Proof of Theorem 1.1

We will work in this section with the annuli decomposition defined in the previous section. We already defined in Section 2.3 the set $\mathcal{N}$ of boundary components of $\mathcal{U}$, which we will denote here generically by T^2 (instead of S because they are tori). We also defined the subclass $\mathcal{N}^o$ as those tori T^2 in $\mathcal{N}$ for which $o \in M(T^2)$ and observed that they were linearly ordered, i.e., $\mathcal{N}^o = \{T_0^{2o}, T_1^{2o}, \ldots\}$, with $T_i^{2o} \ll T_{i'}^{2o}$ if $i < i'$. For later convenience, we further divide $\mathcal{N} \setminus \mathcal{N}^o$ into two subclasses denoted by $\mathcal{N}^{\blacklozenge}$ and $\mathcal{N}^{\lozenge}$; $\mathcal{N}^{\blacklozenge}$ (resp. $\mathcal{N}^{\lozenge}$) is defined as the set of tori in $\mathcal{N} \setminus \mathcal{N}^o$ for which $M(T^2)$ is a solid torus (resp. not a solid torus). Tori in $\mathcal{N}^{\blacklozenge}$ (resp. $\mathcal{N}^{\lozenge}$) will be denoted as $T^{2\blacklozenge}$ (resp. $T^{2\lozenge}$). For every T^2 in $\mathcal{N}$ there is a unique piece $U_{k,l}$ (including the possibility of U_{k_0-2}) such that $T^2 \in U_{k,l}$ and $U_{k,l} \subset M(T^2)$. In this way, the indexes k, l are univocally defined and we can write $k(T^2), l(T^2)$. We will continue using the notation $g_k = g/10^{2k}$; in particular, we will use $g_{k(T^2)}$.

The following proposition is crucial for the proof of the Theorem 1.1. Observe that the statement is suitable to be used in an iterative argument, as will be the case when we use it in the proof of Theorem 1.1.

Proposition 3.1. *There exits ϵ^*, ℓ^*, k^* such that if for a $T_1^{2\blacklozenge} \in \mathcal{N}^{\blacklozenge}$ with $k(T_1^{2\blacklozenge}) \geq k^*$, we have that*

- **H1**. *$(U_{k(T_1^{2\blacklozenge}), l(T_1^{2\blacklozenge})}, g_{k(T_1^{2\blacklozenge})})$ is ϵ^*-close in the GH-metric to an interval and*
- **H2**. *there is a curve $\mathscr{C}_1 \subset T_1^{2\blacklozenge}$ non-contractible in $T_1^{2\blacklozenge}$ but contractible in $M(T_1^{2\blacklozenge})$ such that $\mathrm{length}_{g_{k(T_1^{2\blacklozenge})}}(\mathscr{C}_1) \leq \ell^*$,*

then $U_{k(T_1^{2\blacklozenge}),l(T_1^{2\blacklozenge})}$ *is not the only* $U_{k,l}$*-piece of* $M(T_{1;m}^{2\blacklozenge})$ *and, if we denote by* $T_2^{2\blacklozenge}$ *the second boundary component of* $U_{k(T_1^{2\blacklozenge}),l(T_1^{2\blacklozenge})}$, *we have*

C1. $(U_{k(T_2^{2\blacklozenge}),l(T_2^{2\blacklozenge})}, g_{k(T_2^{2\blacklozenge})})$ *is* $2\epsilon^*/3$*-close in the GH-metric to an interval and*

C2. *there is a curve* $\mathscr{C}_2 \subset T_2^{2\blacklozenge}$ *non-contractible in* $T_2^{2\blacklozenge}$ *but contractible in* $M(T_2^{2\blacklozenge})$ *such that* $length_{g_{k(T_2^{2\blacklozenge})}}(\mathscr{C}_2) \leq 2\ell^*/3$.

Proof. By contradiction, assume that for every $\epsilon^*_m = 1/m, \ell^*_m = 1/m$ and $k^*_m = m$, $m = 1, 2, 3, \ldots$, there is $T_{1;m}^{2\blacklozenge} \in \mathcal{N}^{\blacklozenge}$ with $k(T_{1;m}^{2\blacklozenge}) \geq k^*_m$ such that

$\bar{\mathbf{H}}\mathbf{1}$. $(U_{k(T_{1;m}^{2\blacklozenge}),l(T_{1;m}^{2\blacklozenge})}, g_{k(T_{1;m}^{2\blacklozenge})})$ is ϵ^*_m-close in the GH-metric to an interval and

$\bar{\mathbf{H}}\mathbf{2}$. there is a curve $\mathscr{C}_{1;m} \subset T_{1;m}^{2\blacklozenge}$ non-contractible in $T_{1;m}^{2\blacklozenge}$ but contractible in $M(T_{1;m}^{2\blacklozenge})$ such that $length_{g_{k(T_{1;m}^{2\blacklozenge})}}(\mathscr{C}_{1;m}) \leq \ell^*_m$,

but that, if it is not that $U_{k(T_{1;m}^{2\blacklozenge}),l(T_{1;m}^{2\blacklozenge})} = M(T_{1;m}^{2\blacklozenge})$, then, after denoting by $T_{2;m}^{2\blacklozenge}$ to the second boundary component of $U_{k(T_{1;m}^{2\blacklozenge}),l(T_{1;m}^{2\blacklozenge})}$, one of the following two assertions does not hold:

$\bar{\mathbf{C}}\mathbf{1}$. $(U_{k(T_{2;m}^{2\blacklozenge}),l(T_{2;m}^{2\blacklozenge})}, g_{k(T_{2;m}^{2\blacklozenge})})$ is $2\epsilon^*_m/3$-close in the GH-metric to an interval, or

$\bar{\mathbf{C}}\mathbf{2}$. there is a curve $\mathscr{C}_{2;m} \subset T_{2;m}^{2\blacklozenge}$ non-contractible in $T_{2;m}^{2\blacklozenge}$ but contractible in $M(T_{2;m}^{2\blacklozenge})$ such that $length_{g_{k(T_{2;m}^{2\blacklozenge})}}(\mathscr{C}_{2;m}) \leq 2\ell^*_m/3$.

We will show that this leads to an impossibility. Such impossibility will come directly as the result of proving the following three steps.

• **Step A**. *Let* $T_{1;m}^{2\blacklozenge}$ *be a sequence satisfying* $\bar{\mathbf{H}}\mathbf{1}$ *and* $\bar{\mathbf{H}}\mathbf{2}$. *Then*

$$rad_{g_{k(T_{1,m}^{2\blacklozenge})}}(M(T_{1;m}^{2\blacklozenge})) \xrightarrow{m\to\infty} \infty.$$

Step **A** shows that there is m_1 such that for every $m \geq m_1$, $U_{k(T_{1;m}^{2\blacklozenge}),l(T_{1;m}^{2\blacklozenge})}$ is not the only piece of $M(T_{1;m}^{2\blacklozenge})$ (because if so, then, by *item 1* of Definition 1, $rad_{g_k(T_{1;m}^{2\blacklozenge})}(M(T_{1;m}^{2\blacklozenge})) \leq 10^3$). The statement of Step **B** below assumes $m \geq m_1$.

• **Step B**. $(m \geq m_1)$. *Let* $T_{2;m}^{2\blacklozenge}$ *be the second component of the piece* $U_{k(T_{1;m}^{2\blacklozenge}),l(T_{1;m}^{2\blacklozenge})}$. *Then there is a covering sequence to a subsequence of*

$$(U_{k(T_{1;m}^{2\blacklozenge}),l(T_{1;m}^{2\blacklozenge})} \cup U_{k(T_{2;m}^{2\blacklozenge}),l(T_{2;m}^{2\blacklozenge})}, g_{k(T_{1;m}^{2\blacklozenge})}),$$

converging in $C^{1,\beta}$ to a flat $\mathbb{T}^2$-symmetric metric product on $\mathbb{T}^2 \times I_{1,2}$ for some interval $I_{1,2}$. That is, the limit metric on $\mathbb{T}^2 \times I_{1,2}$ is of the form $dx^2 + \tilde{h}_0$ with $\tilde{h}_0$ an (x-independent) $\mathbb{T}^2$-symmetric metric on $\mathbb{T}^2$.

• **Step C**. *There is $m_2 \geq m_1$ such that for all $m \geq m_2$ and m in the subsequence of Step* **B**, *then* $\bar{\mathbf{C}}\mathbf{1}$ *and* $\bar{\mathbf{C}}\mathbf{2}$ *hold.*

From now until the end of the proof of the proposition and to simplify notation, we let

$$U_{1;m} = U_{k(T^{2\blacklozenge}_{1;m}),l(T^{2\blacklozenge}_{1;m})}, \quad M_{1;m} = M(T^{2\blacklozenge}_{1;m}),$$
$$g_{1;m} = g_{k(T^{2\blacklozenge}_{1;m})}, \quad k_{1;m} = k(T^{2\blacklozenge}_{1;m}).$$

Proof of Step A. Assume on the contrary that $rad_{g_{1;m}}(M_{1;m}) \leq R_0$. Then it is simple to see[12] that $M_{1;m}$ must be a subset of an annulus $A(10^{k_{1;m}-1}, 10^{k_{1;m}+k_\bullet})$ for some $k_\bullet > 0$ independent of m. On the other hand $rad_{g_{1;m}}(M_{1;m}) \geq 10^2 - 10^{-1} > 90$ (because of *item 2* of Definition 1 applied to $U_{1;m}$). Under these hypothesis, we obtain that

1) (using $\bar{\mathbf{H}}\mathbf{1}$) a subsequence of the sequence of solid tori $(M_{1;m}, g_{1;m})$ (indexed still by "m") metrically collapses to a compact interval[13] I of length $|I|$ greater or equal to 90, and
2) (using $\bar{\mathbf{H}}\mathbf{2}$) for every ℓ_0 we have $\lim_{m\to\infty} length_{g_{1;m}}(\mathscr{C}_{1;m}) \leq \ell_0$.

We can then apply Proposition 2.3[14] to conclude that $|I| \geq L_0$ for any L_0 and therefore that $|I| = \infty$, contradicting the compactness of the interval I. ◀

We recount briefly the setup and terminology before we go into Step **B**. Let $T^{2\blacklozenge}_{2;m}$ be the second boundary component of $U_{1;m}$, and let $U_{2;m} := U_{k(T^{2\blacklozenge}_{2;m}),l(T^{2\blacklozenge}_{2;m})}$ be the $U_{k,l}$-piece, other than $U_{1;m}$, having $T^{2\blacklozenge}_{2;m}$ as a boundary component. Of course, $k_{2;m} := k(T^{2\blacklozenge}_{2;m}) = k(T^{2\blacklozenge}_{1;m}) + 2 = k_{1;m} + 2$. Following the same pattern of notation as before, we let

$$U_{1,2;m} = U_{1;m} \cup U_{2;m}, \quad g_{2;m} = g_{k(T^{2\blacklozenge}_{2;m})}.$$

[12]For any $p \in M_{1;m}$, $d^{\mathbb{R}^3}_{g_{1;m}}(p, o)$ is less than or equal to $d^{\mathbb{R}^3}_{g_{1;m}}(p, T^{2\blacklozenge}_{1;m}) + diam_{g_{1;m}}(T^{2\blacklozenge}_{1;m}) + d^{\mathbb{R}^3}_{g_{1;m}}(T^{2\blacklozenge}_{1;m}, o)$ which is less than or equal to $R_0 + diam_{g_{1;m}}(T^{2\blacklozenge}_{1;m}) + 1$. But the $g_{1;m}$-diameter of $T^{2\blacklozenge}_{1;m}$ tends to zero (by $\bar{\mathrm{H}}1$), and so we can assume that it is less than or equal to some D_0.

[13]It must converge to an interval and not a two-orbifold (the only two options) because $(M_{1;m}, g_{1;m})$ contains $(U_{1;m}, g_{1;m})$, which by $\bar{\mathbf{H}}\mathbf{1}$ converges to an interval.

[14]To apply Proposition 2.3, use as M_m in its statement the manifold $M_m := \mathcal{T}_{d^{\mathbb{R}^3}_{g_{1;m}}}(M_{1;m}, 10^{-2})$. It is direct that $(M_m, g_{1;m})$ is a volume-collapsing sequence.

Proof of Step B. To this end first note that $(U_{1,2;m}, g_{1;m})$ collapses metrically to an interval[15] to be denoted by $I_{1,2}$; $(U_{1;m}, g_{1;m})$ collapses to I_1 and $(U_{2;m}, g_{2;m})$ collapses to I_2, and we have $I_{1,2} = I_1 \cup I_2$ and $|I_{1,2}| = |I_1| \cup |I_2|$. Without loss of generality, we assume that $T^{2\blacklozenge}_{1;m}$ collapses to the left boundary point of the interval I_1 (or, the same, of $I_{1,2}$) as an interval in $\mathbb{R}$. Further, following Proposition 2.4 and Lemma 2.1 (see also Proposition 4.1 in the appendix for a technical point on the explicit form of the limit), there is a subsequence (indexed again by m) and a covering sequence $\pi_m : \tilde{U}_{1,2;m} \to U_{1,2;m}$ such that

$$(\tilde{U}_{1,2;m}, \tilde{g}_{1;m}) \xrightarrow{C^{1,\beta}} (\mathbb{T}^2 \times I_{1,2}, \tilde{g}_1 = dx^2 + \tilde{h}_1), \tag{9}$$

where, for $x \in I_{1,2}$, $\tilde{h}_1(x) := \tilde{h}_1|_{\mathbb{T}^2\times\{x\}}$ is a $\mathbb{T}^2$-symmetric Riemannian metric. Note that because the convergence (9) is in $C^{1,\beta}$, the "path" $x \to \tilde{h}_1(x)$ is C^1. Therefore, the second fundamental forms $\tilde{\Theta}_1(x) := \tilde{\Theta}_1|_{\mathbb{T}^2\times\{x\}} = \big(\frac{1}{2}\partial_x\tilde{h}\big)\big|_{\mathbb{T}^2\times\{x\}}$ of the slices $\mathbb{T}^2 \times \{x\}$ define a continuous "path" of $\mathbb{T}^2$-symmetric, symmetric two-tensors. Denote the mean curvatures by $\tilde{\theta}_1(x) := tr_{\tilde{h}_1(x)}\tilde{\Theta}_1(x)$. Moreover, also from Proposition 2.4 and Lemma 2.1, there are C^1-fibrations $f_m : U_{1,2;m} \to I_{1,2}$ such that

$$\pi_m^{-1}(f_m^{-1}(x)) \xrightarrow{C^1} \mathbb{T}^2 \times \{x\}. \tag{10}$$

The C^1 convergence here is not optimal for the argumentation below, as we want to have control on the second fundamental forms of the fibers. However, in the technical Proposition 4.2, which we prove in the appendix, it is shown that in this situation f_m can indeed be chosen to achieve convergence in C^2 in (10). *We will assume that this is the case from now on.*

We want to prove that $\tilde{h}_1(x) = \tilde{h}_0$. This will follow directly from the next two claims and the identity $\partial_x\tilde{h}_1(x) = 2\tilde{\Theta}_1(x)$.

Claim 1: If $\tilde{\theta}_1(x) = 0$ at every slice of $\mathbb{T}^2 \times I_{1,2}$, then $\tilde{\Theta}_1(x) = 0$ at every slice of $\mathbb{T}^2 \times I_{1,2}$.

Claim 2: $\tilde{\theta}_1(x) = 0$ at every slice of $\mathbb{T}^2 \times I_{1,2}$.

We prove first *Claim 1*. Let $\varphi_m : \mathbb{T}^2 \times I_{1,2} \to \tilde{U}_{1,2;m}$ be a sequence of diffeomorphisms such that $\varphi_m^*(\tilde{g}_{1;m})$ converges in $C^{1,\beta}$ to $\tilde{g}_1$. Then we can write[16]

$$\varphi_m^*(\tilde{g}_{1;m}) = \alpha_m^2 dx^2 + \tilde{h}_{1;m}(x), \tag{11}$$

[15] Again, this is so because $(U_{1;m}, g_{1;m})$, with $U_{1;m} \subset U_{1,2;m}$, collapses metrically to an interval.

[16] If necessary, φ_m can be slightly modified to avoid cross terms, as in the metric expression (11).

where, as $m \to \infty$, the functions $\alpha_m : \mathbb{T}^2 \times I \to \mathbb{R}^+$ converge in C^1 to the constant function one on $\mathbb{T}^2 \times I_{1,2}$ and the metrics $\tilde{h}_{1;m}$ converge in C^1 to $\tilde{h}_1$. Let $\tilde{\Theta}_{1;m}(x)$ and $\tilde{\theta}_{1;m}(x)$ be the second fundamental forms and mean curvatures of the slices $\mathbb{T}^2 \times \{x\}$, as slices in $(\mathbb{T}^2 \times I_{1,2}, \varphi_m^*(\tilde{g}_{1;m}))$. Then

$$\partial_x \tilde{\theta}_{1;m} = -\Delta_{\tilde{h}_{1;m}} \alpha_m + \big(|\tilde{\Theta}_{1;m}|^2_{\tilde{h}_{1;m}} + Ric_{\tilde{g}_{1;m}}(\mathfrak{n}, \mathfrak{n})\big)\alpha_m, \tag{12}$$

where $\Delta_{\tilde{h}_{1;m}}$ is the $\tilde{h}_{1;m}$-Laplacian on the slices $\mathbb{T}^2 \times \{x\}$ and $\mathfrak{n}$ is the unit normal field to the slices. Let $\zeta(x)$ be a C^1 non-negative real function of one variable with support in $I_{1,2}$, and consider the volume measure on $\mathbb{T}^2 \times I_{1,2}$, given by $dV_m = dA_{\tilde{h}_{1;m}} dx$, where $dA_{\tilde{h}_{1;m}}$ is the area element of $\tilde{h}_{1;m}$ on every slice. Multiplying (12) by ζdV_m and integrating, we obtain the following: for the integral of the left-hand side and after integration by parts in the variable x,

$$-\int_{\mathbb{T}^2 \times I_{1,2}} \big((\partial_x \zeta)\tilde{\theta}_{1;m} + \zeta 2\alpha_m (\tilde{\theta}_{1;m})^2\big) dV_m, \tag{13}$$

where we used $\partial_x dA_{\tilde{h}_{1;m}}/dA_{\tilde{h}_{1;m}} = \alpha_m \tilde{\theta}_m$, and, for the integral of the first term of the right-hand side exactly the value zero because ζ is constant over every slice. As $\alpha_m \xrightarrow{m\to\infty} 1$ and $\tilde{\theta}_{1;m} \xrightarrow{m\to\infty} \tilde{\theta}_1$ (in C^1 and C^0 resp. and all over $\mathbb{T}^2 \times I_{1,2}$), we conclude that if $\tilde{\theta}_1 = 0$, then (13) goes to zero, and that this is so for any ζ. Therefore, the integral of the second term in the right hand side of (12), namely,

$$\int_{\mathbb{T}^2 \times I_{1,2}} \zeta \big(|\tilde{\Theta}_{1;m}|^2_{\tilde{h}_{1;m}} + Ric_{\tilde{g}_{1;m}}(\mathfrak{n}, \mathfrak{n})\big)\alpha_m dV_m,$$

must go to zero independently of ζ. But $Ric_{\tilde{g}_{1;m}}(\mathfrak{n}, \mathfrak{n}) \geq 0$ for every m, and thus, in the limit, we must have $\int \zeta |\tilde{\Theta}_1|_{\tilde{h}_0} dA_{\tilde{h}_0} dx = 0$ for every ζ. Hence $\tilde{\Theta}_1 = 0$ as claimed.

We prove now *Claim 2*. We show first the impossibility of having, for some $\bar{x}$, $\tilde{\theta}_1(\bar{x}) < 0$. After that, we prove the impossibility of having $\tilde{\theta}_1(\bar{x}) > 0$. To do so, we will appeal to the following standard fact. Fact 1: *Let $S \subset M$ be a hypersurface on a manifold M with a unit-normal field $\mathfrak{n}$. Let $p \in M$ and γ a geodesic segment starting at S in the direction of $\mathfrak{n}$, ending at p and with $dist(p, S) = length(\gamma)$. If $\theta|_S \leq \theta_0 < 0$ and $Ric \geq 0$ all over a neighborhood of γ, then $length(\gamma) \leq 2/|\theta_0|$.*

• *Suppose that, for some $\bar{x}$, $\tilde{\theta}_1(\bar{x}) < 0$.* Then by (10) we conclude that there is $m_2 \geq m_1$ such that for every $m \geq m_2$ we have $\theta_{1;m}|_{f_m^{-1}(\bar{x})} < \tilde{\theta}_1(\bar{x})/2$, where $\theta_m(x)$ is the mean curvature of $f_m^{-1}(x)$—namely, $\pi_m^*(\theta_m) = \tilde{\theta}_m$. But note that the solid torus $M(f_m^{-1}(\bar{x}))$ lies inside $M(T^{2\blacklozenge}_{1;m})$, which is a region of non-negative Ricci, that $\partial M(f_m^{-1}(\bar{x}))$ is $f_m^{-1}(\bar{x})$, and finally by Step **A** that $rad_{g_{1;m}}(M(f_m^{-1}(\bar{x}))) \to \infty$. This easily contradicts Fact 1, as then for any $m \geq m_2$ there is a point p_m and a geodesic segment in $M(f_m^{-1}(\bar{x}))$ starting at $f_m^{-1}(\bar{x})$ and ending at p_m of $g_{1,m}$-length equal to $rad_{g_{1;m}}(M(f_m^{-1}(\bar{x})))$ and therefore realizing the $g_{1;m}$-distance from p_m to $f_m^{-1}(\bar{x})$.

• *Suppose that, for some* $\bar{x}$, $\tilde{\theta}_1(\bar{x}) > 0$. Again by (10), we conclude that there is $m_2' \geq m_1$ such that for every $m \geq m_2'$ we have $\theta_{1;m}|_{f_m^{-1}(\bar{x})} > \tilde{\theta}_1(\bar{x})/2$. We will prove that there is a sequence of geodesic segments η_m, for $m \geq m_3$, lying entirely inside $\mathbb{R}^3 \setminus (M(T^{2\blacklozenge}_{1;m})^\circ \cup \overline{B_g(o, r_0)})$, starting at $T^{2\blacklozenge}_{1;m}$ and ending at a point p_m and with

$$d^{\mathbb{R}^3}_{g_{1;m}}(p_m, T^{2\blacklozenge}_{1;m}) = length_{g_{1;m}}(\eta_m),$$
$$length_{g_{1;m}}(\eta_m) \xrightarrow{m\to\infty} \infty.$$

About this sequence we make two crucial remarks: first, the geodesic η_m will lie entirely in the open set $\mathbb{R}^3 \setminus \overline{B_g(o, r_0)}$ where the Ricci curvature is non-negative; second, the mean curvature at the initial point of η_m in $T^{2\blacklozenge}_{1;m}$, and in the direction η_m' (which is opposite to the one used to define $\tilde{\theta}_1(x)$) is less than or equal to $-\tilde{\theta}_1(\bar{x})/2 < 0$. That $\tilde{\theta}_1(\bar{x})$ cannot be positive, contrary to what was assumed, will follow directly from these two remarks and Fact 1. We move then to prove the existence of such sequence.

Recall that *a ray* is an infinite-length geodesic diffeomorphic to $[0, \infty) = \mathbb{R}^+ \cup \{0\}$ minimizing the distance between any two of its points. Let $\mathfrak{R}_{r_0}$ be the set of rays ξ in $(\mathbb{R}^3, g)$ starting at a *base point* $b(\xi)$ in $\partial B_g(o, r_0)$ and lying entirely inside the closed set $\mathbb{R}^3 \setminus B_g(o, r_0)$. The family $\mathfrak{R}_{r_0}$ is easily seen to be non-empty and the union of the rays in $\mathfrak{R}_{r_0}$ to be a closed set in $\mathbb{R}^3$. Moreover, observe the following simple fact about $\mathfrak{R}_{r_0}$ to be used later. Fact 2: *Consider a sequence* γ_j *of geodesic segments lying entirely in* $\mathbb{R}^3 \setminus B_g(o, r_0)$, *having one of its end points in* $\partial B_g(o, r_0)$ *and minimizing the distance between its two extreme points. If* $length_g(\gamma_j) \to \infty$, *then there is a subsequence of* γ_j *converging (on compact sets of* $\mathbb{R}^3$*) to a ray in* $\mathfrak{R}_{r_0}$.

Let $\mathcal{P}_L$ be the set of points in the rays of $\mathfrak{R}_{r_0}$ lying at a g-distance L from the base point of the ray to which they belong—more precisely,

$$\mathcal{P}_L = \{p \in \xi \in \mathfrak{R}_{r_0} / d^{\mathbb{R}^3}_g(p, b(\xi)) = L\}.$$

Now, for every m there is $L_m > 0$ sufficiently big with the following properties[17] :

P1. $\mathcal{P}_{L_m} \subset \big(\mathbb{R}^3 \setminus M(T^{2\blacklozenge}_{1;m})\big)$ and

[17] If $M(T^{2\blacklozenge}_{1;m}) \subset B_{g_{k_{1;m}}}(o, \bar{L}_m)$, then take $L_m = m\bar{L}_m 10^{k_{1;m}}$.

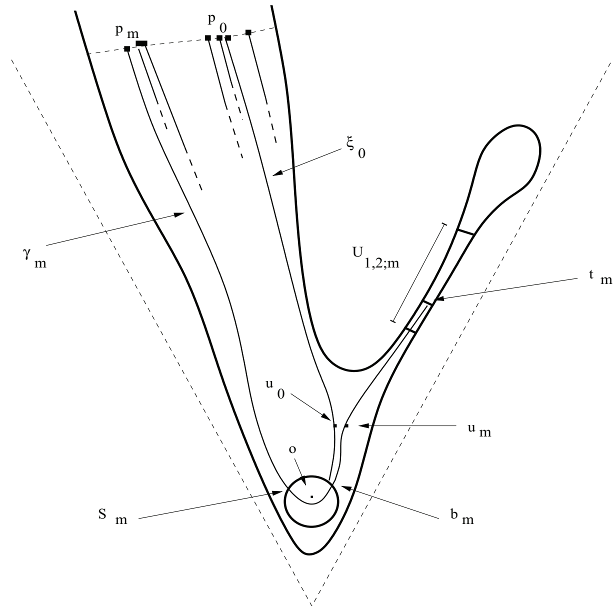

Figure 4. Representation of the construction in the proof of Step **B**. In terms of length, it more economic to go from p_0 to t_m using the path $p_0 \xrightarrow{\text{by}\xi_0} u_0 \xrightarrow{\text{by short curve}} u_m \xrightarrow{\text{by}\gamma_m} t_m$, rather than going from p_0 to t_m along γ_m.

P2. $d^{\mathbb{R}^3}_{g_{1;m}}(\mathcal{P}_{L_m}, T^{2\blacklozenge}_{1;m}) \geq m$ (note that the distance is with respect to $g_{1;m}$).

Let γ_m be a geodesic segment from a point t_m in $T^{2\blacklozenge}_{1;m}$ to a point p_m in $\mathcal{P}_{L_m}$ and realizing the $g_{1;m}$-distance between the closed sets $T^{2\blacklozenge}_{1;m}$ and $\mathcal{P}_{L_m}$ of $\mathbb{R}^3$. Because of **P1**, such segment must lie entirely in $\mathbb{R}^3 \setminus M(T^{2\blacklozenge}_{1;m})^\circ$, but it is a priori not evident that it will not intersect $B_g(o, r_0)$. We show now that there is $m_3 \geq m_2'$ such that for every $m \geq m_3$, $\gamma_m \cap \overline{B_g(o, r_0)} = \emptyset$. With this information and **P2**, we can conclude that $\eta_m := \gamma_m$ is the sequence we claimed for and the *claim* 2 will be finished. Suppose on the contrary that there is a subsequence (denoted again by γ_m) such that $\gamma_m \cap \overline{B_g(o, r_0)} \neq \emptyset$. In this case, $\gamma_m \cap \overline{B_g(o, r_0)}$, as a closed set in γ_m, has a point b_m nearest to t_m and a point s_m nearest to p_m. Let $\hat{\gamma}_m$ be the piece of γ_m enclosed between t_m and b_m. Obviously $\hat{\gamma}_m$ lies inside $\mathbb{R}^3 \setminus B_g(o, r_0)$. Therefore, as commented above, the sequence $\hat{\gamma}_m$ has a subsequence (denoted again by $\hat{\gamma}_m$) converging to a ray ξ_0 (on compact sets of $\mathbb{R}^3$). Let u_0 be a point in ξ_0 at a g-distance $4r_0$ from the base point $b(\xi_0)$ at $\partial B_g(o, r_0)$. Let u_m be a sequence of points in $\hat{\gamma}_m$ converging to u_0. Then for every $\epsilon > 0$ there is $m(\epsilon)$ such that for any

$m \geq m(\epsilon)$ we have

$$d_g^{\mathbb{R}^3}(u_0, u_m) \leq \epsilon, \text{ and } 4r_0 - \epsilon \leq d_g^{\mathbb{R}^3}(u_m, b_m) \leq 4r_0 + \epsilon.$$

Let p_0 be in ξ_0 at a g-distance L_m from $b(\xi_0)$, which, by definition, is a point in $\mathcal{P}_{L_m}$. Then, if $m \geq m(\epsilon)$, we can write

$$\begin{aligned} d_g^{\mathbb{R}^3}(\mathcal{P}_{L_m}, T^{2\blacklozenge}_{1;m}) &\leq d_g^{\mathbb{R}^3}(p_0, t_m) \leq d_g^{\mathbb{R}^3}(p_0, u_0) + d_g^{\mathbb{R}^3}(u_0, u_m) + d_g^{\mathbb{R}^3}(u_m, t_m) \\ &\leq L_m - 4r_0 + \epsilon + d_g^{\mathbb{R}^3}(u_m, t_m). \end{aligned}$$

On the other hand,

$$\begin{aligned} d_g^{\mathbb{R}^3}(\mathcal{P}_m, T^{2\blacklozenge}_{1;m}) &= d_g^{\mathbb{R}^3}(p_m, t_m) \geq d_g^{\mathbb{R}^3}(p_m, s_m) + d_g^{\mathbb{R}^3}(b_m, t_m) \\ &\geq L_m - 2r_0 + d_g^{\mathbb{R}^3}(b_m, t_m) \geq L_m - 2r_0 + d_g^{\mathbb{R}^3}(u_m, t_m), \end{aligned}$$

where we used that $d_g^{\mathbb{R}^3}(p_m, s_m) \geq L_m - 2r_0$, which is easily deduced from the fact that, because $p_m \in \mathcal{P}_{L_m}$, we have $d_g^{\mathbb{R}^3}(p_m, \partial B_g(o, r_0)) \leq L_m$. The two equations before lead readily to the inequality $2r_0 \leq \epsilon$, which is impossible if one choses for instance $\epsilon = r_0$. A representation of the construction can be seen in Figure 4. This finishes the proof of *Claim 2* and therefore of Step **B**. ◀

Proof of Step C. We work here with the subsequence of Step **B**, but to simplify notation still use the subindex m. On $U_{1,2;m}$ define the C^1 vector field $W_m = \nabla f_m / |\nabla f_m|^2$, and on $\tilde{U}_{1,2;m}$ define the lifted function $\tilde{f}_m = f_m \circ \pi_m$ and the lifted vector field $\tilde{W}_m = \nabla \tilde{f}_m / |\nabla \tilde{f}_m|^2$. W_m and $\tilde{W}_m$ define flows ψ_m and $\tilde{\psi}_m$ on $U_{1,2;m}$ and $\tilde{U}_{1,2;m}$, respectively. Because $df_m(W_m) = 1$ and $d\tilde{f}_m(\tilde{W}_m) = 1$, the flows ψ_m and $\tilde{\psi}_m$ take fibers into fibers; that is if $x_1,\ x_2 \in I$, then

$$\psi_m(x_2 - x_1, -) : f_m^{-1}(x_1) \to f_m^{-1}(x_2)$$

and

$$\tilde{\psi}_m(x_2 - x_1, -) : \tilde{f}_m^{-1}(x_1) \to \tilde{f}_m^{-1}(x_2).$$

Fix $x_0 \in I_{1,2}$. Let $\tilde{\chi}_m : \mathbb{T}^2 \to \tilde{f}_m^{-1}(x_0)$ be chosen (to be concrete) such that as $m \to \infty$ and as $\tilde{f}_m^{-1}(x_0) \xrightarrow{C^2} \mathbb{T}^2 \times \{x_0\}$, χ_m converges in C^2 to the "identity" diffeomorphism: $t \in \mathbb{T}^2 \to (t, x_0) \in \mathbb{T}^2 \times I_{1,2}$. With the help of $\tilde{\chi}_m$ and $\tilde{\psi}_m$, one can define C^2-diffeomorphisms

$$\tilde{\varphi}_m : \mathbb{T}^2 \times I_{1,2} \to \tilde{U}_{1,2;m}, \text{ as } \tilde{\varphi}_m(t, x) = \tilde{\psi}_m(x - x_0, \tilde{\chi}_m(t)),$$

for which we have $\tilde{\varphi}_m(\mathbb{T}^2 \times \{x\}) = \tilde{f}_m^{-1}(x)$ and $d\tilde{\varphi}_m(\partial_x) = \tilde{W}_m$. Moreover, as $\tilde{W}_m$ is perpendicular to the fibers, we have the following form of the pull-back metric:

$$\tilde{\varphi}_m^* \tilde{g}_{1;m} = \alpha_m^2 dx^2 + \tilde{h}_{1;m}(x),$$

where α_m and $\tilde{h}_{1;m}(x)$ (may be different from those in Step **B**, but we name them the same) converge in C^1 to the function identically one and $\tilde{h}_0$, respectively. We inspect now the behavior of the length of curves on fibers when we translate them along ∂_x. Let $\tilde{\mathscr{C}}_{x_1}$ be a curve on $\mathbb{T}^2 \times \{x_1\}$, and let $\tilde{\mathscr{C}}_x$ be the transported of $\tilde{\mathscr{C}}_{x_1}$ by ∂_x to $\mathbb{T}^2 \times \{x\}$. Then, as $\partial_x \tilde{h}_m = 2\alpha_{1,m}\tilde{\Theta}_{1;m}$, we obtain the direct estimate

$$|\partial_x length_{\tilde{h}_{1;m}(x)}(\tilde{\mathscr{C}}_x)| \leq \frac{\big(\sup_{\mathbb{T}^2\times\{x\}} |\tilde{\Theta}_{1;m}|\big)}{2} length_{\tilde{h}_{1;m}(x)}(\tilde{\mathscr{C}}_x). \tag{14}$$

However, because of Step **B**, $|\tilde{\Theta}_{1;m}|_{\tilde{h}_{1;m}} \xrightarrow{m\to\infty} 0$ (uniformly on $\mathbb{T}^2 \times I_{1,2}$) and we deduce that *for every* $1 > \nu > 0$ *there is* $m(\nu)$ *such that if* $m \geq m(\nu)$ *and* $x_1, x_2 \in I_{1,2}$*, then*

$$\begin{aligned}(1-\nu)length_{\tilde{h}_{1;m}(x_1)}(\tilde{\mathscr{C}}_{x_1}) &\leq length_{\tilde{h}_{1;m}(x_2)}(\tilde{\mathscr{C}}_{x_2}) \\ &\leq (1+\nu)length_{\tilde{h}_{1;m}(x_1)}(\tilde{\mathscr{C}}_{x_1}).\end{aligned} \tag{15}$$

Now, from (15) and noting that the result of transporting a curve $\mathscr{C}_{x_1} \subset f_m^{-1}(x_1)$ (closed or not) by W_m to a curve $\mathscr{C}_{x_2} \subset f_m^{-1}(x_2)$ is the same as the result of lifting $\mathscr{C}_{x_1}$ to an (equal length) curve $\tilde{\mathscr{C}}_{x_1} \subset \mathbb{T}^2 \times \{x_1\}$ by means of $\pi_m \circ \tilde{\varphi}_m$, transporting it by ∂_x to a curve $\tilde{\mathscr{C}}_{x_2}$, and then pushing it down to an (equal length) curve $\mathscr{C}_{x_2} \subset f_m^{-1}(x_2)$, we deduce that if $m \geq m(\nu)$ and $x_1, x_2 \in I_{1,2}$, then

$$\begin{aligned}(1-\nu)length_{h_{1;m}(x_1)}(\mathscr{C}_{x_1}) &\leq length_{h_{1;m}(x_2)}(\mathscr{C}_{x_2}) \\ &\leq (1+\nu)length_{h_{1;m}(x_1)}(\mathscr{C}_{x_1}).\end{aligned} \tag{16}$$

We are ready to prove that there is m_2 such that if $m \geq m_2$ then $\bar{\mathbf{C}}\mathbf{1}$ and $\bar{\mathbf{C}}\mathbf{2}$ holds. We prove first $\bar{\mathbf{C}}\mathbf{1}$ and then $\bar{\mathbf{C}}\mathbf{2}$.

• First, since the $h_{1;m}(x)$-diameters of the fibers $f_m^{-1}(x)$, here denoted by $\Gamma_{1;m}(x)$, are realized by the length of geodesic segments (inside the fiber), then we obtain from (16)

$$1-\nu \leq \frac{\Gamma_{1;m}(x_1)}{\Gamma_{1;m}(x_2)} \leq 1+\nu, \tag{17}$$

for any $x_1, x_2 \in I_{1,2}$ and $m \geq m(\nu)$. Second, in exactly the same way that we proved (6) in the example of Section 2.4.1, one can prove the following statement: *Given* Λ_1 *there are* ν_0 *and* Γ_0 *such that for any Riemannian manifold* (V, g_V) *with* $|Ric_{g_V}| \leq \Lambda_1$ *and with a* $\mathbb{T}^2$*-fibration* $f_V : V \to I_V$ $(|I_V| \geq 1)$ *for which*

$$1-\nu_0 \leq \frac{\Gamma_V(x_1)}{\Gamma_V(x_2)} \leq 1+\nu_0,\ x_1, x_2 \in I_V, \text{ and } \sup_{x\in I_V} \Gamma_V(x) \leq \Gamma_0,$$

where $\Gamma_V(x) = diam(f_V^{-1}(x))$*, we have*

$$\frac{1}{6}\inf_{x\in I_V} \Gamma_V(x) \leq dist_{GH}(V, I_V) \leq \frac{2}{3}\sup_{x\in I_V} \Gamma_V(x). \tag{18}$$

Now, take $\Lambda_1 = 100\Lambda_0$, where Λ_0 is the coefficient that we assumed in the quadratic curvature decay of g—that is, in $|Ric_g| \leq \Lambda_0/r^2$. Let $\nu_0 = \nu_0(\Lambda_1)$ and $\Gamma_0 = \Gamma_0(\Lambda_1)$. Choose $\nu \leq \min\{1/4, \nu_0\}$ and $m_2 \geq m(\nu)$ (as defined above) and sufficiently big that for any $m \geq m_2$ we have $\sup_{x\in I_{1,2}} \Gamma_{1;m}(x) \leq \Gamma_0$. If as in $\bar{\mathbf{H}}\mathbf{1}$, $(U_{1;m}, g_{1;m})$ is ϵ^*-close in the GH-metric to $(I_1, |\ |)$, then by the first inequality of (18) (applied[18] to $V = U_{1;m}$ and $g_V = g_{1;m}$) and by (17), we have

$$\sup_{x\in I_2} \Gamma_{1;m}(x) \leq \frac{6}{1-\nu}\epsilon^*. \tag{19}$$

Hence by (19) and the second inequality of (18) (applied[19] to $V = U_{2;m}$ and $g_V = g_{2;m}$), and recalling that $g_{2;m} = \frac{1}{10^2} g_{1;m}$ (implying $\Gamma_{2;m}(x) = \Gamma_{1;m}(x)/10$), we obtain

$$dist_{GH}((U_{2;m}, g_{2;m}), (I_2, |\ |)) \leq \frac{2}{3}\frac{1}{10}\frac{6}{(1-\nu)}\epsilon^* \leq \frac{2}{3}\epsilon^*,$$

where the last inequality is because $\nu \leq 1/4$. This shows that $\bar{\mathbf{C}}\mathbf{1}$ holds.

• Suppose, as in $\bar{\mathbf{H}}\mathbf{2}$, that there is a closed $\mathscr{C}_{1;m} \in T^{2\blacklozenge}_{1;m}$ for which it is $length_{g_{1;m}}(\mathscr{C}_{1;m}) \leq \ell^*_m$. Let x_1 be the left point of the interval I_1, and let x_2 be the left point of the interval I_2. Then the curve $\mathscr{C}_{1;m}$ belongs to the fiber $f_m^{-1}(x_1)$. Let $\mathscr{C}_{2;m}$ be the transport of $\mathscr{C}_{1;m}$ by W_m to $f_m^{-1}(x_2) = T^{2\blacklozenge}_{2;m}$. By (15) we have

$$length_{g_{2;m}}(\mathscr{C}_{2;m}) = \frac{1}{10} length_{g_{1;m}}(\mathscr{C}_{2;m}) \leq \frac{4}{3}\frac{1}{10} length_{g_{1;m}}(\mathscr{C}_{1;m}) \leq \frac{2}{3}\ell^*.$$

This shows that $\bar{\mathbf{C}}\mathbf{2}$ holds. ◀

q.e.d.

We are ready to prove Theorem 1.1.

Proof of Theorem 1.1. We will work with the annuli decomposition of Section 2.5. The key to the proof of Theorem 1.1 is to show that if $i \geq i_0$, for some $i_0 \geq 0$, then the manifold $M(T^{2o}_{i+1}, T^{2o}_i)$ is a IIB-manifold. Once this is shown, the proof of Theorem 1.1 is as follows. Let $\mathbb{S}^2_{\bar{r}} = \partial\mathbb{B}^3(o, \bar{r})$ be the "coordinate sphere" of radius $\bar{r}$ in $\mathbb{R}^3$, and let $\bar{r}$ be large enough that $\mathbb{S}^2_{\bar{r}} \subset \mathbb{R}^3 \setminus M(T^{2o}_{i_0})$. As[20]

$$\mathbb{R}^3 \setminus M(T^{2o}_{i_0})^\circ = \bigcup_{i=i_0}^{\infty} M(T^{2o}_{i+1}, T^{2o}_i), \tag{20}$$

[18]Note that $|Ric_{g_{k_{1;m}}}| \leq 100\Lambda_0$ on $U_{1;m}$.

[19]Note that $|Ric_{g_{k_{2;m}}}| \leq 100\Lambda_0$ on $U_{2;m}$.

[20]Note from the properties of annuli decompositions that for any sequence T^2_j of pairwise different tori in $\mathcal{N}$ we have $d^{\mathbb{R}^3}_g(o, T^2_j) \to \infty$. In particular, $d^{\mathbb{R}^3}_g(o, T^{2o}_i) \to \infty$ as $i \to \infty$. This justifies equation (20).

then we have

$$\mathbb{S}^2_{\bar r} \subset \bigcup_{i=i_0}^{i=i_1} M(T^{2o}_{i+1}, T^{2o}_i), \tag{21}$$

for some $i_1 > i_0 > 0$. By Proposition 2.1, the right-hand side of (21) is a IIB-manifold if every one of its summands is a IIB-manifold. Therefore, $\mathbb{S}^2_{\bar r}$ bounds a ball in $\cup_{i=i_0}^{i=i_1} M(T^{2o}_{i+1}, T^{2o}_i)$ and so bounds a ball in $\mathbb{R}^3 \setminus \{o\}$ because $\mathbb{R}^3 \setminus \{o\}$ contains $\cup_{i=i_0}^{i=i_1} M(T^{2o}_{i+1}, T^{2o}_i)$. But $\mathbb{S}^2_{\bar r}$ does not bound a ball in $\mathbb{R}^3 \setminus \{o\}$, and we reach a contradiction.

We move then to prove that there is $i_0 \geq 0$ such that for any $i \geq i_0$, $M(T^{2o}_{i+1}, T^{2o}_i)$ is a IIB-manifold.

Define i_0 such that, for every piece $U_{k,l} \subset \mathbb{R}^3 \setminus M(T^{2o}_{i_0})^\circ$, we have ($\epsilon^*, \ell^*, k^*$ below are as in Proposition 3.1)

1) $k \geq k^*$,
2) $(U_{k,l}, g_k)$ is either ϵ^*-close in the GH-metric to either an interval or a two-orbifold, and
3) if $(U_{k,l}, g_k)$ is ϵ^*-close to a two-orbifold then the g_k-length of the fibers $\mathscr{C}$ of the Seifert structure is less than or equal to ℓ^*, i.e. $length_{g_k}(\mathscr{C}) \leq \ell^*$.

We will use such an i_0 from now on and show that if $i \geq i_0$, then $M(T^{2o}_{i+1}, T^{2o}_i)$ is a IIB-manifold. Some notation now. If a piece $U_{k,l}$ in $\mathbb{R}^3 \setminus M(T^{2o}_{i_0})^\circ$ is ϵ^*-close to an interval, then we say that the piece is of type $I(\epsilon^*)$, and if it is not and therefore is ϵ^*-close to a two-orbifold, then we say that the piece is of type $II(\epsilon^*)$.

Let $i \geq i_0$:

• If $M(T^{2o}_{i+1}, T^{2o}_i)$ does not contain a piece of type $I(\epsilon^*)$, then the manifold $M(T^{2o}_{i+1}, T^{2o}_i)$ is a union of Seifert manifolds (with Seifert structures coinciding at any intersection) and therefore a Seifert manifold with two boundary components, T^{2o}_{i+1} and T^{2o}_i. It follows that in this case $M(T^{2o}_{i+1}, T^{2o}_i)$ is a IIB-manifold.

• If $M(T^{2o}_{i+1}, T^{2o}_i)$ contains a piece of type $I(\epsilon^*)$, then we can distinguish two cases, as follows.

(i) $M(T^{2o}_{i+1}, T^{2o}_i)$ is itself a piece of type $I(\epsilon^*)$ (in this case, the only $U_{k,l}$-piece) and therefore diffeomorphic to $\mathbb{T}^2 \times I$ and thus a IIB-manifold, or

(ii) $M(T^{2o}_{i+1}, T^{2o}_i)$ is not a piece of type $I(\epsilon^*)$, in which case the only piece $U_{k,l} \subset M(T^{2o}_{i+1}, T^{2o}_i)$ having T^{2o}_{i+1} and T^{2o}_i as boundary components (see Section 2.3) is of type $II(\epsilon^*)$ and has at least a third boundary component.

We discuss now case (ii) and show that $M(T^{2o}_{i+1}, T^{2o}_i)$ is also in this case a IIB-manifold. Denote by $\mathscr{U}_{i+1,i}$ to the union of all the pieces $U_{k,l}$ in $M(T^{2o}_{i+1}, T^{2o}_i)$ of type $II(\epsilon^*)$, and by $\widehat{\mathscr{U}_{i+1,i}}$ to the only connected component of $\mathscr{U}_{i+1,i}$ containing T^{2o}_{i+1} and T^{2o}_i. The $\widehat{\mathscr{U}_{i+1,i}}$ is a union of Seifert manifolds (with Seifert structure coinciding at any intersection) and therefore a Seifert manifold itself. Let $\widehat{\mathscr{N}}_{i+1,i}$ be the set of boundary components of $\widehat{\mathscr{U}_{i+1,i}}$ other than T^{2o}_{i+1} and T^{2o}_i, and observe that any torus T^2 in $\widehat{\mathscr{N}}_{i+1,i}$ is $\ll$ than T^{2o}_{i+1} but is not related in the order $\ll$ to T^{2o}_i (otherwise it would be one of the T^{2o}_i's). Now, the tori T^2 in $\widehat{\mathscr{N}}_{i+1,i}$ are either of type $T^{2\blacklozenge}$ or of type $T^{2\lozenge}$—namely, either $M(T^2)$ is a solid torus or not (see beginning of Sec. 3). A $T^{2\blacklozenge}$ in $\widehat{\mathscr{N}}_{i+1,i}$ is the boundary of a $U_{k,l}$-piece of type $II(\epsilon^*)$, and, because $i \geq i_0$ and the definition of i_0, the fibers $\{\mathscr{C}\}$ of the Seifert structure of such piece have $g_{k(T^{2\blacklozenge})}$-length less than or equal to ℓ^*. In particular the fibers $\{\mathscr{C}\}$ on $T^{2\blacklozenge}$ (which as closed curves are non-contractible in $T^{2\blacklozenge}$) have $g_{k(T^{2\blacklozenge})}$-length less than or equal to ℓ^*. Summarizing, we would have $k(T^{2\blacklozenge}) \geq k^*$ (because $i \geq i_0$) and:

H1'. $(U_{k(T^{2\blacklozenge}),l(T^{2\blacklozenge})}, g_{k(T^{2\blacklozenge})})$ is ϵ^*-close in the GH-metric to an interval.

H2'. There is a curve $\mathscr{C} \subset T^{2\blacklozenge}$ (indeed, anyone of the $\{\mathscr{C}\}$'s) non-contractible in $T^{2\blacklozenge}$ such that $length_{g_{k(T^{2\blacklozenge})}}(\mathscr{C}) \leq \ell^*$.

Therefore, and crucially, if the fibers $\{\mathscr{C}\}$ in $T^{2\blacklozenge}$ are contractible inside $M(T^{2\blacklozenge})$, then applying Proposition 3.1 iteratively we would obtain a consecutive sequence of pieces of type $I(\epsilon^*)$ extending to infinity, i.e., a $\mathbb{T}^2 \times \mathbb{R}^+$-end, which is not possible because then $M(T^{2\blacklozenge})$ would not be compact[21] . We conclude that for every $T^{2\blacklozenge}$ in $\widehat{\mathscr{N}}_{i+1,i}$ the fibers $\{\mathscr{C}\}$ are non-contractible inside $M(T^{2\blacklozenge})$. This implies, as was comment at the end of Section 2.2, that the Seifert structure of $\widehat{\mathscr{U}_{i+1,i}}$ can be extended to every $M(T^{2\blacklozenge})$. Hence the manifold

$$\widehat{\mathscr{U}_{i+1,i}} \bigcup \left[\bigcup_{T^{2\blacklozenge} \in \widehat{\mathscr{N}}_{i+1,i}} M(T^{2\blacklozenge}) \right]$$

is a Seifert manifold, has at least two boundary components, and is thus a IIB-manifold. Finally, as was explained in Section 2.2, every manifold $M(T^{2\lozenge})$, with $T^{2\lozenge} \in \widehat{\mathscr{N}}_{i+1,i}$, is IIB. Therefore,

$$M(T^{2o}_{i+1}, T^{2o}_i) = \widehat{\mathscr{U}_{i+1,i}} \bigcup \left[\bigcup_{T^{2\blacklozenge} \in \widehat{\mathscr{N}}_{i+1,i}} M(T^{2\blacklozenge}) \right] \bigcup \left[\bigcup_{T^{2\lozenge} \in \widehat{\mathscr{N}}_{i+1,i}} M(T^{2\lozenge}) \right]$$

[21] Alternatively, it would imply the existence of an embedded torus (a section of such end) dividing $\mathbb{R}^3$ into two unbounded connected components, which is not possible.

is a IIB-manifold by Proposition 2.1. This finishes the proof of Theorem 1.1. q.e.d.

4. Appendix

4.1. Remarks on manifolds and convergence. A three-manifold M is $C^{k+1,\beta}$, $k \geq 1$, $0 < \beta < 1$ if it is a topological manifold provided with an atlas with transition functions in $C^{k+1,\beta}$. A *Riemannian* three-manifold (M,g) is $C^{k,\beta}$ if M is $C^{k+1,\beta}$ and the entries of g in every coordinate system of the $C^{k+1,\beta}$ atlas of M are $C^{k,\beta}$ functions.

A sequence of $C^{k,\beta}$ Riemannian manifolds (M_i, g_i) converges in $C^{k,\beta}$ to a $C^{k,\beta}$ Riemannian manifold (M,g) if there are $C^{k+1,\beta}$-diffeomoprhisms $\varphi_i : M \to M_i$ such that the entries of $\varphi_i^* g_i$ in every coordinate system of the atlas of M converge in $C^{1,\beta}$ to the entries of g in the coordinate system.

There are norms that we will use that do not depend on the coordinates. In particular, on a $C^{k,\beta}$ Riemannian manifold (M,g) one can define the $C_g^{k'+1}$-norm, $k' \leq k$, of functions as usual as

$$\|f\|_{C_g^{k'+1}} = \sup_{x \in M} \left(\sum_{j=0}^{j=k'+1} |\nabla^{(j)} f|(x) \right),$$

where $\nabla^{(j)}$ is the operator resulting from applying ∇ j-times. Note that $\nabla^{(j)} f = \nabla^{(j-1)} df$ and that the $C_g^{k'+1}$ norm of f involves only derivatives of g up to order k'. In particular the space C_g^2 is well defined on a $C^{1,\beta}$ Riemannian manifold. Moreover, one easily has the following property: If (M_i, g_i) converges in $C^{1,\beta}$ to (M,g) (via diffeomorphisms φ_i) and f_i is a sequence of functions in M_i, then there is i_0 such that for any $i \geq i_0$ we have $\|\varphi_i^* f_i\|_{C_g^2} \leq 2\|f_i\|_{C_{g_i}^2}$ (here, $\varphi_i^* f_i = f_i \circ \varphi_i$).

4.2. Some technical propositions. The following theorem would be standard if we were working in the smooth category. With low regularity there are some points to check.

Proposition 4.1. *Let (M,g) be a compact $C^{1,\beta}$-Riemannian manifold with boundary. Suppose that $\phi : \mathbb{T}^2 \times M \to M$ is a continuous and free action by isometries. Then there exists a $C^{2,\beta}$-diffeomorphism $\varphi : M \to \mathbb{T}^2 \times I$ such that $\varphi^* g = dx^2 + h(x)$ where $h(x)$ is a $C^{1,\beta}$-path of $\mathbb{T}^2$-symmetric, and therefore flat metrics in $\mathbb{T}^2$.*

Proof. By [**21**, Theorem 6], the set of orbits $\mathbb{T}^2(p) = \{\phi(t,p), t \in \mathbb{T}^2\}$, $p \in M$, is a foliation of M by C^1-embedded tori. Let $\mathbb{T}_1^2 \neq \mathbb{T}_2^2$ be two leaves, and let γ_{12} be a geodesic segment realizing the distance between them and therefore perpendicular to them. As the action is by isometries, the set $\{\phi(t, \gamma_{12}), t \in \mathbb{T}^2\}$ is a foliation of the region enclosed by $\mathbb{T}^2_1$ and $\mathbb{T}^2_2$ by geodesic segments realizing the distance between $\mathbb{T}^2_1$ and $\mathbb{T}^2_2$ and perpendicular to them. As in this argumentation the leaves $\mathbb{T}^2_1$ and $\mathbb{T}^2_2$ are arbitrary, it follows that any inextensible geodesic perpendicular to one leaf is also perpendicular to any other leaf. Let $\gamma(x)$, x the arc-length, be one of such geodesics. Define $\varphi : \mathbb{T}^2 \times I \to M$, $|I| = length(\gamma)$, as $\varphi(t, x) = \phi(t, \gamma(x))$. By [**21**, See (D) on p. 402 in particular] the map φ is a C^1 diffeomorphism. We have $\varphi^* g = dx^2 + h(x)$, where $h(x)$ is a C^0-path of $\mathbb{T}^2$-symmetric metrics in $\mathbb{T}^2$. Let (y, z) be (local) flat coordinates on $\mathbb{T}^2$ that together with x form (local and C^1) coordinates. The standard Laplacian acting on certain functions f at least can be computed in the coordinates (x, y, z) as $\Delta f = [\det h]^{-1/2}(\partial_i(g^{ij}[\det h]^{\frac{1}{2}}\ \partial_j f))$ (because $\det h$ is just C^0). Such is the case[22] when $f = x, y$ or $\int^x [\det h]^{-1/2} dx$. As $\det h = \det h\,(x)$, the coordinates y and z are harmonic (and C^1) and therefore from standard elliptic regularity also $C^{2,\beta}$ in M (recall for this that M is $C^{2,\beta}$ and g is $C^{1,\beta}$). It remains to see the regularity of x. Define a new coordinate by $\bar{x} = \int^x [\det h]^{-1/2} dx$. Then $\bar{x}$ is harmonic and because it is C^1, by standard elliptic regularity again, it is $C^{2,\beta}$ in M. Therefore, $(\bar{x}, y, z)$ is a harmonic and $C^{2,\beta}$ coordinate system. Hence in these coordinates the metric coefficients g_{ij} are of class $C^{1,\beta}$. Thus $[\det h]^{1/2}$ is of class $C^{1,\beta}$, and because $x(\bar{x}) = \int^{\bar{x}} [\det h]^{1/2} d\bar{x}$, we deduce that x is also $C^{2,\beta}$ in M. q.e.d.

Proposition 4.2. *Suppose that a sequence (U_m, g_m) with $|Ric_{g_m}| \leq \Lambda_0$ collapses metrically to $(I, |\ |)$. Then there is a covering subsequence $(\tilde{U}_{m_j}, \tilde{g}_{m_j})$ (with covering maps π_{m_j}) converging in $C^{1,\beta}$ to a $\mathbb{T}^2$-symmetric space $(\mathbb{T}^2 \times I, dx^2 + \tilde{h}(x))$, and there is a sequence of functions $f_{m_j} : U_{m_j} \to \mathbb{R}$, such that $f_{m_j} \circ \pi_{m_j} : \mathbb{T}^2 \times I \to \mathbb{R}$ converges in C^2 to the coordinate function x. In particular, fixed a value of x, $\pi^{-1}_{m_j}(f^{-1}_{m_j}(x))$ converges in C^2 to the slice $\mathbb{T}^2 \times \{x\}$.*

Proof. The first part of the claim—i.e., the existence of the covering subsequence—is known to us from Lemma 2.1. Thus assume that

$$(\tilde{U}_{m_j}, \tilde{g}_{m_j}) \xrightarrow{C^{1,\beta}} (\mathbb{T}^2 \times I, \tilde{g} = dx^2 + \tilde{h}(x)).$$

Following [**5**] (see also [**12**, p. 336]), for every $\epsilon > 0$ there are[23] smoothings $g^{\epsilon}_{m_j}$ of g_{m_j} such that

[22] To justify the Δf in these cases, multiply by a smooth and arbitrary test function of compact support and integrate by parts.

[23] We remark that this useful smoothing procedure has been used recurrently in [**10**] as it greatly simplifies the arguments. Our use does not differ much from the purposes it was used there.

$$dist_{Lip}(g_{m_j}, g^{\epsilon}_{m_j}) \leq \epsilon,\ |Ric_{g^{\epsilon}_{m_j}}| \leq 2\Lambda_0,\ \text{and} \qquad (22)$$
$$|\nabla^{(k)}_{g^{\epsilon}_{m_j}} Ric_{g^{\epsilon}_{m_j}}| \leq \Lambda_k(\epsilon), k \geq 1,$$

where $dist_{Lip}$ is the Lipschitz distance (see [**13**, **12**])[24] . Moreover, we have the following two properties for fixed ϵ.

E1. There is a subsequence of $(\tilde{U}_{m_j}, \tilde{g}^{\epsilon}_{m_j})$ (indexed with m_j again but depending on ϵ) converging in C^{∞} and via diffeomorphisms χ_j to $(\mathbb{T}^2 \times I, \tilde{g}^{\epsilon} = dx^2 + \tilde{h}^{\epsilon}(x))$. Hence, as discussed in Section 4.1, there is $j_0(\epsilon)$ such that for every $j \geq j_0(\epsilon)$ and sequence of functions F_j on $\tilde{U}_{m_j}$ we have $\|\chi^*_j F_j\|_{C^2_{\tilde{g}^{\epsilon}}} \leq 2\|F_j\|_{C^2_{\tilde{g}^{\epsilon}_{m_j}}}$.

E2. From Lemma 1.6 [**11**, p. 336], there are fibrations $f^{\epsilon}_{m_j} : U_{m_j} \to I$ such that, for all $k \geq 1$, $\|f^{\epsilon}_{m_j} \circ \pi_{m_j}\|_{C^k_{\tilde{g}^{\epsilon}_{m_j}}} \leq C'_k(\epsilon)$. Moreover, $f^{\epsilon}_{m_j} \circ \pi_{m_j}$ converges in C^1 to the function x in $(\mathbb{T}^2 \times I, dx^2 + \tilde{h}^{\epsilon}(x))$, and, because of the estimates before, the convergence is also in C^{∞}. In particular, $\lim \|\chi^*_j(\pi_{m_j} \circ f_{m_j}) - x\|_{C^2_{\tilde{g}^{\epsilon}}} \xrightarrow{j \to \infty} 0$

And if we make $\epsilon \to 0$, we have, because of the first two terms of (22), the following property.

E3. As $\epsilon \to 0$, the spaces $(\mathbb{T}^2 \times I, dx^2 + \tilde{h}^{\epsilon}(x))$ converge in $C^{1,\beta'}$ ($\beta' < \beta$) and via diffeomorphisms φ_{ϵ} to $(\mathbb{T}^2 \times I, dx^2 + \tilde{h}(x))$. Moreover, by Proposition 4.1 the C^2-coordinates x in them converge in C^2 to the (by Proposition 4.1) C^2-coordinate x in the limit space.

From **E1** and **E3** we immediately obtain that, for every $\epsilon(i) = 1/i$, $i = 1, 2, 3, \ldots$ one can find $m_{j(i)}$ with $j(i) \geq j_0(\epsilon(i))$, in such a way that the subsequence $(\tilde{U}_{m_{j(i)}}, \tilde{g}^{\epsilon(i)}_{m_{j(i)}})$ converges in $C^{1,\beta'}$ and via the diffeomorphisms $\chi_{j(i)} \circ \varphi_{\epsilon(i)}$ to $(\mathbb{T}^2 \times I, dx^2 + \tilde{h}(x))$. Then we have

$$\begin{aligned}
&\|\varphi^*_{\epsilon(i)}\chi^*_{j(i)}(\pi_{m_{j(i)}} \circ f^{\epsilon(i)}_{m_{j(i)}}) - x\|_{C^2_{\tilde{g}}} \leq \\
&\qquad \leq \|\varphi^*_{\epsilon(i)}\chi^*_{j(i)}(\pi_{m_{j(i)}} \circ f^{\epsilon(i)}_{m_{j(i)}}) - \varphi^*_{\epsilon(i)}x + \varphi^*_{\epsilon(i)}x - x\|_{C^2_{\tilde{g}}} \\
&\qquad \leq 2\|\chi^*_{j(i)}(\pi_{m_{j(i)}} \circ f^{\epsilon(i)}_{m_{j(i)}}) - x\|_{C^2_{\tilde{g}^{\epsilon(i)}_{m_{j(i)}}}} + \|\varphi^*_{\epsilon(i)}x - x\|_{C^2_{\tilde{g}}},
\end{aligned}$$

where the last term tends to zero as $i \to \infty$. q.e.d.

Proof of Lemma 2.1. The result is a straightforward consequence of the assumption that $(M_i, g_i) \in \mathcal{M}(\mathfrak{N}_0)$ for some fixed $\mathfrak{N}_0$ and the results in [**10**]. There are, however, some technical points that are better

[24]Note that what makes these estimates useful is that they are independent from the injectivity radius.

to clarify, and these have to do with the fact that several metrics are involved at the same time. Fukaya's proofs of course will not be repeated here, and we refer the reader to his articles for full information.

For every "i" let $\breve{M}_i^\circ(p_i)$ be the connected component of M_i° containing p_i, and let $d_i = d_{g_i}^{M_i}$. We let $M_i^{\underline{\epsilon}}(p_i) := \breve{M}_i^\circ(p_i) \setminus \mathcal{T}_{d_i}(\partial M_i, \underline{\epsilon})$ and similarly for $M_i^{\overline{\epsilon}}(p_i)$. From Proposition 2.2 we can take a subsequence (index again by "i") such that $(M_i^{\underline{\epsilon}}(p_i), d_i)$ converges to a compact metric space $(X^{\underline{\epsilon}}, d^{\underline{\epsilon}})$. The subsequence can be chosen in such a way that $M_i^{\overline{\epsilon}}(p_i)$ converges (as a compact set) to $X^{\overline{\epsilon}} \subset X^{\underline{\epsilon}}$. We keep using this sequence in the following.

Following [**10**] [25] , for every $x \in X^{\overline{\epsilon}}$ there is $\delta(x) \leq (\overline{\epsilon} - \underline{\epsilon})/2$ such that $(B_{d^{\underline{\epsilon}}}(x, \delta(x)), d^{\underline{\epsilon}})$ is locally isometric to a model space **I.a**, **I.b**, **II.a**, or **II.b**. Consider then in $B_{d^{\underline{\epsilon}}}(x, \delta(x))$ the corresponding Riemannian metric and denote it by $g^{\underline{\epsilon}}$. In addition to this information, there is a sequence of points $q_i \in M_i^{\overline{\epsilon}}(p_i)$ with $q_i \to x$, such that $(\overline{B_{g_i}(q_i, \delta(x))}, g_i)$ converges in the GH-topology to $(\overline{B_{d^{\underline{\epsilon}}}(x, \delta(x))}, g^{\underline{\epsilon}})$.

Now, using the compactness of $X^{\overline{\epsilon}}$, one can pick points $x_1, \ldots, x_J$ in $X^{\overline{\epsilon}}$ such that the balls $B_{d^{\underline{\epsilon}}}(x_j, \delta(x_j)/4)$, $j = 1, \ldots, J$, cover $X^{\overline{\epsilon}}$. Assume that p_i converges to a point x_0, that points $p_{j,i}$ converge to x_j, and that the union $\bigcup_{j=1}^{j=J} B_{d^{\underline{\epsilon}}}(x_j, \delta(x_j)/4)$ is connected (otherwise, take the connected component of the union containing x_0). Then it is direct to check that

$$(\bigcup_{j=1}^{j=J} \overline{B_{g_i}(p_{j,i}, \delta(x_j))}, g_i) \xrightarrow{\text{GH}} (\bigcup_{j=1}^{j=J} \overline{B_{d^{\underline{\epsilon}}}(x_j, \delta(x_j))}, g^{\underline{\epsilon}}).$$

Then one can use the local construction in [**10**, p. 9] (which is based on [**9**]) to find C^1 functions

$$f_i : \bigcup_{j=1}^{j=J} \overline{B_{g_i}(p_{j,i}, 3\delta(x_j)/4)} \to X^{\underline{\epsilon}},$$

(but non-surjective) satisfying the properties in Theorem 0.12 of [**10**] and with range covering $\bigcup_{j=1}^{j=J} \overline{B_{d^{\underline{\epsilon}}}(x_j, \delta(x_j)/2)}$. The Ω_i's and the space (X, d) are finally defined as

$$\Omega_i := f_i^{-1}(\bigcup_{j=1}^{j=J} \overline{B_{d^{\underline{\epsilon}}}(x_j, \delta(x_j)/2)}),$$

$$(X, d) := (\bigcup_{j=1}^{j=J} \overline{B_{d^{\underline{\epsilon}}}(x_j, \delta(x_j)/2)}, g^{\underline{\epsilon}}),$$

[25]Recall the discussion after the statement of Lemma 2.1.

with (X, d) satisfying **D1** and **D2** by construction. We then have $(\Omega_i, g_i) \xrightarrow{\text{GH}} (X, d)$ and $f_i : (\Omega_i, g_i) \to (X, d)$ with the properties **I1** which correspond in our case to properties (0.13.1) and (0.13.2) of Theorem 0.12 of [**10**].

We discuss now how to show **I2** in *case* **D1**. The *case* **D2** is done along similar lines. and as we will not use it in this article, the proof is left to the reader. Take covers $(\tilde{\Omega}_i, \tilde{g}_i)$ to have the injectivity radius at one point controlled away from zero. Leave aside for a moment the issue of the existence of such cover. As $|Ric_{\tilde{g}_i}| \leq \Lambda_0$, we can take a convergent subsequence, say, to $(\tilde{\Omega}, \tilde{g})$. The group of Deck-covering transformations of $\tilde{\Omega}_i$ converge necessarily to a closed group G of isometries of the limit space $(\tilde{\Omega}, \tilde{g})$. On the other hand, for any $x \in X \setminus Sing(X)$, the fiber $\pi_i^{-1}(f_i^{-1}(x))$ that covers the torus $f_i^{-1}(x)$ under π_i converges to a torus, say, $\tilde{T}^2(x) \subset \tilde{\Omega}$. The group G acts effectively[26] by isometries on $\tilde{T}^2(x)$ and its quotient is a point. It follows that G is a torus.

To show that there are covers as mentioned before, observe that, from Lemma 2.1, any "sufficiently collapsed" manifold (of bounded diameter and curvature) must possess at least one small and non-contractible loop. Now, in *case* **D1**, the manifolds Ω_i are diffeomorphic to either $\mathbb{T}^2 \times \mathbb{I}$ or $\mathbb{B}^2 \times \mathbb{S}^1$, whose fundamental groups are $\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Z}$, respectively. In either case, one can then take (controlled) covers having no non-contractible and small loops. In this way, the cover is necessarily non-collapsed. q.e.d.

References

[1] J. W. Alexander, *On the subdivision of 3-space by a polyhedron.*, vol. 10(1), Proc. Natl. Acad. Sci., USA, 1924, pp. 6–8.

[2] Michael T. Anderson, *On the topology of complete manifolds of nonnegative Ricci curvature*, Topology **29** (1990), no. 1, 41–55. MR **1046624 (91b:**53041)

[3] ______, *On stationary vacuum solutions to the Einstein equations*, Ann. Henri Poincaré **1** (2000), no. 5, 977–994. MR **1806984 (2002a:**53085)

[4] ______, *On long-time evolution in general relativity and geometrization of 3-manifolds*, Comm. Math. Phys. **222** (2001), no. 3, 533–567. MR **1888088 (2003d:**53113)

[5] Josef Bemelmans, Min-Oo, and Ernst A. Ruh, *Smoothing Riemannian metrics*, Math. Z. **188** (1984), no. 1, 69–74. MR **767363 (85m:**58184)

[6] Jeff Cheeger and Tobias H. Colding, *Lower bounds on Ricci curvature and the almost rigidity of warped products*, Ann. of Math. (2) **144** (1996), no. 1, 189–237. MR **1405949 (97h:**53038)

[7] Jeff Cheeger, Mikhail Gromov, and Michael Taylor, *Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of*

[26]The action is effective because if an isometry of G leaves every point of $\tilde{T}^2(x)$ invariant, then it must be the identity as an isometry in $\tilde{\Omega}$.

complete Riemannian manifolds, J. Differential Geom. **17** (1982), no. 1, 15–53. MR **658471 (84b:**58109)

[8] Jeff Cheeger and Gang Tian, *On the cone structure at infinity of Ricci flat manifolds with Euclidean volume growth and quadratic curvature decay*, Invent. Math. **118** (1994), no. 3, 493–571. MR **1296356 (95m:**53051)

[9] Kenji Fukaya, *Collapsing Riemannian manifolds to ones of lower dimensions*, J. Differential Geom. **25** (1987), no. 1, 139–156. MR **873459 (88b:**53050)

[10] ———, *A boundary of the set of the Riemannian manifolds with bounded curvatures and diameters*, J. Differential Geom. **28** (1988), no. 1, 1–21. MR **950552 (89h:**53090)

[11] ———, *Collapsing Riemannian manifolds to ones with lower dimension. II*, J. Math. Soc. Japan **41** (1989), no. 2, 333–356. MR **984756 (90c:**53103)

[12] ———, *Hausdorff convergence of Riemannian manifolds and its applications*, Recent topics in differential and analytic geometry, Adv. Stud. Pure Math., vol. 18, Academic Press, Boston, MA, 1990, pp. 143–238. MR **1145256 (92k:**53076)

[13] Misha Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, english ed., Modern Birkhäuser Classics, Birkhäuser Boston Inc., Boston, MA, 2007, Based on the 1981 French original, With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates. MR **2307192 (2007k:**53049)

[14] S. W. Hawking, *Gravitational instantons*, Phys. Lett. A **60** (1977), no. 2, 81–83. MR 0465052 (57 #4965)

[15] Peter Li, *Large time behavior of the heat equation on complete manifolds with nonnegative Ricci curvature*, Ann. of Math. (2) **124** (1986), no. 1, 1–21. MR **847950 (87k:**58259)

[16] W. B. Raymond Lickorish, *An introduction to knot theory*, Graduate Texts in Mathematics, vol. 175, Springer-Verlag, New York, 1997. MR **1472978 (98f:**57015)

[17] Gang Liu, *3-manifolds with nonnegative Ricci curvature*, Invent. Math. **193** (2013), no. 2, 367–375. MR 3090181

[18] Zhong-dong Liu, *Ball covering on manifolds with nonnegative Ricci curvature near infinity*, Proc. Amer. Math. Soc. **115** (1992), no. 1, 211–219. MR **1068127 (92h:**53046)

[19] John Lott and Zhongmin Shen, *Manifolds with quadratic curvature decay and slow volume growth*, Ann. Sci. École Norm. Sup. (4) **33** (2000), no. 2, 275–290. MR **1755117 (2002e:**53049)

[20] J. Milnor, *A note on curvature and fundamental group*, J. Differential Geometry **2** (1968), 1–7. MR 0232311 (38 #636)

[21] S. B. Myers and N. E. Steenrod, *The group of isometries of a Riemannian manifold*, Ann. of Math. (2) **40** (1939), no. 2, 400–416. MR 1503467

[22] G. Perelman, *Manifolds of positive Ricci curvature with almost maximal volume*, J. Amer. Math. Soc. **7** (1994), no. 2, 299–305. MR **1231690 (94f:**53077)

[23] ———, *A complete Riemannian manifold of positive Ricci curvature with Euclidean volume growth and nonunique asymptotic cone*, Comparison geometry (Berkeley, CA, 1993–94), Math. Sci. Res. Inst. Publ., vol. 30, Cambridge Univ. Press, Cambridge, 1997, pp. 165–166. MR **1452873 (98e:**53067)

[24] Peter Petersen, *Riemannian geometry*, second ed., Graduate Texts in Mathematics, vol. 171, Springer, New York, 2006. MR **2243772 (2007a:**53001)

[25] Martin Reiris, *Stationary solutions and asymptotic flatness I*, Classical Quantum Gravity **31** (2014), no. 15, 155012, 33. MR 3233266

[26] ______, *Stationary solutions and asymptotic flatness II*, Classical Quantum Gravity **31** (2014), no. 15, 155013, 18. MR 3233267

[27] Richard Schoen and Shing Tung Yau, *Complete three-dimensional manifolds with positive Ricci curvature and scalar curvature*, Seminar on Differential Geometry, Ann. of Math. Stud., vol. 102, Princeton Univ. Press, Princeton, N.J., 1982, pp. 209–228. MR **645740 (83k:**53060)

[28] Peter Scott, *The geometries of* 3*-manifolds*, Bull. London Math. Soc. **15** (1983), no. 5, 401–487. MR **705527 (84m:**57009)

[29] Friedhelm Waldhausen, *On irreducible* 3*-manifolds which are sufficiently large*, Ann. of Math. (2) **87** (1968), 56–88. MR 0224099 (36 #7146)

[30] Shing Tung Yau, *Some function-theoretic properties of complete Riemannian manifold and their applications to geometry*, Indiana Univ. Math. J. **25** (1976), no. 7, 659–670. MR 0417452 (54 #5502)

[31] Shun-Hui Zhu, *A finiteness theorem for Ricci curvature in dimension three*, J. Differential Geom. **37** (1993), no. 3, 711–727. MR **1217167 (94f:**53071)

Max Planck Institute for Gravitational Physics
Albert Einstein Institute
Science Park Potsdam-Golm
Am Mhlenberg 1
D-14476 Potsdam-Golm
Germany
E-mail address: martin.reiris@aei.mpg.de