PASTING DIFFEOMORPHISMS OF $\mathbf{R}^n$

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THEOREM (Munkres [10; 8.1]). *Let h be any orientation preserving C^k diffeomorphism $(1 \leq k \leq \infty)$ of $\mathbf{R}^n$ onto itself. Then there is a C^k diffeomorphism $\tilde{h}$ of $\mathbf{R}^n$ onto itself which coincides with the identity near the origin and which coincides with h near infinity.*

COROLLARY. *If f, g are C^k diffeomorphisms of $\mathbf{R}^n$ onto itself such that gf^{-1} is orientation preserving, then there is a C^k diffeomorphism $\tilde{f}$ of $\mathbf{R}^n$ onto itself which coincides with f near the origin and which coincides with g near infinity.*

Proof. Let $h = gf^{-1}$ and apply the theorem. Then $\tilde{f} = \tilde{h}f$ works.

We call $\tilde{f}$ a *pasting* of f and g. Two questions arise: (1) If f and g are close to each other in some sense, can $\tilde{f}$ be chosen to be close to f and g? (2) Since $\tilde{f}$ coincides with f in a neighborhood of the origin, is it also possible to paste in cases where f and/or g fail to be differentiable on a subset of a smaller neighborhood of the origin? The method of proof, which Munkres used, does not seem to be useful for answering these questions, especially the second. We shall show that the first question has a general affirmative answer, but the notion of closeness used is not the usual one. These results are described in Section 1. The second question does not always have an affirmative answer. For technical reasons, we must restrict ourselves to the case $n \neq 4$. After this restriction, there is still an obstruction to pasting. These results are described in Section 2. In the final section, we compare our results with results which have been obtained on the differentiable Schoenflies problem. This leads to a result about what can happen to a piecewise differentiable homeomorphism when one attempts to smooth it by a convolution. An application is also made to Munkres' obstruction theory. While we do not bother to restate each of our results in the context of homeomorphisms of manifolds, which are diffeomorphisms on the complement of a discrete set, it is clear that these results apply in that context as well. In particular, every theorem has an analogue for homeomorphisms of the n-sphere, which have a common fixed point.

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1. Approximating diffeomorphisms

Let M be a complete Riemannian manifold of class C^∞, let $C^k(M)$ denote the class of k times continuously differentiable mapping from M to M $(1 \leq k \leq \infty)$, and let $\mathrm{Diff}^k(M)$ denote the subset of C^k diffeomorphisms of M onto itself.

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If $f \in C^k(M)$ and $\varepsilon : M \to (0, \infty)$ is a given continuous function, then $g \in C^k(M)$ is called a C^k *metric ε-approximation* to f provided that

$$\rho(f(x), g(x)) < \varepsilon(x),$$
$$\| Df(x) - Dg(x) \| < \varepsilon(x), \cdots, \| D^k f(x) - D^k g(x) \| < \varepsilon(x)$$

for all x in M (the metric ρ and the various norms are induced by the Riemannian metric [5; Chapter 1]). The topology of metric approximation on $C^k(M)$ is metrizable for M compact, but is not even first countable for M noncompact.

For the remainder of this section, we restrict ourselves to the case $M = \mathbf{R}^n$ with the usual metric. We want to study another notion of approximation which arises quite naturally in the study of diffeomorphisms. Since $\mathrm{Diff}^k(\mathbf{R}^n)$ has a group structure, it seems natural to wonder how close g comes to operating like f. Specifically, we shall say that g is a C^k *operational ε-approximation* to f provided that gf^{-1} is a C^k metric ε-approximation to the identity. Since the unit ball about the identity linear transformation I consists entirely of nonsingular linear transformations, it follows immediately that if $g \in C^1(R^n)$ is a C^1 operational 1-approximation to f, then g has nonsingular differential everywhere.

THEOREM 1.1. *If $f \in \mathrm{Diff}^k(\mathbf{R}^n)$ and if $g \in C^k(\mathbf{R}^n)$ is a C^k operational δ-approximation to f $(0 < \delta < 1)$, then there is an $h \in \mathrm{Diff}^k(\mathbf{R}^n)$ which coincides with g on any specified neighborhood of the origin, which coincides with f on a neighborhood of infinity, and which is a C^k operational ε-approximation to f, where $\varepsilon = \min\{1, 2\delta\}$.*

LEMMA 1.2. *Given any $k > 0$, $\gamma > 0$ and any closed ball B_r^n of radius r about the origin, there is a smooth (C^∞) function $\alpha : \mathbf{R}^n \to [0, 1]$ such that $\alpha \mid B_r^n = 1$, $\alpha \mid (\mathbf{R}^n - B_{r'}^n) = 0$ $(r' > r)$, and such that for $m = 0, 1, \cdots, k$,*

$$\sum_{q=0}^{m} \binom{m}{q} \| D^q \alpha(x) \| \le 1 + \gamma.$$

Proof. Let $\beta : \mathbf{R} \to [0, 1]$ be a smooth function such that $\beta(t) = 1$ for $t \le 0$ and $\beta(t) = 0$ for $t \ge 1$. Then there is a constant K such that for all t in $\mathbf{R}$ and for $m = 0, 1, \cdots, k$, we have

$$\sum_{q=1}^{m} \binom{m}{q} \left| \frac{d^q \beta(t)}{dt^q} \right| \le \sum_{q=1}^{k} \binom{k}{q} \left| \frac{d^q \beta(t)}{dt^q} \right| \le K.$$

For $0 < \sigma < 1$, define $\beta_\sigma(t) = \beta(\sigma t)$. Then

$$\sum_{q=0}^{m} \binom{m}{q} \left| \frac{d^q \beta_\sigma(t)}{dt^q} \right| \le \sum_{q=0}^{k} \binom{k}{q} \left| \frac{d^q \beta_\sigma(t)}{dt^q} \right|$$
$$\le \beta(\sigma t) + \sigma \sum_{q=1}^{k} \binom{k}{q} \left| \frac{d^q \beta(\sigma t)}{dt^q} \right| \le 1 + \sigma K.$$

Choose $\sigma > 0$ so small that $\sigma K \le \gamma$. Then $\alpha(x) = \beta_\sigma(\| x \| - r)$ has the desired properties.

Proof of 1.1. Consider

$$\tilde{h}(x) = \alpha(x)gf^{-1}(x) + [1 - \alpha(x)]x$$

where $\alpha : \mathbf{R}^n \to \mathbf{R}$ is as in 1.2 where γ has been chosen so that $(1 + \gamma)\delta \leq \min\{1, 2\delta\}$. Then $\tilde{h}$ is a C^k function which coincides with gf^{-1} near the origin and which coincides with the identity near infinity. Our candidate for the diffeomorphism is $h = \tilde{h}f$. We first check that h is an operational ε-approximation to f. Now $hf^{-1} = \tilde{h}$, and

$$D\tilde{h}(x) = D\alpha(x)[gf^{-1}(x) - x] + \alpha(x)[Dgf^{-1}(x) - I] + I,$$

$$D^m\tilde{h}(x) = \sum_{q=0}^{m} \binom{m}{q} D^q\alpha(x)D^{m-q}[gf^{-1}(x) - x], \quad m > 1,$$

(cf. [1; 1.3]). Thus

$$\| D\tilde{h}(x) - I \| \leq \| D\alpha(x) \| \; \| gf^{-1}(x) - x \| + \alpha(x) \| Dgf^{-1}(x) - I \|,$$

$$\| D^m\tilde{h}(x) \| \leq \sum_{q=0}^{m} \binom{m}{q} \| D^q\alpha(x) \| \; \| D^{m-q}[gf^{-1}(x) - x] \|.$$

Now $\| gf^{-1}(x) - x \|$, $\| Dgf^{-1}(x) - I \|$, $\cdots$, $\| D^k gf^{-1}(x) \|$ are all bounded uniformly by δ, and

$$\sum_{q=0}^{m} \binom{m}{q} \| D^q\alpha(x) \| < 1 + \gamma$$

for $m = 1, 2, \cdots, k$ by the choice of α. Therefore, h is an operational ε-approximation to f. In particular, h has a nonsingular differential at each point in $\mathbf{R}^n$. To conclude the proof, we must show that $h \in \mathrm{Diff}^k(\mathbf{R}^n)$. This fact follows by the next lemma.

LEMMA 1.3. *Let f be a homeomorphism of $\mathbf{R}^n$ onto itself and let $h \in C^k(\mathbf{R}^n)$ have nonsingular differential at every point and coincide with f in a neighborhood of infinity. Then $h \in \mathrm{Diff}^k(\mathbf{R}^n)$.*

Proof. We must show that $h^{-1}(y)$ is a singleton for every y in $\mathbf{R}^n$. If $h^{-1}(y) = \emptyset$ for some y, then there is a sufficiently large ball about the origin such that $hf^{-1} \mid B$ is a mapping of B into itself which is the identity on the boundary and which is not onto. This would violate the nonretraction property. Now $h^{-1}(y)$ is discrete and bounded for every y in $\mathbf{R}^n$. If $h^{-1}(y) = \{x_1, \cdots, x_m\}$, then since y is a regular value, the degree of h is given by

$$\deg(h) = \textstyle\sum_{i=1}^{m} \operatorname{sgn} \det Dh(x_i)$$

where $\operatorname{sgn} \det Dh(x) = \pm 1$ for every x in $\mathbf{R}^n$ (cf. [7; §5]). Now since h coincides with the homeomorphism f near infinity, then $\deg h = \pm 1$. Thus if $m > 1$, then there must be integers i, j such that $\det Dh(x_i) > 0$ and $\det Dh(x_j) < 0$. But then on any arc from x_i to x_j, there must be a point at which $\det Dh$ vanishes, contrary to hypothesis.

COROLLARY 1.4. *If $f \in \mathrm{Diff}^k(\mathbf{R}^n)$ and $g \in C^k(\mathbf{R}^n)$ is a C^1 operational δ-approximation to f $(0 < \delta < 1)$, then $g \in \mathrm{Diff}^k(\mathbf{R}^n)$.*

COROLLARY 1.5. *If $f \in \mathrm{Diff}^k(\mathbf{R}^n)$, and if $g \in C^k(\mathbf{R}^n)$ is a C^k operational δ-approximation to f $(0 < \delta < 1)$, then there is an $h \in \mathrm{Diff}^k(\mathbf{R}^n)$ which coincides with f on any specified neighborhood of the origin, which coincides with g on a neighborhood of infinity, and which is a C^k operational ε-approximation to f where $\varepsilon = \min\{1, 2\delta\}$.*

Proof. As in the proof of 1.1, take

$$\tilde{h}(x) = \alpha(x)x + [1 - \alpha(x)]fg^{-1}(x).$$

Then $h = \tilde{h}f$ is a C^k operational ε-approximation to f which coincides with f near the origin and which coincides with g near infinity. By 1.4, g is a diffeomorphism; so by 1.3, h is a diffeomorphism.

The next theorem shows that the topology of C^k metric approximation and the topology of C^k operational approximation are equivalent. However, we shall see from the proof that even if one kind of approximation is uniform (constant ε), the other kind of approximation will usually be equivalent only if we allow a variable bound ($\delta(x)$ nonconstant). This will cause a mild inconvenience in Section 3. We wish to thank the referee for showing us that this notion of equivalence holds when $k > 1$.

THEOREM 1.6. *Let $f \in \mathrm{Diff}(\mathbf{R}^n)$ and let $\varepsilon : \mathbf{R}^n \to (0, \infty)$ be given and continuous. Then*

(A) *there is a continuous function $\delta : \mathbf{R}^n \to (0, \infty)$ such that if $g \in C^k(\mathbf{R}^n)$ is a C^k metric δ-approximation to f, then g is a C^k operational ε-approximation to f;*

(B) *there is a continuous function $\delta : \mathbf{R}^n \to (0, \infty)$ such that if $g \in C^k(\mathbf{R}^n)$ is a C^k operational δ-approximation to f, then g is a C^k metric ε-approximation to f.*

Proof. Assume that $g \in C^k(\mathbf{R}^n)$ is a C^k metric δ-approximation to f for some $\delta : \mathbf{R}^n \to (0, \infty)$, i.e.

$$\| g(x) - f(x) \| < \delta(x), \| D(g - f)(x) \| < \delta(x), \cdots, \| D^k(g - f)(x) \| < \delta(x).$$

Then by the composite mapping formula [1; 1.4],

$$\begin{aligned} \| D^r(gf^{-1}(y) - y] \| &= \| D^r(g - f) \circ f^{-1}(y) \| \\ &= \| \textstyle\sum_{q=1}^r \sum \sigma_q D^q(g - f)(x)[D^{i_1}f^{-1}(y), \cdots, D^{i_q}f^{-1}(y)] \\ &\leq \delta(x)[\textstyle\sum \sigma_q \| D^{i_1}f^{-1}(y) \| \cdots \| D^{i_q}f^{-1}(y) \|] \\ &\leq \delta \circ f^{-1}(y)P_r(y) \end{aligned}$$

for $r = 1, \cdots, k$ and where P_r is a positive function of y. Therefore, g is a C^k operational ε-approximation to f provided that

$$\delta \circ f^{-1}(y) \leq \varepsilon(y)/(1 + P_1(y) + \cdots + P_k(y))$$

Conversely, suppose that $g \in C^k(\mathbf{R}^n)$ is a C^k operational δ-approximation to f for some $\delta : \mathbf{R}^n \to (0, \infty)$, i.e.

$$\| gf^{-1}(y) - y \| < \delta(y), \| Dgf^{-1}(y) - I \| < \delta(y), \cdots, \| D^k gf^{-1}(y) \| < \delta(y).$$

By the composite mapping formula,

$$\begin{aligned} D^r[gf^{-1}(y) - y] &= D^r(g - f)(x)[Df^{-1}(y), \cdots, Df^{-1}(y)] \\ &\quad + \textstyle\sum_{q=1}^{r-1} \sum \sigma_q D^q(g - f)(x)[D^{i_1}f^{-1}(y), \cdots, D^{i_q}f^{-1}(y)], \end{aligned}$$

i.e.

$$\begin{aligned} &D^r(g - f)(x)[Df^{-1}(y), \cdots, Df^{-1}(y)] \\ &\quad = D^r[gf^{-1}(y) - y] - \textstyle\sum_{q=1}^{r-1} \sum \sigma_q D^q(g - f)(x)[D^{i_1}f^{-1}(y), \cdots, D^{i_q}f^{-1}(y)]. \end{aligned}$$

Now $Df^{-1}(y)$ is nonsingular.

LEMMA 1.7. *Let $A : V \times \cdots \times V \to \mathbf{R}$ be r-multilinear and symmetric and let $B : V \to V$ be linear and nonsingular. Then*

$$\| A \| \leq \| A[B, \cdots, B] \| \, \| B^{-1} \|^r.$$

Proof. We observe that if $M : \mathbf{R}^n \times \cdots \times \mathbf{R}^n \to R$ is r-multilinear and symmetric and if $L : \mathbf{R}^n \to \mathbf{R}^n$ is linear, then $\| M[L, \cdots, L] \| \leq \| M \| \, \| L \|^r$. For

$$\begin{aligned} \frac{\| M[L(x), L(y), \cdots, L(z)] \|}{\| x \| \, \| y \| \cdots \| z \|} &= \frac{\| M[L^T L(x), y, \cdots, Lz] \|}{\| x \| \, \| y \| \cdots \| z \|} \\ &= \frac{\| M[(L^T)^{r-1} Lx, y, \cdots, z] \|}{\| x \| \, \| y \| \cdots \| z \|} \\ &\leq \frac{\| M[\cdot, y, \cdots, z] \|}{\| y \| \cdots \| z \|} \| L^T \|^{r-1} \| L \| \\ &\leq \| M \| \, \| L \|^r. \end{aligned}$$

Applying this result to $M = A[B, \cdots, B]$, $L = B^{-1}$ we obtain

$$\begin{aligned} \| A \| &= \| A[BB^{-1}, \cdots, BB^{-1}] \| \\ &= \| M[B^{-1}, \cdots, B^{-1}] \| \leq \| A[B, \cdots, B] \| \, \| B^{-1} \|^r. \end{aligned}$$

Returning to the proof of 1.6, we apply 1.7 to the formula for

$$D^r(g - f)(x)[Df^{-1}(y), \cdots, Df^{-1}(y)]$$

and obtain

$$\begin{aligned} &\| D^r(g - f)(x) \| / \| Df(x) \|^r \\ &\quad \leq d(y) + \textstyle\sum_{q=1}^{r-1} \sum \sigma_q \| D^q(g - f)(x) \| \, \| D^{i_1}f^{-1}(x) \| \cdots \| D^{i_1}f^{-1}(x) \|. \end{aligned}$$

Thus by an easy induction on r, we establish

$$\| D^r(g - f)(x) \| \leq \delta \circ f(x) Q_r(x)$$

where Q_r is a positive function in x. Therefore, g is a C^k metric ϵ-approximation to f provided that

$$\delta \circ f(x) \leq \epsilon(x)/(1 + Q_1(x) + \cdots + Q_k(x)).$$

2. Homeomorphisms which are almost diffeorphisms

We shall now consider the case of homeomorphisms of $\mathbf{R}^n$ onto itself which are C^k diffeomorphisms on some neighborhood of infinity ($1 \leq k \leq \infty$). The following notation will be useful:

B_r^n = closed ball of radius r about the origin,
S_r^{n-1} = boundary of B_r^n,
$A_{r,r'}^n = B_{r'}^n - B^{\circ n}_r = \bigcup_{r \leq \rho \leq r'} S_\rho^{n-1}$.

We shall often write B^n for B_1^n and S^{n-1} for S_1^{n-1}. In all cases, the differentiable structure is the one inherited from the usual differentiable structure on $\mathbf{R}^n$. Our pasting can be described roughly as follows: Given f, g homeomorphisms of $\mathbf{R}^n$ which are diffeomorphisms outside of B_r^n, we find $r' > r$ such that $f(B_r^n)$ is interior to $g(B_{r'}^n)$. We show that the region bounded by $f(S_r^{n-1})$ and $g(S_{r'}^{n-1})$ is a differentiably embedded annulus A. The pasting h of f and g is defined by taking

$$h \mid B_r^n = f \mid B_r^n, \quad h \ (\mathbf{R}^n - B_{r'}^n) = g \mid (\mathbf{R}^n - B_{r'}^n),$$

and $h : A_{r,r'}^n \to A$ is defined by executing an isotopy between $f \mid S_r^{n-1}$ and $g \mid S_{r'}^{n-1}$ across the annulus. The possible corners along S_r^n and $S_{r'}^{n-1}$ are smoothed.

The smoothing of the corners presents no difficulty. Showing that A is an annulus is straightforward for $n \neq 4$ (for $n = 4$, it is equivalent to determining the uniqueness of the combinatorial structure for $\mathbf{R}^4$). However, the desired isotopy may not exist, and hence there is an obstruction to our construction. This obstruction is measured in terms of the group Γ_n of smoothings of the combinatorial manifold S^n.

We begin the technical treatment of this procedure by recalling the essential features of the isotopy problem. Now the notion of restriction defines a homomorphism

$$\rho : \mathrm{Diff}^k\ (B^n) \to \mathrm{Diff}^k\ (S^{n-1}).$$

Under the C^k metric approximation topology, the path components of these spaces are precisely the C^k regular isotopy classes. The groups of path components are denoted by

$$\pi_0[\mathrm{Diff}^k\ (B^n)] \quad \text{and} \quad \pi_0[\mathrm{Diff}^k\ (S^{n-1})].$$

Now $\pi_0[\mathrm{Diff}^k\ (S^{n-1})]$ is abelian and Im (ρ) is a normal subgroup of $\mathrm{Diff}^k\ (S^{n-1})$ [10; 1.4 and 1.7]. Also, $f \in \mathrm{Diff}^k\ (S^{n-1})$ is regularly isotopic to the identity if and only if f is orientation preserving and $f \in \mathrm{Im}\ (\rho)$ [10; 1.6]. Define

$$\Gamma_n = \frac{\pi_0\ [\mathrm{Diff}^k\ (S^{n-1})]}{\rho_* \pi_0\ [\mathrm{Diff}^k\ (B^n)]} \cong \frac{\mathrm{Diff}^k\ (S^{n-1})}{\rho\ [\mathrm{Diff}^k\ (B^n)]}$$

By [10; 1.7 and 1.8] these two definitions of Γ_n are equivalent and independent of k ($1 \leq k \leq \infty$). Thus Γ_n is the group of obstructions to extending diffeomorphisms of S^{n-1} to diffeomorphisms of B^n. It is also the group of obstructions to the existence of a regular isotopy of $f \in \mathrm{Diff}^k(S^{n-1})$ to the identity. Finally, for $f \in \mathrm{Diff}^k(S^{n-1})$ and orientation preserving we can form the smooth manifold $B^n \cup_f B^n$ by identifying the boundaries under f. This manifold is a topological sphere which is combinatorially equivalent to S^n. However, this manifold is differentiably equivalent to S^n if and only if f is regularly isotopic to the identity. Consequently, Γ_n also represents the group of differentiable structures compatible with the usual combinatorial structure for S^n. Now it is known that S^n has a unique combinatorial structure for $n \neq 4$ [12], and it is known that $\Gamma_n = 0$ for $n = 12$ and $n \leq 6$ ([11] and [4]), but that Γ_n is nonzero for $n > 6$ and $n \neq 12$ [4]. Thus in the latter range, we shall encounter obstructions to smoothing.

LEMMA 2.1. *If $f : S^{n-1} \to \mathbf{R}^n$ is a C^k embedding and if $n \neq 4$, then* Im (f) *bounds a submanifold which is C^k diffeomorphic to B^n.*

Proof. By the Schoenflies theorem (due to Morton Brown [2]) the bounded complement of Im (f) is a topological n-disk. Since f is a C^k embedding, it is a C^k manifold. Thus for $n \neq 4$, it is C^k diffeomorphic to B^n [6; pages 108–110], [10; 6.3 and 6.7].

COROLLARY 2.2. *If $f : \mathbf{R}^n \to \mathbf{R}^n$ is a homeomorphism which is a diffeomorphism on the complement of B_r^n, and if $n \neq 4$, then f determines an element of Γ_n.*

Proof. By the Cerf-Palais lemma [6; 9.6] there is a C^k isotopy of $\mathbf{R}^n$ which carries $f(S_r^{n-1})$ onto S_r^{n-1}. The induced element $[f] \in \Gamma_n$ does not depend on the choice of isotopy.

COROLLARY 2.3 (DIFFERENTIABLE ANNULUS THEOREM). *Let*

$$f, g : S^{n-1} \to S^n$$

be C^k embeddings with disjoint images ($1 \leq k \leq \infty$). If $n \neq 4$, then the submanifold A which is bounded by Im (f) *and* Im (g) *is C^k diffeomorphic to* $S^{n-1} \times [0, 1]$.

Proof. Let M_f (M_g) denote the n-disk bounded by Im (f) (Im (g)) which does not contain Im (g) (Im (f), respectively). Applying the Cerf-Palais lemma twice, we obtain a C^k isotopy of S^n which carries M_f and M_g onto the distinct polar caps. This isotopy carries A onto the remaining annulus.

Remark. For $n = 4$, the conclusions of 2.2 and 2.3 remain valid under the hypothesis that Im (f) bounds a standard 4-disk in $\mathbf{R}^4$.

LEMMA 2.4 (Munkres [10; 6.11]). *Let M be a C^k manifold without boundary and let W be a neighborhood of $M \times 0$ in $M \times [0, 1]$. If f is a C^k embedding of W into $M \times [0, 1]$ such that $f \mid (M \times 0) =$ id, then there is a C^k embedding*

$\tilde{f}: W \to f(W)$ such that $\tilde{f}$ coincides with the identity on some neighborhood of $M \times 0$ and $\tilde{f}$ coincides with f on some neighborhood of the complement of W.

COROLLARY 2.5. *If $f, g \in \mathrm{Diff}^k(M)$ are regularly isotopic, then the isotopy f_t can be chosen so that $f_t(x) = (f(x), t)$ for $0 \leq t < \varepsilon$ and $f_t(x) = (g(x), t)$ for $1 - \varepsilon < t \leq 1$, and for all x in M.*

COROLLARY 2.6. *Suppose that $f: \mathbf{R}^n \to \mathbf{R}^n$ is a homeomorphism which is a C^k diffeomorphism on the neighborhood $A^n_{r',r''}$ of S^{n-1}_r $(r' < r < r'')$, and suppose that there is given a tubular neighborhood $h: S^{n-1} \times \mathbf{R} \to \mathbf{R}^n$ of $f(S^{n-1}_r)$ such that $f(A^n_{r',r''}) \subset \mathrm{Im}\,(h)$. Then there is a homeomorphism $\tilde{f}: \mathbf{R}^n \to \mathbf{R}^n$ such that*

(1) *$\tilde{f}$ coincides with f on a neighborhood of $\mathbf{R}^n - \mathring{A}^n_{r',r''}$,*
(2) *$\tilde{f} \mid A^n_{r',r''}$ is a C^k diffeomorphism,*
(3) *$\tilde{f}(\rho, \theta) = h[h^{-1}f(r, \theta), \rho - r]$ for $\mid \rho - r \mid$ sufficiently small*

(We use spherical coordinates (ρ, θ) on $\mathbf{R}^n$, $\rho \geq 0, \theta \in S^{n-1}$).

Proof. It suffices to construct $\tilde{f} \mid B^n_r$. Define

$$g: A^n_{r',r} \to \mathbf{R}^n$$

by $g(\rho, \theta) = h[h^{-1}f(r, \theta), \rho - r]$. Then g is a C^k embedding of $A^n_{r',r}$ into $\mathrm{Im}\,(h)$ and

$$g^{-1}f: A^n_{r',r} \to A^n_{r',r}$$

satisfies the hypotheses of 2.4. Thus there is an embedding

$$\tilde{g}: A^n_{r',r} \to A^n_{r',r}$$

such that $\tilde{g}$ coincides with $g^{-1}f$ near $S^{n-1}_{r'}$ and $\tilde{g}$ coincides with the identity near S^{n-1}_r. Then $g\tilde{g}$ satisfies (1), (2), and (3). Since $g\tilde{g}$ coincides with f near $S^{n-1}_{r'}$, it can be extended to $\tilde{f}$ by $\tilde{f} \mid B^n_{r'} = f \mid B^n_{r'}$.

THEOREM 2.7. *Let f, g be homeomorphisms of $\mathbf{R}^n$ onto itself such that fg^{-1} is orientation preserving and such that f and g are C^k diffeomorphisms on the complement of $B^n_{r-\varepsilon}$. If $n \neq 4$ and if $[f] = [g]$ in Γ_n, then there is a homeomorphism h of $\mathbf{R}^n$ onto itself such that*

$$h \mid B^n_{r-\varepsilon'} = f \mid B^n_{r-\varepsilon'}, \quad h \mid (\mathbf{R}^n - B^n_{r'+\varepsilon'}) = g \mid (\mathbf{R}^n - B^n_{r'+\varepsilon'})$$

for some $r' > r$ and $\varepsilon' < \varepsilon$, and h is a C^k diffeomorphism on the complement of $B^n_{r-\varepsilon}$.

Proof. Choose $r' > r$ such that $f(B^n_r)$ is interior to $g(B^n_{r'})$. By 2.2, $[f]$ and $[g]$ are defined, and there is an isotopy F mapping $A^n_{r,r'}$ into $S^{n-1} \times [r', r'']$ if and only if $[f] = [g]$. By 2.5, this isotopy can be chosen to be constant near the edges. By 2.3, there is an embedding

$$\phi: S^{n-1} \times (r - \varepsilon, r' + \varepsilon) \to \mathbf{R}^n$$

such that $\phi \mid S^{n-1} \times [r, r']$ maps onto the annular region between $f(S^{n-1}_r)$

and $g(S_{r'}^{n-1})$, $\phi \mid S^{n-1} \times (r - \varepsilon, r + \varepsilon)$ is a tubular neighborhood of $f(S_r^{n-1})$, and $\phi \mid S^{n-1} \times (r' - \varepsilon, r' + \varepsilon)$ is a tubular neighborhood of $g(S_{r'}^{n-1})$. Apply 2.6 to f at S_r^{n-1} and to g at $S_{r'}^{n-1}$ with respect to the ϕ tubular neighborhoods. Define $h : \mathbf{R}^n \to \mathbf{R}^n$ by

$$h \mid B_r^n = \tilde{f} \mid B_r^n, \quad h \mid A_{r,r'}^n = \phi \circ F, \quad h \mid (\mathbf{R}^n - B_{r'}^n) = \tilde{g} \mid (\mathbf{R}^n - B_{r'}^n).$$

Since $F, \tilde{f}, \tilde{g}$ are radially constant with respect to the ϕ tubular neighborhoods near S_r^{n-1} and $S_{r'}^{n-1}$, h is a C^k diffeomorphism except in $B_{r-\varepsilon}^n$ where it coincides with f.

COROLLARY 2.8. *If $n = 12$ or $n < 7$ and $n \neq 4$, then there is no obstruction to pasting.*

Remark. If any such pasting is possible, then by applying the Cerf-Palais lemma twice as in 2.2, we obtain an isotopy between $f \mid S_r^{n-1}$ and $g \mid S_{r'}^{n-1}$. Consequently, $[f] = [g]$ is necessary as well as sufficient for pasting.

3. Associated problems

Let $\tilde{f} : S^{n-1} \to \mathbf{R}^n$ be a C^k embedding $(1 \leq k \leq \infty)$. When does there exist a C^r embedding $f : B^n \to \mathbf{R}^n$ which extends $\tilde{f}$? This is the differentiable Schoenflies problem. It has been studied extensively by M. Morse and W. Huebsch. A survey of their results and an extensive bibliography can be found in [8]. For our purposes, their main results are the following.

(A) $\tilde{f}$ always has an extension to a homeomorphism $f : B^n \to \mathbf{R}^n$ which is a C^k diffeomorphism on the complement of the origin [8; 2.1].

(B) A necessary and sufficient condition for $\tilde{f}$ to have an extension to a C^k diffeomorphism $f : B^n \to \mathbf{R}^n$ is that $\tilde{f}$ be regularly isotopic to id: $S^{n-1} \to \mathbf{R}^n$ [8; 2.2].

Using 2.7 and 2.8, we obtain the following modification of (B):

THEOREM 3.1. *Let $\tilde{f} : S^{n-1} \to \mathbf{R}^n$ be a C^k embedding which preserves orientation. Suppose that either $n \neq 4$ or $n = 4$ and $\tilde{f}(S^3)$ bounds a standard smooth 4-disk in $\mathbf{R}^4$. Then $[\tilde{f}] \in \Gamma_n$ is defined, and $\tilde{f}$ has an extension to a C^k diffeomorphism $f : B^n \to \mathbf{R}^n$ if and only if $[\tilde{f}] = 0$. In particular for $n = 12$ or $n < 7$ $(n \neq 4)$ the extension always exists.*

Proof. By (A) we form an extension f of $\tilde{f}$ which is a diffeomorphism except at 0. We try to apply 2.7 to paste f to the identity on $B_{1/2}^n$ and we are successful if and only if $[\tilde{f}] = 0$.

Remark. We can also prove 3.1 by using (B), 2.1, and the Cerf-Palais lemma. Thus in one sense, 2.7 is a generalization of (B).

The viewpoint of 3.1 also gives insight into Munkres' obstruction theory for smoothing piecewise differentiable homeomorphisms. Let f be given as in (A). Before Munkres' theory can be applied, it is necessary that a rather rigid uniformity condition [10; 2.2 and 2.5] be satisfied on a neighborhood of 0.

A major consequence of this condition is that $f(S_r^{n-1})$ bounds a star-like neighborhood of the origin for all sufficiently small $r > 0$. In particular, this means that $[f]$ is always defined and is the obstruction to smoothing. As we have seen, no such restriction is necessary for $n \neq 4$ and for $n = 4$ all that is needed is that $f(S_r^{n-1})$ bounds a standard smooth 4-disk (of course, to find a weaker analytic condition which insures this may be extremely difficult). However, if it were known that B^4 had a unique differentiable structure, then isolated singular points of homeomorphisms could be removed without any additional restriction. Now in case f fails to be a diffeomorphism along some subcomplex L with dimension > 0, the same ideas apply, except that some mild restriction on f near L would probably be required, e.g., Munkres' Lipschitz condition [10; 2.1], but hopefully that would be all that is required.

Returning to the f in (A), we know that the singularity cannot be removed when $[f] \neq 0$. What happens if we try to force f to be smooth, i.e., if we smooth f by a convolution

$$f_\varepsilon(x) = \int_{\mathbf{R}^n} f(x + \varepsilon y)\phi(y)\, dy$$

where ϕ is a smooth function with support B_1^n and $\int \phi(y)\, dy = 1$?

THEOREM 3.2. *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a homeomorphism which is a C^1 diffeomorphism on the complement of the origin such that $[f] \neq 0$. Then for any $1 > r > 0$, there is an $\varepsilon_0 > 0$ such that for each $0 < \varepsilon < \varepsilon_0$, $Df_\varepsilon(x)$ is singular for some x in B_r^n and nonsingular for x in $\mathbf{R}^n - B_1^n$.*

Proof. We show that f_ε can be made to approximate f so well that a proof similar to 1.1 works. Unfortunately, since f_ε is a C^1 metric approximation to f (with constant ε') and since in 1.6, $\delta(x)$ cannot be taken to be constant even though ε' is constant, then we cannot apply 1.1 directly. Let $\alpha : \mathbf{R}^n \to [0, 1]$ be a smooth function such that $\alpha \mid B_r^n = 1$ and $\alpha \mid (\mathbf{R}^n - B_r^n) = 0$. Then there is a $K > 1$ such that

$$\| D\alpha(x) \| + \alpha(x) \| Df^{-1}(x) \| < K$$

for all x in $A_{r,1}^n$. Choose ε_0 so small that for $0 < \varepsilon < \varepsilon_0$, $\| f_\varepsilon(y) - f(y) \| < K^{-1}$ and $\| D_\varepsilon f(y) - Df(y) \| < K^{-1}$ for all x in $f^{-1}(A_{r,1}^n)$ [9; §4]. Define

$$g(x) = \alpha(x) f_\varepsilon f^{-1}(x) + [1 - \alpha(x)]x.$$

Then for $y = f^{-1}(x)$ and x in $A_{r,1}^n$ we have

$$\begin{aligned}
&\| Dg(x) - I \| \\
&\quad = \| D\alpha(x) \| \| f_\varepsilon f^{-1}(x) - x \| + \alpha(x) \| Df_\varepsilon f^{-1}(x) - I \| \\
&\quad \leq \| D\alpha(x) \| \| f_\varepsilon(y) - f(y) \| + \alpha(x) \| Df_\varepsilon(y) - Df(y) \| \| Df^{-1}(x) \| \\
&\quad < K^{-1}K.
\end{aligned}$$

Thus g coincides with $f_\varepsilon f^{-1}$ on B_r^n, g coincides with the identity on the complement of B_1^n, and g has nonsingular differential on the complement of B_r^n, i.e., $\tilde{f} = gf$ coincides with f_ε on B_r^n, with f on the complement of B_1^n, and $\tilde{f}$ has nonsingular differential on the complement of B_r^n. If $\tilde{f}$ has nonsingular differential at every point, then by 1.3, $\tilde{f}$ would be a diffeomorphism which coincides with f near infinity, an impossibility. Thus $\tilde{f} \mid B_r^n = f \mid B_r^n$ must have a singular differential at some point.

We call $h : \mathbf{R}^n \to \mathbf{R}^n$ a regular Lipschitz homeomorphism if there are positive constants k, K such that for all x, y in $\mathbf{R}^n$,

$$k\| x - y \| \leq \| h(x) - h(y) \| \leq K\| x - y \|.$$

This condition is equivalent to requiring that h is a homeomorphism and that h and h^{-1} both satisfy uniform Lipschitz conditions. It is easy to verify that in this case, any smoothing of h by convolution satisfies

$$\| h_\varepsilon(x) - h_\varepsilon(y) \| \leq K\| x - y \|.$$

When attempting to maintain the lower bound, one always seems to encounter technical difficulties. We shall now show that in some dimensions (when $\Gamma_n \neq 0$) there is a real obstruction. Thus by the nature of analysis, it seems unlikely that such an inequality can be preserved under convolution in any dimension greater than one. We shall now construct an example of a regular Lipschitz homeomorphism $h : \mathbf{R}^n \to \mathbf{R}^n$ (n chosen so that $\Gamma_n \neq 0$) such that h_ε fails to satisfy any such lower bound, i.e., if h_ε is a homeomorphism, then h_ε^{-1} fails to satisfy a Lipschitz condition. This failure occurs not at infinity, but at some point x_0 near the origin.

Let $f : S^{n-1} \to S^{n-1}$ be a diffeomorphism with $[f] \neq 0$. Using chord-length distance on S^{n-1}, there are constants $0 < k \leq 1 \leq K$ such that for every x, y in S^{n-1}

$$k\| x - y \| \leq \| f(x) - f(y) \| \leq K\| x - y \|.$$

Define $h : \mathbf{R}^n \to \mathbf{R}^n$ by $h(0) = 0$ and $h(x) = \| x \| f(x/\| x \|)$ for $x \neq 0$. Now $h(S_r^{n-1}) \subset S_r^{n-1}$ for every r; so the above inequalities are satisfied by h for $(x, y) = (x, 0)$. Suppose $\| y \| \geq \| x \| > 0$. Then

$$\begin{aligned} \| h(x) - h(y) \| &\leq \| \, \| x \| f(x/\| x \|) - \| y \| f(y/\| y \|) \, \| \\ &\leq K\| x \| \| (x/\| x \|) - (y/\| y \|) \| + | \, \| x \| - \| y \| \, | \\ &\leq 2\,K \max \{ | \, \| x \| - \| y \| \, |, \ \| x - (\| x \|/\| y \|) y \| \} \\ &\leq 2K\| x - y \| \end{aligned}$$

and

$$\begin{aligned} \| h(x) - h(y) \| &\geq \| \, \| x \| f(x/\| x \|) - \| y \| f(y/\| y \|) \, \| \\ &\geq \max \{ | \, \| x \| - \| y \| \, |, \ \| x \| \| f(x/\| x \|) - f(y/\| y \|) \| \} \\ &\geq k \max \{ | \, \| x \| - \| y \| \, |, \ \| x - (\| x \|/\| y \|) y \| \} \\ &\geq (k/\sqrt{2}) \| x - y \| \end{aligned}$$

where the second and fourth inequalities follow by comparing the length of the diagonal of the regular trapezoid

$$\{x,\ (\|x\|/\|y\|)y,\ y,\ (\|y\|/\|x\|)x\}$$

with the lengths of its edges. Thus we see that h is a regular Lipschitz homeomorphism. Using 3.2 and the fact that $[h] = [f] \neq 0$, we conclude that for any smoothing h_ϵ, there is a point x_0 near the origin where $Dh_\varepsilon(x_0)$ is singular. For $v \in \ker Dh_\varepsilon(x_0)$, we can make

$$\| h_\varepsilon(x_0 + tv) - h_\varepsilon(x_0) \|/\| (x_0 + tv) - x_0 \|$$

arbitrarily small by choosing $t > 0$ sufficiently small.

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