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SPACELIKE SURFACES OF CONSTANT MEAN CURVATURE ± 1 IN DE SITTER 3-SPACE $\mathbb{S}^3_1(1)$

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ABSTRACT. It is shown that spacelike surfaces of constant mean curvature ± 1 (abbreviated as CMC ± 1) in de Sitter 3-space $\mathbb{S}^3_1(1)$ can be constructed from holomorphic curves in $\mathbb{P}\,\mathrm{SL}(2;\mathbb{C}) = \mathrm{SL}(2;\mathbb{C})/\{\pm\,\mathrm{id}\}$ via a Bryant type representation formula. This Bryant type representation formula is used to investigate an explicit one-to-one correspondence, the so-called Lawson correspondence, between spacelike CMC ± 1 surfaces in de Sitter 3-space $\mathbb{S}^3_1(1)$ and spacelike maximal surfaces in Lorentz 3-space $\mathbb{E}^3_1$. The hyperbolic Gauss map of spacelike surfaces in $\mathbb{S}^3_1(1)$, which is a close analogue of the classical Gauss map, is considered. It is shown that the hyperbolic Gauss map plays an important role in the study of spacelike CMC ± 1 surfaces in $\mathbb{S}^3_1(1)$. In particular, the relationship between the holomorphicity of the hyperbolic Gauss map and spacelike CMC ± 1 surfaces in $\mathbb{S}^3_1(1)$ is studied.

1. Introduction

In the study of minimal surfaces in Euclidean 3-space $\mathbb{E}^3$, the Gauss map and the Weierstrass representation formula play crucial roles. Details of their roles in minimal surface theory can be found in many places in the literature; for example, see [13] and [15].

In [5], R. L. Bryant proved a representation formula for surfaces of constant mean curvature 1 in hyperbolic 3-space $\mathbb{H}^3(-1)$, which is an analogue of the classical Weierstrass representation formula for minimal surfaces in $\mathbb{E}^3$. Locally, Bryant's representation formula can also be written in terms of holomorphic data as the classical Weierstrass representation formula. In fact, long before Bryant discovered his formula, L. Bianchi [3] pointed out that local surfaces of constant mean curvature 1 in hyperbolic 3-space admit a Weierstrass representation formula. A very readable account of Bianchi's ideas can be found in de Lima and Roitman [12]. Although they are essentially equivalent to each other, Bryant's representation formula is different from Bianchi's.

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Using his representation formula, Bryant showed that constant mean curvature 1 surfaces in $\mathbb{H}^3(-1)$ are locally isometric to minimal surfaces in $\mathbb{E}^3$ and have many properties in common with minimal surfaces in $\mathbb{E}^3$. This is an expected result due to the so-called *Lawson correspondence* introduced by H. Blaine Lawson. In [10], Lawson showed that there exists a one-to-one correspondence between minimal surfaces in $\mathbb{E}^3$ and constant mean curvature 1 surfaces in $\mathbb{H}^3(-1)$ and that the corresponding surfaces satisfy the same Gauss-Codazzi equations. However, there are striking differences between such constant mean curvature 1 surfaces in $\mathbb{H}^3(-1)$ and minimal surfaces in $\mathbb{E}^3$. For example, the total curvature of constant mean curvature 1 surfaces in $\mathbb{H}^3(-1)$ need not have a certain quantization, while the total curvature of minimal surfaces in $\mathbb{E}^3$ always does. Bryant also showed that the *hyperbolic Gauss map*, an analogue of the classical Gauss map, of constant mean curvature 1 surfaces (of finite total curvature) may not be holomorphically extended across the finite number of punctures, in contrast to the classical Gauss map of minimal surfaces in $\mathbb{E}^3$.

In [16], B. Palmer proved that there exists a Lawson correspondence between certain constant mean curvature spacelike surfaces in different Lorentzian space forms. In particular, there is a one-to-one correspondence between spacelike maximal surfaces in Lorentz 3-space $\mathbb{E}^3_1$ and spacelike surfaces of constant mean curvature ± 1 in de Sitter 3-space $\mathbb{S}^3_1(1)$. On the other hand, O. Kobayashi [9] and L. McNertney [14] gave a Weierstrass type representation formula for spacelike maximal surfaces in $\mathbb{E}^3_1$. Hence one might expect a Bryant type representation formula for spacelike surfaces of constant mean curvature ± 1 in $\mathbb{S}^3_1(1)$.

In this paper, motivated by Bryant's results, we prove a Bryant type representation formula for spacelike surfaces of constant mean curvature ± 1 in $\mathbb{S}^3_1(1)$, which is an analogue of the Weierstrass type representation formula in Kobayashi [9] and McNertney [14] (see Section 3).

In Section 4, we study an explicit one-to-one correspondence between spacelike surfaces of constant mean curvature ± 1 in $\mathbb{S}^3_1(1)$ and spacelike maximal surfaces in $\mathbb{E}^3_1$ using the Bryant type representation formula.

An analogue of the hyperbolic Gauss map[1] can be defined for spacelike surfaces in $\mathbb{S}^3_1(1)$ and plays an important role in studying spacelike surfaces of constant mean curvature ± 1 in $\mathbb{S}^3_1(1)$. We study the hyperbolic Gauss map along with secondary and generalized Gauss maps in Section 5. In Section 7, the relationship between the holomorphicity of the hyperbolic Gauss map and spacelike surfaces of constant mean curvature ± 1 is studied using the Lax system.

[1]The hyperbolic Gauss map was introduced by C. Epstein in [6] and used by R. L. Bryant [5] to study constant mean curvature 1 surfaces in $\mathbb{H}^3(-1)$.

In Section 6, a duality property of spacelike surfaces of constant mean curvature 1 in $\mathbb{S}^3_1(1)$ is obtained, which is analogous to that of M. Umehara and K. Yamada [22].

In Section 8, we consider an Umehara-Yamada type parametrization [19] of the Bryant type representation formula and show that it can be deformed to the Weierstrass type representation formula for maximal surfaces in $\mathbb{E}^3_1$. Although the main idea comes from M. Umehara and K. Yamada's paper [19], we use settings similar to those in the paper [1] by R. Aiyama and K. Akutagawa.

Some examples of spacelike CMC ±1 surfaces in $\mathbb{S}^3_1(1)$ are presented in the appendix (Section 9).

2. Spacelike surfaces in de Sitter 3-space $\mathbb{S}^3_1(1)$

Let $\mathbb{E}^4_1$ be the Lorentz 4-space with rectangular coordinates $x_0,\ x_1,\ x_2,\ x_3$ and the standard Lorentzian metric $\langle\ ,\ \rangle$ of signature $(-,+,+,+)$ given by the quadratic form

$$-(x_0)^2 + (x_1)^2 + (x_2)^2 + (x_3)^2.$$

The de Sitter space $\mathbb{S}^3_1(1)$ is a complete Lorentzian 3-manifold of constant sectional curvature 1 that can be realized as the hyperboloid of one sheet in $\mathbb{E}^4_1$:

$$\mathbb{S}^3_1(1) = \{(x_0, x_1, x_2, x_3) \in \mathbb{E}^3_1 : -(x_0)^2 + (x_1)^2 + (x_2)^2 + (x_3)^2 = 1\}.$$

Let M be an oriented 2-manifold and $f : M \longrightarrow \mathbb{S}^3_1(1)$ be an immersion. The immersion is said to be *spacelike* if the induced metric I on M is Riemannian (positive definite). The induced metric I determines a conformal structure $\mathcal{C}_I$ on M (and hence M and f can be regarded as a Riemann surface and conformal immersion).

Let (x, y) be an isothermal coordinate system with respect to the conformal structure $\mathcal{C}_I$. Then the first fundamental form I can be written in terms of (x, y) as

$$I = ds^2 = e^u\{(dx)^2 + (dy)^2\}.$$

Let $z = x + iy$. Then $(z, \bar{z})$ defines a complex coordinate system with respect to the conformal structure $\mathcal{C}_I$. The first fundamental form I can also be written in terms of $(z, \bar{z})$ as

$$I = ds^2 = e^u dz \otimes d\bar{z}.$$

In terms of complex coordinates z and $\bar{z}$, the differential operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ are given by

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),\ \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{2.1}$$

Using the differential operators (2.1), we can compute

$$\langle f_z, f_z\rangle = \langle f_{\bar z}, f_{\bar z}\rangle = 0,\ \langle f_z, f_{\bar z}\rangle = \frac{1}{2}e^u.$$

Let N be a unit normal vector field of M. Then

$$\langle N, N\rangle = -1,\ \langle f_z, N\rangle = \langle f_{\bar z}, N\rangle = 0.$$

The quadratic 1-form $Qdz \otimes dz := \langle f_{zz}, N\rangle dz \otimes dz$ is called Hopf differential. Abusing the terminology, we will simply call the coefficient $Q = \langle f_{zz}, N\rangle$ Hopf differential. It is well-known (for example, see [4]) that a surface immersed in hyperbolic 3-space $\mathbb{H}^3(-1)$, $f : M \longrightarrow \mathbb{H}^3(-1)$, has constant mean curvature if and only if the Hopf differential is holomorphic, i.e., $Q_{\bar z} = 0$. This is still true for spacelike immersions into de Sitter 3-space $\mathbb{S}^3_1(1)$, $f : M \longrightarrow \mathbb{S}^3_1(1)$, as shown in Section 7 (see equation (7.2)). The second fundamental form II of M derived from N is

$$II = -\langle df, dN\rangle = Qdz \otimes dz + He^u dz \otimes d\bar z + \bar Q d\bar z \otimes d\bar z,$$

where H is the mean curvature of M. The mean curvature H is computed as $\langle f_{z\bar z}, N\rangle = \frac{1}{2}He^u$.

Let M be a simply-connected Riemann surface and $f : M \longrightarrow \mathbb{S}^3_1(1)$ a spacelike surface[2] with unit normal vector field N. Then we can find an orthonormal frame field $\mathcal{F}$ defined by

$$\mathcal{F} = (N, e^{-u/2}f_x, e^{-u/2}f_y, f) : M \longrightarrow \mathrm{SO}(3,1)^+,$$

where $\mathrm{SO}(3,1)^+$ is the identity component of the special Lorentz group

$$\mathrm{SO}(3,1) = \{\mathcal{A} \in GL(4;\mathbb{R}) : |\mathcal{A}| = 1,\ \langle \mathcal{A}\mathbf{v}, \mathcal{A}\mathbf{w}\rangle = \langle \mathbf{v},\ \mathbf{w}\rangle, \mathbf{v}, \mathbf{w} \in \mathbb{E}^4_1\}.$$

Let $(x_0, x_1, x_2, x_3) \in \mathbb{E}^4_1$. Then $\mathbf{v} = (x_0, x_1, x_2, x_3)$ can be identified with the 2×2 hermitian matrix

$$\begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix}. \tag{2.2}$$

The standard basis elements $e_0,\ e_1,\ e_2,\ e_3$ for $\mathbb{E}^4_1$ can be identified with the Pauli spin matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $\mathbf{v}, \mathbf{w} \in \mathbb{E}^4_1$. Then

$$\langle \mathbf{v}, \mathbf{w}\rangle = -\frac{1}{2}\,\mathrm{tr}(\mathbf{v}\sigma_3\mathbf{w}^t\sigma_3).$$

In particular,

$$\langle \mathbf{v}, \mathbf{v}\rangle = -\frac{1}{2}\,\mathrm{tr}(\mathbf{v}\sigma_3\mathbf{v}^t\sigma_3) = -\det\mathbf{v},$$

[2]Throughout this paper, we will assume the smoothness of spacelike immersions in $\mathbb{S}^3_1(1)$.

i.e., the above identification is, in fact, an isometry.

The complex special linear group $\mathrm{SL}(2;\mathbb{C})$ acts isometrically on $\mathbb{E}^4_1$ via the group action

$$\mu : \mathrm{SL}(2;\mathbb{C}) \times \mathbb{E}^4_1 \longrightarrow \mathbb{E}^4_1; \mu(g)\mathbf{v} = g\mathbf{v}g^*.$$

This action is transitive on $\mathbb{S}^3_1(1)$. The isotropy at $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is $\mathrm{SU}(1,1)$ and $\mathbb{S}^3_1(1)$ is represented as the symmetric space[3] $\mathbb{S}^3_1(1) = \mathrm{SL}(2;\mathbb{C})/\mathrm{SU}(1,1)$. The action μ induces a double covering $\mathrm{SL}(2;\mathbb{C}) \longrightarrow \mathrm{SO}(3,1)^+$ of the identity component of the Lorentz group $\mathrm{SO}(3,1)$. By using this double covering we can find a lift F (called a coordinate frame) of $\mathcal{F}$ to $\mathrm{SL}(2;\mathbb{C})$:

$$\mu(F)(\sigma_0, \sigma_1, \sigma_2, \sigma_3) = \mathcal{F}.$$

Let M be a Riemann surface and $f : M \longrightarrow \mathbb{S}^3_1(1)$ a spacelike immersion. Then there exists a local framing $F : U \longrightarrow \mathrm{SL}(2;\mathbb{C})$ (which is unique up to the sign), where U is an oriented and simply connected open set in M, such that

$$\begin{array}{lllllllll} e_0 & = & \mu(F)(\sigma_0) & = & F\sigma_0F^* & = & FF^* & = & N, \\ e_1 & = & \mu(F)(\sigma_1) & = & F\sigma_1F^* & = & e^{-u/2}f_x, & & \\ e_2 & = & \mu(F)(\sigma_2) & = & F\sigma_2F^* & = & e^{-u/2}f_y, & & \\ e_3 & = & \mu(F)(\sigma_3) & = & F\sigma_3F^* & = & f. & & \end{array} \tag{2.3}$$

Following Cartan's formalism, there exist unique connection 1-forms $\{\omega^\beta_\alpha : \alpha, \beta = 0, 1, 2, 3\}$ such that

$$de_\alpha = e_\beta \omega^\beta_\alpha. \tag{2.4}$$

We use the index range $1 \leq i, j, k \leq 3$ and denote by ω^i the connection form ω^i_0. Then equation (2.4) can be written as

$$\begin{aligned} de_0 &= e_i\omega^i, \\ de_i &= e_0\omega^i + e_j\omega^j_i, \\ \omega^i &= \omega^i_0 = \omega^0_i, \\ 0 &= \omega^i_j + \omega^j_i. \end{aligned}$$

Note that for this framing F of the immersion f we have

$$\omega^1 \wedge \omega^1_3 + \omega^2 \wedge \omega^2_3 = 0, \tag{2.5}$$

$$\omega^3 = 0. \tag{2.6}$$

[3]This symmetric space representation explains Remark 8.4.

For our adapted framing F, Cartan's structure equations include (2.5), (2.6) and the following five equations:

$$d\omega^1 + \omega^1_2 \wedge \omega^2 = 0, \tag{2.7}$$

$$d\omega^2 + \omega^2_1 \wedge \omega^1 = 0, \tag{2.8}$$

$$d\omega^1_2 + \omega^1_3 \wedge \omega^3_2 = -\omega^1 \wedge \omega^2 \quad \text{(Gauss equation)}, \tag{2.9}$$

$$d\omega^1_3 + \omega^1_2 \wedge \omega^2_3 = 0 \text{ (Codazzi equations)}, \tag{2.10}$$

$$d\omega^2_3 + \omega^2_1 \wedge \omega^1_3 = 0, \tag{2.11}$$

which in short can be written as

$$d\omega^i = -\omega^i_j \wedge \omega^j,$$
$$d\omega^i_j = -\omega^i_k \wedge \omega^k_j - \omega^i \wedge \omega^j.$$

Denote by ds^2 the metric in $\mathbb{S}^3_1(1)$ induced by the canonical Lorentzian metric in $\mathbb{E}^4_1$. Then

$$e_3^*(ds^2) = \langle de_3, de_3 \rangle = -(\omega^3)^2 + (\omega^1_3)^2 + (\omega^2_3)^2. \tag{2.12}$$

This gives rise to an indefinite metric in the oriented orthonormal frame bundle $\mathcal{F}$ of $\mathbb{S}^3_1(1)$.

Let $f : M \longrightarrow \mathbb{S}^3_1(1)$ be a spacelike immersion and F a local framing from an open set in M to $\mathrm{SL}(2;\mathbb{C})$ associated with the immersion f. Let $\Omega := F^{-1}dF \in \mathfrak{sl}(2;\mathbb{C})$. The Gauss and Codazzi equations are equivalent to Maurer-Cartan equation

$$d\Omega + \Omega \wedge \Omega = 0, \tag{2.13}$$

which is the null curvature (integrability) condition of the Maurer-Cartan form Ω.

The Maurer-Cartan form $\Omega = F^{-1}dF$ can be written as the following equation:

$$F^{-1}dF = \frac{1}{2}F^* \begin{pmatrix} \omega^3 + i\omega^2_1 & (\omega^1 - \omega^1_3) + i(\omega^2 - \omega^2_3) \\ (\omega^1 + \omega^1_3) - i(\omega^2 + \omega^2_3) & -(\omega^3 + i\omega^2_1) \end{pmatrix}. \tag{2.14}$$

Here, F^* denotes the pull-back map $F^* : \mathfrak{sl}(2;\mathbb{C})^* \longrightarrow T^*M$.

Proposition 2.1. *Let $f : M \longrightarrow \mathbb{S}^3_1(1)$ be a spacelike immersion. If $\{N = e_0, e_1, e_2\}$ forms an adapted frame field along f, then the mean curvature H and Gaussian curvature K of f satisfy the following equations:*

$$\omega^1 \wedge \omega^2 = (K+1)\omega^1_3 \wedge \omega^2_3, \tag{2.15}$$

$$\omega^2 \wedge \omega^1_3 + \omega^2_3 \wedge \omega^1 = 2H\omega^1_3 \wedge \omega^2_3. \tag{2.16}$$

3. Bryant type representation formula for spacelike surfaces of constant mean curvature 1 in $\mathbb{S}^3_1(1)$

Definition 3.1. Let M be a Riemann surface. Then a holomorphic map $F : M \longrightarrow \mathrm{SL}(2;\mathbb{C})$ is said to be *null* if $F^*(\phi) = 0$ or equivalently $\det(F^{-1}dF) = 0$, where ϕ is the Cartan-Killing form[4] $\phi = -4\det(g^{-1}dg)$.

We now show the following Bryant [5] type representation formula for spacelike surfaces of constant mean curvature 1 in de Sitter 3-space $\mathbb{S}^3_1(1)$. From now on we simply abbreviate the term "constant mean curvature 1" by CMC 1.

Theorem 3.2 (Bryant type representation formula). *Let M be a Riemann surface and $F : M \longrightarrow \mathrm{SL}(2;\mathbb{C})$ a holomorphic null immersion. Assume the pull-back metric $e_3^*(ds^2) = -(\omega^3)^2 + (\omega^1_3)^2 + (\omega^2_3)^2$ is nondegenerate, where ds^2 is the induced metric in $\mathbb{S}^3_1(1)$. Then*

$$f := e_3 \circ F = F\sigma_3F^* : M \longrightarrow \mathbb{S}^3_1(1) \tag{3.1}$$

is a spacelike CMC[5] 1 immersion. Conversely, let M be an oriented open simply connected Riemann surface and $f : M \longrightarrow \mathbb{S}^3_1(1)$ a spacelike CMC 1 immersion. Then there exists a holomorphic null immersion $F : M \longrightarrow \mathrm{SL}(2;\mathbb{C})$ such that $f = e_3 \circ F$. Moreover, F is unique up to right multiplication by $g \in \mathrm{SU}(1,1)$.

Proof. To simplify our calculations, we use the notation $\omega = \omega^1_3 + i\omega^2_3$ and $\pi = \omega^1 - i\omega^2$. Assume that $F : M \longrightarrow \mathrm{SL}(2;\mathbb{C})$ is a holomorphic null immersion. By equation (2.14),

$$\begin{aligned} F^{-1}dF &= \frac{1}{2}F^*\begin{pmatrix} \omega^3 + i\omega^2_1 & (\omega^1 - \omega^1_3) + i(\omega^2 - \omega^2_3) \\ (\omega^1 + \omega^1_3) - i(\omega^2 + \omega^2_3) & -(\omega^3 + i\omega^2_1) \end{pmatrix} \\ &= \frac{1}{2}F^*\begin{pmatrix} \omega^3 + i\omega^2_1 & \bar{\pi} - \omega \\ \pi + \bar{\omega} & -(\omega^3 + i\omega^2_1) \end{pmatrix}. \end{aligned}$$

Let

$$\begin{aligned} F^*(\omega^3 + i\omega^2_1) &= 2\alpha, \\ F^*(\pi + \bar{\omega}) &= 2\beta, \\ F^*(\bar{\pi} - \omega) &= 2\gamma, \end{aligned} \tag{3.2}$$

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[4]The quadratic Cartan-Killing from ϕ in $\mathfrak{sl}(2;\mathbb{C})$ is, in fact, $\phi = -8\det(g^{-1}dg)$. Here, it has been rescaled to $-4\det(g^{-1}dg)$.

[5]Since the sign of the mean curvature H depends upon the orientation of a surface, i.e., the orientation of the unit normal vector field N, the representation formula may as well define a CMC -1 spacelike immersion in $\mathbb{S}^3_1(1)$.

where α, β, γ are holomorphic 1-forms on M. Since F is null, we have

$$\begin{aligned} F^*(\phi) &= 4(\alpha^2 + \beta\gamma) = 0, \\ F^*(\omega^3) &= \alpha + \overline{\alpha}, \\ F^*(\omega) &= \overline{\beta} - \gamma. \end{aligned}$$

Now let $f = e_3 \circ F$. Then

$$\begin{aligned} ds_f^2 &:= f^*(ds^2) \\ &= F^* \circ e_3^*(ds^2) \\ &= F^*(-(\omega^3)^2 + \omega \otimes \overline{\omega}) \\ &= -(\alpha + \overline{\alpha})^2 + (\overline{\beta} - \gamma) \otimes (\beta - \overline{\gamma}) \\ &= -2\alpha \otimes \overline{\alpha} + \beta \otimes \overline{\beta} + \gamma \otimes \overline{\gamma}. \end{aligned}$$

By assumption, $e_3^*(ds^2)$ is nondegenerate; hence $ds_f^2 \neq 0$. Since F is an immersion, the last expression defines a positive definite metric. We show that for the immersion $f : M \longrightarrow \mathbb{S}_1^3(1)$, $H \equiv 1$.

Let $U \subset M$ be a simply connected open set in which there exists a smooth 1-form[6] η of type $(1, 0)$ such that $ds_f^2 = \eta \otimes \overline{\eta}$. Clearly, M is covered by such open sets. There exist functions A, B, C defined in U such that

$$\begin{aligned} F^*(\omega^3 + i\omega_1^2) &= 2A\eta, \\ F^*(\pi + \overline{\omega}) &= 2B\eta, \\ F^*(\overline{\pi} - \omega) &= 2C\eta, \end{aligned} \tag{3.3}$$

and

$$A^2 + BC = 0, \ -2A\overline{A} + B\overline{B} + C\overline{C} = 1. \tag{3.4}$$

In the open set U, $ds_f^2 = (-2A\overline{A} + B\overline{B} + C\overline{C})\eta \otimes \overline{\eta} = \eta \otimes \overline{\eta}$. Since $A^2 + BC = 0$, we see that there exist smooth functions p, q defined in U (unique up to replacement by $(-p, -q)$) such that

$$\begin{aligned} A &= pq, \\ B &= q^2, \\ C &= -p^2, \end{aligned}$$

and

$$-2pq\overline{pq} + q^2\overline{q}^2 + p^2\overline{p}^2 = (p\overline{p} - q\overline{q})^2 = 1 \text{ in } U.$$

[6]Such a smooth 1-form η can be written as $\eta = e^{\frac{u}{2}} dz$, where $u(z, \bar{z})$ is a solution to the elliptic Liouville equation $u_{z\bar{z}} = -e^{-u}$.

By the continuity of $p\overline{p}-q\overline{q}$, either $p\overline{p}-q\overline{q}=1$ in U or $p\overline{p}-q\overline{q}=-1$ in U. Without loss of generality, we may assume that $p\overline{p}-q\overline{q}=1$ in U. Recall that the Lie group $\mathrm{SU}(1,1)\subset \mathrm{SL}(2;\mathbb{C})$ is defined as follows:

$$\begin{aligned}\mathrm{SU}(1,1) &= \{\mathrm{U}\in \mathrm{SL}(2;\mathbb{C}) : \mathrm{U}\sigma_3\mathrm{U}^* = \sigma_3\}\\ &= \left\{\begin{pmatrix} p & \overline{q}\\ q & \overline{p}\end{pmatrix}\in \mathrm{SL}(2;\mathbb{C}) : p,q\in\mathbb{C}\right\}.\end{aligned}$$

Let $h : M \longrightarrow \mathrm{SU}(1,1)$ be a map defined by

$$h(z)=\begin{pmatrix} p(z) & \overline{q}(z)\\ q(z) & \overline{p}(z)\end{pmatrix}$$

for each $z\in M$. Then

$$\begin{aligned}h\sigma_3h^* &= \begin{pmatrix} p & \overline{q}\\ q & \overline{p}\end{pmatrix}\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix} \overline{p} & \overline{q}\\ q & p\end{pmatrix}\\ &= \begin{pmatrix} p\overline{p}-q\overline{q} & p\overline{q}-\overline{q}p\\ q\overline{p}-\overline{p}q & q\overline{q}-p\overline{p}\end{pmatrix}\\ &= \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}\\ &= \sigma_3,\end{aligned}$$

and so, $e_3\circ(Fh)=e_3\circ F$. Moreover,

$$\begin{aligned}(Fh)^{-1}d(Fh) &= (h^{-1}F^{-1})((dF)h+Fdh)\\ &= h^{-1}(F^{-1}dF)h+h^{-1}dh\\ &= h^{-1}\begin{pmatrix} pq & -p^2\\ q^2 & -pq\end{pmatrix}\eta h+h^{-1}dh\\ &= \begin{pmatrix} \overline{p}dp-\overline{q}dq & \overline{p}d\overline{q}-\overline{q}d\overline{p}-\eta\\ -qdp+pdq & -qd\overline{a}+pd\overline{p}\end{pmatrix}\\ &= \frac{1}{2}(Fh)^*\begin{pmatrix} \omega^3+i\omega_1^2 & \overline{\pi}-\omega\\ \pi+\overline{\omega} & -(\omega^3+i\omega_1^2)\end{pmatrix}.\end{aligned}$$

It follows that $(Fh)^*(\omega^3)=0$ and $(Fh)^*(\omega)=\eta$. Thus, $Fh : U\longrightarrow \mathrm{SL}(2;\mathbb{C})$ is an oriented adapted framing in U along the immersion $f=e_3\circ F=e_3(Fh)$.

Using equation (2.16) in Proposition 2.1 it is easy to show that f has constant mean curvature $H\equiv 1$ in U iff $(Fh)^*(\pi+\overline{\omega})$ is a 1-form of type $(1,0)$. Since $A\eta$, $B\eta$, $C\eta$ are holomorphic 1-forms and η is a 1-form of type $(1,0)$ in U, we see that p/q is a meromorphic function in U. Similarly q/p is also a meromorphic function in U. The 1-form $pdq-qdp$ is of type $(1,0)$ since it can be written as

$$pdq - qdp = \begin{cases} p^2 d\left(\dfrac{q}{p}\right) & \text{where } p \neq 0, \\ -q^2 d\left(\dfrac{p}{q}\right) & \text{where } q \neq 0. \end{cases}$$

Hence $(Fh)^*(\pi+\overline{\omega}) = (Fh)^*(\pi)+\overline{\eta} = 2(pdq-qdp)$ is of type $(1,0)$. Therefore $H \equiv 1$ in U and so $H \equiv 1$ in M as stated.

To prove the converse, let M be an oriented and simply connected open Riemann surface. Let $f : M \longrightarrow \mathbb{S}^3_1(1)$ be a smooth spacelike immersion with CMC $H \equiv 1$. There exists a 1-form η of type $(1,0)$ in M such that $ds^2_f = \eta \otimes \overline{\eta}$. We can choose a lifting $g : M \longrightarrow \mathrm{SL}(2;\mathbb{C})$ such that the associated frame field $\{e_3(g)\}$ is adapted with $g^*(\omega) = -\eta$. Then

$$g^{-1}dg = \frac{1}{2}g^*\begin{pmatrix} \omega^3 + i\omega^2_1 & (\omega^1 - \omega^1_3) + i(\omega^2 - \omega^2_3) \\ (\omega^1 + \omega^1_3) - i(\omega^2 + \omega^2_3) & -(\omega^3 + i\omega^2_1) \end{pmatrix}.$$

Let $\rho = g^*(\omega^2_1)$ and $\xi = g^*(\pi+\overline{\omega})$. Then ξ is a 1-form of type $(1,0)$ (since the mean curvature of f is 1) and

$$g^{-1}dg = \frac{1}{2}\begin{pmatrix} i\rho & \overline{\xi} - 2\eta \\ \xi & -i\rho \end{pmatrix}.$$

Consider the $\mathfrak{su}(1,1)$-valued 1-form in M

$$\mu = \frac{1}{2}\begin{pmatrix} i\rho & \overline{\xi} \\ \xi & -i\rho \end{pmatrix}.$$

It follows that μ satisfies the differential equation $d\mu = -\mu \wedge \mu$ since both ξ and η are 1-forms of type $(1,0)$. The equation satisfies the integrability condition; hence, by the Frobenius Theorem, there exists a smooth map $h : M \longrightarrow \mathrm{SU}(1,1)$ (unique up to left translation by a constant in $\mathrm{SU}(1,1)$) such that $\mu = h^{-1}dh$. Since $h \in \mathrm{SU}(1,1)$, h can be written as

$$h = \begin{pmatrix} p & \overline{q} \\ q & \overline{p} \end{pmatrix}$$

for some smooth functions p and q defined in M. Set $F = gh^{-1}$. Then

$$\begin{aligned} F^{-1}dF &= (gh^{-1})^{-1}d(gh^{-1}) \\ &= h(g^{-1}dg)h^{-1} + hdh^{-1} \\ &= h(h^{-1}dh - \eta\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})h^{-1} + hdh^{-1} \\ &= \begin{pmatrix} pq & -p^2 \\ q^2 & -pq \end{pmatrix}\eta. \end{aligned}$$

Since η is a 1-form of type $(1,0)$, $dF = -F\begin{pmatrix} pq & p^2 \\ q^2 & pq \end{pmatrix}\eta$ is a 1-form of type $(1,0)$; hence $\overline{\partial}F = \frac{\partial F}{\partial \overline{z}}d\overline{z} = 0$, i.e., F is holomorphic. Clearly F is null $(\det F^{-1}dF = 0)$ and satisfies

$$e_3 \circ F = e_3 \circ (gh^{-1}) = e_3 \circ g = f.$$

From this we get $f_* = (e_3 \circ F)_* = e_{3*} \circ F_*$. Since f_* and e_{3*} are one-to-one, F_* is also one-to-one, i.e., F is an immersion.

Now we show the uniqueness of the holomorphic lifting of f up to right multiplication by constants in $\mathrm{SU}(1,1)$. Let $F_1, F_2 : M \longrightarrow \mathrm{SL}(2;\mathbb{C})$ be two holomorphic liftings of f. Then $f : e_3 \circ F_1 = e_3 \circ F_2$, i.e., $F_1\sigma_3F_1^* = F_2\sigma_3F_2^*$. Solving this equation for F_1, we get

$$F_1 = F_2\sigma_3(F_1^{-1}F_2)^*\sigma_3^{-1}.$$

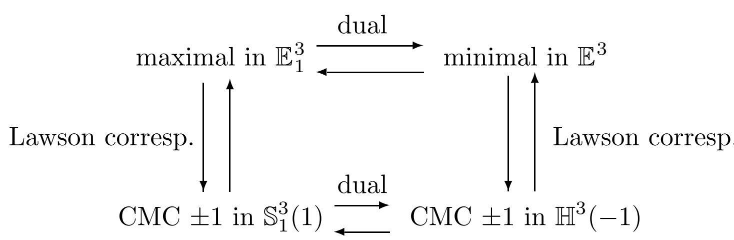
Let $h := \sigma_3(F_1^{-1}F_2)^*\sigma_3^{-1}$. Then $F_1 = F_2h$ and

$$\begin{aligned} F_1\sigma_3F_1^* = F_2\sigma_3F_2^* &\Longrightarrow F_2(h\sigma_3h^*)F_2^* = F_2\sigma_3F_2^* \\ &\Longrightarrow h\sigma_3h^* = \sigma_3 \\ &\Longrightarrow h \in \mathrm{SU}(1,1). \end{aligned}$$

Hence $h : M \longrightarrow \mathrm{SU}(1,1)$ is an antiholomorphic map of M into $\mathrm{SL}(2;\mathbb{C})$. However, $\mathrm{SU}(1,1)$ is a totally real submanifold of $\mathrm{SL}(2;\mathbb{C})$, so h must be constant. □

Remark 3.3. One can immediately see that this theorem is a close analogue of the Weierstrass type representation formula by O. Kobayashi [9] and L. McNertney [14] if we replace $\mathbb{S}^3_1(1)$ by $\mathbb{E}^3_1$, $\mathrm{SL}(2;\mathbb{C})$ by $\mathbb{C}^3$, $e_3 : \mathrm{SL}(2;\mathbb{C}) \longrightarrow \mathbb{S}^3_1(1)$ by $\mathrm{Re} : \mathbb{C}^3 \longrightarrow \mathbb{E}^3_1$, the Cartan-Killing form ϕ by the natural complex inner product in $\mathbb{C}^3$, and finally $H \equiv 1$ by $H \equiv 0$ in the Theorem 3.2.

Remark 3.4. A holomorphic null immersion $F : M \longrightarrow \mathrm{SL}(2;\mathbb{C})$ induces both a CMC ±1 surface in hyperbolic 3-space $\mathbb{H}^3(-1)$ by Bryant's representation formula [5] and a spacelike CMC ±1 surface in de Sitter 3-space $\mathbb{S}^3_1(1)$ by Theorem 3.2. In fact, the following *duality diagram* holds:



Remark 3.5. In Theorem 3.2, we have excluded the compact simply connected Riemann surface, i.e., the Riemann sphere S^2. The reason is that *there is no non-zero non-constant holomorphic* 1*-form globally defined in* S^2.

4. A correspondence between spacelike CMC ± 1 surfaces in $\mathbb{S}^3_1(1)$ and spacelike maximal surfaces in $\mathbb{E}^3_1$

The hyperbolic 2-space $\mathbb{H}^2(-1)$ can be described as

$$\mathbb{H}^2(-1) = \{x \in \mathrm{Herm}(2) : x_3 = 0,\ \det x = 1,\ x_0 > 0\}$$

by identifying $\mathbb{E}^4_1$ with the collection Herm(2) of 2×2 hermitian matrices (2.2). Since the Lie group SU(1,1) acts on $\mathbb{H}^2(-1)$ isometrically and transitively,

$$\mathbb{H}^2(-1) = \mathrm{SU}(1,1)/\mathrm{U}(1) = \{hh^* : h \in \mathrm{SU}(1,1)\}.$$

$\mathbb{H}^2(-1)$ can also be regarded as the Poincaré open disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with the metric $ds^2 = 4|dz|^2/(1-|z|^2)^2$ through the stereographic projection $\Psi : \mathbb{H}^2(-1) \longrightarrow \mathbb{D}$ from $-e_0 = (-1,0,0) \in \mathbb{E}^3_1$. Let $(x_0, x_1, x_2) \in \mathbb{H}^2(-1)$. Then

$$\Psi(x_0, x_1, x_2) = \left(0, \frac{x_1}{x_0+1}, \frac{x_2}{x_0+1}\right) \cong \frac{x_1 + ix_2}{x_0+1} \in \mathbb{D}.$$

Given $h = \begin{pmatrix} p & \overline{q} \\ q & \overline{p} \end{pmatrix} \in \mathrm{SU}(1,1)$, we have

$$hh^* = \begin{pmatrix} p\overline{p} + q\overline{q} & 2p\overline{q} \\ 2\overline{p}q & p\overline{p}+q\overline{q} \end{pmatrix} = \begin{pmatrix} x_0 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 \end{pmatrix}$$

and

$$\Psi(x_0, x_1, x_2) = \Psi(hh^*) = \frac{\overline{q}}{\overline{p}} \in \mathbb{D} \Longrightarrow \frac{q}{p} \in \mathbb{D}.$$

Let $F : M \longrightarrow \mathrm{SL}(2;\mathbb{C})$ be a holomorphic null immersion with $F = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix}$ and $F_1F_4 - F_2F_3 = 1$. Then by Theorem 3.2, $f = F\sigma_3 F^* : M \longrightarrow \mathbb{S}^3_1(1)$ defines a spacelike CMC ± 1 surface in $\mathbb{S}^3_1(1)$. Let $U \subset M$ be a simply connected open subset of M. Recall that the holomorphic 1-form $F^{-1}dF$ can be written as

$$(4.1)\qquad F^{-1}dF = \begin{pmatrix} pq & -p^2 \\ q^2 & -pq \end{pmatrix}\eta = \begin{pmatrix} q/p & -1 \\ q^2/p^2 & -q/p \end{pmatrix} p^2\eta = \begin{pmatrix} p/q & -p^2/q^2 \\ 1 & -p/q \end{pmatrix} q^2\eta \text{ in } U.$$

Here, η is a 1-form of type $(1,0)$ defined in U such that $ds^2_f = \eta \otimes \eta$ and p, q are smooth functions defined in U such that p/q and q/p are meromorphic functions and $|p|^2 - |q|^2 = 1$. Let $g = q/p$ and $\omega = p^2\eta$. Then $g : U \longrightarrow \mathbb{D}$ is a holomorphic function and ω is a holomorphic 1-form defined in U. The map g is called the *secondary Gauss map*[7] of the local spacelike CMC ± 1 surface $f : U \longrightarrow \mathbb{S}^3_1(1)$. We adopted the name *secondary Gauss map*[8] from

[7]This Gauss map is, in fact, closely related to the classical Gauss map. This will be discussed in Section 5.

[8]R. Aiyama and K. Akutagawa [1] called such a Gauss map the *adjusted Gauss map*.

M. Umehara and K. Yamada (see, for example, [20] or [21]). Since η is of type $(1,0)$, it can be written as $\eta = e^{u/2}dz$, where $u : U \longrightarrow \mathbb{R}$ is a real-valued function defined in U. By letting $h = p^2 e^{u/2}$, we get $\omega = hdz$ and

$$F^{-1}dF = \begin{pmatrix} g & -1 \\ g^2 & -g \end{pmatrix} \omega = \begin{pmatrix} g & -1 \\ g^2 & -g \end{pmatrix} hdz. \tag{4.2}$$

The induced metric of the local spacelike CMC ± 1 immersion f is

$$ds_f^2 = \eta \otimes \overline{\eta} = |h|^2(1-|g|^2)^2|dz|^2 \text{ in } U \subset M.$$

Corresponding to this spacelike CMC ± 1 immersion f, there exists a local spacelike maximal immersion ψ in $\mathbb{E}^3_1$ with Weierstrass data (g, ω) given by the integral formula

$$\psi(\zeta) = \mathrm{Re} \int_{\zeta_0}^{\zeta} (2gh, (1+g^2)h, -i(1-g^2)h)dz, \ z \in U, \tag{4.3}$$

by the Weierstrass type formula in Kobayashi [9] and McNertney [14]. The induced metric of the spacelike maximal immersion is

$$ds_\psi^2 = |h|^2(1-|g|^2)^2|dz|^2 \text{ in } U.$$

One can find another spacelike maximal immersion that corresponds to the same spacelike CMC ± 1 immersion in $\mathbb{S}^3_1(1)$. This time, in equation (4.1), let $g = p/q$ and $\omega = q^2\eta$. Then $g : U \longrightarrow \mathbb{C} \setminus \mathbb{D}$ is a meromorphic function and ω is a holomorphic 1-form defined in U. This function g is also called the *secondary Gauss map.* Note that this secondary Gauss map is the *inversion* of the previous one. By letting $h = q^2 e^{u/2}$, we get $\omega = hdz$ and

$$F^{-1}dF = \begin{pmatrix} g & -g^2 \\ 1 & -g^2 \end{pmatrix} \omega = \begin{pmatrix} g & -g^2 \\ 1 & -g^2 \end{pmatrix} hdz. \tag{4.4}$$

The induced metric of f is again $ds_f^2 = |h|^2(1-|g|^2)^2|dz|^2$ in $U \subset M$. Then we find a corresponding spacelike maximal immersion given by the integral formula (4.3) with the new Weierstrass data (g, ω).

Conversely, let ψ be a spacelike maximal immersion from a Riemann surface M into $\mathbb{E}^3_1$. Then, as asserted in [9], ψ can be described *locally* as the integral formula

$$\psi(\zeta) = \mathrm{Re} \int_{\zeta_0}^{\zeta} (2gh, (1+g^2)h, -i(1-g^2)h)dz, \ z \in U,$$

where U is a simply connected open subset of M and $g : U \longrightarrow \mathbb{C} \setminus \{z \in \mathbb{C} : |z| = 1\}$ is a holomorphic or a meromorphic functions. By the continuity of g, either $|g(z)| < 1$ in U or $|g(z)| > 1$ in U. In either case, we consider the following initial value problem:

$$F^{-1}dF = \begin{pmatrix} g & -1 \\ g^2 & -g \end{pmatrix} hdz \text{ if } |g(z)| < 1$$

or

$$F^{-1}dF = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} hdz \text{ if } |g(z)| > 1,$$

with initial value condition, for instance, $F(z_0) = \sigma_0$. Here, σ_0 is the 2×2 identity matrix $\sigma_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, as before. Note that the equation satisfies the integrability condition, i.e., $\Omega = F^{-1}dF$ is a solution of the Maurer-Cartan equation (2.13). Hence there exists a unique solution $F : U \longrightarrow \mathrm{SL}(2;\mathbb{C})$, which is a holomorphic null immersion, to the initial value problem. Theorem 3.2, then, yields a spacelike surface $f : F\sigma_3 F^* : U \longrightarrow \mathbb{S}^3_1(1)$ of CMC ± 1 with the induced metric $ds^2_f = |h|^2(1-|g|^2)^2|dz|^2$. Therefore we see that there is a one-to-one correspondence between spacelike CMC ±1 surfaces in $\mathbb{S}^3_1(1)$ and spacelike maximal surfaces in $\mathbb{E}^3_1$. This correspondence is not a coincidence and is expected. In fact, B. Palmer [16] proved the following Lorentzian version of Lawson correspondence [10]: *There exists a one-to-one correspondence between spacelike CMC H_0 surfaces in $\mathbb{E}^3_1$ and spacelike CMC $\pm\sqrt{H_0^2+1}$ surfaces in $\mathbb{S}^3_1(1)$.* In particular, there is a one-to-one correspondence between spacelike maximal surfaces ($H_0 = 0$) in $\mathbb{E}^3_1$ and spacelike CMC ±1 surfaces in $\mathbb{S}^3_1(1)$. One must note that this correspondence is local.

5. The Gauss map of spacelike CMC ±1 surfaces in $\mathbb{S}^3_1(1)$

In this section, we consider an analogue of the hyperbolic Gauss map (see [5] or [6]) of spacelike CMC ±1 surfaces in $\mathbb{S}^3_1(1)$ and investigate the relationship between the secondary, hyperbolic, and generalized Gauss maps.

Let $f : M \longrightarrow \mathbb{S}^3_1(1)$ be a spacelike surface in $\mathbb{S}^3_1(1)$. At each base point $e_3 = f(m) \in \mathbb{S}^3_1(1)$, $e_0 \in \mathrm{T}_{e_3}\mathbb{S}^3_1(1)$ is an oriented unit normal vector to the tangent plane $f_*(\mathrm{T}_m M)$. The oriented *timelike geodesic* in $\mathbb{S}^3_1(1)$ emanating from e_3, which is tangent to the normal vector $e_0(m)$, asymptotically approaches the ideal boundary S^2_∞ at exactly two points $[e_0+e_3]$, $[e_0-e_3] \in \mathrm{S}^2_\infty$. The orientation allows us to call $[e_0+e_3]$ the initial point and $[e_0-e_3]$ the terminal point. Define a map $G : M \longrightarrow \mathrm{S}^2_\infty$ by $G(m) = [e_0+e_3](m)$ for each $m \in M$. This map is an analogue of the hyperbolic Gauss map of surfaces in hyperbolic 3-space $\mathbb{H}^3(-1)$ and we will also call it the *hyperbolic Gauss map*. By identifying S^2_∞ and the Riemann sphere $\mathbb{C}\cup\{\infty\}$ with the canonical conformal structure, we can consider the holomorphicity of the hyperbolic Gauss map.

Suppose that M is an oriented open simply connected Riemann surface and $f : M \longrightarrow \mathbb{S}^3_1(1)$ is a spacelike CMC ±1 surface in $\mathbb{S}^3_1(1)$. Then, by Theorem 3.2, there exists a holomorphic null immersion $F : M \longrightarrow \mathrm{SL}(2;\mathbb{C})$ such that $f = e_3 \circ F = F\sigma_3 F^*$. As we have seen earlier, the holomorphic 1-form $F^{-1}dF$ can be locally written as

$$F^{-1}dF = \begin{pmatrix} pq & -p^2 \\ q^2 & -pq \end{pmatrix} \eta,$$

in some simply connected open set $U \subset M$. Here, η is a 1-form of type $(1,0)$ in U and p, q are smooth functions in U such that q/p is meromorphic as before. Define $h : U \longrightarrow \mathrm{SU}(1,1)$ by $h = \begin{pmatrix} p & \overline{q} \\ q & \overline{p} \end{pmatrix}$. Then the mapping $Fh : U \longrightarrow \mathrm{SL}(2;\mathbb{C})$ is a local adapted framing in U, and we can compute

$$f = e_3 \circ F = e_3 \circ (Fh) = (Fh)\sigma_3(Fh)^* = F\sigma_3 F^* \text{ in } U.$$

Now,

$$\begin{aligned}(e_0+e_3)(Fh) &= (Fh)(e_0+e_3)(Fh)^* \\ &= Fh(e_0+e_3)h^*F^* \\ &= 2F\begin{pmatrix} p & \overline{q} \\ q & \overline{p}\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix}\begin{pmatrix} \overline{p} & \overline{q} \\ q & p\end{pmatrix}F^* \\ &= 2F\begin{pmatrix} p \\ q\end{pmatrix}\left(F\begin{pmatrix} p \\ q\end{pmatrix}\right)^* \in N^{3+} \text{ or } N^{3-} \cong \mathrm{S}^2_\infty.\end{aligned}$$

Here,

$$N^{3+} = \{x \in \mathbb{E}^4_1 : x = (x_0,x_1,x_2,x_3),\ \langle x,x\rangle = 0,\ x_0 > 0\}$$

is the future light cone and

$$N^{3-} = \{x \in \mathbb{E}^4_1 : x = (x_0,x_1,x_2,x_3),\ \langle x,x\rangle = 0,\ x_0 < 0\}$$

is the past light cone, either of which can be identified with the ideal boundary S^2_∞. Let us write $F = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4\end{pmatrix}$. Then

$$F\begin{pmatrix} p \\ q\end{pmatrix} = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4\end{pmatrix}\begin{pmatrix} p \\ q\end{pmatrix} = \begin{pmatrix} F_1p + F_2q \\ F_3p + F_4q\end{pmatrix}.$$

Hence,

$$\begin{aligned}G = [e_0+e_3] &= [F_1p+F_2q, F_3p+F_4q] \in \mathbb{CP}^1 \\ &= [dF_1, dF_3] \\ &= \left[1, \frac{dF_3}{dF_1}\right] \cong \frac{dF_3}{dF_1} \in \mathbb{C}\cup\{\infty\}.\end{aligned}$$

Similarly, we also get

$$G = [dF_2, dF_4] \cong \frac{dF_4}{dF_2} \in \mathbb{C}\cup\{\infty\}.$$

If $G = [e_0 - e_3]$, then $G = \bar{d}F_3/\bar{d}F_1 = \bar{d}F_4/\bar{d}F_2$, where $\bar{d}F := F\begin{pmatrix} \bar{p}\bar{q} & -\bar{p}^2 \\ \bar{q}^2 & -\bar{p}\bar{q}\end{pmatrix}\bar{\eta}$.

As seen in Section 4, the holomorphic 1-form $F^{-1}dF$ can also be written as

$$F^{-1}dF = \begin{pmatrix} g & -1 \\ g^2 & -g\end{pmatrix}\omega$$

in a simply connected open set $U \subset M$. Here, $g : U \longrightarrow \mathbb{D}$ is the secondary Gauss map and $\omega = hdz$ for some holomorphic map h defined in U. Then,

$$dF = F \begin{pmatrix} g & -1 \\ g^2 & -g \end{pmatrix} hdz = \begin{pmatrix} F_1 g + F_2 g^2 & -(F_1 + F_2 g) \\ F_3 g + F_4 g^2 & -(F_3 + F_4 g) \end{pmatrix} hdz,$$

and the hyperbolic Gauss map can be written in terms of the secondary Gauss map g as

$$G = [e_0 + e_3] = [F_1 p + F_2 q, F_3 p + F_4 q] = [F_1 + F_2 g, F_3 + F_4 g].$$

Hence,

$$G = \frac{F_3 + F_4 g}{F_1 + F_2 g},$$

i.e., $G(z) = \varphi_z \circ g(z)$, where $\varphi_z : \mathbb{C} \longrightarrow \mathbb{C}$ is the Möbius transformation given by

$$\varphi_z(\zeta) = \frac{F_3 + F_4 \zeta}{F_1 + F_2 \zeta}.$$

Similarly, if $F^{-1}dF$ is given by

$$F^{-1}dF = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \omega,$$

where $g : U \longrightarrow \mathbb{C} \setminus \mathbb{D}$ is the secondary Gauss map and $\omega = hdz$ is a holomorphic 1-form defined in some simply connected open set $U \subset M$, then

$$G = \frac{F_3 g + F_4}{F_1 g + F_2},$$

i.e, $G(z) = \psi_z \circ g(z)$, where $\psi_z : \mathbb{C} \longrightarrow \mathbb{C}$ is the Möbius transformation given by

$$\psi_z(\zeta) = \frac{F_3(z)\zeta + F_4(z)}{F_1(z)\zeta + F_2(z)}.$$

This relates the secondary Gauss map to the hyperbolic Gauss map. Hence we see that if the spacelike immersion $f : M \longrightarrow \mathbb{S}^3_1(1)$ has CMC ± 1, then the hyperbolic Gauss map is holomorphic[9].

We now consider the generalized Gauss map of spacelike surfaces in $\mathbb{E}^4_1$ and investigate its relationships to the hyperbolic Gauss map (and to the secondary Gauss map as well). The main ideas presented here are based upon the work of R. Aiyama and K. Akutagawa [1]. Let $G(2, \mathbb{E}^4_1)$ be the Grassmannian of oriented spacelike 2-planes in $\mathbb{E}^4_1$. The oriented spacelike 2-plane spanned by two spacelike vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ is denoted by $[\mathbf{v}_1 \wedge \mathbf{v}_2]$. The Grassmannian $G(2, \mathbb{E}^4_1)$ has the following complex structure: Let $\mathbb{C}^4_1$ denote the complexification of $\mathbb{E}^4_1$ endowed with complex linear coordinates $\mathbf{w} = (w_0, w_1, w_2, w_3)$

[9]It is interesting to ask if the converse is also true. That is, is it true that *if the hyperbolic Gauss map of a spacelike immersion $f : M \longrightarrow \mathbb{S}^3_1$ is holomorphic, then f has CMC ± 1?* This question is answered in Section 7.

and the indefinite Hermitian product $\langle \mathbf{w}, \mathbf{w}\rangle = -|w_0|^2 + |w_1|^2 + |w_2|^2 + |w_3|^2$. The complexified Lorentz space $\mathbb{C}^4_1$ can be identified with the 2×2 matrix

$$\underline{\mathbf{w}} = \begin{pmatrix} w_0 + w_3 & w_1 + iw_2 \\ w_1 - iw_2 & w_0 - w_3 \end{pmatrix},$$

and hence

$$\langle \mathbf{w}, \mathbf{w}\rangle = \frac{1}{2}\operatorname{tr}\left\{ \underline{\mathbf{w}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \underline{\mathbf{w}}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

If $\mathbf{w}$ is a real vector in $\mathbb{E}^4_1 \subset \mathbb{C}^4_1$, then this identification coincides with the one given by equation (2.2). Let $\mathbb{CP}^3_1$ be the complex projective space of the spacelike lines in $\mathbb{C}^4_1$ and $[\mathbf{w}]$ or $[\underline{\mathbf{w}}]$ denote the (spacelike) line through the origin 0 in $\mathbb{C}^4_1 \cong \mathfrak{gl}(2, \mathbb{C})$ such that $\langle \mathbf{w}, \mathbf{w}\rangle > 0$. The complex quadric $\mathbb{Q}^2_1$ in $\mathbb{CP}^3_1$ is the algebraic variety

$$\mathbb{Q}^2_1 := \{[\mathbf{w}] \in \mathbb{CP}^3_1 : \det \mathbf{w} = -(w_0)^2 + (w_1)^2 + (w_3)^2 + (w_3)^2 = 0\}.$$

The Grassmannian $G(2, \mathbb{E}^4_1)$ can be identified with the complex quadric $\mathbb{Q}^2_1$ by the natural correspondence

$$\mathbb{Q}^2_1 \longrightarrow G(2, \mathbb{E}^4_1);\ [\mathbf{w}] \longmapsto [\operatorname{Re} \mathbf{w} \wedge \operatorname{Im} \mathbf{w}].$$

$G(2, \mathbb{E}^4_1)$ can also be realised as a homogeneous space: $\mathrm{SL}(2; \mathbb{C})$ acts transitively on $G(2, \mathbb{E}^4_1) \cong \mathbb{Q}^2_1$ by the group action

$$g \cdot [\mathbf{v}_1 \wedge \mathbf{v}_2] := [g \cdot \mathbf{v}_1 \wedge g \cdot \mathbf{v}_2],$$

where $g \in \mathrm{SL}(2; \mathbb{C})$ and $[\mathbf{v}_1 \wedge \mathbf{v}_2] \in G(2, \mathbb{E}^4_1)$. The above action can also be written as

$$g \cdot [\mathbf{v}_1 + i\mathbf{v}_2] := [g(\underline{\mathbf{v}}_1 + i\underline{\mathbf{v}}_2)g^*].$$

The isotropy group at $[e_1 \wedge e_2] = [\sigma_1 + i\sigma_2] = [\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)]$ is

$$\mathbb{C}^* = \left\{ \begin{pmatrix} w & 0 \\ 0 & \frac{1}{w} \end{pmatrix} : w \in \mathbb{C} \setminus \{0\} \right\}.$$

Thus,

$$G(2, \mathbb{E}^4_1) \cong \mathbb{Q}^2_1 = \left\{ \left[g \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} g^* \right] \in \mathbb{CP}^3_1 : g \in \mathrm{SL}(2; \mathbb{C})) \right\} \cong \mathrm{SL}(2; \mathbb{C})/\mathbb{C}^*.$$

We now use the following complex coordinates in $G(2, \mathbb{E}^4_1)$:

$$\begin{aligned} \phi = (\phi_1, \phi_2) : G(2, \mathbb{E}^4_1) &\longrightarrow \hat{\mathbb{C}} \times \hat{\mathbb{C}}; \\ \left[g \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} g^* \right] &\longmapsto \left(\frac{g_{21}}{g_{11}}, \frac{g_{22}}{g_{12}} \right),\ \text{for } g = (g_{ij}) \in \mathrm{SL}(2; \mathbb{C}). \end{aligned}$$

This map ϕ is bijective to $\hat{\mathbb{C}} \times \hat{\mathbb{C}} \setminus \{(\zeta, \zeta) | \zeta \in \hat{\mathbb{C}}\}$. SL(2; $\mathbb{C}$) acts conformally on each $\hat{\mathbb{C}}$ by the Möbius transformation

$$g[\zeta] = \frac{g_{21} + g_{22}\zeta}{g_{11} + g_{12}\zeta} \text{ for } g = (g_{ij}) \in \mathrm{SL}(2; \mathbb{C}),\ \zeta \in \hat{\mathbb{C}}.$$

Note that $\phi_1(g) = g[0]$ and $\phi_2(g) = g[\infty]$. Let $f : M \longrightarrow \mathbb{E}^4_1$ be a spacelike immersion from a Riemann surface with complexified isothermic coordinate $z = x + iy$. The *generalized Gauss map* of f is then identified by

$$\mathcal{G} := [f_x \wedge f_y] = [f_x + if_y] = [f_{\bar{z}}] : M \longrightarrow G(2, \mathbb{E}^4_1) \cong \mathbb{Q}^2_1.$$

Let $\mathcal{G}_1 := \phi_1 \circ \mathcal{G}$ and $\mathcal{G}_2 := \phi_2 \circ \mathcal{G}$. Let $f : M \longrightarrow \mathbb{S}^3_1(1) \subset \mathbb{E}^4_1$ be a conformal spacelike immersion from an oriented and simply connected open Riemann surface M into $\mathbb{S}^3_1(1)$, with $ds^2_f = e^u|dz|^2$. Then there exists an adapted framing $F : M \longrightarrow \mathrm{SL}(2; \mathbb{C})$ of f such that $e_1 \circ F = e^{-u/2} f_x$ and $e_2 \circ F = e^{-u/2} f_y$. Let $F = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix}$. Then the generalized Gauss map $\mathcal{G}$ of f can be written as

$$\mathcal{G} = [(e_1 \circ F) \wedge (e_2 \circ F)] = \left[F \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} F^* \right] : M \longrightarrow G(2, \mathbb{E}^4_1) \cong \mathbb{Q}^2_1$$

and $\mathcal{G}_1 = F_3/F_1 = F[0]$, $\mathcal{G}_2 = F_4/F_2 = F[\infty]$. On the other hand, the hyperbolic Gauss map $G : M \longrightarrow \mathrm{S}^2_\infty$ of f is given by

$$\begin{aligned} G = [e_0 + e_3](F) &= \left[F \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} F^* \right] \\ &= \left[\begin{pmatrix} F_1 \\ F_3 \end{pmatrix} \begin{pmatrix} \overline{F_1} & \overline{F_3} \end{pmatrix} \right] \longleftrightarrow \frac{F_3}{F_1} = \mathcal{G}_1. \end{aligned}$$

That is, the hyperbolic Gauss map is identified with the first projection $\mathcal{G}_1$ of the generalized Gauss map $\mathcal{G}$. By taking the opposite orientation of normal geodesics (i.e., the opposite orientation of the Riemann surface M), the hyperbolic Gauss map is defined to be $G = [e_0 - e_3]$. In this case, the hyperbolic Gauss map is identified with the second projection $\mathcal{G}_2 = F_4/F_2$ of the generalized Gauss map $\mathcal{G}$.

We have seen that the hyperbolic Gauss map G is also given by $G = dF_3/dF_1$ or $G = dF_4/dF_2$. Hence the first and second projections of the generalized Gauss map can be explicitly written as $\mathcal{G}_1 = dF_3/dF_1 = dF_4/dF_2$ and $\mathcal{G}_2 = \bar{d}F_3/\bar{d}F_1 = \bar{d}F_4/\bar{d}F_2$, respectively.

6. A duality property of spacelike CMC ± 1 surfaces in $\mathbb{S}^3_1(1)$

In [22], M. Umehara and K. Yamada defined the dual surfaces of CMC 1 surfaces in hyperbolic 3-space and studied their properties. In this section, we also consider the dual surfaces of spacelike CMC ± 1 surfaces in $\mathbb{S}^3_1(1)$ and study properties similar to those in the hyperbolic case.

DEFINITION 6.1 (The dual spacelike CMC ±1 surfaces in $\mathbb{S}^3_1(1)$). Let $F : M \longrightarrow \mathrm{SL}(2;\mathbb{C})$ be a holomorphic null immersion. Then we easily see that $F^{-1} : M \longrightarrow \mathrm{SL}(2;\mathbb{C})$ is a holomorphic null immersion as well. By Theorem 3.2, the map $f^\sharp := e_3 \circ F^{-1} = F^{-1}\sigma_3(F^{-1})^* : M \longrightarrow \mathbb{S}^3_1(1)$ defines a spacelike CMC ±1 surface in $\mathbb{S}^3_1(1)$. This surface $f^\sharp$ is said to be the dual surface of f. It follows immediately from $(F^{-1})^{-1} = F$ that $(f^\sharp)^\sharp = f$.

Let $F : M \longrightarrow \mathrm{SL}(2;\mathbb{C})$ be a holomorphic null immersion with $F = \left(\begin{smallmatrix} F_1 & F_2 \\ F_3 & F_4 \end{smallmatrix}\right)$. In Section 5, we have seen that the hyperbolic Gauss map G can be locally written as

$$G = \frac{F_3 + F_4 g}{F_1 + F_4 g}$$

if $|g(z)| < 1$ or

$$G = \frac{F_3 g + F_4}{F_1 g + F_2}$$

if $|g(z)| > 1$. Let $G_0 := -(F_1 + F_2 g)$ and $G_1 := -(F_3 + F_4 g)$, where $|g(z)| < 1$. Then by direct computation we see that

$$FdF^{-1} = \begin{pmatrix} G & -1 \\ G^2 & -G \end{pmatrix} H dz, \tag{6.1}$$

where $G = G_1/G_0$ and $H = -(G_0)^2 h$. Similarly, if we let $G_0 := F_1 g + F_2$ and $G_1 := F_3 g + F_4$ where $|g(z)| > 1$, then

$$FdF^{-1} = \begin{pmatrix} G^\spadesuit & -(G^\spadesuit)^2 \\ 1 & -G^\spadesuit \end{pmatrix} H^\spadesuit dz,$$

where $G^\spadesuit = \frac{1}{G} = \frac{G_0}{G_1}$ and $H^\spadesuit = -(G_1)^2 h$.

THEOREM 6.2. *Let $f : M \longrightarrow \mathbb{S}^3_1(1)$ be a spacelike CMC ±1 immersion with Weierstrass data (g, ω) and Hopf differential*[10] *$Q = \omega \otimes dg$. Then the hyperbolic Gauss map $G^\sharp$, the Weierstrass data $(g^\sharp, \omega^\sharp)$ and the Hopf differential $Q^\sharp$ of the dual surface $f^\sharp$ are, respectively, given by*

$$G^\sharp = g,\ g^\sharp = G,\ \omega^\sharp = -\frac{Q}{dG},\ Q^\sharp = -Q \ \textit{if}\ |g(z)| < 1$$

and

$$G^{\spadesuit\sharp} = g,\ g^\sharp = G^\spadesuit,\ \omega^\sharp = -\frac{Q}{dG^\spadesuit},\ Q^\sharp = -Q \ \textit{if}\ |g(z)| > 1.$$

[10]One can easily see that this holomorphic 2-form is actually the same as the Hopf differential. For more details see, for example, [5] or [19].

Proof. We prove the statement when $|g(z)| < 1$ in M. Let $F = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix}$. Then from $G = dF_3/dF_1 = dF_4/dF_2$ we get the equation $dF_3 dF_2 - dF_1 dF_4 = 0$. Thus,

$$G = \frac{dF_3}{dF_1} = \frac{dF_3}{dF_1} \frac{\left(F_4 - \frac{dF_4}{dF_3} F_3\right)}{\left(F_4 - \frac{dF_4}{dF_3} F_3\right)} = \frac{F_4 dF_3 - F_3 dF_4}{F_4 dF_1 - F_3 dF_2}.$$

Set $F^\sharp = F^{-1}$. Since $FF^{-1} = I_2$, $d(FF^{-1}) = (dF)F^{-1} + FdF^{-1} = 0$. We compute

$$\begin{aligned}
(F^\sharp)^{-1} dF^\sharp &= FdF^{-1} \\
&= -(dF)F^{-1} \\
&= -\begin{pmatrix} dF_1 & dF_2 \\ dF_3 & dF_4 \end{pmatrix} \begin{pmatrix} F_4 & -F_2 \\ -F_3 & F_1 \end{pmatrix} \\
&= -\begin{pmatrix} F_4 dF_1 - F_3 dF_2 & -F_2 dF_1 + F_1 dF_2 \\ F_4 dF_3 - F_3 dF_4 & -F_2 dF_3 + F_1 dF_4 \end{pmatrix} \\
&= \begin{pmatrix} g^\sharp & -1 \\ {g^\sharp}^2 & -g^\sharp \end{pmatrix} \omega^\sharp.
\end{aligned}$$

This implies

$$g^\sharp = \frac{(g^\sharp)^2 \omega}{g^\sharp \omega} = \frac{F_4 dF_3 - F_3 dF_4}{F_4 dF_1 - F_3 dF_2} = G$$

and $h^\sharp = H$. Replacing F by $F^\sharp$, we also get $G^\sharp = g$. The induced metric of the dual surface $f^\sharp$ is given by

$$\begin{aligned}
ds^2 &= |h^\sharp|^2 (1 - |g^\sharp|^2)^2 |dz|^2 \\
&= |H|^2 (1 - |G|^2)^2 |dz|^2.
\end{aligned}$$

We now prove that $\omega^\sharp = -Q/dG$ and $Q^\sharp = -Q$. Since $G = G_1/G_0$, $dG = (G_0 dG_1 - G_1 dG_0)/(G_0)^2$. Recall that $G_0 = -(F_1 + F_2 g)$ and $G_1 = -(F_3 + F_4 g)$. Then,

$$\begin{aligned}
G_0 dG_1 - G_1 dG_0 &= \{(F_1 + F_2 g) dF_3 - (F_3 + F_4 g) dF_1\} \\
&\quad + \{(F_1 + F_2 g) dF_4 - (F_3 + F_4 g) dF_2\} g \\
&\quad + (F_1 F_4 - F_3 F_2) dg + (F_2 F_4 - F_4 F_2) g dg \\
&= dg.
\end{aligned}$$

Thus,

$$-\frac{Q}{dG} = -\frac{\omega \otimes dg}{dg/(G_0)^2} = -(G_0)^2 \omega = -(G_0)^2 h dz = H dz = \omega^\sharp,$$

and

$$Q^\sharp = \omega^\sharp \otimes dg^\sharp = H dz \otimes dG = -(G_0)^2 h dz \otimes \frac{dg}{(G_0)^2} = -\omega \otimes dg = -Q.$$

Similarly, we can prove the statement in the case when $|g(z)| > 1$ in M. □

7. Spacelike CMC ±1 surfaces in $\mathbb{S}^3_1(1)$ and the holomorphicity of the hyperbolic Gauss map

In this section, we assume the same geometric setting of spacelike surfaces in $\mathbb{S}^3_1(1)$ as in Section 2.

Let $f : M \longrightarrow \mathbb{S}^3_1(1)$ be a spacelike surface in $\mathbb{S}^3_1(1)$ and $F : M \longrightarrow \mathrm{SL}(2;\mathbb{C})$ a local framing satisfying (2.3). Then the Maurer-Cartan form $\Omega = F^{-1}dF \in \mathfrak{sl}(2;\mathbb{C})$ can be written as

$$F^{-1}dF = \mathsf{U}dz + \mathsf{V}d\bar{z},$$

where

$$\mathsf{U} = \begin{pmatrix} -\frac{u_z}{4} & e^{-u/2} \\ \frac{(H-1)}{2}e^{u/2} & \frac{u_z}{4} \end{pmatrix}, \quad \mathsf{V} = \begin{pmatrix} \frac{u_{\bar{z}}}{4} & \frac{(H+1)}{2}e^{u/2} \\ e^{-u/2}\bar{Q} & -\frac{u_{\bar{z}}}{4} \end{pmatrix}.$$

The local framing F satisfies the *Lax equations*

$$\begin{cases} F_z = F\mathsf{U}, \\ F_{\bar{z}} = F\mathsf{V}. \end{cases} \tag{7.1}$$

The compatibility condition $F_{z\bar{z}} = F_{\bar{z}z}$ gives

$$\mathsf{U}_{\bar{z}} - \mathsf{V}_z - [\mathsf{U}, \mathsf{V}] = 0,$$

which can be written as the following Gauss-Codazzi equations:

$$u_{z\bar{z}} - \frac{(H^2-1)}{2}e^u + 2Q\bar{Q}e^{-u} = 0, \tag{7.2}$$

$$Q_{\bar{z}} = 2H_z e^u. \tag{7.3}$$

From equation (7.3) we immediately see that the mean curvature H is constant if and only if the Hopf differential Q is holomorphic.

Let $F : M \longrightarrow \mathrm{SL}(2;\mathbb{C})$ be a local framing of a spacelike surface $f : M \longrightarrow \mathbb{S}^3_1(1)$ satisfying

$$\begin{aligned} e_0 &= N = F\sigma_0 F^* = FF^*, \\ e_1 &= \frac{f_x}{|f_x|} = F\sigma_1 F^*, \\ e_2 &= \frac{f_y}{|f_y|} = F\sigma_2 F^*, \\ e_3 &= f = F\sigma_3 F^*. \end{aligned}$$

In [7], S. Fujimori, S. Kobayashi and W. Rossman gave an alternative form of the Bryant type representation formula for spacelike CMC 1 surfaces in $\mathbb{S}^3_1(1)$ based on the above geometric settings. Namely, they showed:

THEOREM 7.1. *Any conformal spacelike CMC* 1 *immersion from a simply-connected Riemann surface into* $\mathbb{S}^3_1(1)$ *can be written as* $F\sigma_3 F^*$ *for some* antiholomorphic *map* $F \in \mathrm{SL}(2;\mathbb{C})$ *satisfying the equation* (4.2) *or* (4.4).

Note that, with the orientation we are using in this section, spacelike CMC -1 surfaces are induced by holomorphic framing while spacelike CMC 1 surfaces are induced by antiholomorphic framing.

Using the Lax system, we can prove the following proposition that relates the holomorphicity of hyperbolic Gauss maps to that of spacelike CMC ±1 surfaces in $\mathbb{S}^3_1(1)$.

PROPOSITION 7.2. *Let* $f : M \longrightarrow \mathbb{S}^3_1(1)$ *be a spacelike immersion and* G *its hyperbolic Gauss map. Then:*

(1) *The hyperbolic Gauss map* $G := [e_0 + e_3] = [N + f]$ *(when* G *preserves the orientation) is holomorphic if and only if* f *is totally umbilic.*
(2) *The hyperbolic Gauss map* $G := [e_0 - e_3] = [N - f]$ *(when* G *reverses the orientation) is holomorphic if and only if* f *satisfies* $H = -1$.

Recall from Section 5 that the hyperbolic Gauss map $[e_0 + e_3]$ is identified with $\mathcal{G}_1 = F_{21}/F_{11}$ and $[e_0 - e_3]$ is identified with $\mathcal{G}_2 = F_{22}/F_{12}$, where the local framing of f is given by $F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \in \mathrm{SL}(2;\mathbb{C})$. Here, we say that the hyperbolic Gauss map $[e_0+e_3]$ (resp. $[e_0-e_3]$) is *holomorphic* if $\mathcal{G}_1$ (resp. $\mathcal{G}_2$) is holomorphic.

Proof. From the Lax system (7.1),

$$\begin{aligned} F_{\bar z} &= FV \\ &= \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \begin{pmatrix} \frac{u_{\bar z}}{4} & \frac{1}{2}e^{u/2}(H+1) \\ e^{-u/2}\bar Q & -\frac{u_{\bar z}}{4} \end{pmatrix} \\ &= \begin{pmatrix} F_{11}\frac{u_{\bar z}}{4} + F_{12}e^{-u/2}\bar Q & \frac{1}{2}F_{11}e^{u/2}(H+1) - F_{12}\frac{u_{\bar z}}{4} \\ F_{21}\frac{u_{\bar z}}{4} + F_{22}e^{-u/2}\bar Q & \frac{1}{2}F_{21}e^{u/2}(H+1) - F_{22}\frac{u_{\bar z}}{4} \end{pmatrix}. \end{aligned}$$

The hyperbolic Gauss map $\mathcal{G}_1 = F_{21}/F_{11}$ is holomorphic, i.e., $(\mathcal{G}_1)_{\bar z} = 0$, if and only if

$$(F_{21})_{\bar z}F_{11} - F_{21}(F_{11})_{\bar z} = e^{-u/2}\bar Q = 0.$$

Hence we see that $\mathcal{G}_1$ is holomorphic if and only if the Hopf differential vanishes, i.e., f is totally umbilic.

Simlilarly, the hyperbolic Gauss map $\mathcal{G}_2 = F_{22}/F_{12}$ is holomorphic if and only if

$$(F_{22})_{\bar{z}}F_{12} - F_{22}(F_{12})_{\bar{z}} = -\frac{1}{2}e^u(H+1) = 0.$$

Therefore $\mathcal{G}_2$ is holomorphic if and only if $H = -1$. □

Corollary 7.3. *Let $f : M \longrightarrow \mathbb{S}^3_1(1)$ be a spacelike immersion. Then f is totally umbilic and satisfies $H = -1$ if and only if the hyperbolic Gauss map of f is constant.*

8. The Umehara-Yamada type parametrization and deformation of representation formula for spacelike CMC $\pm c$ surfaces in $\mathbb{S}^3_1(c^2)$

In [19], Umehara and Yamada showed that minimal surfaces in $\mathbb{R}^3$ are the limit surfaces of CMC c ($c > 0$) surfaces in hyperbolic 3-space $\mathbb{H}^3(-c^2)$ of constant sectional curvature $-c^2$ as $\mathrm{SL}(2;\mathbb{C})$ collapses into $\mathbb{C}^3$. In this section, we derive a similar result that a parametrized Bryant type representation formula in Theorem 8.2 can be deformed to the Weierstrass type representation formula (4.3) for spacelike maximal surfaces in $\mathbb{E}^3_1$. We first reformulate the Bryant type representation formula (3.1) as follows.

Definition 8.1 (de Sitter 3-Space $\mathbb{S}^3_1(c^2)$). Let $\mathbb{S}^3_1(c^2)$ be the de Sitter 3-space of radius $1/c$ in $\mathbb{E}^4_1$, i.e.,

$$\mathbb{S}^3_1(c^2) = \{(t, x_1, x_2, x_3) \in \mathbb{E}^4_1 : -t^2 + x_1^2 + x_2^2 + x_3^2 = 1/c^2\}.$$

This de Sitter 3-space has constant positive sectional curvature c^2.

Theorem 8.2 (Reformulation of Theorem 3.2). *Let M be an open simply connected Riemann surface with a base point $z_0 \in M$ and α an $\mathfrak{sl}(2;\mathbb{C})$-valued holomorphic 1-form in M. Suppose that α satisfies the following two conditions:*

$$\det\alpha = 0, \tag{8.1}$$

$$\mathrm{tr}\{\widetilde{\alpha}\sigma_3\alpha^*\sigma_3\} > 0, \tag{8.2}$$

where $\widetilde{\alpha}$ is the cofactor matrix of α. Then there exists a unique holomorphic immersion $F : M \longrightarrow \mathrm{SL}(2;\mathbb{C})$ such that

1. *$F(z_0) = \sigma_0$,*
2. *$F^{-1}dF = \alpha$,*
3. *$f = \frac{1}{c}F\sigma_3F^* : M^2 \longrightarrow \mathbb{S}^3_1(c^2)$ is a conformal spacelike CMC $\pm c$ immersion.*

Proof. Let $\alpha := F^{-1}dF$ and $F(z_0) = \sigma_0$. The equation $F^{-1}dF = \alpha$ satisfies the integrability condition; hence $F^{-1}dF = \alpha$ with the initial value condition $F(z_0) = \sigma_0$ has a unique solution $F : M \longrightarrow \mathrm{SL}(2;\mathbb{C})$. Since $F^{-1}dF = \alpha$ is a holomorphic 1-form, F is a holomorphic map. From the condition $\det \alpha = 0$, we have $\det(F^{-1}dF) = 0$, that is, F is null. Recall that the holomorphic 1-form $F^{-1}dF = \alpha$ can be locally written as

$$\alpha = \begin{pmatrix} pq & -p^2 \\ q^2 & -pq \end{pmatrix} \eta,$$

where $p,\ q$ are smooth functions such that $|p|^2 - |q|^2 = 1$, and η is a 1-form of type $(1,0)$. Then the cofactor matrix $\widetilde{\alpha}$ of α is

$$\widetilde{\alpha} = \begin{pmatrix} -pq & p^2 \\ -q^2 & pq \end{pmatrix} \eta$$

and

$$\alpha^* = \begin{pmatrix} \bar{p}\bar{q} & \bar{q}^2 \\ -\bar{p}^2 & -\bar{p}\bar{q} \end{pmatrix} \bar{\eta}.$$

Hence,

$$\widetilde{\alpha}\sigma_3\alpha^*\sigma_3 = \begin{pmatrix} -|p|^2|q|^2 + |p|^4 & -p\bar{q} \\ \bar{p}q & |q|^4 + -|p|^2|q|^2 \end{pmatrix} \eta \otimes \bar{\eta}$$

and

$$ds_f^2 = \frac{1}{c^2}\eta \otimes \bar{\eta} = \frac{1}{c^2}\operatorname{tr}\{\widetilde{\alpha}\sigma_3\alpha^*\sigma_3\} > 0$$

by the assumption. This means that F is an immersion. Therefore, by Theorem 3.2, $f = \frac{1}{c^2}F\sigma_3F^* :\ M \longrightarrow \mathbb{S}^3_1(c^2)$ is a conformal spacelike CMC $\pm c$ immersion. □

REMARK 8.3. The representation formula in Theorem 8.2 had previously been discovered by R. Aiyama and K. Akutagawa and is stated in their paper [1] without proof. The author is grateful to J. Inoguchi [8] and W. Rossman [18] for pointing out to him this work of R. Aiyama and K. Akutagawa, which was unknown to him at that time. The author thanks R. Aiyama for providing him her papers including [1].

REMARK 8.4 (SU(1,1) ambiguity). Let $b \in \mathrm{SU}(1,1)$. Then the $\mathfrak{sl}(2;\mathbb{C})$-valued holomorphic 1-form $b\alpha b^*$ induces the same immersion f.

REMARK 8.5 (An isometric perturbation of spacelike CMC $\pm c$ immersions in $\mathbb{S}^3_1(c^2)$). Let $F : M \longrightarrow \mathrm{SL}(2;\mathbb{C})$ be a holomorphic null immersion. Then, by Theorem 3.2, $f = \frac{1}{c}F\sigma_3F^*$ is a spacelike CMC $\pm c$ immersion in $\mathbb{S}^3_1(c^2)$. If $b \in \mathrm{SU}(1,1)$, then $bFb^* : M \longrightarrow \mathrm{SL}(2;\mathbb{C})$ is also a holomorphic null immersion. Thus, by Theorem 3.2,

$$\frac{1}{c}(bFb^*)\sigma_3(bFb^*)^* = b\left(\frac{1}{c}F\sigma_3F^*\right)b^* = bfb^*$$

is a spacelike CMC $\pm c$ immersion in $\mathbb{S}^3_1(c^2)$. We show that

$$\operatorname{tr}\{[\widetilde{(bFb^*)^{-1}d(bFb^*)}]\sigma_3[(bFb^*)^{-1}d(bFb^*)]^*\sigma_3\} = \operatorname{tr}\{(\widetilde{F^{-1}dF})\sigma_3(F^{-1}dF)^*\sigma_3\},$$

and hence that the induced metrics df_f^2 and $df_{bfb^*}^2$ are the same. That is, the spacelike CMC $\pm c$ immersion f and its perturbation bfb^* for $b \in \mathrm{SU}(1,1)$ are isometric to each other. We first compute

$$\begin{aligned}(bFb^*)^{-1}d(bFb^*) &= (b^*)^{-1}F^{-1}b^{-1}b(dF)b^* \\ &= (\sigma_3 b\sigma_3)F^{-1}(\sigma_3 b^*\sigma_3)b(dF)b^* \\ &= (\sigma_3 b\sigma_3)F^{-1}dFb^* \\ &= (b^*)^{-1}(F^{-1}dF)b^*.\end{aligned}$$

If A and B are two nonsingular matrices, then $\widetilde{AB} = \widetilde{B}\widetilde{A}$. If A is a nonsingular matrix with $\det A = 1$, then $\widetilde{A} = A^{-1}$. With these properties, we get

$$\begin{aligned}&[\widetilde{(bFb^*)^{-1}d(bFb^*)}]\sigma_3[(bFb^*)^{-1}d(bFb^*)]^*\sigma_3 \\ &\qquad = [(b^*)^{-1}(\widetilde{F^{-1}dF})b^*]\sigma_3[(b^*)^{-1}F^{-1}dFb^*]^* \\ &\qquad = [(b^*)^{-1}(\widetilde{F^{-1}dFb^*})]\sigma_3[b(F^{-1}dF)^*b^{-1}]\sigma_3 \\ &\qquad = (b^*)^{-1}[(\widetilde{F^{-1}dF})(b^*\sigma_3 b)(F^{-1}dF)^*b^{-1}\sigma_3] \\ &\qquad = (b^*)^{-1}[(\widetilde{F^{-1}dF})\sigma_3(F^{-1}dF)^*b^{-1}\sigma_3].\end{aligned}$$

Since $\operatorname{tr}(AB) = \operatorname{tr}(BA)$,

$$\begin{aligned}&\operatorname{tr}\{[\widetilde{(bFb^*)^{-1}d(bFb^*)}]\sigma_3[(bFb^*)^{-1}d(bFb^*)]^*\sigma_3\} \\ &\qquad = \operatorname{tr}\{(b^*)^{-1}[(\widetilde{F^{-1}dF})\sigma_3(F^{-1}dF)^*b^{-1}\sigma_3]\} \\ &\qquad = \operatorname{tr}\{(\widetilde{F^{-1}dF})(b^*\sigma_3 b)(F^{-1}dF)^*(b^{-1}\sigma_3(b^*)^{-1})\} \\ &\qquad = \operatorname{tr}\{(\widetilde{F^{-1}dF})\sigma_3(F^{-1}dF)^*\sigma_3\}.\end{aligned}$$

Remark 8.6. The converse of Theorem 8.2 also holds, i.e., for a conformal spacelike CMC $\pm c$ immersion $f : M \longrightarrow \mathbb{S}^3_1(c^2)$ there exists a holomorphic $\mathfrak{sl}(2;\mathbb{C})$-valued 1-form α in M satisfying

$$\det\alpha = 0,$$
$$\operatorname{tr}\{\widetilde{\alpha}\sigma_3\alpha^*\sigma_3\} > 0.$$

Moreover, α is unique up to the change described in Remark 8.4.

Definition 8.7 (Tasaki-Umehara-Yamada deformation of Lie groups). Let $I \subset \mathbb{R}$ be an open interval and $\{G_t\}$ a 1-parameter family of connected Lie groups of dimension n. Then an $(n{+}1)$-dimensional real analytic differentiable structure on the set $\mathcal{L} = \{(t,a) :\ t \in I,\ a \in G_t\}$ is called a *real analytic deformation of Lie groups* if it satisfies the following three conditions:

(1) Each G_t is an analytic submanifold of $\mathcal{L}$.
(2) The multiplications of G_t are real analytic with respect to t.
(3) The curve $\imath(t) = (t, e_t)(t \in I)$ is real analytic with respect to the parameter t.

Let $\mathcal{L} = \{(t,a) :\ t \in I,\ a \in G_t\}$ be such a real analytic deformation of Lie groups. We denote by θ_t the Maurer-Cartan form of G_t. The Lie algebra of G_t is considered as n-dimensional Euclidean space $\mathbb{R}^n$ with a Lie bracket $[\ ,\ ]_t$. The following lemma holds.

LEMMA 8.8 (Tasaki-Umehara-Yamada). *Let M be a simply connected differentiable manifold and z_0 a base point. Let $\{\alpha_t\}_{t\in I}$ be a real analytic 1-parameter family of $\mathbb{R}^n$-valued 1-form of M such that*

$$d\alpha_t + \frac{1}{2}[\alpha_t \wedge \alpha_t]_t = 0.$$

Then there exists a unique real analytic 1-parameter family of maps $f_t :\ M \longrightarrow \mathcal{L}(t \in I)$ such that

(1) $f_t(M) \subset G_t$,
(2) $f_t^*\theta_t = \alpha_t$,
(3) $f_t(z_0) = \imath(t)$.

Proof. $d\alpha_t + \frac{1}{2}[\alpha_t \wedge \alpha_t]_t = 0$ is the integrability condition of the system $f_t^*\theta_t = \alpha_t$ with the initial condition $f_t(z_0) = \imath(t)$. Hence, by the Frobenius Theorem, the lemma holds. □

Let $p(c) :\ \mathbb{S}^3_1(c^2) \setminus \{x_3 = -1/c\} \longrightarrow \mathbb{E}^3_1$ be the map defined by

$$p(c)(t, x_1, x_2, x_3) = \frac{1}{1 + x_3}(t, x_1, x_2),\ (t, x_1, x_2, x_3) \in \mathbb{S}^3_1(c^2) \setminus \{x_3 = -1/c\}.$$

This map is, in fact, the stereographic projection from the point $(0, 0, 0, -1/c)$.

LEMMA 8.9. *Let $\{G(c)\}_{c\geq 0}$ be a 1-parameter family of Lie groups defined by*

$$G(c) = \begin{cases} \mathrm{SL}(2;\mathbb{C}) & (c \neq 0), \\ \mathbb{C}^3 & (c = 0). \end{cases}$$

There is a real analytic structure on the set $\mathcal{L} = \{(c,a) : c \in [0,\infty),\ a \in G(c)\}$ satisfying:

(1) *Each $G(c)$ is an analytic submanifold of $\mathcal{L}$.*
(2) *The curve $\imath(c) = (c, \mathrm{id})(c \in [0,\infty))$ in $\mathcal{L}$ is real analytic.*
(3) *The multiplications of $G(c)$ are real analytic with respect to the parameter c.*

Proof. Note that $G(c)$ can be identified with

$$\mathbb{S}^3_1(c^2)^{\mathbb{C}} = \{(t,\zeta) \in \mathbb{R}\times\mathbb{C}^3 :\ -t^2+\zeta_1^2+\zeta_2^2+\zeta_3^2 = 1/c^2\}$$

by identifying $(t,\zeta)\in\mathbb{S}^3_1(c^2)^{\mathbb{C}}$ with the 2×2 hermitian matrix

$$\nu(c)(t,\zeta) = ic\begin{pmatrix} t+\zeta_3 & \zeta_1+i\zeta_2 \\ \zeta_1-i\zeta_2 & t-\zeta_3\end{pmatrix} \in \mathrm{SL}(2;\mathbb{C}).$$

In $V_c := \mathbb{S}^3_1(c^2)^{\mathbb{C}} \setminus \{\zeta_3 = -1/c\}$, we define a map

$$2p(c)^{\mathbb{C}} :\ \hat{V}_c \longrightarrow W_c := \mathbb{R}\times\mathbb{C}^2 \setminus \{4+c^2(-t^2+\zeta_1^2+\zeta_2^2)=0\}$$

by

$$2p(c)^{\mathbb{C}}(t,\zeta_1,\zeta_2,\zeta_3) = \frac{2}{1+c\zeta_3}(t,\zeta_1,\zeta_2).$$

Hence,

$$\nu(c)\circ(2p(c)^{\mathbb{C}})^{-1} :\ W_c \longrightarrow V_c(:=\nu(c)(\hat{V}_c)\subset G(c)=\mathrm{SL}(2;\mathbb{C})).$$

Let $\xi^c := \nu(c)\circ(2p(c)^{\mathbb{C}})^{-1}(\xi)$ for $\xi\in W_c\subset\mathbb{R}\times\mathbb{C}^2$. Also, let $V_1 = \{(c,a)\in\mathcal{L} :\ a\in V_c \text{ if } c\neq 0\}$ and $V_2 = (0,\infty)\times\mathrm{SL}(2;\mathbb{C})$. Then the coordinates (c,ξ^c) on V_1 and the canonical coordinates on V_2 define a real analytic structure on $\mathcal{L}$. This real analytic structure on $\mathcal{L}$ satisfies (1), (2) and (3). □

For each $\xi\in W_c$,

$$\xi^c = \begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix} + ic\begin{pmatrix}\xi_1 & \xi_2+i\xi_3 \\ \xi_2-i\xi_3 & \xi_1\end{pmatrix} + O(c^2),$$
$$\lim_{c\to 0}(c,\xi^c) = (0,\xi) \text{ in } \mathcal{L},$$

and

$$\begin{aligned}&\lim_{c\to 0}(c,\xi^c\sigma_3(\xi^c)^*)\\ &= \lim_{c\to 0}(c,\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + c\begin{pmatrix}\xi_1+\bar{\xi}_1 & \xi_2+\bar{\xi}_2+i(\xi_3+\bar{\xi}_3) \\ \xi_2+\bar{\xi}_2-i(\xi_3+\bar{\xi}_3) & \xi_1+\bar{\xi}_1\end{pmatrix} + O(c^2))\\ &= (0,\xi+\bar{\xi}) \text{ in } \mathcal{L}.\end{aligned}$$

Let M be a Riemann surface and $f_0 : M \longrightarrow \mathbb{E}^3_1$ a conformal spacelike maximal immersion. Then the $\mathbb{C}^3$-valued holomorphic 1-form $\alpha = \partial f_0 = (\alpha_1,\alpha_2,\alpha_3)$ satisfies the following properties:

$$(\alpha_1)^2+(\alpha_2)^2-(\alpha_3)^2 = 0, \tag{8.3}$$

$$\alpha_1\overline{\alpha_1}+\alpha_2\overline{\alpha_2}+\alpha_3\overline{\alpha_3} > 0. \tag{8.4}$$

Conversely, when M is simply connected, the Weierstrass type representation formula (4.3) asserts that a $\mathbb{C}^3$-valued holomorphic 1-form α determines a conformal spacelike maximal immersion $f_0 : M \longrightarrow \mathbb{E}^3_1$ such that $\alpha = \partial f_0$.

We denote the first and second fundamental forms of f_0 by ds_0^2 and h_0, respectively. For each real number $c \in \mathbb{R}^*$ we define a symmetric covariant tensor of type $(0, 2)$ by

$$h_c = h_0 + c \cdot ds_0^2.$$

Then, by the fundamental theorem of the surfaces, there exists an isometric immersion $\widetilde{f}_c : (M, ds_0^2) \longrightarrow \mathbb{S}_1^3(c^2)$ whose second fundamental form is h_c. We call these immersions $\{\widetilde{f}_c\}_{c\in\mathbb{R}^*}$ a *canonical* 1*-parameter family of immersions* associated with the minimal immersion f_0. The following theorem shows that O. Kobayashi's representation formula is a limit of the Bryant type representation formula in Theorem 8.2 as $c \to 0$.

Theorem 8.10. *Let M be a simply connected Riemann surface and $z_0 \in M$ a base point. Let α be a holomorphic $\mathbb{C}^3$-valued* 1*-form in M satisfying* (8.3) *and* (8.4)*. Then for each $c \in \mathbb{R}^*$, the holomorphic $\mathfrak{sl}(2;\mathbb{C})$-valued* 1*-form*

$$\alpha(c) = c\begin{pmatrix} \alpha_3 & \alpha_1 + i\alpha_2 \\ \alpha_1 - i\alpha_2 & \alpha_3 \end{pmatrix}$$

satisfies the assumptions (8.1) *and* (8.2) *of Theorem 8.2. Moreover, the conformal spacelike immersion $\widetilde{f}_c = \frac{1}{c}F_c\sigma_3F_c^* : M^2 \longrightarrow \mathbb{S}_c^3(F_c(z_0) = I_2)$ with constant mean curvature $H = \pm c$ induced from $\alpha(c)$ by Theorem 8.2 has the following properties:*

(1) *There exists a conformal spacelike maximal immersion f_0 such that $f_0 = \lim_{c\to 0} 2p(c) \circ \widetilde{f}_c$ and $\alpha = \partial f_0$.*
(2) *The* 1*-parameter family of immersions $f_c : M \longrightarrow \mathbb{E}_1^3$ defined by*

$$f_c = \begin{cases} 2p(c) \circ \widetilde{f}_c & (c \neq 0), \\ f_0 & (c = 0) \end{cases}$$

is real analytic with respect to the parameter $c \in \mathbb{R}$.
(3) $f_c(z_0) = 0$.
(4) $\{\widetilde{f}_c\}_{c\in\mathbb{R}^*}$ *coincides with the canonical* 1*-parameter family of immersions associated with $\widetilde{f}_0$.*

Remark 8.11. The above theorem implies that the secondary Gauss map of each f_c is nothing but the Gauss map of f_0.

Proof. The first fundamental form $ds^2(c)$ and the Hopf differential $Q(c)$ are given by

$$ds^2(c) = \mathrm{trace}\{\widetilde{\alpha}\sigma_3\alpha^*\sigma_3\}, \tag{8.5}$$

$$Q(c) = (\alpha_1 - i\alpha_2)d\left(\frac{\alpha_3}{\alpha_1 - i\alpha_2}\right). \tag{8.6}$$

The assertion (4) is straightforward because $ds^2(c)$ and Q do not depend on the parameter c. Let $\alpha(0) = \alpha$. Then $\{\alpha(c)\}_{c\geq 0}$ is a 1-parameter family of

$\mathfrak{g}(c)$-valued 1-forms on M^2, where $\mathfrak{g}(c)$ is the Lie algebra of $G(c)$. This 1-parameter family $\{\alpha(c)\}_{c\geq 0}$ is real analytic with respect to c. Let $F_0 = \frac{1}{2} f_0$. Then $F_0 : M \longrightarrow \mathbb{E}^3_1 \subset \mathbb{C}^3$ and $F_c : M \longrightarrow \mathrm{SL}(2;\mathbb{C})(c \neq 0)$ satisfy $2dF_0 = \alpha(0)$ and $(F_c)^{-1} dF_c = \alpha(c)$. Hence Lemma 8.8 implies that the 1-parameter family of maps $(c, F_c) : M \longrightarrow \mathcal{L}(c \geq 0)$ is real analytic with respect to c, and thus

$$\lim_{c\to 0}(c, F_c) = (0, F_0) \text{ in } \mathcal{L},$$
$$\lim_{c\to 0}(c, F_c\sigma_3 F_c^*) = (0, F_0 + \overline{F_0}) = (0, f_0) \text{ in } \mathcal{L}.$$

For each $c > 0$, we get $\nu(c)(\widetilde{f}_c) = F_c\sigma_3 F_c^*$ and

$$2p(c) \circ \widetilde{f}_c = (2p(c))^{\mathbb{C}}(\widetilde{f}_c) = (2p(c))^{\mathbb{C}} \circ \nu(c)^{-1}(F_c\sigma_3 F_c^*).$$

Therefore, $\lim_{c\to 0} 2p(c) \circ \widetilde{f}_c = f_0$. □

9. Appendix: Some examples of spacelike CMC ±1 surfaces in $\mathbb{S}^3_1(1)$

In this appendix, we present some examples of spacelike CMC ±1 surfaces in $\mathbb{S}^3_1(1)$.

Let us consider the following stereographic projections in order to view the isometric images of spacelike CMC ±1 surfaces in $\mathbb{S}^3_1(1)$ into the exterior $\mathrm{Ext}\,\mathbb{H}^2(-1) = \{(x_0, x_1, x_2) \in \mathbb{E}^3_1 : -x_0^2 + x_1^2 + x_2^2 > -1\}$ of hyperbolic 2-space[11] $\mathbb{H}^2(-1)$.

Let $\wp_+ : \mathbb{S}^3_1(1)\setminus\{x_3 = -1\} \longrightarrow \mathbb{E}^3_1\setminus\mathbb{H}^2(-1)$ be the stereographic projection from $-e_3 = (0, 0, 0, -1)$. Then

$$\wp_+(x_0, x_1, x_2, x_3) = \left(\frac{x_0}{1 + x_3}, \frac{x_1}{1 + x_3}, \frac{x_2}{1 + x_3}\right). \tag{9.1}$$

Let $\wp_- : \mathbb{S}^3_1(1) \setminus \{x_3 = 1\} \longrightarrow \mathbb{E}^3_1 \setminus \mathbb{H}^2(-1)$ be the stereographic projection from $e_3 = (0, 0, 0, 1)$. Then

$$\wp_-(x_0, x_1, x_2, x_3) = \left(\frac{x_0}{1 - x_3}, \frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3}\right). \tag{9.2}$$

Cut $\mathbb{S}^3_1(1)$ into two halves by the hyperplane $x_3 = 0$. Denote by $\mathbb{S}^3_1(1)_+$ (resp. $\mathbb{S}^3_1(1)_-$) the half containing $e_3 = (0, 0, 0, 1)$ (resp. $-e_3 = (0, 0, 0, -1)$). Then $\wp_+ : \mathbb{S}^3_1(1)_+ \longrightarrow \mathrm{Ext}\,\mathbb{H}^2(-1)$ and $\wp_- : \mathbb{S}^3_1(1)_- \longrightarrow \mathrm{Ext}\,\mathbb{H}^2(-1)$.

[11] Usually, the upper hyperboloid $\{(x_0, x_1, x_2) \in \mathbb{E}^3_1 : -x_0^2 + x_1^2 + x_2^2 = -1 \text{ and } x_0 > 0\}$ is called the hyperbolic 2-space $\mathbb{H}^2(-1)$. The upper hyperboloid is isometrically diffeomorphic to the Poincaré model of hyperbolic 2-space via the stereographic projection from $-e_0 = (-1, 0, 0)$ as seen in Section 4. In this section, we regard the hyperboloid of two sheets $\{(x_0, x_1, x_2) \in \mathbb{E}^3_1 : -x_0^2 + x_1^2 + x_2^2 = -1\}$ as hyperbolic 2-space $\mathbb{H}^2(-1)$.

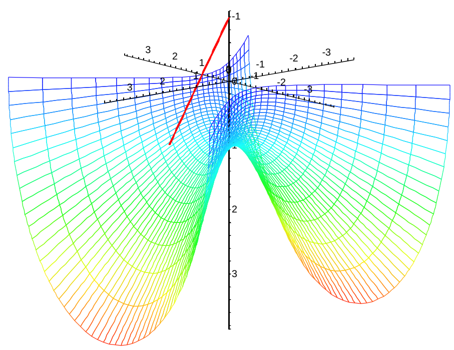


FIGURE 1. $\wp_+ : \mathbb{S}^3_1(1)_+ \longrightarrow \mathrm{Ext}\,\mathbb{H}^2(-1)$

EXAMPLE 9.1 (Spacelike Enneper cousin in $\mathbb{S}^3_1(1)$). Let $(g,h) = (z,1)$. Then using the Umehara-Yamada type representation (4.2), we set up the following initial value problem:

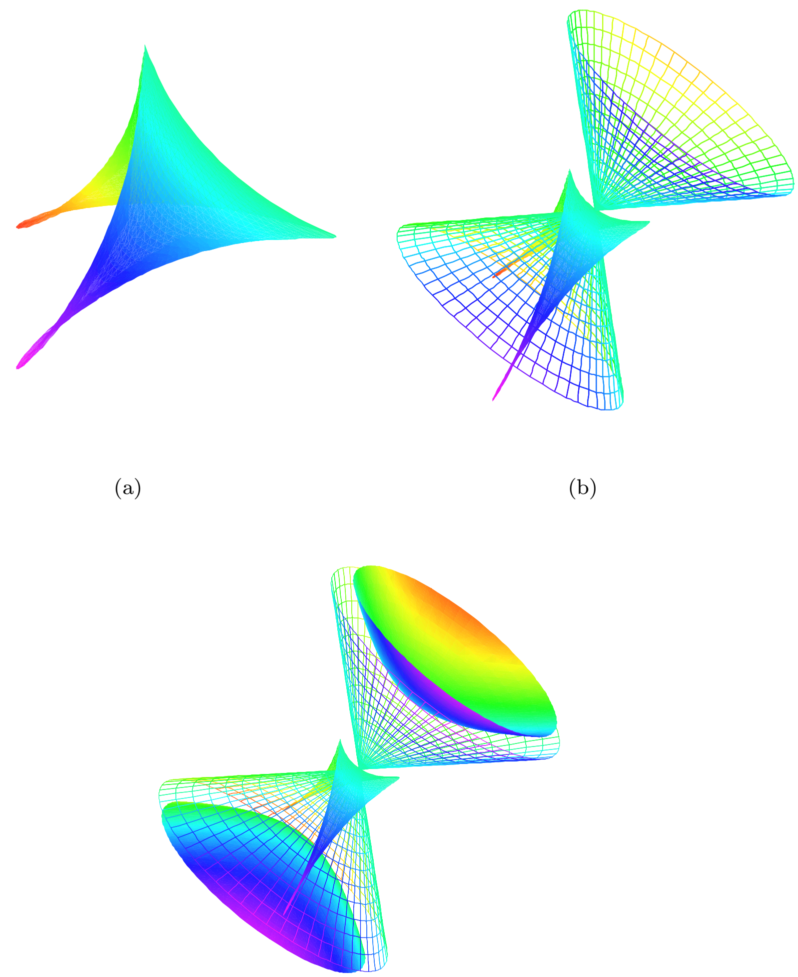
$$F^{-1}dF = \begin{pmatrix} z & -1 \\ z^2 & -z \end{pmatrix} dz,\ F(0) = \sigma_0.$$

This initial value problem has a unique solution

$$F = \begin{pmatrix} z\sin z + \cos z & -\sin z \\ -z\cos z + \sin z & \cos z \end{pmatrix}, \tag{9.3}$$

which is a holomorphic null immersion into $\mathrm{SL}(2;\mathbb{C})$. The Bryant type representation formula (3.1) then yields a spacelike CMC ± 1 surface in $\mathbb{S}^3_1(1)$. The resulting surface corresponds to the spacelike Enneper surface in $\mathbb{E}^3_1$ under the Lawson correspondence. For this reason, the resulting surface is called *spacelike Enneper cousin* in $\mathbb{S}^3_1(1)$.

Figure 2 shows the spacelike Enneper cousin projected into $\mathrm{Ext}\,\mathbb{H}^2(-1)$ via $\wp_+$.



(a) (b)

(c)

FIGURE 2. Spacelike Enneper cousin projected into $\text{Ext}\,\mathbb{H}^2(-1)$ via $\wp_+$ with light cone and the boundary $\mathbb{H}^2(-1)$ in $\mathbb{E}^3_1$

EXAMPLE 9.2 (Spacelike catenoid cousin in $\mathbb{S}^3_1(1)$). Let $(g, h) = (1/z, 1)$. Then, using the Umehara-Yamada type representation (4.2), we set up the following initial value problem:

$$F^{-1}dF = \begin{pmatrix} 1/z & -1 \\ 1/z^2 & -1/z \end{pmatrix} dz, \; F(1) = \sigma_0.$$

This initial value problem has a unique solution

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22}, \end{pmatrix}, \tag{9.4}$$

where

$$F_{11} = -\frac{3-\sqrt{5}}{2}\left(\frac{1}{2} + \frac{3}{10}\sqrt{5}\right) z^{-\frac{1-\sqrt{5}}{2}},$$
$$F_{12} = -\frac{3-\sqrt{5}}{2}\left(\frac{1}{2} + \frac{3}{10}\sqrt{5}\right) z^{\frac{1+\sqrt{5}}{2}} - \frac{3+\sqrt{5}}{2}\left(\frac{1}{2} - \frac{3}{10}\sqrt{5}\right) z^{\frac{1-\sqrt{5}}{2}},$$
$$F_{21} = \frac{\sqrt{5}}{5}(z^{-\frac{1-\sqrt{5}}{2}} - z^{-\frac{1+\sqrt{5}}{2}}),$$
$$F_{22} = \left(\frac{1}{2} - \frac{3}{10}\sqrt{5}\right) z^{\frac{1+\sqrt{5}}{2}} + \left(\frac{1}{2} + \frac{3}{10}\sqrt{5}\right) z^{\frac{1-\sqrt{5}}{2}}.$$

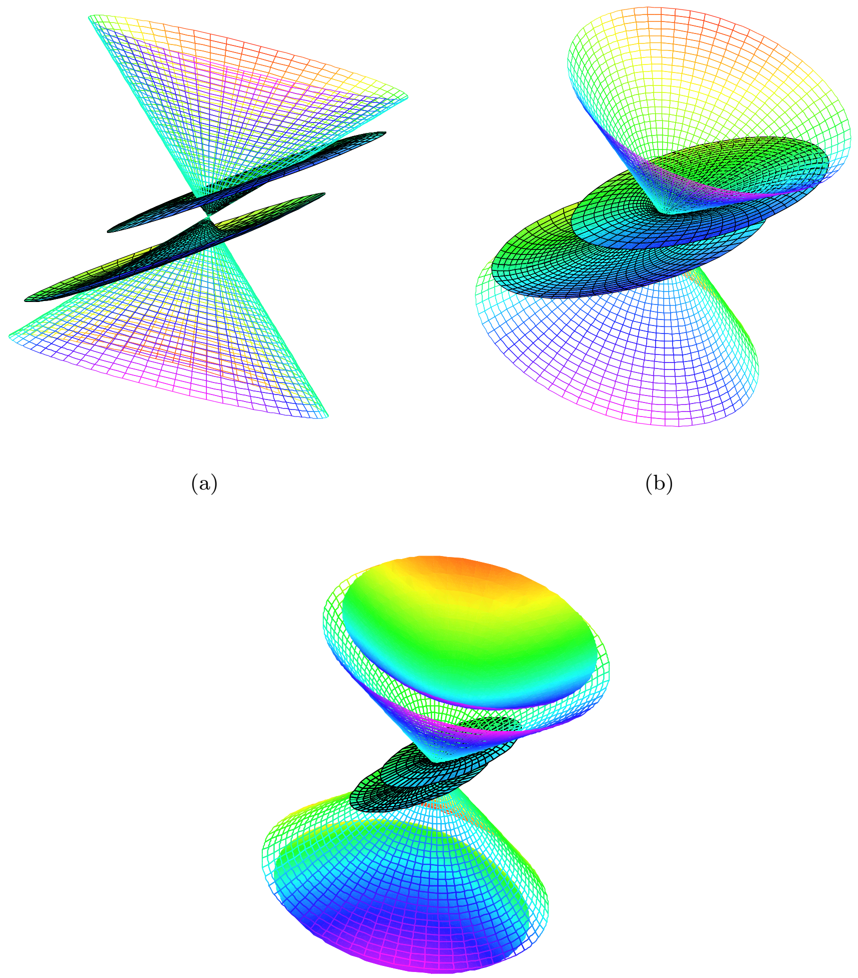
The map F is a holomorphic null immersion into $\mathrm{SL}(2;\mathbb{C})$ and the Bryant type representation formula (3.1) yields a spacelike CMC ± 1 surface in $\mathbb{S}^3_1(1)$. The resulting surface corresponds to the spacelike catenoid in $\mathbb{E}^3_1$ under the Lawson correspondence. For this reason, the resulting surface is called *spacelike catenoid cousin* in $\mathbb{S}^3_1(1)$.

Figure 3 shows different views of the spacelike catenoid cousin projected into $\mathrm{Ext}\,\mathbb{H}^2(-1)$ via $\wp_+$.

K. Akutagawa [2] and J. Ramanathan [17] proved the following theorem.

THEOREM 9.3. *Let M be a complete spacelike surface in $\mathbb{S}^3_1(1)$ with constant mean curvature ± 1. Then M is a totally umbilic flat surface. Moreover, M is a parabolic type surface of revolution.*

By Theorem 9.3 and Corollary 7.3, the hyperbolic Gauss map of complete CMC ± 1 spacelike surfaces is constant. Hence they can be regarded as analogs of the *horospheres* in hyperbolic 3-space $\mathbb{H}^3(-1)$.



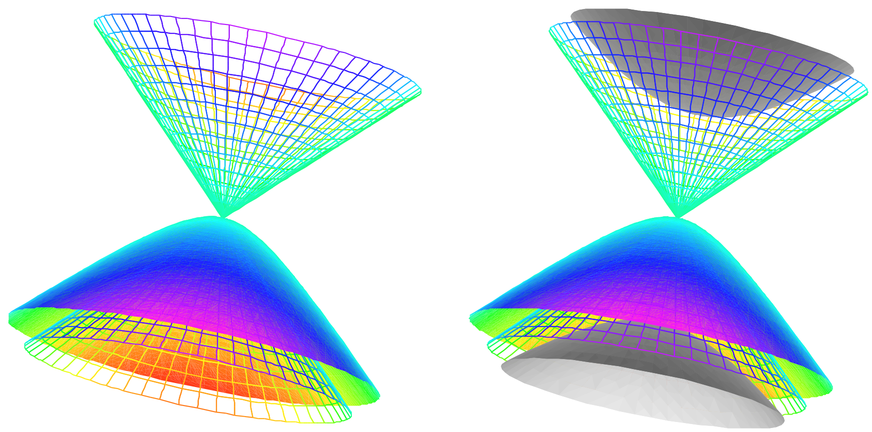
(a) (b)

(c)

FIGURE 3. Spacelike catenoid cousin projected into $\operatorname{Ext}\mathbb{H}^2(-1)$ via $\wp_+$ with light cone and the boundary $\mathbb{H}^2(-1)$ in $\mathbb{E}^3_1$

EXAMPLE 9.4 (Horosphere type spacelike surfaces in $\mathbb{S}^3_1(1)$). Let $(G,H) = (0,1)$. Then using equation (6.1), we set up the initial value problem:

$$FdF^{-1} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} dz, \;\; F(0) = \sigma_0.$$



(a) (b)

FIGURE 4. Horosphere type spacelike surface (9.5) projected into $\mathrm{Ext}\,\mathbb{H}^2(-1)$ via $\wp_+$ with light cone and the boundary $\mathbb{H}^2(-1)$ in $\mathbb{E}^3_1$

This initial value problem has a unique solution

$$F = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix},$$

which is a holomorphic null immersion into $\mathrm{SL}(2;\mathbb{C})$. The Bryant type representation formula (3.1) yields a horosphere type spacelike surface in $\mathbb{S}^3_1(1)$

$$f = F\sigma_3 F^* = \begin{pmatrix} 1-|z|^2 & -z \\ -\bar{z} & -1 \end{pmatrix}. \tag{9.5}$$

Figure 4 shows the horosphere type spacelike surface (9.5) into $\mathrm{Ext}\,\mathbb{H}^2(-1)$ via $\wp_+$.

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