EXTENSIONS OF BRANDT SEMIGROUPS AND APPLICATIONS[1]

BY

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Clifford gave a general means of finding all possible extensions of a (weakly reductive) semigroup S by a semigroup T with zero [2]. However, as in group theory, it is generally difficult to give an explicit determination of the extensions for special types of semigroups. This has been done for only two cases: (1) S completely simple and T arbitrary. (2) S a group and T a completely 0-simple semigroup [2]. The first is due to Clifford [1] and the second to Munn [2]. In [3], Warne determined when the extensions of a completely 0-simple semigroup by a completely 0-simple semigroup are determined by a partial homomorphism.

The main result of this paper is the determination of all extensions of a Brandt semigroup by an arbitrary semigroup. We first use this theorem to determine when an extension of a Brandt semigroup by a regular 0-bisimple semigroup is given by a partial homomorphism. We then use the theorem to find the number of extensions of a Brandt semigroup by a simple group (with zero) in a certain case.

Let S and T be disjoint semigroups, T having a zero element 0. A semigroup V will be called an (ideal) extension of S by T if it contains S as an ideal, and if the Rees factor semigroup V/S [1] is isomorphic with T.

Let V be an extension of a semigroup S by a semigroup T with zero; we will use the following notations. If S has a zero, it is denoted by 0 (0 is then automatically the zero of V). The zero of T is denoted by $0'$. Multiplication in V is denoted by $\circ$, while multiplication in S or T is denoted simply by juxtaposition. The elements of S are denoted by lower case and the elements of T by capital roman letters. The set of non-zero elements of any semigroup P with zero is denoted by P^*.

If V is an extension of S by T (with zero) we say that V is determined by a partial homomorphism if there exists a partial homomorphism $\pi : T^* \to S$ such that for all $A, B \in T^*$, $c, d \in S$

$$\begin{aligned} A \circ B &= AB && \text{if} \quad AB \neq 0' \\ &= (A\pi)(B\pi) && \text{if} \quad AB = 0'; \end{aligned}$$

$$A \circ c = (A\pi)c; \qquad c \circ A = c(A\pi); \qquad c \circ d = c\,d.$$

When S is a Brandt semigroup we write $S = M^0(G; I, I; \Delta)$ and let e be the identity of G [2].

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If A is any non-empty set, $\mathcal{I}_A$ will denote the full symmetric inverse semigroup on A [2]. We denote multiplication in $\mathcal{I}_A$ by juxtaposition. If $\alpha \in \pi_I$, the rank of α is the cardinal number of the range of α. If J is any set, $| J |$ denotes its cardinal number.

For all concepts and notations not defined in this paper the reader is referred to [2].

THEOREM 1. *Let V be an extension of a Brandt semigroup S by an arbitrary semigroup T with zero. Let S be given the Rees representation $S = M^0(G; I, I; \Delta)$. Then there exists a partial homomorphism $w : A \to w_A$ of T^* into $\mathcal{I}_I$, the full symmetric inverse semigroup on I. Let s_A and t_A denote the domain and range of w_A respectively. If $AB = 0'$; either $t_A \cap s_B = \square$ or $t_A \cap s_B$ is a single element $d_{A,B}$. For each A in T^*, there exists a mapping ψ_A of s_A into the group G such that for $AB \neq 0'$*

$$(*) \qquad (i\psi_A)(iw_A\psi_B) = i\psi_{AB} \qquad \text{for all} \quad i \in s_{AB}.$$

The products in V are given by

$$(1) \quad \text{(a)} \quad A \circ B = AB \qquad \text{if} \quad AB \neq 0' \quad \text{in} \quad T,$$

$$\text{(b)} \quad A \circ B = 0 \quad (\text{in} \quad S) \quad \text{if} \quad AB = 0' \quad (\text{in} \quad T) \quad \text{and} \quad t_A \cap s_B = \square,$$

$$\text{(c)} \quad A \circ B = ((d_{A,B}\, w_A^{-1}\psi_A)(d_{A,B}\,\psi_B);\, d_{A,B}\, w_A^{-1},\, d_{A,B}\, w_B)$$
$$\text{if} \quad AB = 0' \ (\text{in } T) \ \text{and} \ t_A \cap s_B = d_{A,B}.$$

$$(2) \qquad (a; i, j) \circ A = (a(j\psi_A); i, jw_A) \quad \text{if} \quad j \in s_A$$
$$= 0 \qquad \text{if} \quad j \bar{\in} s_A$$
$$0 \circ A = 0$$

$$(3) \qquad A \circ (a; i, j) = ((iw_A^{-1}\psi_A)a; iw_A^{-1}, j) \quad \text{if} \quad i \in t_A$$
$$= 0 \qquad \text{if} \quad i \bar{\in} t_A$$
$$A \circ 0 = 0$$

Conversely, let S be a Brandt semigroup and T be a semigroup with zero such that $S \cap T = \square$. If we are given the mappings w and ψ_A described above and define product $\circ$ in the class sum of S and T^* by (1)–(3), then V is an extension of S by T.

Proof. Let V be an extension of $S = M^0(G; I, I; \Delta)$ by a semigroup T with zero. Let

$$s_A = \{i \in I \,|(a; k, i) \circ A \neq 0 \quad \text{for all} \quad k \in I, a \in G\}.$$

If $i \in s_A$, $(e; i, i) \circ A = (z; k, l)$, $z \in G$, $k, l \in I$.

Thus

$$(e; i, i) \circ A = (e; i, i)(z; k, l)$$

and $i = k$, i.e., $(e; i, i) \circ A = (z; i, 1)$. Hence we may write

$$(e; i, i) \circ A = (i\psi_A ; i, iw_A)$$

where w_A maps s_A into I and ψ_A maps s_A into G. Now, if $j \in s_A$,

$$(a; i, j) \circ A = ((a; i, j) \circ (e; j, j)) \circ A = (a; i, j)((e, j, j) \circ A)$$
$$= (a; i, j)(j\psi_A ; j, jw_A) = (a(j\psi_A); i, jw_A).$$

We note that 0 is the zero of V. Hence, this yields (2).

Let A in T^* and let

$$t_A = \{i \in I \mid A \circ (a; i, j) \neq 0 \quad \text{for all} \quad j \in I,\, a \in G\}.$$

Similarly as above, we obtain

$$(3') \qquad \begin{aligned} A \circ (a; i, j) &= ((i\phi_A)a; i\gamma_A , j) && \text{if} \quad i \in t_A \\ &= 0 && \text{if} \quad i \bar{\in} t_A \end{aligned}$$
$$A \circ 0 = 0,$$

where γ_A maps t_A into I and ϕ_A maps t_A into G.

Let $i \in s_A$ and let $j = iw_A$. Then

$$((e; i, i) \circ A) \circ (e; j, i) = (i\psi_A ; i, iw_A) \circ (e; j, i) = (i\psi_A ; i, i).$$

Hence

$$(e; i, i) \circ (A \circ (e; j, i)) = (i\psi_A ; i, i)$$

which implies that $j \in t_A$. Similarly

$$(i\psi_A ; i, i) = (e; i, i) \circ (j\phi_A ; j\gamma_A , i)$$

which implies that $j\gamma_A = i$. After multiplying we obtain $i\psi_A = j\phi_A$. Summarizing, $i \in s_A$ and $j = iw_A$ implies $j \in t_A$, $j\gamma_A = i$, $i\psi_A = j\phi_A = iw_A\phi_A$.

The range of w_A is t_A and $iw_A\gamma_A = i$. Similarly, $j \in t_A$ and $j\gamma_A = i$ implies that $i \in s_A$ and $iw_A = j$. Hence the range of γ_A is s_A and $j\gamma_A w_A = j$. Thus w_A and γ_A are mutually inverse 1-1 mappings of s_A onto t_A and t_A onto s_A respectively. Consequently we may write $\gamma_A = w_A^{-1}$, and, if $j \in t_A$, we will write $j\phi_A = jw_A^{-1}\psi_A$. Hence (3′) reduced to (3).

Suppose that $A, B \in T^*$ and $AB \neq 0'$. We have $i \in s_A$ and $iw_A \in s_B$ if and only if $i \in (t_A \cap s_B)w_A^{-1}$, where $(t_A \cap s_B)w_A^{-1}$ is the domain of $w_A\, w_B$ (multiplication in $\mathcal{I}_I$). We obtain

$$((e; i, i) \circ A) \circ B = (i\psi_A ; i, iw_A) \circ B = ((i\psi_A)(iw_A\, \psi_B); i, iw_A\, w_B)$$

and

$$(e; i, i) \circ A \circ B = ((i\psi_{AB}); i, iw_{AB}).$$

Hence $i \in (t_A \cap s_B)w_A^{-1}$ if and only if $i \in s_{AB}$. In this case $iw_{AB} = iw_A\, w_B$. This means $w_{AB} = w_A\, w_B$ (in $\mathcal{I}_I$). Thus $w : A \to w_A$ is a partial homomorphism of T^* into $\mathcal{I}_I$. If $AB \neq 0'$ and $i \in s_{AB}$, $i\psi_{AB} = i\psi_A \, iw_A \, \psi_B$. $(*)$ is satisfied.

Suppose that A, $B \in T^*$ and $AB = 0'$. Then $A \circ B \in S$. Suppose that $A \circ B = 0$. If $s_B \cap t_A \neq \square$, there exists $i \in s_B \cap t_A$ and $j = iw_A^{-1}$ implies that $j \in s_A$ and $jw_A \in s_B$. Hence $((e; j, j) \circ A) \circ B \neq 0$ contradicting the fact that $(e; j, j) \circ (A \circ B) = 0$. Thus, $s_B \cap t_A = \square$. Next suppose that $A \circ B \in S^*$. Let $A \circ B = (g_{A,B}, i_{A,B}, k_{A,B})$. Then

$$(4) \qquad \begin{aligned} (e; i_{A,B}, i_{A,B})(A \circ B) &= (e; i_{A,B}, i_{A,B})(g_{A,B}; i_{A,B}, k_{A,B}) \\ &= (g_{A,B}; i_{A,B}, k_{A,B}), \end{aligned}$$

$$(4') \qquad \begin{aligned} (e; i_{A,B}, i_{A,B}) \circ A) \circ B &= ((i_{A,B}\psi_A); i_{A,B}, i_{A,B} w_A) \circ B \\ &= ((i_{A,B}\psi_A)(i_{A,B} w_A \psi_B); i_{A,B}, i_{A,B} w_A w_B). \end{aligned}$$

Consequently $i_{A,B} \in s_A$, $i_{A,B} w_A \in s_B$, i.e., $i_{A,B} w_A \in s_B \cap t_A$. Suppose that $i \in s_B \cap t_A$. If $j = iw_A^{-1}$, then $jw_A \in s_B$ and $j \in s_A$. Hence

$$\begin{aligned} ((e; j, j) \circ A) \circ B &= ((j\psi_A); j, jw_A) \circ B = ((j\psi_A)(jw_A \psi_B); j, jw_A w_B) \\ &= (e; j, j)(A \circ B) = (e; j, j)(g_{A,B}; i_{A,B}, k_{A,B}). \end{aligned}$$

Thus $j = i_{A,B}$ and $i = jw_A = i_{A,B} w_A$ so that $s_B \cap t_A = i_{A,B} w_A$. Therefore $AB = 0'$ implies that either $s_B \cap t_A = \square$ or $s_B \cap t_A$ contains a single element. More specifically $A \circ B = 0$ (in S) implies that $s_B \cap t_A = \square$ while $A \circ B$ in S^* implies that $s_B \cap t_A = d_{A,B}$. In particular (1) has also been established. For if $AB = 0'$ and $s_B \cap t_A = \square$ then $A \circ B = 0$ and if $AB = 0'$ and $s_B \cap t_A = d_{A,B}$, then by (4) and (4')

$$g_{A,B} = (i_{A,B}\psi_A)(i_{A,B} w_A \psi_B) \quad \text{and} \quad k_{A,B} = i_{A,B} w_A w_B .$$

Since $d_{A,B} = i_{A,B} w_A$,

$$d_{A,B} w_A^{-1} = i_{A,B} \quad \text{and} \quad k_{A,B} = d_{A,B} w_A^{-1} w_A w_B = d_{A,B} w_B ,$$

and (1)(c) is established.

We next establish the converse. It is only necessary to verify the associative law.

Case I. T^*T^*S. First suppose that A, B in T^* and $AB \neq 0'$ (in T). Note that $i \in t_{AB}$ if and only if $i \in t_B$ and $iw_B^{-1} \in t_A$. Thus either $(AB) \circ (a; i, j)$ and $A \circ (B \circ (a; i, j))$ are both equal to 0 or both are different from 0. We consider the latter case.

$$\begin{aligned} (AB) \circ (a; i,j) &= ((iw_{AB}^{-1} \psi_{AB})a; iw_{AB}^{-1}, j) \\ A \circ (B \circ (a; i, j)) &= A \circ ((iw_B^{-1}\psi_B)a; iw_B^{-1}, j) \\ &= ((iw_B^{-1}w_A^{-1}\psi_A)(iw_B^{-1}\psi_B)a; iw_B^{-1}w_A^{-1}, j). \end{aligned}$$

We have $iw_B^{-1}w_A^{-1} = iw_{AB}^{-1}$ since w is a partial homomorphism. If $i \in t_{AB}$,

$iw_{AB}^{-1} \in s_{AB}$ and by $(*)$, we obtain

$$(iw_{AB}^{-1}\,\psi_A)(iw_{AB}^{-1}\,w_A\,\psi_B) = iw_{AB}^{-1}\,\psi_{AB}\,.$$

Consequently

$$(iw_B^{-1}w_A^{-1}\psi_A)(iw_B^{-1}\psi_B) = iw_{AB}^{-1}\,\psi_{AB}\,.$$

Hence $(AB)\circ(a; i, j) = A\circ(B\circ(a; i, j))$ if $AB \neq 0'$ (in T).
Next, we suppose that $AB = 0'$. If $s_B \cap t_A = \square$, $(A\circ B)\circ(a; i, j) = 0$. In addition $A\circ(B\circ(a; i, j)) = 0$ since $i \in t_B$ and $iw_B^{-1} \in t_A$ would imply that $i \in t_{AB}$, i.e., $iw_B^{-1} \in t_A \cap s_B$.

Suppose that $AB = 0'$ and $s_B \cap t_A = d_{A,B}$. Then

$$(A\circ B)\circ(a; i, j) = ((d_{A,B}\,w_A^{-1}\psi_A)(d_{A,B}\,\psi_B); d_{A,B}\,w_A^{-1}, d_{A,B}\,w_B)(a; i, j).$$

Now $d_{A,B}\,w_B = i$ iff $i \in t_B$ and $iw_B^{-1} \in t_A$. Hence $(A\circ B)\circ(a; i, j)$ and $A\circ(B\circ(a; i, j))$ are both equal to zero or both are not equal to zero. In the latter case

$$\begin{aligned}(A\circ B)\circ(a; i, j) &= ((d_{A,B}\,w_A^{-1}\psi_A)(d_{A,B}\,\psi_B)a; d_{A,B}\,w_A^{-1}, j)\\ A\circ(B\circ(a; i, j)) &= A\circ((iw_B^{-1}\psi_B)a; iw_B^{-1}, j)\\ &= ((iw_B^{-1}w_A^{-1}\psi_A)(iw_B^{-1}\psi_B)a; iw_B^{-1}w_A^{-1}, j).\end{aligned}$$

Since $d_{A,B}\,w_A^{-1} = iw_B^{-1}w_A^{-1}$ and $d_{A,B} = iw_B^{-1}$, we obtain

$$(A\circ B)\circ(a; i, j) = A\circ(B\circ(a; i, j)).$$

Case II. ST^*T^*; this case is treated similarly as Case I.

Case III. ST^*S. We have

$$((a; i, j)\circ A)\circ(b; k, 1) = (a(j\psi_A); i, jw_A)(b; k, l) \quad \text{if } j \in s_A$$

$$(a(j\psi_A); i, jw_A)(b; k, l) = (a(j\psi_A)b; i, l) \quad \text{if } k = jw_A\,,$$

i.e.,

$$((a; i, j)\circ A)(b; k, l) \neq 0 \quad \text{if and only if} \quad j \in s_A \text{ and } k = jw_A\,.$$

Similarly $(a; i, j)(A\circ(b; k, l)) \neq 0$ if and only if $k \in t_A$ and $j = kw_A^{-1}$.

Now $j \in s_A$ and $k = jw_A$ if and only if $k \in t_A$ and $kw_A^{-1} = j$. Hence

$$((a; i, j)\circ A)(b; k, l) \text{ and } (a; i, j)(A\circ(b; k, l))$$

are both equal to zero or both are different from zero. In the latter case,

$$\begin{aligned}((a; i, j)\circ A)(b; k, l) &= (a(j\psi_A)b; i, l)\\ &= (a(kw_A^{-1}\psi_A)b; i, l) = (a; i, j)(A\circ(b; k, l)).\end{aligned}$$

Case IV. SST^*. We have

$$((a; i, j)(b; k, l))\circ A = (ab; i, l)\circ A \quad \text{if } j = k, \tag{5}$$

$$(ab; i, l)\circ A = ((ab)(l\psi_A); i, lw_A) \quad \text{if } l \in s_A\,. \tag{6}$$

Consequently

$$((a; i, j)(b; k, l)) \circ A \neq 0 \quad \text{if and only if} \quad j = k \text{ and } l \in s_A .$$

Further

$$(7) \qquad (a; i, j)((b; k, l) \circ A) = (a; i, j)(b(l\psi_A); k, lw_A) \quad \text{if } l \text{ in } s_A ,$$

$$(8) \qquad (a; i, j)(b(l\psi_A); k, lw_A) = ((ab)(l\psi_A); i, lw_A) \quad \text{if } j = k.$$

Hence

$$(a, i, j)(b, k, l) \circ A) \neq 0 \quad \text{if and only if} \quad l \in s_A \text{ and } j = k.$$

Thus $((a; i, j)(b; k, l)) \circ A$ and $(a; i, j)((b; k, l) \circ A)$ are either both equal to zero or both are different from zero and in the latter case we have equality by (5), (6), (8) and (7).

Case V. T^*SS; this case is treated similarly as Case IV.

We have verified associativity for T^*T^*S, ST^*T^*, ST^*S, SST^* and T^*SS. $T^*T^*T^*$ and T^*ST^* are a consequence of the established cases by [1, Theorem 1, p. 166].

Remark. An extension of a Brandt semigroup by an arbitrary semigroup always exists [1].

The following general result shows us that an extension of a Brandt semigroup by an inverse semigroup must be an inverse semigroup.

THEOREM 2. *Let V be an extension of a semigroup S by a semigroup T. Then, V is an inverse semigroup if and only if S and T are inverse semigroups.*

Proof. Suppose S and T are inverse semigroups. Since S and T are regular, V is regular. Thus, each principal left ideal and each principal right ideal of V has an idempotent generator [2, Lemma 1.13, p. 27]. Suppose $eV = fV$. Then, $e = f \circ x, f = e \circ y$ where x, y in V. Thus, $f \circ e = f \circ (f \circ x = f \circ x = e, e \circ f = e \circ (e \circ y) = e \circ y = f$. If e, f in T^*, $e = f$. If e, f in S, $e = f$ [2, Th. 1.17, p. 28]. The cases, $e \in S, f \in T^*$ and $e \in T^*, f \in S$ are impossible. Thus, $e = f$ and every principal right ideal of V has a unique idempotent generator. Similarly, each principal left ideal of V has a unique idempotent generator. Thus V is an inverse semigroup [2, p. 28, Th. 1.17]. Suppose V is an inverse semigroup. Clearly T is regular. If a in S, there exists x in V such that $a \circ x \circ a = a$. Now $a \mathcal{R} e$ for some $e \in V$. Thus $e = a \circ z$ for z in V. Hence $e \in S$ and $ea = a$. Thus

$$a = a \circ x \circ a = a \circ x \circ (e \circ a) = a \circ (x \circ e) \circ a = a(xe)a$$

and S is regular.

It is easily seen that the idempotents in S and in T commute and hence S and T are inverse semigroups [2].

LEMMA 1. Let $\mathcal{I}_F$ *be the full symmetric inverse semigroup on any set F.*

Then if $A, B \in \mathcal{I}_F$, $A \mathcal{L} B$ *if and only if range of* A = *range of* B.

$A \mathcal{R} B$ *if and only if domain of* A = *domain of* B,

$A \mathcal{D} B$ *if and only if rank of* A = *rank of* B.

We next use Theorem 1 to determine when the extensions of a Brandt semigroup by a regular 0-bisimple semigroup are given by a partial homomorphism.

THEOREM 3. *An extension* V *of a Brandt semigroup* S *by a regular* 0*-bisimple semigroup* T *is given by a partial homomorphism if and only if there exists an idempotent* E *in* T^* *such that there is at most one idempotent of* S^* *under* E.

Proof. Let V be an extension of S by T satisfying the conditions of the theorem. Now, since $A \to w_A$ is a partial homomorphism of T^* into $\mathcal{I}_I$, if $E^2 = E$ in T^*, $w_E w_E = w_E$, i.e. w_E is an idempotent of $\mathcal{I}_I$. This means w_E is the identity transformation on $s_E = t_E$ [2, p. 29]. Then by $(*)$ of Theorem 1, we have $i\psi_E i\psi_E = i\psi_E$, i.e., $i\psi_E = e$, the identity of G, for all $i \in s_E$. Thus by (2) and (3) of Theorem 1, if $i \in s_E$, $(e; i, i) < E$. If $i \bar{\in} s_E$, $(e; i, i) \circ E = 0$. Hence either $s_E = \square$ or s_E is a single element. If $s_E = \square$, there exists no idempotent of S^* under E. If s_E is a single element, there exists precisely one idempotent of S^* under E.

Since T is 0-bisimple, T^*w is contained in a single $\mathcal{D}$-class of $\mathcal{I}_I$. If $A \in T^*$, $A\ D\ E$ and $|s_A| = |s_E|$ by Lemma 1. Therefore if $s_E = \square$, $s_A = \square$ for all $A \in T^*$, and if s_E is a single element, s_A is a single element for all $A \in T^*$. Let us first consider the case $s_A = \square$ for all $A \in T^*$. Clearly $t_A = \square$ for all A in T^*. Let $A\theta = 0$ for all $A \in T^*$. Hence by Theorem 1, V is given by the partial homomorphism θ of T^* into S. In the second case, write $w_A = (s_A, t_A)$. The multiplication in T^*w is then given as follows:

$$\begin{aligned} w_A w_B = (s_A, t_A)(s_B, t_B) &= (s_A, t_B) \quad \text{if } t_A = s_B \\ &= 0(\text{in } \mathcal{I}_I) \quad \text{if } t_A \neq s_B. \end{aligned}$$

With ψ_A as in Theorem 1, if $i \in s_A$ let $i\psi_A = s_A \psi_A = \chi_A$. By (2), Theorem 1,

$$(e; s_A, s_A) \circ A = (s_A \psi_A; s_A, s_A w_A).$$

Using this expression it is easily shown that $A \to \chi_A$ is a mapping of T^* into G.

If $i \in s_{AB} (AB \neq 0')$, $i = s_{AB} = s_A$ and $iw_A = t_A = s_B$. Hence $*$ of Theorem 1 becomes, $\chi_A \chi_B = \chi_{AB}$, i.e. $A \to \chi_A$ is a partial homomorphism of T^* into G.

The following statements are consequences of Theorem 1.

If $AB = 0'$ in T and $s_B \cap t_A = \square$, i.e. $s_B \neq t_A$,

$$(\chi_A; s_A, t_A)(\chi_B; s_B, t_B) = 0 = A \circ B.$$

If $AB = 0'$ and $s_B \cap t_A = d_{A,B}$, i.e. $t_A = s_B$, then

$$\begin{aligned} A \circ B = ((d_{A,B} w_A^{-1})\psi_A\, d_{A,B}\psi_B; d_{A,B} w_A^{-1}, d_{A,B} w_B) &= (s_A \psi_A s_B \psi_B; s_A, t_B) \\ = (\chi_A \chi_B; s_A, t_B) &= (\chi_A; s_A, t_A)(\chi_B; s_B, t_B). \end{aligned}$$

If $j = s_A$,

$$(a; i, j) \circ A = (a; (j\psi_A); i, jw_A) = (a(s_A\psi_A); i, t_A) = (a; i, j)(\chi_A; s_A, t_A).$$

If $j \neq s_A$,

$$(a; i, j) \circ A = 0 = (a; i, j)(\chi_A; s_A, t_A).$$

Similarly, $A \circ (a; i, j) = (\chi_A; s_A, t_A)(a; i, j)$.

Now, define $A\theta = (\chi_A; s_A, t_A)$. It remains only to show that θ is a partial homomorphism of T^* into S.

Since $s_{AB} = s_A$, $s_B = t_A$, and $t_{AB} = t_B$, if $AB \neq 0'$;

$$A\theta B\theta = (\chi_A; s_A, t_A)(\chi_B; s_B, t_B)$$
$$= (\chi_A\chi_B; s_A, t_B) = (\chi_{AB}; s_{AB}, t_{AB}) = (AB)\theta.$$

To establish the converse suppose that V is given by a partial homomorphism θ. Let E be any idempotent of T^*. If $e \leq E$, $e \leq E\theta$ and hence $e = 0$ or $e = E\theta$ since S is completely 0-simple.

We now apply Theorem 1 to give the number of extensions of a finite Brandt semigroup by certain simple groups (with zero). If $|I| = n$ let D_i, $i = 0, 1, 2, \cdots, n$, denote the $\mathfrak{D}$-classes of $\mathcal{I}_I$. D_i is the collection of elements of rank i (Lemma 1). Let G_r be the symmetric group on r symbols.

THEOREM 4. *If S is a finite Brandt semigroup and T^* is a simple group with $|T^*| > \max(|I|, |G|)$, then there are $2^{|I|}$ extensions of S by T.*

Proof. Let $n = |I|$. Let w be the homomorphism of T^* into $\mathcal{I}_I$ of Theorem 1 and suppose $T^*w \subseteq D_r$ for $r \geq 1$. If w is an isomorphism, $|T^*| \leq |G_r| \leq n!$ [3] contradicting the hypothesis. Thus for any $A \in T^*$, w_A is an idempotent of D_r and hence is the identity transformation on some set M_k of r elements. There are $\binom{n}{r}$ such sets. If we define θ_i by $A\theta_i = i\psi_A$, $i \in M_k$, $A \in T^*$, then θ_i, for each $i \in M_k$, is a homomorphism of T^* into G by $(*)$, Theorem 1. Since $|T^*| > |G|$, each θ_i is trivial. Thus by (2) and (3) of Theorem 1, there are at most $\binom{n}{r}$ extensions of S by T^*, such that $T^*w \subseteq D_r$. Hence the number of extensions of S by T cannot exceed

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n = 2^{|I|}.$$

Conversely if we let $i\psi_A = e$, the identity of G, for all $A \in T^*$, $i \in I$ and let T^*w run through the idempotents of $\mathcal{I}_I$ we obtain $2^{|I|}$ extensions of S by T by Theorem 1.

Remark (Added in proof). It is only necessary to assume that $|I|$ is finite in the statement of Theorem 4.

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