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Hörmander's type conditions for evolution in spaces of (small) Gevrey functions

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Abstract. We give Hörmander's type necessary and/or sufficient conditions for the existence of solutions of the Cauchy problem for (overdetermined) systems of linear partial differential operators with constant coefficients, in spaces of (small) Gevrey functions.

Key words: overdetermined systems, Hörmander's type conditions, (small) Gevrey functions, Phragmén-Lindelöf principles.

Introduction

In this paper we continue the study of the Cauchy problem for (overdetermined) systems of linear partial differential operators with constant coefficients in spaces of (small) Gevrey functions, as considered in [BN2].

We split $\mathbb{R}^N \simeq \mathbb{R}^k_t \times \mathbb{R}^n_x$ and consider either the class $\gamma^{(s)}$ of (small) Gevrey functions of order $s > 1$ in both variables t and x, or, allowing different scales of regularity in t and x, the topological tensor products $\tilde{\gamma}^{(s)}$ and $\gamma^{(r,s)}$, the first not requiring ultradifferentiability in the t-variables, the second requiring ultradifferentiability of order $r > 1$ in t and $s > 1$ in x.

For a pair $K_1 \subset K_2$ of closed convex subsets of $\mathbb{R}^N$ with $K_1 \subset \mathbb{R}^n_x$, let us denote by $\gamma^*_{K_1}$ and $\gamma^*_{K_2}$ one of the above spaces of (ultra)differentiable Whitney functions on K_1 and K_2 respectively.

Then, for an $a_1 \times a_0$ matrix $A_0(D)$ of linear partial differential operators with constant coefficients, we are concerned with the following (*overdetermined*) *Cauchy problem*:

$$\begin{cases} \text{given } f \in (\gamma^*_{K_2})^{a_1} \text{ with } A_1(D)f = 0, \text{ and } g \in (\gamma^*_{K_1})^{a_0} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{such that } A_0(D)g = f|_{K_1}, \\ \text{find } u \in (\gamma^*_{K_2})^{a_0} \text{ such that } A_0(D)u = f \text{ and } u|_{K_1} = g\,, \end{cases} \tag{0.1}$$

where the rows of $A_1(D)$ give a basis for the integrability conditions for $A_0(D)$ (see §1; cf. also [AN1], [N2], [BN1], [BN2]).

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As in [BN2], we shall say that the pair (K_1, K_2) is of evolution (resp.: of causality, hyperbolic) for $A_0(D)$ in the class γ^* if the Cauchy problem (0.1) admits at least a solution (resp.: at most one solution, one and only one solution).

In this paper we obtain necessary and/or sufficient conditions for evolution, generalizing a result of Hörmander in [Hö3], where he gave a necessary and sufficient condition for evolution for the pair consisting of a hyperplane of $\mathbb{R}^N$ and one of the half-spaces it bounds, in the scalar case ($a_0 = a_1 = 1$) and in the space of C^∞ functions.

In [BN1] we generalized this result of Hörmander to the case of overdetermined systems and initial data on an affine subspace Σ of $\mathbb{R}^N$ of arbitrary codimension, always in the C^∞ class. We obtained a necessary condition for evolution (H), which naturally generalizes the one of [Hö3], and which is equivalent to it when Σ is a hyperplane and $A_0(D)$ is a scalar operator. We obtained then a stronger condition (H') which turns out to be sufficient, but not necessary, for evolution, and which coincides with the previous condition (H) in the hyperplane case.

Here we generalize these results of [BN1] to the case of (small) Gevrey functions, obtaining a necessary condition $(H)^s$ for evolution (see §2.1), and a sufficient stronger condition $(H')^s$, which coincides with $(H)^s$ when Σ is a hyperplane (see §2.3). When $n = 1$ the characteristic varieties are algebraic curves, and we then obtain a condition $(h)^s$ which is necessary and/or sufficient for evolution, according to the $s > 1$ for which $(h)^s$ is satisfied (see §2.2). In particular, for $s \in \mathbb{Q}$, when $k = 1$ or $n = 1$ we find that Hörmander's type condition $(H)^s$ is necessary and sufficient for evolution.

We finally give some examples of applications of the above results.

1. Preliminaries and notation

We briefly collect here some basic notion and results that we shall need in the following. We also refer to [N2], [BN1], [BN2], [B] for more details.

1.1. Algebraic setting

We denote by $\mathcal{P} = \mathbb{C}[\theta_1, \ldots, \theta_N]$ the unitary commutative ring of polynomials with complex coefficients in N indeterminates.

Given then a $\mathbb{C}$-linear space $\mathcal{F}$ of (ultra)differentiable (Whitney) functions defined on a subset of $\mathbb{R}^N_z$, we consider $\mathcal{F}$ as a unitary $\mathcal{P}$-module by letting $p(\theta) \in \mathcal{P}$ act on $f \in \mathcal{F}$ as

$$p(\theta)f = fp(\theta) = p(D)f,$$

by the formal substitution $\theta_j \to D_j = \frac{1}{i}\frac{\partial}{\partial z_j}$ for $j = 1, \dots, N$. We call such an $\mathcal{F}$ a *differential* $\mathcal{P}$*-module*.

Given an $a_1 \times a_0$ matrix $A_0(D)$ of linear partial differential operators in $\mathbb{R}^N$ with constant coefficients, and given a differential $\mathcal{P}$-module $\mathcal{F}$, we consider the system

$$\begin{cases} u \in \mathcal{F}^{a_0} \\ A_0(D)u = f \in \mathcal{F}^{a_1}. \end{cases} \tag{1.1}$$

In order to solve the system above, we must take into account the necessary integrability conditions that f must satisfy. To this aim we insert the $\mathcal{P}$-homomorphism ${}^tA_0(\theta) : \mathcal{P}^{a_1} \to \mathcal{P}^{a_0}$ into a Hilbert resolution

$$0 \longrightarrow \mathcal{P}^{a_d} \xrightarrow{{}^tA_{d-1}(\theta)} \mathcal{P}^{a_{d-1}} \longrightarrow \cdots \longrightarrow \mathcal{P}^{a_2} \xrightarrow{{}^tA_1(\theta)} \mathcal{P}^{a_1} \xrightarrow{{}^tA_0(\theta)} \mathcal{P}^{a_0} \\ \longrightarrow \mathcal{M} \longrightarrow 0$$

of the unitary finitely generated $\mathcal{P}$-module $\mathcal{M} = \operatorname{coker}{}^tA_0(\theta)$, where the rows of the matrix $A_1(D)$ give a system of generators for the module of all the integrability conditions for f that can be expressed in terms of partial differential operators.

The existence of a non-trivial map ${}^tA_1(\theta) : \mathcal{P}^{a_2} \to \mathcal{P}^{a_1}$ such that the sequence

$$\mathcal{P}^{a_2} \xrightarrow{{}^tA_1(\theta)} \mathcal{P}^{a_1} \xrightarrow{{}^tA_0(\theta)} \mathcal{P}^{a_0}$$

is a complex, means that the system (1.1) is *overdetermined*.

From the isomorphisms

$$\begin{cases} \operatorname{Ext}^0_{\mathcal{P}}(\mathcal{M}, \mathcal{F}) \simeq \{f \in \mathcal{F}^{a_0} : A_0(D)f = 0\} \\ \operatorname{Ext}^j_{\mathcal{P}}(\mathcal{M}, \mathcal{F}) \simeq \dfrac{\ker\left(A_j(D) : \mathcal{F}^{a_j} \to \mathcal{F}^{a_{j+1}}\right)}{\operatorname{Image}\left(A_{j-1}(D) : \mathcal{F}^{a_{j-1}} \to \mathcal{F}^{a_j}\right)} \quad \text{for } j \geq 1 \end{cases}$$

we thus obtain that uniqueness and/or existence of solutions of (1.1), for every right hand side $f \in \mathcal{F}^{a_1}$ satisfying the integrability condition $A_1(D)f = 0$, are strictly related to the vanishing of the cohomology groups $\operatorname{Ext}^0_{\mathcal{P}}(\mathcal{M}, \mathcal{F})$ and/or $\operatorname{Ext}^1_{\mathcal{P}}(\mathcal{M}, \mathcal{F})$.

By means of duality, the Fourier-Laplace transform and the Ehrenpreis fundamental principle this will lead the study of the Cauchy problem for overdetermined systems to the study of the validity of a Phragmén-Lindelöf principle (see Theorems 1.4 and 1.5).

For this purpose we first need some further algebraic notion related to the Cauchy problem. Let

$$\Sigma = \{(t,x) \in \mathbb{R}^k_t \times \mathbb{R}^n_x \simeq \mathbb{R}^N_z : t=0\} \simeq \mathbb{R}^n_x \subset \mathbb{R}^N$$

and denote by $\theta = (\tau,\zeta) = (\tau_1,\ldots,\tau_k,\zeta_1,\ldots,\zeta_n) \in \mathbb{C}^k_\tau \times \mathbb{C}^n_\zeta \simeq \mathbb{C}^N_\theta$ the dual coordinates of $z=(t,x)=(t_1,\ldots,t_k,x_1,\ldots,x_n) \in \mathbb{R}^k_t \times \mathbb{R}^n_x$.

We can consider $\mathcal{P}_n = \mathbb{C}[\zeta_1,\ldots,\zeta_n]$ as a unitary subring of $\mathcal{P} = \mathbb{C}[\tau_1,\ldots,\tau_k,\zeta_1,\ldots,\zeta_n]$. Given then a $\mathcal{P}$-module $\mathcal{M}$ we denote by $(\mathcal{M})_n$ the set $\mathcal{M}$ considered as a $\mathcal{P}_n$-module by change of base ring.

We say that: Σ is *formally non-characteristic* for $\mathcal{M}$ if $(\mathcal{M})_n$ is a $\mathcal{P}_n$-module of finite type; Σ is *quasi-free* for $\mathcal{M}$ if $(\mathcal{M})_n$ is a torsion free $\mathcal{P}_n$-module.

To every prime ideal $\wp$ of $\mathcal{P}$ we associate the affine algebraic variety

$$V(\wp) = \{\theta = (\tau,\zeta) \in \mathbb{C}^k \times \mathbb{C}^n : p(\theta) = 0 \ \forall p \in \wp\}.$$

We denote by $\mathrm{Supp}(\mathcal{M})$ and $\mathrm{Ass}(\mathcal{M})$ respectively the support of $\mathcal{M}$ and the set of all prime ideals associated to $\mathcal{M}$. Then we recall that a necessary and sufficient condition in order that Σ is formally non-characteristic and quasi-free is that the following conditions are satisfied (cf. [BN2], [B]):

i) there exists real constants λ, b such that

$$|\tau| \leq \lambda(1+|\zeta|)^b \quad \forall(\tau,\zeta) \in V(\wp), \quad \forall \wp \in \mathrm{Supp}(\mathcal{M}), \tag{1.2}$$

ii) the natural projection maps

$$\begin{aligned} \pi_n : V(\wp) &\longrightarrow \mathbb{C}^n_\zeta \\ (\tau,\zeta) &\longmapsto \zeta \end{aligned}$$

are surjective, finite and dominant, for all $\wp \in \mathrm{Ass}(\mathcal{M})$.

The smallest number b such that (1.2) is valid for some $\lambda > 0$ is called the *reduced order* of Σ for $\mathcal{M}$, and is denoted by p_o (this number exists and is rational by the Tarski-Seidenberg theorem).

1.2. Spaces of (small) Gevrey functions

We shall be concerned, in the following, with spaces of (small) Gevrey (Whitney) functions. In the study of evolution from the affine subspace Σ it is natural to consider also spaces of (Whitney) functions satisfying different regularity requirements in the t and the x variables. This can be done by considering topological tensor product spaces. We briefly recall here the main notions, referring to [BN2] for more details.

For an open subset Ω of $\mathbb{R}^N$, the space of (small) Gevrey functions of order $s > 1$ in Ω is defined by

$$\gamma^{(s)}(\Omega) = \Big\{ f \in \mathcal{E}(\Omega) : \forall K \subset\subset \Omega \ \ \forall \varepsilon > 0 \ \exists c > 0 : \\ \sup_K |D^\alpha f| \le c\varepsilon^{|\alpha|}(|\alpha|!)^s \ \ \forall \alpha \in \mathbb{N}^N \Big\},$$

where $\mathcal{E}(\Omega)$ is the space of complex valued smooth functions on Ω.

Then we split $\mathbb{R}^N \simeq \mathbb{R}^k_t \times \mathbb{R}^n_x$ and consider, as topological tensor product spaces (cf. [BN2], [B]), the classes $\tilde{\gamma}^{(s)}$ of smooth functions which belong uniformly, with their t-derivatives, as functions of the x-variables, to the (small) Gevrey class of order $s > 1$, and $\gamma^{(r,s)}$ of (small) Gevrey functions of order $r > 1$ in t and $s > 1$ in x:

$$\tilde{\gamma}^{(s)}(\Omega) = \Big\{ f \in \mathcal{E}(\Omega) : \forall K \subset\subset \Omega \ \ \forall \varepsilon > 0 \ \ \forall \beta \in \mathbb{N}^k \ \exists c > 0 : \\ \sup_K |D_t^\beta D_x^\alpha f(t,x)| \le c\varepsilon^{|\alpha|}(|\alpha|!)^s \ \ \forall \alpha \in \mathbb{N}^n \Big\},$$

$$\gamma^{(r,s)}(\Omega) = \Big\{ f \in \mathcal{E}(\Omega) : \forall K \subset\subset \Omega \ \ \forall \varepsilon > 0 \ \exists c > 0 : \\ \sup_K |D_t^\beta D_x^\alpha f(t,x)| \le c\varepsilon^{|\alpha|+|\beta|}(|\beta|!)^r(|\alpha|!)^s \ \ \forall \beta \in \mathbb{N}^k \ \ \forall \alpha \in \mathbb{N}^n \Big\}.$$

We have Paley-Wiener-type theorems (cf. [K1], [K2], [BN2]):

Theorem 1.1 *Let K be a compact convex subset of $\mathbb{R}^N$.*

A necessary and sufficient condition for an entire function $U \in \mathcal{O}(\mathbb{C}^N)$ to be the Fourier-Laplace transform of an element $u \in \left(\tilde{\gamma}^{(s)}(\mathbb{R}^N)\right)'$ (resp.: $u \in \left(\gamma^{(r,s)}(\mathbb{R}^N)\right)'$) with support contained in K is that there exists constants $c, L, L' \ge 0$ such that

$$|U(\tau,\zeta)| \le c(1+|\tau|)^{L'} \exp\{L|\zeta|^{1/s} + H_K(\operatorname{Im}\tau, \operatorname{Im}\zeta)\} \\ \forall (\tau,\zeta) \in \mathbb{C}^k \times \mathbb{C}^n \simeq \mathbb{C}^N$$

$$(resp.: |U(\tau,\zeta)| \leq c\exp\{L'|\tau|^{1/r} + L|\zeta|^{1/s} + H_K(\operatorname{Im}\tau, \operatorname{Im}\zeta)\} \quad \forall(\tau,\zeta) \in \mathbb{C}^k \times \mathbb{C}^n),$$

where $H_K(\cdot) = \sup_{x\in K}\langle x, \cdot\rangle$ *is the supporting function of* K.

We finally consider, for a locally closed subset F of $\mathbb{R}^N$, the spaces of (small) Gevrey Whitney functions on F : $\gamma_F^{(s)}$, $\tilde{\gamma}_F^{(s)}$ and $\gamma_F^{(r,s)}$, defined by the exact sequence

$$0 \longrightarrow \mathcal{I}(F,\Omega) \cap \gamma^*(\Omega) \longrightarrow \gamma^*(\Omega) \longrightarrow \gamma_F^* \longrightarrow 0,$$

where Ω is an open neighbourhood of F in $\mathbb{R}^N$ and $\mathcal{I}(F,\Omega)$ is the space of smooth functions in Ω which vanish with all their derivatives on F (here $\gamma^*(\Omega)$ and γ_F^* denote one of the spaces $\gamma^{(s)}(\Omega)$, $\tilde{\gamma}^{(s)}(\Omega)$, $\gamma^{(r,s)}(\Omega)$ and, respectively, $\gamma_F^{(s)}$, $\tilde{\gamma}_F^{(s)}$, $\gamma_F^{(r,s)}$).

These spaces are endowed with the natural quotient space topology.

1.3. The overdetermined Cauchy problem and the Phragmén-Lindelöf principle for evolution

Let $K_1 \subset K_2$ be a pair of closed convex subsets of $\mathbb{R}^N \simeq \mathbb{R}_t^k \times \mathbb{R}_x^n$ with $K_1 \subset \Sigma \simeq \mathbb{R}_x^n$, and $A_0(D)$ an $a_1 \times a_0$ matrix of linear partial differential operators with constant coefficients.

If γ^* is one of the classes of (small) Gevrey (Whitney) functions previously considered, we are interested in the following (*overdetermined*) *Cauchy problem*:

$$\begin{cases} \text{find } u \in (\gamma_{K_2}^*)^{a_0} \text{ s.t.} \\ A_0(D)u = f \in (\gamma_{K_2}^*)^{a_1} \\ u|_{K_1} = g \in (\gamma_{K_1}^*)^{a_0} \end{cases} \tag{1.3}$$

for all initial data $(f,g) \in (\gamma_{K_2}^*)^{a_1} \times (\gamma_{K_1}^*)^{a_0}$ satisfying the *compatibility conditions*

$$\begin{cases} A_0(D)g = f|_{K_1} \\ A_1(D)f = 0. \end{cases} \tag{1.4}$$

Such a set of initial data satisfying (1.4) will be called *compatible*.

Definition 1.2 We say that the pair (K_1, K_2) is of *evolution* (resp.: of *causality*, *hyperbolic*) for $A_0(D)$ (or for $\mathcal{M} = \operatorname{coker}{}^tA_0(\theta)$) in the class γ^*, if the above Cauchy problem (1.3) admits at least a solution (resp.: at most one solution, one and only one solution), for each set of compatible initial data.

Remark 1.3 By Whitney's extension theorem every datum $g \in (\gamma^*_{K_1})^{a_0}$ extends to an element $\tilde{g} \in (\gamma^*_{K_2})^{a_0}$, and hence we can reduce to the Cauchy problem (1.3) with zero initial data ($g = 0$). In this case the compatibility conditions (1.4) reduce to:

$$\begin{cases} f \in (\gamma^*_{K_2})^{a_1} \cap \mathcal{I}(K_1, K_2) \\ A_1(D) f = 0, \end{cases}$$

where $\mathcal{I}(K_1, K_2)$ is the ideal of Whitney functions on K_2 which identically vanish with all their derivatives on K_1.

In [BN2] we proved the following:

Theorem 1.4 *Let $\mathcal{M}$ be a unitary finitely generated $\mathcal{P}$-module and consider, for $\wp \in \mathrm{Ass}(\mathcal{M})$, the associated affine algebraic varieties*

$$V = V(\check{\wp}) = \{\theta \in \mathbb{C}^N : p(-\theta) = 0 \ \forall p \in \wp\}. \tag{1.5}$$

Let $K_1 \subset K_2$ be a pair of closed convex subsets of $\mathbb{R}^N \simeq \mathbb{R}^k_t \times \mathbb{R}^n_x$ with $K_1 \subset \Sigma \simeq \mathbb{R}^n_x$, and fix two increasing sequences $\{K_1^{(\alpha)}\}_{\alpha \in \mathbb{N}}$ and $\{K_2^{(\alpha)}\}_{\alpha \in \mathbb{N}}$ of compact convex subsets of $\mathbb{R}^N$ with $K_1 = \bigcup_\alpha K_1^{(\alpha)}$ and $K_2 = \bigcup_\alpha K_2^{(\alpha)}$.

Then the following statements are equivalent:

(1) *the pair (K_1, K_2) is of evolution for $\mathcal{M}$ in the class $\tilde{\gamma}^{(s)}$ with $s > 1$;*

(2) *the pair (K_1, K_2) is of evolution for $\mathcal{P}/\wp$ in the class $\tilde{\gamma}^{(s)}$ with $s > 1$, for all $\wp \in \mathrm{Ass}(\mathcal{M})$;*

(3) $\mathrm{Ext}^1_{\mathcal{P}}(\mathcal{M}, \tilde{\gamma}^{(s)}_{K_2} \cap \mathcal{I}(K_1, K_2)) = 0$;

(4) $\mathrm{Ext}^1_{\mathcal{P}}(\mathcal{P}/\wp, \tilde{\gamma}^{(s)}_{K_2} \cap \mathcal{I}(K_1, K_2)) = 0$ *for all* $\wp \in \mathrm{Ass}(\mathcal{M})$;

(5) *the homomorphism*

$$\mathrm{Ext}^0_{\mathcal{P}}(\mathcal{M}, \tilde{\gamma}^{(s)}_{K_2}) \longrightarrow \mathrm{Ext}^0_{\mathcal{P}}(\mathcal{M}, \tilde{\gamma}^{(s)}_{K_1})$$

is onto;

(6) *the homomorphisms*

$$\mathrm{Ext}^0_{\mathcal{P}}(\mathcal{P}/\wp, \tilde{\gamma}^{(s)}_{K_2}) \longrightarrow \mathrm{Ext}^0_{\mathcal{P}}(\mathcal{P}/\wp, \tilde{\gamma}^{(s)}_{K_1})$$

are onto for all $\wp \in \mathrm{Ass}(\mathcal{M})$;

(7) *the dual homomorphism*

$$\left(\mathrm{Ext}^0_{\mathcal{P}}(\mathcal{M},\tilde{\gamma}^{(s)}_{K_1})\right)' \longrightarrow \left(\mathrm{Ext}^0_{\mathcal{P}}(\mathcal{M},\tilde{\gamma}^{(s)}_{K_2})\right)'$$

has a closed image;

(8) *the dual homomorphisms*

$$\left(\mathrm{Ext}^0_{\mathcal{P}}(\mathcal{P}/\wp,\tilde{\gamma}^{(s)}_{K_1})\right)' \longrightarrow \left(\mathrm{Ext}^0_{\mathcal{P}}(\mathcal{P}/\wp,\tilde{\gamma}^{(s)}_{K_2})\right)' \tag{1.6}$$

have a closed image, for all $\wp \in \mathrm{Ass}(\mathcal{M})$;

(9) *the following Phragmén-Lindelöf principle for evolution holds on* $V = V(\check{\wp})$, *for all* $\wp \in \mathrm{Ass}(\mathcal{M})$:

$\widetilde{(Ph-L)}^{(s)}$

$$\begin{cases} \forall\alpha \in \mathbb{N}\ \exists\beta\in\mathbb{N},\ c>0 \ \ \textit{such that} \\ \textit{if}\ \ f\in\mathcal{O}(V)\ \textit{satisfies, for some}\ \alpha_f\in\mathbb{N},\ c_f>0: \\ \begin{cases} |f(\theta)| \le (1+|\tau|)^{\alpha}\exp\{\alpha|\zeta|^{1/s}+H_{K_2^{(\alpha)}}(\operatorname{Im}\theta)\} \ \ \forall\theta=(\tau,\zeta)\in V \\ |f(\theta)| \le c_f(1+|\tau|)^{\alpha_f}\exp\{\alpha_f|\zeta|^{1/s}+H_{K_1^{(\alpha_f)}}(\operatorname{Im}\theta)\} \\ \hfill \forall\theta=(\tau,\zeta)\in V \end{cases} \\ \textit{then it also satisfies}: \\ |f(\theta)| \le c(1+|\tau|)^{\beta}\exp\{\beta|\zeta|^{1/s}+H_{K_1^{(\beta)}}(\operatorname{Im}\theta)\} \ \ \forall\theta=(\tau,\zeta)\in V; \end{cases}$$

(10) *the following Phragmén-Lindelöf principle for plurisubharmonic functions holds on* $V = V(\check{\wp})$, *for all* $\wp \in \mathrm{Ass}(\mathcal{M})$:

$$\begin{cases} \forall\alpha \in \mathbb{N}\ \exists\beta\in\mathbb{N},\ c>0 \ \ \textit{such that} \\ \textit{if}\ \ u\in P(V)\ \textit{satisfies, for some}\ \alpha_u\in\mathbb{N},\ c_u>0: \\ \begin{cases} |u(\theta)| \le \alpha\log(1+|\tau|)+\alpha|\zeta|^{1/s}+H_{K_2^{(\alpha)}}(\operatorname{Im}\theta) \\ \hfill \forall\theta=(\tau,\zeta)\in V \\ |u(\theta)| \le c_u+\alpha_u\log(1+|\tau|)+\alpha_u|\zeta|^{1/s}+H_{K_1^{(\alpha_u)}}(\operatorname{Im}\theta) \\ \hfill \forall\theta=(\tau,\zeta)\in V \end{cases} \\ \textit{then it also satisfies}: \\ |u(\theta)| \le c+\beta\log(1+|\tau|)+\beta|\zeta|^{1/s}+H_{K_1^{(\beta)}}(\operatorname{Im}\theta) \\ \hfill \forall\theta=(\tau,\zeta)\in V. \end{cases} \tag{1.7}$$

Theorem 1.5 *Let $\mathcal{M}$ be a unitary finitely generated $\mathcal{P}$-module and consider, for $\wp \in \mathrm{Ass}(\mathcal{M})$, the associated affine algebraic varieties $V = V(\check{\wp})$ defined by* (1.5).

Let $K_1 \subset K_2$ be a pair of closed convex subsets of $\mathbb{R}^N \simeq \mathbb{R}^k_t \times \mathbb{R}^n_x$ with $K_1 \subset \Sigma \simeq \mathbb{R}^n_x$, and fix two increasing sequences $\{K_1^{(\alpha)}\}_{\alpha\in\mathbb{N}}$ and $\{K_2^{(\alpha)}\}_{\alpha\in\mathbb{N}}$ of compact convex subsets of $\mathbb{R}^N$ with $K_1 = \bigcup_\alpha K_1^{(\alpha)}$ and $K_2 = \bigcup_\alpha K_2^{(\alpha)}$.

Then the following statements are equivalent:

(1) *the pair (K_1, K_2) is of evolution for $\mathcal{M}$ in the class $\gamma^{(r,s)}$ of (small) Gevrey functions of order $r > 1$ in t and $s > 1$ in x;*

(2) *the pair (K_1, K_2) is of evolution for $\mathcal{P}/\wp$ in the class $\gamma^{(r,s)}$ with $r, s > 1$, for all $\wp \in \mathrm{Ass}(\mathcal{M})$;*

(3) $\mathrm{Ext}^1_{\mathcal{P}}(\mathcal{M}, \gamma^{(r,s)}_{K_2} \cap \mathcal{I}(K_1, K_2)) = 0$;

(4) $\mathrm{Ext}^1_{\mathcal{P}}(\mathcal{P}/\wp, \gamma^{(r,s)}_{K_2} \cap \mathcal{I}(K_1, K_2)) = 0$ *for all $\wp \in \mathrm{Ass}(\mathcal{M})$;*

(5) *the homomorphism*

$$\mathrm{Ext}^0_{\mathcal{P}}(\mathcal{M}, \gamma^{(r,s)}_{K_2}) \longrightarrow \mathrm{Ext}^0_{\mathcal{P}}(\mathcal{M}, \gamma^{(r,s)}_{K_1})$$

is onto;

(6) *the homomorphisms*

$$\mathrm{Ext}^0_{\mathcal{P}}(\mathcal{P}/\wp, \gamma^{(r,s)}_{K_2}) \longrightarrow \mathrm{Ext}^0_{\mathcal{P}}(\mathcal{P}/\wp, \gamma^{(r,s)}_{K_1})$$

are onto for all $\wp \in \mathrm{Ass}(\mathcal{M})$;

(7) *the dual homomorphism*

$$\Big(\mathrm{Ext}^0_{\mathcal{P}}(\mathcal{M}, \gamma^{(r,s)}_{K_1})\Big)' \longrightarrow \Big(\mathrm{Ext}^0_{\mathcal{P}}(\mathcal{M}, \gamma^{(r,s)}_{K_2})\Big)'$$

has a closed image;

(8) *the dual homomorphisms*

$$\Big(\mathrm{Ext}^0_{\mathcal{P}}(\mathcal{P}/\wp, \gamma^{(r,s)}_{K_1})\Big)' \longrightarrow \Big(\mathrm{Ext}^0_{\mathcal{P}}(\mathcal{P}/\wp, \gamma^{(r,s)}_{K_2})\Big)' \tag{1.8}$$

have a closed image, for all $\wp \in \mathrm{Ass}(\mathcal{M})$;

(9) *the following Phragmén-Lindelöf principle for evolution holds on $V = V(\check{\wp})$, for all $\wp \in \mathrm{Ass}(\mathcal{M})$:*

$$(Ph-L)^{(r,s)} \begin{cases} \forall\alpha\in\mathbb{N}\ \exists\beta\in\mathbb{N},\ c>0 \quad \textit{such that} \\ \textit{if } f\in\mathcal{O}(V) \textit{ satisfies, for some } \alpha_f\in\mathbb{N},\ c_f>0: \\ \begin{cases} |f(\theta)|\le\exp\{\alpha|\tau|^{1/r}+\alpha|\zeta|^{1/s}+H_{K_2^{(\alpha)}}(\operatorname{Im}\theta)\} \\ \hfill \forall\theta=(\tau,\zeta)\in V \\ |f(\theta)|\le c_f\exp\{\alpha_f|\tau|^{1/r}+\alpha_f|\zeta|^{1/s}+H_{K_1^{(\alpha_f)}}(\operatorname{Im}\theta)\} \\ \hfill \forall\theta=(\tau,\zeta)\in V \end{cases} \\ \textit{then it also satisfies}: \\ |f(\theta)|\le c\exp\{\beta|\tau|^{1/r}+\beta|\zeta|^{1/s}+H_{K_1^{(\beta)}}(\operatorname{Im}\theta)\} \\ \hfill \forall\theta=(\tau,\zeta)\in V; \end{cases}$$

(10) *the following Phragmén-Lindelöf principle for plurisubharmonic functions holds on $V=V(\check{\wp})$, for all $\wp\in\operatorname{Ass}(\mathcal{M})$:*

$$\begin{cases} \forall\alpha\in\mathbb{N}\ \exists\beta\in\mathbb{N},\ c>0 \quad \textit{such that} \\ \textit{if } u\in P(V) \textit{ satisfies, for some } \alpha_u\in\mathbb{N},\ c_u>0: \\ \begin{cases} |u(\theta)|\le\alpha|\tau|^{1/r}+\alpha|\zeta|^{1/s}+H_{K_2^{(\alpha)}}(\operatorname{Im}\theta) \quad \forall\theta=(\tau,\zeta)\in V \\ |u(\theta)|\le c_u+\alpha_u|\tau|^{1/r}+\alpha_u|\zeta|^{1/s}+H_{K_1^{(\alpha_u)}}(\operatorname{Im}\theta) \\ \hfill \forall\theta=(\tau,\zeta)\in V \end{cases} \\ \textit{then it also satisfies}: \\ |u(\theta)|\le c+\beta|\tau|^{1/r}+\beta|\zeta|^{1/s}+H_{K_1^{(\beta)}}(\operatorname{Im}\theta) \quad \forall\theta=(\tau,\zeta)\in V. \end{cases} \tag{1.9}$$

Remark 1.6 When K_1 or K_2 are compact, we can take in Theorems 1.4 and 1.5 the constant sequences $K_1^{(\alpha)}=K_1$ or $K_2^{(\alpha)}=K_2$, respectively.

In [BN2] we proved analogous theorems also for hyperbolicity and causality. However, we consider here only the evolution case.

Using the Phragmén-Lindelöf principles above we shall prove, in the following sections, Hörmander's type necessary and/or sufficient conditions for existence of solutions of the (overdetermined) Cauchy problem with initial data on a formally non-characteristic and quasi-free affine subspace of $\mathbb{R}^N$.

It will be usefull, to this aim, the following classical Phragmén-Lindelöf principle:

Proposition 1.7 *Let A, C, D, L be some fixed non-negative constants.*

Then, if u is a plurisubharmonic function in $\mathbb{C}^N$ satisfying, for some positive constants B_u, m_u, L_u

$$\begin{cases} u(\theta) \leq A|\theta| + B_u + m_u \log(1+|\theta|) + L_u|\theta|^{1/s} & \forall \theta \in \mathbb{C}^N \\ u(\vartheta) \leq C\log(1+|\vartheta|) + L|\vartheta|^{1/s} + D & \forall \vartheta \in \mathbb{R}^N, \end{cases}$$

it must also satisfy:

$$u(\theta) \leq A|\operatorname{Im}\theta| + 4NC\log(1+|\theta|) + \ell|\theta|^{1/s} + D \quad \forall \theta \in \mathbb{C}^N,$$

where

$$\ell = \frac{2N}{\cos\frac{\pi}{2s}} L.$$

The proof of this proposition follows from standard arguments, similar to those we shall use in the proof of Proposition 2.10, and will therefore be omited (see also [B], Theorem 3.5.8). The constants ℓ and $C' = 4NC$ above are not really the "best" constants, but we do not need sharper estimates of them here.

2. The Cauchy problem with data on a formally non-characteristic and quasi-free affine subspace of $\mathbb{R}^N$

In [Hö3] Hörmander proved a necessary and sufficient condition in order that the pair consisting of a hyperplane of $\mathbb{R}^N$ and one of the half-spaces it bounds is of evolution, in the C^∞ class, for a $\mathcal{P}$-module of the form $\mathcal{P}/\mathcal{I}$, where $\mathcal{I} = (p)$ is a principal ideal generated by a polynomial $p \in \mathcal{P}$.

In [BN1] we extended this result to the case of overdetermined systems and initial data on an affine subspace of $\mathbb{R}^N$ of arbitrary codimension, showing how conditions generalizing the one in [Hö3] are related to the Phragmén-Lindelöf principle.

In this paper we give analogous results in the class of (small) Gevrey functions.

Let $\Sigma \simeq \mathbb{R}^n_x \subset \mathbb{R}^N \simeq \mathbb{R}^k_t \times \mathbb{R}^n_x$, and $\Gamma \subset \mathbb{R}^k_t$ a closed convex cone with non-empty interior and vertex in 0.

The following lemmas will be usefull in the sequel:

Lemma 2.1 *Let Σ be formally non-characteristic and quasi-free for the unitary finitely generated $\mathcal{P}$-module $\mathcal{M}$.*

For each pair $F_1 \subset F_2$ of closed convex subsets of Σ, we have that if *the pair* $(F_2, \Gamma\times F_2)$ *is of evolution for* $\mathcal{M}$ *in the class* $\tilde{\gamma}^{(s)}$ *(resp.: in* $\gamma^{(r,s)}$*), then also the pair* $(F_1, \Gamma\times F_1)$ *is of evolution for* $\mathcal{M}$ *in the class* $\tilde{\gamma}^{(s)}$ *(resp.: in* $\gamma^{(r,s)}$*), for* $r, s > 1$.

Proof. By the assumtpion that Σ is formally non-characteristic and quasi-free for $\mathcal{M}$, all compatible data on F_1 and $\Gamma\times F_1$ can be extended to compatible data on F_2 and $\Gamma\times F_2$ (cf. also [BN2], [B]). □

Lemma 2.2 *Let* Σ *be formally non-characteristic and quasi-free for the unitary finitely generated* $\mathcal{P}$*-module* $\mathcal{M}$*, with reduced order* p_o.

For each closed convex subset F *of* Σ*, we have that if the pair* $(F, \Gamma\times F)$ *is of evolution for* $\mathcal{M}$ *in the class* $\tilde{\gamma}^{(s)}$*, then the pair* $(F, \Gamma\times F)$ *is of evolution for* $\mathcal{M}$ *also in the classes* $\gamma^{(r,s)}$ *for all* $r \geq p_o s$ *(with* $r, s > 1$*).*

Proof. It easily follows from the Phragmén-Lindelöf principle for evolution (Theorems 1.4 and 1.5).

Let, indeed, $\{K_\alpha\}_{\alpha\in\mathbb{N}}$ be an increasing sequence of compact convex subsets of $\mathbb{R}^N$ with $\bigcup_{\alpha\in\mathbb{N}} K_\alpha = F$, and note that the supporting function of $K_2^{(\alpha)} = (\Gamma\cap B_k(0,\alpha))\times K_\alpha = (\Gamma\cap\{|t|\leq\alpha\})\times K_\alpha$ is given by

$$H_{K_2^{(\alpha)}}(\operatorname{Im}\tau, \operatorname{Im}\zeta) = H_{K_\alpha}(\operatorname{Im}\zeta) + \alpha\kappa_\Gamma(\tau), \quad (\tau,\zeta)\in\mathbb{C}^k\times\mathbb{C}^n,$$

where

$$\kappa_\Gamma(\tau) = \sup_{\substack{t\in\Gamma\\ |t|\leq 1}}\langle t, \operatorname{Im}\tau\rangle, \quad \tau\in\mathbb{C}^k. \tag{2.1}$$

Fix $\wp\in\operatorname{Ass}(\mathcal{M})$ and $V = V(\check{\wp})$. Let $r\geq p_o s$ and $u\in P(V)$ satisfying

$$\begin{cases} u(\theta)\leq\alpha|\tau|^{1/r}+\alpha|\zeta|^{1/s}+H_{K_\alpha}(\operatorname{Im}\zeta)+\alpha\kappa_\Gamma(\tau) & \forall\theta=(\tau,\zeta)\in V\\ u(\theta)\leq c_u+\alpha_u|\tau|^{1/r}+\alpha_u|\zeta|^{1/s}+H_{K_{\alpha_u}}(\operatorname{Im}\zeta) & \forall\theta=(\tau,\zeta)\in V.\end{cases}$$

By assumption, (1.2) is satisfied on V for $b = p_o$, and hence

$$|\tau|^{1/r}\leq c(1+|\zeta|^{1/s}) \quad \forall(\tau,\zeta)\in V$$

for some $c > 0$, if $r\geq p_o s$. Therefore:

$$\begin{cases} u(\theta)\leq\alpha(1+c)|\zeta|^{1/s}+H_{K_\alpha}(\operatorname{Im}\zeta)+\alpha\kappa_\Gamma(\tau)+\alpha c & \forall\theta=(\tau,\zeta)\in V\\ u(\theta)\leq(c_u+\alpha_u c)+\alpha_u(1+c)|\zeta|^{1/s}+H_{K_{\alpha_u}}(\operatorname{Im}\zeta) & \forall\theta=(\tau,\zeta)\in V.\end{cases}$$

Since the pair $(F, \Gamma \times F)$ is of evolution for $\mathcal{M}$ in $\tilde{\gamma}^{(s)}$ by assumption, by the Phragmén-Lindelöf principle for evolution (Theorem 1.4) u must then satisfy an estimate of the form

$$\begin{aligned} u(\theta) &\leq \beta' \log(1 + |\tau|) + \beta'|\zeta|^{1/s} + H_{K_{\beta'}}(\operatorname{Im}\zeta) + c' \\ &\leq \beta' r|\tau|^{1/r} + \beta'|\zeta|^{1/s} + H_{K_{\beta'}}(\operatorname{Im}\zeta) + c' \quad \forall \theta = (\tau, \zeta) \in V, \end{aligned}$$

for some $\beta' \in \mathbb{N}$ and $c' > 0$ depending only on α and c.

It follows that the pair $(F, \Gamma \times F)$ is of evolution for $\mathcal{M}$ also in the class $\gamma^{(r,s)}$ with $r \geq p_o s$, by Theorem 1.5. □

Remark 2.3 The condition that Σ is quasi-free is indeed not necessary in Lemma 2.2, since it can be proved that (1.2) is equivalent to the only assumption that Σ is formally non-characteristic for $\mathcal{M}$ (cf. [BN2]).

2.1. A necessary Hörmander's type condition for evolution

We give in this subsection a necessary condition for evolution, which naturally generalizes the one given by Hörmander in [Hö3].

Let $\mathcal{M}$ be a unitary finitely generated $\mathcal{P}$-module, and assume that $\Sigma \simeq \mathbb{R}^n_x$ is formally non-characteristic and quasi-free for $\mathcal{M}$. Then, as we already recalled in §1.1, for each $\wp \in \operatorname{Ass}(\mathcal{M})$ the natural projection map

$$\pi_n : V(\check{\wp}) \ni (\tau, \zeta) \longmapsto \zeta \in \mathbb{C}^n$$

is surjective, finite and dominant, and there is a smallest positive constant p_o such that, for some $k > 0$:

$$|\tau| \leq k(1 + |\zeta|)^{p_o} \quad \forall (\tau, \zeta) \in V = V(\check{\wp}). \tag{2.2}$$

Moreover, there is a proper affine algebraic variety $Z \subset \mathbb{C}^n$ such that $V \setminus \pi_n^{-1}(Z)$ is smooth and

$$V \setminus \pi_n^{-1}(Z) \ni (\tau, \zeta) \xmapsto{\pi_n} \zeta \in \mathbb{C}^n \setminus Z$$

is an m-sheeted covering of $\mathbb{C}^n \setminus Z$.

For $s > 1$ we consider then the following Hörmander's type condition on V:

$$
(H)^s \quad \begin{cases} \exists R, c_1, c_2 > 0 \quad \text{such that for every } \rho \in \mathbb{R}^n \\ \text{with } \; B_n(\rho, R|\rho|^{1/s}) = \{|\zeta - \rho| \le R|\rho|^{1/s}\} \subset \mathbb{C}^n \setminus Z \\ \text{and for every connected component } \omega \text{ of } \pi_n^{-1}(B_n(\rho, R|\rho|^{1/s})) \\ \text{there is } \; \theta = (\tau, \zeta) \in \omega \text{ such that} \\ \kappa_\Gamma(\tau) \le c_1|\zeta|^{1/s} + c_2, \end{cases}
$$

where $\kappa_\Gamma(\tau)$ is defined by (2.1).

Remark 2.4 Comparing condition $(H)^s$ above with condition (H) in [BN1] for the C^∞ case, we note that for the class of (small) Gevrey functions here we require not only a growth of the form $|\zeta|^{1/s}$ for $\kappa_\Gamma(\tau)$ (in (H) we had bounded $\kappa_\Gamma(\tau)$), but also a growth of the form $R|\rho|^{1/s}$ for the radius of the ball (in (H) we had a fixed radius R).

Remark 2.5 Note that condition $(H)^s$ implies condition $(H)^{s'}$ for all $1 < s' \le s$.

We have the following:

Theorem 2.6 *Let $\mathcal{M}$ be a unitary finitely generated $\mathcal{P}$-module, for which $\Sigma \simeq \mathbb{R}^n_x$ is formally non-characteristic and quasi-free, with reduced order p_o.*

Let F be a closed convex subset of Σ with a non-empty interior in $\mathbb{R}^n_x$ and Γ a closed convex cone of $\mathbb{R}^k_t$ with non-empty interior and vertex in 0.

Then a necessary condition in order that the pair $(F, \Gamma \times F)$ is of evolution for $\mathcal{M}$ in the class $\gamma^{(r,s)}$, for $r, s > 1$ with $r \ge p_o s$, is that condition $(H)^q$ is satisfied on $V(\check{\wp})$ for all $\wp \in \mathrm{Ass}(\mathcal{M})$ and $q \in \mathbb{Q}$ with $1 < q \le s$.

Proof. Let $\overset{\circ}{F}$ denote the non-empty relative interior of F in $\mathbb{R}^n_x$. It is not restrictive to assume $0 \in \overset{\circ}{F}$, so that F contains a closed ball $B_n(0, A)$ of $\mathbb{R}^n_x$, for some $A > 0$. By the assumption that Σ is formally non-characteristic and quasi-free and Lemma 2.1 it is then no restrictive to assume $F = B_n(0, A)$.

The assumption that the pair $(F, \Gamma \times F)$ is of evolution for $\mathcal{M}$ in the class $\gamma^{(r,s)}$ is thus equivalent, by Theorem 1.5 (and Remark 1.6), to the validity of the following Phragmén-Lindelöf principle for plurisubharmonic functions, for all $\wp \in \mathrm{Ass}(\mathcal{M})$:

$\forall \alpha \in \mathbb{N} \; \exists \beta \in \mathbb{N}, \; c > 0$ such that

if $u \in P(V(\check{\wp}))$ satisfies, for some constants $\alpha_u \in \mathbb{N}$ and $c_u > 0$

$$\begin{cases} u(\theta) \leq \alpha|\tau|^{1/r} + \alpha|\zeta|^{1/s} + \alpha\kappa_\Gamma(\tau) + A|\operatorname{Im}\zeta| & \forall\theta = (\tau,\zeta) \in V(\check{\wp}) \\ u(\theta) \leq c_u + \alpha_u|\tau|^{1/r} + \alpha_u|\zeta|^{1/s} + A|\operatorname{Im}\zeta| & \forall\theta = (\tau,\zeta) \in V(\check{\wp}) \end{cases} \tag{2.3}$$

then it also satisfies:

$$u(\theta) \leq c + \beta|\tau|^{1/r} + \beta|\zeta|^{1/s} + A|\operatorname{Im}\zeta| \quad \forall\theta = (\tau,\zeta) \in V(\check{\wp}). \tag{2.4}$$

Let us assume by contradiction that there exists $q \in \mathbb{Q}$ with $1 < q \leq s$ such that condition $(H)^q$ is not satisfied for some $\wp \in \operatorname{Ass}(\mathcal{M})$.

Set $V = V(\check{\wp})$ and, for each $\sigma > 1$:

$$\begin{aligned} \mathcal{U}_\sigma = \big\{(\rho, r) \in \mathbb{R}^n \times \mathbb{R}^+ : B_n(\rho, r|\rho|^{1/\sigma}) \subset \mathbb{C}^n \setminus Z, \\ \exists \text{ a connected component } \omega \text{ of } \pi_n^{-1}(B_n(\rho, r|\rho|^{1/\sigma})) \\ \text{in } V \text{ s.t. } \kappa_\Gamma(\tau) > r|\zeta|^{1/\sigma} + r \quad \forall(\tau,\zeta) \in \omega\big\}. \end{aligned}$$

This set $\mathcal{U}_\sigma$ is semi-algebraic if $\sigma \in \mathbb{Q}$. We consider then

$$f_\sigma(t) = \sup\{r \in \mathbb{R} : (\rho, r) \in \mathcal{U}_\sigma,\ |\rho| = t\},$$

which is a semi-algebraic function when $\sigma \in \mathbb{Q}$.

Condition $(H)^q$ is equivalent to the fact that $f_q(t)$ is bounded for $t \to +\infty$. Assuming the contrary, applying the Tarski-Seidenberg theorem (Theorem A.2.5 of [Hö3]) to the semi-algebraic function $f_q(t)$, we can find $R > 0$, $0 < p \in \mathbb{Q}$ and $t_0 \geq 0$ such that

$$f_q(t) = t^p(2R + o(1)) \quad \text{for} \quad t > t_0. \tag{2.5}$$

We can then find $t_1 \geq t_0$ such that

$$f_s(t) \geq f_q(t) = t^p(2R + o(1)) \quad \forall t > t_1$$

and hence:

$$\begin{aligned} &\forall t > t_1\ \exists\rho_t \in \mathbb{R}^n \ \text{ with } \ |\rho_t| = t,\ B_n(\rho_t, Rt^{p+\frac{1}{s}}) \subset \mathbb{C}^n \setminus Z \\ &\exists \text{ a connected component } \omega_t \text{ of } \ \pi_n^{-1}(B_n(\rho_t, Rt^{p+\frac{1}{s}})) \ \text{ s.t.} \\ &\kappa_\Gamma(\tau) > Rt^p(|\zeta|^{1/s} + 1) \quad \forall(\tau,\zeta) \in \omega_t. \end{aligned} \tag{2.6}$$

Let us now fix a (small) Gevrey function $\chi \in \gamma^{(s')}(\mathbb{R}^n_x)$ with $\operatorname{supp}\chi \subset B_n(0, \frac{A}{2})$ and $\hat{\chi}(0) = 1$, with $s' > 1$ to be choosen in the following. Its Fourier-Laplace transform is characterized by (cf. [K1]):

$$\forall h>0\ \exists c_h>0:\ |\hat{\chi}(\zeta)|\le c_h\exp\left\{-\frac{1}{h}|\zeta|^{1/s'}+\frac{A}{2}|\operatorname{Im}\zeta|\right\}\quad\forall\zeta\in\mathbb{C}^n. \tag{2.7}$$

Consider then the plurisubharmonic functions

$$u_{t,\ell}(\theta)=\log|\hat{\chi}(\zeta-\rho_t)|+\ell|\tau|^{1/r}+\ell|\zeta|^{1/s},\quad \theta=(\tau,\zeta)\in\mathbb{C}^k\times\mathbb{C}^n$$

for $t>t_1$, $\ell\in\mathbb{N}$.

Since $\operatorname{supp}\chi\subset B(0,\frac{A}{2})$ and $\hat{\chi}(0)=1$, we have that $|\hat{\chi}(\zeta)|\le\exp\left\{\frac{A}{2}|\operatorname{Im}\zeta|\right\}$ for all $\zeta\in\mathbb{C}^n$ and hence

$$u_{t,\ell}(\theta)\le\frac{A}{2}|\operatorname{Im}\zeta|+\ell|\tau|^{1/r}+\ell|\zeta|^{1/s}\quad\forall\theta=(\tau,\zeta)\in\mathbb{C}^k\times\mathbb{C}^n.$$

Moreover, by the assumption on Σ, the inequality (2.2) is valid for some $k>0$, and hence, if $r\ge p_o s$, from (2.6):

$$\begin{aligned}&\forall B>0\ \exists t_2\ge t_1\ \text{ s.t. }\ \forall t>t_2\\&u_{t,\ell}(\theta)\le A|\operatorname{Im}\zeta|+B\kappa_\Gamma(\tau)\quad\forall(\tau,\zeta)\in\omega_t.\end{aligned}$$

On the other hand, for $r\ge p_o s$ by (2.7) we can find $h>0$ sufficiently small and $t_3\ge t_2$ such that for all $t>t_3$ and (τ,ζ) near the boundary of ω_t:

$$\begin{aligned}u_{t,\ell}(\theta)&\le\log c_h-\frac{1}{h}|\rho_t+Rt^{p+\frac{1}{s}}z-\rho_t|^{1/s'}+\frac{A}{2}|\operatorname{Im}\zeta|\\&\quad+\ell'|\rho_t+Rt^{p+\frac{1}{s}}z|^{1/s}+\ell'\\&\le A|\operatorname{Im}\zeta|,\end{aligned}$$

for z near the boundary of $B_n(0,1)\subset\mathbb{C}^n$ and for some $\ell'\in\mathbb{N}$, if $1<s'<\min\{s,ps+1\}$.

We define then on V the plurisubharmonic function:

$$v_{t,\ell}(\theta)=\begin{cases}\max\{u_{t,\ell}(\theta),A|\operatorname{Im}\zeta|\} & \theta\in\omega_t\\ A|\operatorname{Im}\zeta| & \theta\in V\setminus\omega_t.\end{cases} \tag{2.8}$$

When $t>t_3$ we have that $v_{t,\ell}\in P(V)$ satisfies:

$$\begin{cases}v_{t,\ell}(\theta)\le A|\operatorname{Im}\zeta|+B\kappa_\Gamma(\tau) & \forall\theta=(\tau,\zeta)\in V\\ v_{t,\ell}(\theta)\le A|\operatorname{Im}\zeta|+\ell|\tau|^{1/r}+\ell|\zeta|^{1/s} & \forall\theta=(\tau,\zeta)\in V,\end{cases}$$

but (2.4) cannot hold true, since for every $\ell \in \mathbb{N}$ and $t > t_3$ we can find $\theta_t = (\tau_t, \zeta_t) \in \omega_t$ with $\pi_n(\theta_t) = \rho_t$ and

$$v_{t,\ell}(\theta_t) \geq \log|\hat{\chi}(0)| + \ell|\tau_t|^{1/r} + \ell|\zeta_t|^{1/s} = \ell|\tau_t|^{1/r} + \ell|\zeta_t|^{1/s}.$$

This contradicts the validity of the Phragmén-Lindelöf principle (2.3)–(2.4), and hence condition $(H)^q$ must be satisfied for all $1 < q \leq s$, on every $V(\check{\wp})$ for $\wp \in \mathrm{Ass}(\mathcal{M})$. □

By Lemma 2.2 and Theorem 2.6 we immediately obtain:

Theorem 2.7 *Let $\mathcal{M}$ be a unitary finitely generated $\mathcal{P}$-module, for which $\Sigma \simeq \mathbb{R}^n_x$ is formally non-characteristic and quasi-free.*

Let F be a closed convex subset of Σ with a non-empty interior in $\mathbb{R}^n_x$ and Γ a closed convex cone of $\mathbb{R}^k_t$ with non-empty interior and vertex in 0.

Then a necessary condition in order that the pair $(F, \Gamma \times F)$ is of evolution for $\mathcal{M}$ in the class $\tilde{\gamma}^{(s)}$, for $s > 1$, is that condition $(H)^q$ is satisfied on $V(\check{\wp})$ for all $\wp \in \mathrm{Ass}(\mathcal{M})$ and $q \in \mathbb{Q}$ with $1 < q \leq s$.

Example 2.8 Let us consider the following system in $\mathbb{R}^2_t \times \mathbb{R}_x$:

$$\begin{cases} \dfrac{\partial}{\partial t_1} - i\left(\dfrac{\partial}{\partial x} - 1\right)^2 \\[2ex] \dfrac{\partial}{\partial t_2} + \dfrac{\partial^3}{\partial x^3}, \end{cases}$$

and set $\Gamma = \{t \in \mathbb{R}^2 : t_1 \geq 0,\ t_2 \geq 0\}$.

The associated affine algebraic variety is given by

$$V = \{(\tau_1, \tau_2, \zeta) \in \mathbb{C}^3 : \tau_1 = (\zeta - i)^2,\ \tau_2 = \zeta^3\}.$$

For $(\tau, \zeta) \in V$ we have that

$$\begin{aligned} \kappa_\Gamma(\tau) &= (\mathrm{Im}\,\tau_1)^+ + (\mathrm{Im}\,\tau_2)^+ \\ &= 2\{\mathrm{Re}\,\zeta(\mathrm{Im}\,\zeta - 1)\}^+ + \{\mathrm{Im}\,\zeta[3(\mathrm{Re}\,\zeta)^2 - (\mathrm{Im}\,\zeta)^2]\}^+. \end{aligned}$$

We can easlity see that condition $(H)^s$ cannot be satisfied for any $s > 1$, $R, c_1, c_2 > 0$ when $\frac{2}{3}\pi < \arg\zeta < \frac{5}{3}\pi$. By Theorems 2.6 and 2.7 it thus follows that any pair of the form $(F, \Gamma \times F)$, for a closed convex subset F of $\Sigma \simeq \mathbb{R}_x$, cannot be of evolution for the given system in any class $\tilde{\gamma}^{(s)}$ or $\gamma^{(r,s)}$.

2.2. The case of algebraic curves

We take, in this subsection, the case $n = 1$. Then, for a unitary finitely generated $\mathcal{P}$-module $\mathcal{M}$, for each $\wp \in \mathrm{Ass}(\mathcal{M})$ the associated affine algebraic variety $V(\check{\wp})$ is an algebraic curve.

We consider the following condition for $\wp \in \mathrm{Ass}(\mathcal{M})$ and $R' > 0$:

$$(h)^s \begin{cases} \exists c_0,\, c_1,\, R > 0 \text{ s.t. on every connected component of} \\ \{(\tau,\zeta) \in V(\check{\wp}) : |\operatorname{Im}\zeta| \leq R|\operatorname{Re}\zeta|^{1/s},\ |\zeta| \geq R'\} \text{ there is a sequence} \\ \{(\tau_\nu,\zeta_\nu)\}_{\nu\in\mathbb{N}} \text{ with } |\zeta_\nu| \to +\infty \text{ for } \nu \to +\infty \text{ and} \\ \kappa_\Gamma(\tau_\nu) \leq c_0|\zeta_\nu|^{1/s} + c_1 \quad \forall \nu \in \mathbb{N}. \end{cases}$$

Remark 2.9 Condition $(h)^s$ implies condition $(h)^{s'}$ for all $1 < s' \leq s$.

We have the following:

Proposition 2.10 *Let $\mathcal{M}$ be a finitely generated unitary $\mathcal{P}$-module, for which $\Sigma \simeq \mathbb{R}_x$ is formally non-characteristic and quasi-free. Let K be a compact convex subset of Σ with non-empty interior in $\mathbb{R}_x$, and Γ a closed convex cone of $\mathbb{R}^k_t$ with non-empty interior and vertex in 0. Then:*

i) *if condition $(h)^s$ is satisfied for some $s > 1$ on $V(\check{\wp})$ for all $\wp \in \mathrm{Ass}(\mathcal{M})$ and for all $R' > 0$, then the pair $(K, \Gamma \times K)$ is of evolution for $\mathcal{M}$ in the classes $\tilde{\gamma}^{(s')}$ for all $s' \in \mathbb{R}$ with $1 < s' \leq s$;*

ii) *if the pair $(K, \Gamma \times K)$ is of evolution for $\mathcal{M}$ in the class $\tilde{\gamma}^{(s)}$ for some $s > 1$, then condition $(h)^q$ is satisfied for all $\wp \in \mathrm{Ass}(\mathcal{M})$, $R' > 0$ and $q \in \mathbb{Q}$ with $1 < q \leq s$.*

Proof. **1)** Let us first prove i). By Remark 2.9 it is sufficient to prove that for every fixed real $s > 1$ condition $(h)^s$ implies evolution in the class $\tilde{\gamma}^{(s)}$.

Fix $\wp \in \mathrm{Ass}(\mathcal{M})$ and let us make some remarks about the irreducible affine algebraic curve $V = V(\check{\wp})$ in $\mathbb{C}^{k+1}$ (cf. also [S], [BN1], [B]).

By assumption the natural projection map

$$\begin{aligned} \pi : V &\longrightarrow \mathbb{C}_\zeta \\ (\tau,\zeta) &\longmapsto \zeta \end{aligned}$$

is surjective, finite and dominant.

The closure $\bar{V}$ of V in $\mathbb{CP}^{k+1}$ is an irreducible projective curve, and π

extends to a surjective, finite and dominant map

$$\bar{\pi} : \bar{V} \to \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}.$$

We note that $\bar{V} \setminus V = \bar{\pi}^{-1}(\infty)$.

The normalization $\tilde{V} \xrightarrow{\sigma} \bar{V}$ is an irreducible smooth projective curve and the birational isomorphism σ is regular. Let $\sigma^{-1} \circ \bar{\pi}^{-1}(\infty) = \{P_1, \dots, P_\ell\}$. Then we can fix pairwise disjoint connected open neighbourhoods $\tilde{V}_1, \dots, \tilde{V}_\ell$ of $P_1, \dots, P_\ell$ respectively in $\tilde{V}$, in such a way that, setting $V_j = \sigma(\tilde{V}_j \setminus \{P_j\}) \subset V$, for $j = 1, \dots, \ell$, we obtain:

(i) $V_j \cap V_h = \emptyset$ for $1 \leq j < h \leq \ell$;

(ii) $\sigma : \tilde{V}_j \setminus \{P_j\} \to V_j$ is biholomorphic;

(iii) $\pi : V_j \to \pi(V_j) \subset \mathbb{C}_\zeta$ is an m_j-sheeted covering for some integer $m_j \geq 1$;

(iv) $\pi(V_j) \cup \{\infty\}$ is an open neighbourhood of ∞ in $\mathbb{CP}^1$;

(v) for each $j \in \{1, \dots, \ell\}$, $\pi^{-1}(\mathbb{R}) \cap V_j$ consists of $2m_j$ connected components.

We can also assume that for a fixed $r > 1$ and every $j = 1, \dots, \ell$ we have:

$$\pi(V_j) = \{\zeta \in \mathbb{C} : |\zeta| > r\}.$$

For each $j = 1, \dots, \ell$ we obtain a Puiseux parametric description of V_j of the form:

$$\begin{cases} \zeta = z^{m_j} \\ \tau_h = \displaystyle\sum_{\alpha \leq \nu(h,j)} c_{h,j,\alpha} z^\alpha, \quad \text{for } h = 1, \dots, k. \end{cases}$$

A) Let us first assume $s \in \mathbb{Q}$. Then the set

$$E_{R,R'} = \{(\tau, \zeta) \in V : |\operatorname{Im} \zeta| \leq R |\operatorname{Re} \zeta|^{1/s}, \ |\zeta| \geq R'\}$$

is semi-algebraic, and for $R' > r$ sufficiently large for each $j = 1, \dots, \ell$ the intersection $V_j \cap E_{R,R'}$ consists of $2m_j$ connected components.

It is not restrictive to assume in the following $R > 1$.

For ν sufficiently large, condition $(h)^s$ implies that for every $j = 1, \dots, \ell$, on each of the $2m_j$ connected components of $V_j \cap E_{R,R'}$ we can find a sequence $\{(\tau_\nu, \zeta_\nu)\}_{\nu \in \mathbb{N}}$ with $|\zeta_\nu| \to +\infty$ for $\nu \to +\infty$ and such that

$$\kappa_\Gamma(\tau_\nu) \leq c_0 |\zeta_\nu|^{1/s} + c_1 \quad \forall \nu \in \mathbb{N}.$$

Let us fix $j \in \{1, \dots, \ell\}$ and omit the index j for simplicity.

Up to rotations, we can thus find $2m$ sequences $\{z_\nu^{(h)}\}_{\nu\in\mathbb{N}}$, for $h = 0, \dots, 2m-1$, such that:

$$\begin{cases} z_\nu^{(h)} \in \mathbb{C} \setminus B(0,r), \ |\operatorname{Im}(z_\nu^{(h)} e^{-i\frac{h\pi}{m}})| \leq R|\operatorname{Re}(z_\nu^{(h)} e^{-i\frac{h\pi}{m}})|^{1/s} \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall \nu \in \mathbb{N}, \ h = 0, \dots, 2m-1 \\ |z_\nu^{(h)}| \nearrow +\infty \ \text{ for } \ \nu \to +\infty, \ \ \forall h = 0, \dots, 2m-1 \\ \kappa_\Gamma(\tau(z_\nu^{(h)})) \leq c_0 |z_\nu^{(h)}|^{m/s} + c_1 \quad \forall \nu \in \mathbb{N}, \ h = 0, \dots, 2m-1. \end{cases}$$

Let

$$\begin{aligned} E = \Big\{(t, z^{(0)}, \dots, z^{(2m-1)}) \in [r, +\infty) \times \mathbb{C}^{2m} : \ |z^{(h)}| \geq t, \\ |\operatorname{Im}(z^{(h)} e^{-i\frac{h\pi}{m}})| \leq R|\operatorname{Re}(z^{(h)} e^{-i\frac{h\pi}{m}})|^{1/s}, \\ \kappa_\Gamma(\tau(z^{(h)})) \leq c_0 |z^{(h)}|^{m/s} + c_1 \ \ \forall h = 0, \dots, 2m-1\Big\}. \end{aligned}$$

This set is semi-algebraic, and by assumption the projection map

$$\pi : E \longrightarrow [r, +\infty)$$

is onto for large t.

Then, by Theorem A.2.8 of [Hö3], we can find $2m$ Puiseux series $z^0(t), \dots, z^{(2m-1)}(t)$ converging for large positive t and such that $(t, z^{(0)}(t), \dots, z^{(2m-1)}(t)) \in E$. We can next extend these curves up to $0 \in \mathbb{C}$, in such a way to obtain $2m$ real analytic curves $w_h(t)$ which divide the $\mathbb{C}_\zeta$-plane into $2m$ connected components, and such that, for some positive constants c_0' and c_1':

$$\begin{aligned} &|\operatorname{Im}(w_h(t) e^{-i\frac{h\pi}{m}})| \leq R|\operatorname{Re}(w_h(t) e^{-i\frac{h\pi}{m}})|^{1/s}, \\ &\kappa_\Gamma(\tau(w_h(t))) \leq c_0' |w_h(t)|^{m/s} + c_1', \end{aligned} \tag{2.9}$$

$$-\frac{\pi}{m}\sigma < \arg w_h(t) e^{-ih\frac{\pi}{m}} < \frac{\pi}{m}\sigma \quad \forall h = 0, \dots, 2m-1, \ \ t \geq 0, \tag{2.10}$$

for $0 < \sigma < 1/2$ to be choosen in the following.

By assumption K is compact and hence contained in some ball $B(0, A) \subset \mathbb{R}_x$ for $A > 0$. By Lemma 2.1 we can then assume, without loss of generality, $K = B(0, A) \subset \mathbb{R}_x$, and prove the validity of the following Phragmén-

Lindelöf principle:

$$\forall\alpha \in \mathbb{N}\ \exists\beta \in \mathbb{N},\ c > 0 \text{ such that}$$
if $u \in P(V)$ satisfies, for some $\alpha_u \in \mathbb{N}$ and $c_u > 0$:
$$\begin{cases} u(\theta) \le \alpha\log(1+|\tau|) + \alpha|\zeta|^{1/s} + \alpha\kappa_\Gamma(\tau) + A|\operatorname{Im}\zeta| \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall\theta = (\tau,\zeta) \in V \\ u(\theta) \le c_u + \alpha_u\log(1+|\tau|) + \alpha_u|\zeta|^{1/s} + A|\operatorname{Im}\zeta| \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall\theta = (\tau,\zeta) \in V \end{cases} \tag{2.11}$$
then it also satisfies:
$$u(\theta) \le c + \beta\log(1+|\tau|) + \beta|\zeta|^{1/s} + A|\operatorname{Im}\zeta| \quad \forall\theta = (\tau,\zeta) \in V. \tag{2.12}$$

To this aim it's enough to prove the following, because of (2.2) and (2.9):

$$\forall\alpha \in \mathbb{N}\ \exists\beta \in \mathbb{N},\ c > 0 \text{ s.t.}$$
if $u \in P(\mathbb{C})$ satisfies, for some $\alpha_u \in \mathbb{N}$ and $c_u > 0$:
$$\begin{cases} u(z) \le A|\operatorname{Im} z^m| + \alpha|z|^{m/s} & \text{for } z = w_h(t),\ h = 0,\dots,2m-1 \\ u(z) \le A|z|^m + \alpha_u|z|^{m/s} + c_u & \text{for } z \in \mathbb{C} \end{cases} \tag{2.13}$$
then it also satisfies:
$$u(z) \le A|\operatorname{Im} z^m| + \beta|z|^{m/s} + c \quad \text{for } z \in \mathbb{C}. \tag{2.14}$$

a) Let us first assume $\alpha_u = 0$. We want to apply the maximum principle to each one of the $2m$ connected components in which $\mathbb{C}_\zeta$ is divided by the curves $w_h(t)$. Since the arguments are the same on each of these components, we give here the proof of the estimate in the component $S_{0,1}$ bounded by $w_0(t)$ and $w_1(t)$.

Let us first work in the sector S bounded by $w_0(t)$ and the half-ray $\{\rho e^{i\frac{\pi}{2m}} : \rho \ge 0\}$.

Consider, for $\varepsilon > 0$, the subharmonic function

$$v_\varepsilon(z) = u(z) - \varepsilon\operatorname{Re}((ze^{-i\varphi})^{m+k}) - L\operatorname{Re}(z^{m/s}) - A\operatorname{Im} z^m - c_u,$$

for some $L > 0$ to be choosen in the following and $k > 0$ and φ such that for $\theta = \arg z$, with $z \in S$:

$$-\frac{\pi}{2} < (m+k)(\theta - \varphi) < \frac{\pi}{2}.$$

Note that by (2.10) we always have that $-\frac{\pi}{2} < \frac{m}{s}\theta < \frac{\pi}{2}$ for $\theta = \arg z$ with $z \in S$, since $s > 1$ and $0 < \sigma < 1/2$.

We shall then choose in the following

$$0 < k < \frac{1-2\sigma}{1+2\sigma}m, \quad \varphi = \frac{1-2\sigma}{4m}\pi.$$

Then, for $z = w_0(t) = \rho_0(t)e^{i\theta_0(t)}$ we have:

$$\begin{aligned}
v_\varepsilon(z) &= u(w_0(t)) - \varepsilon\rho_0^{m+k}(t)\cos[(m+k)(\theta_0(t)-\varphi)] \\
&\qquad - L\rho_0^{m/s}(t)\cos\Big(\frac{m}{s}\theta_0(t)\Big) - A\operatorname{Im} w_0^m(t) - c_u \\
&\le A|\operatorname{Im} w_0^m(t)| + \alpha\rho_0^{m/s}(t) - L\rho_0^{m/s}(t)\cos\Big(\frac{m}{s}\theta_0(t)\Big) - A\operatorname{Im} w_0^m(t) \\
&\le 2AR|\operatorname{Re} w_0^m(t)|^{1/s} + \alpha\rho_0^{m/s}(t) - L\rho_0^{m/s}(t)\cos\frac{\pi}{2s} \\
&\le \rho_0^{m/s}(t)\left(2AR + \alpha - L\cos\frac{\pi}{2s}\right) \\
&\le 0
\end{aligned}$$

if

$$L \ge \frac{2AR+\alpha}{\cos\frac{\pi}{2s}}. \tag{2.15}$$

For $z = \rho e^{i\frac{\pi}{2m}}$:

$$\begin{aligned}
v_\varepsilon(z) &= u(z) - \varepsilon\rho^{m+k}\cos\left[(m+k)\left(\frac{\pi}{2m}-\varphi\right)\right] \\
&\qquad - L\rho^{m/s}\cos\left(\frac{m}{s}\frac{\pi}{2m}\right) - A\rho^m - c_u \\
&\le A\rho^m + c_u - A\rho^m - c_u = 0.
\end{aligned}$$

Moreover, for $z = \rho e^{i\theta} \in S$ we have that

$$2A|z|^m = 2A\rho^m \le \varepsilon\rho^{m+k}\cos[(m+k)(\theta-\varphi)]$$

for

$$|z| = \rho \ge \left(\frac{2A}{\varepsilon \min\limits_{\theta\in[-\frac{\pi}{m}\sigma,\frac{\pi}{2m}]}\cos[(m+k)(\theta-\varphi)]}\right)^{1/k} = R_\varepsilon,$$

and hence, for $z = \rho e^{i\theta} \in S$ with $|z| \geq R_\varepsilon$:

$$\begin{aligned} v_\varepsilon(z) &= u(z) - \varepsilon\rho^{m+k}\cos[(m+k)(\theta-\varphi)] \\ &\qquad - L\rho^{m/s}\cos\left(\frac{m}{s}\theta\right) - A\operatorname{Im} z^m - c_u \\ &\leq A|z|^m + c_u - \varepsilon\rho^{m+k}\cos[(m+k)(\theta-\varphi)] + A|z|^m - c_u \\ &\leq 0. \end{aligned}$$

It follows, by the maximum principle, that

$$v_\varepsilon(z) \leq 0 \quad \forall z \in S, \quad \forall \varepsilon > 0.$$

For $\varepsilon \to 0$ we thus obtain that

$$u(z) \leq L|z|^{m/s} + A|\operatorname{Im} z^m| + c_u \quad \forall z \in S.$$

Arguing in the same way in the other sectors, we have that

$$u(z) \leq A|\operatorname{Im} z^m| + L|z|^{m/s} + c_u \quad \forall z \in \mathbb{C}. \tag{2.16}$$

To eliminate now the constant c_u we consider, for $z \in S_{0,1}$, the plurisubharmonic function

$$w(z) = u(z) - A\operatorname{Im} z^m - ML\operatorname{Re}((ze^{-i\theta'})^{m/s}),$$

with L satisfying (2.15) and $M > 1$ to be choosen in the following, and θ' such that

$$-\frac{\pi}{2} < \frac{m}{s}(\theta - \theta') < \frac{\pi}{2} \quad \forall \theta \in \left[-\frac{\pi}{m}\sigma, \frac{\pi}{m} + \frac{\pi}{m}\sigma\right].$$

To this aim we can take $\theta' = \frac{\pi}{2m}$ for $0 < \sigma < \min\left\{\frac{1}{2}, \frac{s-1}{2}\right\}$.

Then, for $z = w_0(t) = \rho_0(t)e^{i\theta_0(t)}$ or $z = w_1(t) = \rho_1(t)e^{i\theta_1(t)}$:

$$\begin{aligned} w(z) &= u(w_j(t)) - A\operatorname{Im} w_j^m(t) - ML\rho_j^{m/s}(t)\cos\left[\frac{m}{s}(\theta_j(t) - \theta')\right] \\ &\leq A|\operatorname{Im} w_j^m(t)| + \alpha\rho_j^{m/s}(t) - A\operatorname{Im} w_j^m(t) - L\rho_j^{m/s}(t)\cos\frac{1+2\sigma}{2s}\pi \\ &\leq \rho_j^{m/s}(t)\left(2AR + \alpha - L\cos\frac{1+2\sigma}{2s}\pi\right) \\ &\leq 0 \end{aligned}$$

if

$$L \geq \frac{2AR+\alpha}{\cos\dfrac{1+2\sigma}{2s}\pi} \geq \frac{2AR+\alpha}{\cos\dfrac{\pi}{2s}},$$

for $0<\sigma<\min\left\{\frac{1}{2},\frac{s-1}{2}\right\}$.

Moreover, for $z=\rho e^{i\theta}\in S_{0,1}$ we have that

$$c_u \leq L\rho^{m/s}\cos\left[\frac{m}{s}(\theta-\theta')\right] \quad \text{if } \rho \geq \left[\frac{c_u}{L\cos\left(\frac{1+2\sigma}{2s}\pi\right)}\right]^{s/m} = R_u.$$

Therefore, for $z=\rho e^{i\theta}\in S_{0,1}$ with $|z|\geq R_u$, by (2.16):

$$\begin{aligned}
w(z) &= u(z) - A\operatorname{Im} z^m - ML\rho^{m/s}\cos\left[\frac{m}{s}(\theta-\theta')\right]\\
&\leq A|\operatorname{Im} z^m| + L\rho^{m/s} + c_u - A\operatorname{Im} z^m\\
&\qquad - (M-1)L\rho^{m/s}\cos\frac{1+2\sigma}{2s}\pi - L\rho^{m/s}\cos\left[\frac{m}{s}(\theta-\theta')\right]\\
&\leq \rho^{m/s}\left[2AR + L - (M-1)L\cos\frac{1+2\sigma}{2s}\pi\right]\\
&\leq 0
\end{aligned}$$

if

$$M \geq 1 + \frac{2AR+L}{L\cos\dfrac{1+2\sigma}{2s}\pi}.$$

By the maximum principle we thus obtain that

$$w(z)\leq 0 \quad \forall z\in S_{0,1},$$

and hence, arguing in the same way in the other sectors, we finally obtain that

$$u(z) \leq A|\operatorname{Im} z^m| + ML|z|^{m/s} \quad \forall z\in\mathbb{C}.$$

b) Let us now consider the general case $\alpha_u\in\mathbb{R}$ and prove the validity of the Phragmén-Lindelöf principle (2.13)–(2.14).

Since $s>1$, for every $\varepsilon>0$ there is $B_{u,\varepsilon}>0$ such that

$$\alpha_u|z|^{m/s} \leq \varepsilon|z|^m + B_{u,\varepsilon} \quad \forall z\in\mathbb{C}.$$

Then, if $u \in P(\mathbb{C})$ satisfies (2.13):

$$\begin{cases} u(z) \leq A|\operatorname{Im} z^m| + \alpha|z|^{m/s} & \text{for } z = w_h(t),\ h = 0, \dots, 2m-1 \\ u(z) \leq (A+\varepsilon)|z|^m + (c_u + B_{u,\varepsilon}) & \text{for } z \in \mathbb{C}, \end{cases}$$

and hence, by step a):

$$u(z) \leq (A+\varepsilon)|\operatorname{Im} z^m| + ML|z|^{m/s} \quad \forall z \in \mathbb{C}.$$

For $\varepsilon \to 0$ we finally obtain the thesis with $\beta = ML$, for

$$M \geq 1 + \frac{2AR + L}{L\cos\left(\dfrac{1+2\sigma}{2s}\pi\right)}, \quad L \geq \frac{2AR + \alpha}{\cos\left(\dfrac{1+2\sigma}{2s}\pi\right)}$$

with $0 < \sigma < \min\left\{\frac{1}{2}, \frac{s-1}{2}\right\}$.

B) Let us now consider the general case $s \in \mathbb{R}$. For every $\varepsilon > 0$ there is $p \in \mathbb{Q}$ such that

$$p - \varepsilon \leq \frac{1}{s} \leq p.$$

By assumption condition $(h)^s$ is valid, and hence also condition $(h)^{\frac{1}{p}}$ is satisfied by Remark 2.9. By part A) we deduce that the pair $(K, \Gamma \times K)$ with $K = B(0, A)$ is of evolution in the class $\tilde{\gamma}^{(\frac{1}{p})}$.

Then the Phragmén-Lindelöf principle (2.11)–(2.12) is valid, since every $u \in P(V)$ satisfying

$$\begin{aligned} u(\theta) &\leq \alpha\log(1+|\tau|) + \alpha|\zeta|^{1/s} + \alpha\kappa_\Gamma(\tau) + A|\operatorname{Im}\zeta| \\ &\leq \alpha\log(1+|\tau|) + \alpha|\zeta|^p + \alpha\kappa_\Gamma(\tau) + A|\operatorname{Im}\zeta| + \alpha \quad \forall\theta = (\tau,\zeta) \in V \\ u(\theta) &\leq \alpha_u\log(1+|\tau|) + \alpha_u|\zeta|^{1/s} + A|\operatorname{Im}\zeta| + c_u \\ &\leq \alpha_u\log(1+|\tau|) + \alpha_u|\zeta|^p + A|\operatorname{Im}\zeta| + c_u + \alpha_u \quad \forall\theta = (\tau,\zeta) \in V \end{aligned}$$

also satisfies, because of the evolution in the class $\tilde{\gamma}^{(\frac{1}{p})}$,

$$\begin{aligned} u(\theta) &\leq \beta\log(1+|\tau|) + \beta|\zeta|^p + A|\operatorname{Im}\zeta| + c \\ &\leq \beta\log(1+|\tau|) + \beta|\zeta|^{\frac{1}{s}+\varepsilon} + A|\operatorname{Im}\zeta| + c + \beta \quad \forall\theta = (\tau,\zeta) \in V. \end{aligned}$$

For $\varepsilon \to 0$ we obtain (2.12) and hence evolution in the class $\tilde{\gamma}^{(s)}$.

2) Statement ii) easily follows from Theorem 2.7 and the fact that condition $(H)^s$ trivially implies condition $(h)^s$. □

Corollary 2.11 *Under the same assumptions of Proposition* 2.10, *if* $1 < s \in \mathbb{Q}$, *then a necessary and sufficient condition in order that the pair* $(K, \Gamma \times K)$ *is of evolution for* $\mathcal{M}$ *in the class* $\tilde{\gamma}^{(s)}$ *is that* $(h)^s$ *is satisfied on* $V(\check{\wp})$ *for all* $\wp \in \mathrm{Ass}(\mathcal{M})$ *and for all* $R' > 0$.

Also, a necessary and sufficient condition in order that the pair $(K, \Gamma \times K)$ *is of evolution for* $\mathcal{M}$ *in the class* $\tilde{\gamma}^{(s)}$ *is that Hörmander's type condition* $(H)^s$ *is satisfied on* $V(\check{\wp})$ *for all* $\wp \in \mathrm{Ass}(\mathcal{M})$.

Remark 2.12 As a consequence of Corollary 2.11, conditions $(h)^s$ and $(H)^s$ are equivalent when $n = 1$ and $s \in \mathbb{Q}$.

Analogously, in the class $\gamma^{(r,s)}$:

Proposition 2.13 *Let* $\mathcal{M}$ *be a finitely generated unitary* $\mathcal{P}$*-module, for which* $\Sigma \simeq \mathbb{R}_x$ *is formally non-characteristic and quasi-free, with reduced order* p_o. *Let* K *be a compact convex subset of* Σ *with non-empty interior in* $\mathbb{R}_x$, *and* Γ *a closed convex cone of* $\mathbb{R}_t^k$ *with non-empty interior and vertex in* 0. *Then*:

i) *if condition* $(h)^s$ *is satisfied for some* $s > 1$ *on* $V(\check{\wp})$ *for all* $\wp \in \mathrm{Ass}(\mathcal{M})$ *and for all* $R' > 0$, *then the pair* $(K, \Gamma \times K)$ *is of evolution for* $\mathcal{M}$ *in the classes* $\gamma^{(r,s')}$ *for all real* $r, s' > 1$ *with* $1 < s' \leq s$ *and* $r \geq p_o s'$;

ii) *if the pair* $(K, \Gamma \times K)$ *is of evolution for* $\mathcal{M}$ *in the class* $\gamma^{(r,s)}$ *for some* $r, s > 1$ *with* $r \geq p_o s$, *then condition* $(h)^q$ *is satisfied for all* $\wp \in \mathrm{Ass}(\mathcal{M})$, $R' > 0$ *and* $q \in \mathbb{Q}$ *with* $1 < q \leq s$.

Proof. Statement i) follows from Proposition 2.10 and Lemma 2.2. The proof of ii) follows from Theorem 2.6. □

Corollary 2.14 *Under the same assumptions of Proposition* 2.13, *if* $1 < s \in \mathbb{Q}$, *then a necessary and sufficient condition in order that the pair* $(K, \Gamma \times K)$ *is of evolution for* $\mathcal{M}$ *in the class* $\gamma^{(r,s)}$, *for* $r > 1$ *with* $r \geq p_o s$, *is that* $(h)^s$ *is satisfied on* $V(\check{\wp})$ *for all* $\wp \in \mathrm{Ass}(\mathcal{M})$ *and for all* $R' > 0$.

Equivalently, a necessary and sufficient condition in order that the pair $(K, \Gamma \times K)$ *is of evolution for* $\mathcal{M}$ *in the class* $\gamma^{(r,s)}$, *for* $r > 1$ *with* $r \geq p_o s$, *is that Hörmander's type condition* $(H)^s$ *is satisfied on* $V(\check{\wp})$ *for all* $\wp \in \mathrm{Ass}(\mathcal{M})$.

Remark 2.15 We proved in [BN2] (see Theorems 7.2 and 7.4) that a sufficient condition in order that the pair $(K, \Gamma \times K)$ is of evolution for $\mathcal{M}$

in the classes $\tilde{\gamma}^{(s)}$ or $\gamma^{(r,s)}$ (for $r, s > 1$ with $r \geq p_o s$) is that the following Petrowski-type condition for evolution is satisfied for some $c_1, c_2 > 0$ and for all $\wp \in \mathrm{Ass}(\mathcal{M})$:

$$(P)^s \qquad \kappa_\Gamma(\tau) \leq c_1|\xi|^{1/s} + c_2 \quad \forall(\tau,\zeta) \in V(\check{\wp}) \text{ with } \zeta = \xi \in \mathbb{R}.$$

Clearly condition $(P)^s$ implies condition $(h)^s$. However, condition $(h)^s$ does not imply, in general, condition $(P)^s$ (see Example 2.17 below).

Example 2.16 Let us consider, for some integers $p, q \geq 1$, the following system in $\mathbb{R}^2_t \times \mathbb{R}_x$:

$$\begin{cases} \dfrac{\partial}{\partial t_1} + i^{p+1}\dfrac{\partial^p}{\partial x^p} \\ \dfrac{\partial}{\partial t_2} + i^{q+1}\dfrac{\partial^q}{\partial x^q}. \end{cases} \tag{2.17}$$

The associated affine algebraic variety is given by

$$V = \{(\tau_1, \tau_2, \zeta) \in \mathbb{C}^3 : \tau_1 = \zeta^p,\ \tau_2 = \zeta^q\},$$

and the reduced order of the system is $p_o = \max\{p, q\}$. Clearly

$$\mathrm{Im}\,\tau_1 = \mathrm{Im}\,\tau_2 = 0 \quad \forall(\tau,\zeta) \in V \text{ with } \zeta = \xi \in \mathbb{R},$$

and hence every pair of the form $(K, \mathbb{R}^2_t \times K)$, for a compact convex subset K of $\mathbb{R}_x$, is of evolution for the system (2.17) in the classes $\tilde{\gamma}^{(s)}$ and $\gamma^{(r,s)}$ for all $s > 1$ and $r \geq s\max\{p, q\}$.

In the example above, not only condition $(h)^s$, but also condition $(P)^s$ was satisfied. Let us now consider an example of an evolution operator satisfying $(h)^s$, but not $(P)^s$.

Example 2.17 Let us consider the following operator

$$\frac{\partial}{\partial t} + \left(\frac{\partial}{\partial x} + 1\right)^3 \tag{2.18}$$

in $\mathbb{R}^+_t \times \mathbb{R}_x$. The associated affine algebraic variety is given by

$$V = \{(\tau,\zeta) \in \mathbb{C}^2 : \tau = (\zeta + i)^3\},$$

and the reduced order of the system is $p_o = 3$.

Setting $\Gamma = \mathbb{R}_t^+$, we thus obtain that

$$\kappa_\Gamma(\tau) = (\operatorname{Im}\tau)^+ = (3\xi^2 - 1)^+ \quad \forall(\tau,\zeta) \in V \ \text{ with } \ \zeta = \xi \in \mathbb{R},$$

and hence the Petrowski-type condition $(P)^s$ cannot be satisfied for any $s > 1$. However,

$$\kappa_\Gamma(\tau) = 0 \quad \forall(\tau,\zeta) \in V \ \text{ with } \ \operatorname{Im}\zeta = -1,$$

and hence condition $(h)^s$ is satisfied for all $s > 1$.

By Propositions 2.10 and 2.13 it follows that, for every compact convex subset K of $\mathbb{R}_x$, the pair $(K, \mathbb{R}_t^+ \times K)$ is of evolution for the given operator (2.18) in $\tilde{\gamma}^{(s)}$ and $\gamma^{(r,s)}$ with $s > 1$ and $r \geq 3s$.

In the two examples above we had in fact evolution also in the C^∞ class (cf. [BN1]). Let us now consider an example of operator which is of evolution in some (small) Gevrey classes, but not in the C^∞ class.

Example 2.18 The operator $\partial_{tt} - \partial_x$ in $\mathbb{R}_t^+ \times \mathbb{R}_x$ has associated affine algebraic variety

$$V = \{(\tau,\zeta) \in \mathbb{C}^2 : \tau^2 = i\zeta\}$$

and reduced order $p_o = 1/2$. For $(\tau,\zeta) \in V$ we have that

$$(\operatorname{Im}\tau)^+ = \frac{|\xi|^{1/2}}{\sqrt{2}} \quad \forall(\tau,\zeta) \in V \ \text{ with } \ \zeta = \xi \in \mathbb{R}.$$

Setting $\Gamma = \mathbb{R}_t^+$, we thus obtain that condition $(P)^s$ and hence $(h)^s$ are satisfied for all $1 < s \leq 2$.

Therefore, for every compact convex subset K of $\mathbb{R}_x$, the pair $(K, \mathbb{R}_t^+ \times K)$ is of evolution for the given operator in the classes $\tilde{\gamma}^{(s)}$ and $\gamma^{(r,s)}$ for all $1 < s \leq 2$ and $r > 1$ ($p_o s = s/2 \leq 1$ for $1 < s \leq 2$).

Note that the pair $(K, \mathbb{R}_t^+ \times K)$ is not of evolution for the operator $\partial_{tt} - \partial_x$ in the C^∞ class (cf. [BN1]).

We always considered $r \geq p_o s$ in the preceding results about evolution in the class $\gamma^{(r,s)}$. Nevertheless, we shall see, by the following example, that we can have evolution in the class $\gamma^{(r,s)}$ for $s > 1$ even if $1 < r < p_o s$.

Example 2.19 Let us consider the system

$$\begin{cases} \dfrac{\partial^2}{\partial t_1^2} + i\dfrac{\partial}{\partial x} \\ \dfrac{\partial^2}{\partial t_2^2} - \dfrac{\partial^2}{\partial x^2} \end{cases}$$

in $\mathbb{R}_t^2 \times \mathbb{R}_x$, with associated affine algebraic variety

$$V = \{(\tau_1, \tau_2, \zeta) \in \mathbb{C}^3 : \tau_1^2 = \zeta,\ \tau_2^2 = \zeta^2\}$$

and reduced order $p_o = 1$. For $\Gamma = \mathbb{R}_t^2$ we have that

$$\kappa_\Gamma(\tau) \le |\zeta|^{1/2} + |\operatorname{Im}\zeta| \quad \forall(\tau, \zeta) \in V.$$

It follows (cf. [BN2], Theorem 8.1) that the pair $(\mathbb{R}_x, \mathbb{R}_t^2 \times \mathbb{R}_x)$ is hyperbolic, and hence of evolution, for the given system in the classes $\tilde{\gamma}^{(s)}$ and $\gamma^{(r,s)}$ for all $1 < s \le 2$ and $r > 1$.

We thus have evolution also for $1 < r < p_o s = s \le 2$.

In Examples 2.18 and 2.19 we had evolution and also hyperbolicity. Let us now consider an example where we have evolution in some spaces of (small) Gevrey functions (but not in the C^∞ class), but not hyperbolicity.

Example 2.20 Let us consider the operator

$$\frac{\partial^2}{\partial t^2} - 2i\frac{\partial^3}{\partial t \partial^2 x} - \frac{\partial^4}{\partial x^4} - i\frac{\partial}{\partial x}$$

in $\mathbb{R}_t \times \mathbb{R}_x$, with assocaited affine algebraic variety

$$V = \{(\tau, \zeta) \in \mathbb{C}^2 : (\tau - \zeta^2)^2 + \zeta = 0\}.$$

For $(\tau, \zeta) \in V$ we have that $\tau = \zeta^2 \pm i\sqrt{\zeta}$, where $\pm\sqrt{\zeta}$ denote the two complex roots of ζ.

The reduced order of the system is then $p_o = 2$ and

$$|\operatorname{Im}\tau| \le |\xi|^{1/2} \quad \forall(\tau, \zeta) \in V \ \text{ with } \ \zeta = \xi \in \mathbb{R}.$$

Therefore condition $(h)^s$ is satisfied for $\Gamma = \mathbb{R}_t$ and $1 < s \le 2$, and hence for every compact convex subset K of $\mathbb{R}_x$ the pair $(K, \mathbb{R}_t \times K)$ is of evolution for the given operator in the classes $\tilde{\gamma}^{(s)}$ and $\gamma^{(r,s)}$ for $1 < s \le 2$ and $r \ge 2s$.

Note that the pair $(K, \mathbb{R}_t \times K)$ is not of evolution in the C^∞ class (cf. [BN1]) and, moreover, it is not hyperbolic in any class of (small) Gevrey functions (cf. [BN2]).

Let us now show, by the following example, that condition $(h)^s$ is sufficient only for "local evolution", i.e. for evolution of pairs of the form $(K, \Gamma \times K)$ for a compact convex subset K of Σ, but not for pairs of the form $(\Sigma, \Gamma \times \Sigma)$.

Example 2.21 Let us consider the following system in $\mathbb{R}^3_t \times \mathbb{R}_x$:

$$\begin{cases} \dfrac{\partial}{\partial t_1} - i\dfrac{\partial^2}{\partial x^2} \\ \dfrac{\partial}{\partial t_2} + i\dfrac{\partial^2}{\partial x^2} \\ \dfrac{\partial}{\partial t_3} - \dfrac{\partial^2}{\partial x^2}. \end{cases} \tag{2.19}$$

The associated affine algebraic variety is given by

$$V = \{(\tau_1, \tau_2, \zeta) \in \mathbb{C}^3 : \tau_1 = \zeta^2,\ \tau_2 = -\zeta^2,\ \tau_3 = -i\zeta^2\},$$

and the reduced order of the system is $p_o = 2$.

For $\Gamma = \{t \in \mathbb{R}^3 : t_j \geq 0\ \forall j = 1, 2, 3\}$ we obtain that

$$\kappa_\Gamma(\tau) = 0 \quad \forall (\tau, \zeta) \in V \quad \text{with} \quad \zeta = \xi \in \mathbb{R},$$

and hence condition $(h)^s$ is satisfied for all $s > 1$ (also condition $(P)^s$ is satisfied), and for every compact convex subset K of $\mathbb{R}_x$ the pair $(K, \Gamma \times K)$ is of evolution for the given system in the classes $\tilde{\gamma}^{(s)}$ and $\gamma^{(r,s)}$ for all $s > 1$, $r \geq 2s$.

We shall see, however, that the pair $(\mathbb{R}_x, \Gamma \times \mathbb{R}_x)$ is not of evolution for the given system in any class of (small) Gevrey functions.

Consider, indeed, the following sequence of plurisubharmonic functions on $\mathbb{C}^4 \simeq \mathbb{C}^3_\tau \times \mathbb{C}_\zeta$:

$$u_n(\tau, \zeta) = \begin{cases} 0 & \text{if } \operatorname{Im}\zeta \leq 0 \\ \dfrac{1}{n}\operatorname{Im}\zeta & \text{if } 0 \leq \operatorname{Im}\zeta \leq n \\ n\operatorname{Im}\zeta + 1 - n^2 & \text{if } \operatorname{Im}\zeta \geq n. \end{cases}$$

Note that

$$u_n(\tau, \zeta) \leq (\operatorname{Im}\zeta)^2 + 1 \leq \frac{4}{3}\kappa_\Gamma(\tau) + 1 \quad \forall (\tau, \zeta) \in V,$$

since (cf. also [BN1]):

$$\begin{aligned}\kappa_\Gamma(\tau) &= \sum_{j=1}^{3}(\operatorname{Im}\tau_j)^+ = 2|\operatorname{Re}\zeta|\cdot|\operatorname{Im}\zeta| + \left((\operatorname{Im}\zeta)^2 - (\operatorname{Re}\zeta)^2\right)^+ \\ &\geq \frac{3}{4}(\operatorname{Im}\zeta)^2 \quad \forall(\tau,\zeta)\in V.\end{aligned}$$

We can thus find constants $A_u, c_u > 0$ such that

$$\begin{cases} u_n(\tau,\zeta) \leq \frac{4}{3}\kappa_\Gamma(\tau) + 1 & \forall(\tau,\zeta)\in V \\ u_n(\tau,\zeta) \leq A_n|\operatorname{Im}\zeta| + c_n & \forall(\tau,\zeta)\in V, \end{cases}$$

but we cannot find fixed constants $A, L, c > 0$ such that

$$u_n(\tau,\zeta) \leq A|\operatorname{Im}\zeta| + L|\tau|^{1/r} + L|\zeta|^{1/s} + c \quad \forall(\tau,\zeta)\in V.$$

This means that the Phragmén-Lindelöf principles for evolution (1.7) and (1.9) are violated, and hence the pair $(\mathbb{R}_x, \Gamma\times\mathbb{R}_x)$ is not of evolution for the given system (2.19) neither in the class $\tilde{\gamma}^{(s)}$ nor in the class $\gamma^{(r,s)}$, for any $r, s > 1$.

2.3. A sufficient Hörmander's type condition for evolution

We use here the same notation as in §2.1, and consider the following Hörmander's type condition:

$$(H')^s \quad \begin{cases} \exists R, r, c_1, c_2 > 0 \ \text{ such that} \\ \text{for every } \rho\in\mathbb{R}^n \text{ with } |\rho|\geq r \text{ and } B_n(\rho, R|\rho|^{1/s}) \subset \mathbb{C}^n\setminus Z \\ \text{and for every connected component } \omega \text{ of } \pi_n^{-1}(B_n(\rho, R|\rho|^{1/s})) \\ \text{there is } B_n(\zeta_\rho, r)\subset B_n(\rho, R|\rho|^{1/s}) \ \text{ such that} \\ \kappa_\Gamma(\tau) \leq c_1|\zeta|^{1/s} + c_2 \quad \forall(\tau,\zeta)\in\pi_n^{-1}(B_n(\zeta_\rho, r))\cap\omega. \end{cases}$$

Remark 2.22 Condition $(H')^s$ implies condition $(H')^{s'}$ for all real $1 < s' \leq s$.

Remark 2.23 Condition $(H')^s$ is in general stronger than condition $(H)^s$, and it can be proved, as in Lemma 12.8.8 of [Hö3], that it is equivalent to $(H)^s$ if $n = N-1$.

Theorem 2.24 *Let $\mathcal{M}$ be a unitary finitely generated $\mathcal{P}$-module, for which $\Sigma \simeq \mathbb{R}^n_x$ is formally non-characteristic and quasi-free. Let K be a compact* *convex subset of Σ with a non-empty interior in $\mathbb{R}^n_x$ and Γ a closed convex cone of $\mathbb{R}^k_t$ with non-empty interior and vertex in* 0.

Then, if condition $(H')^s$ is satisfied for some $s > 1$ on all $V(\check{\wp})$ for $\wp \in \mathrm{Ass}(\mathcal{M})$, it follows that the pair $(K, \Gamma \times K)$ is of evolution for $\mathcal{M}$ in the classes $\tilde{\gamma}^{(s')}$ for all $1 < s' \leq s$.

Proof. By Remark 2.22 it is sufficient to prove that condition $(H')^s$ is sufficient for evolution in the class $\tilde{\gamma}^{(s)}$. Moreover, by Lemma 2.1 it is not restrictive to assume $K = B_n(0, A)$ for some $A > 0$, since K is compact. Up to translations, we can also assume $c_2 = 0$.

Let us fix $\wp \in \mathrm{Ass}(\mathcal{M})$, set $V = V(\check{\wp})$ and prove the validity of the Phragmén-Lindelöf principle for evolution (2.11)–(2.12).

Note that if condition $(H')^s$ is valid for some $R > 0$, then it is also satisfied for any larger R. This means that it will not be restrictive to choose in the following $R > r$ sufficiently large.

By Hadamard's three-circles theorem (cf. [Hö3], Lemma 12.8.6; [A]), we can then fix $R_1 > 0$ with $R_1 < R$ such that $B_n(\zeta_\rho, r) \subset B_n(\rho, R_1|\rho|^{1/s})$ for all $|\rho| \geq r$, and find $\delta = \delta(R) \in (0,1)$ such that for every plurisubharmonic function u satisfying (2.11):

$$u(\tau(\zeta), \zeta) \leq (1-\delta) \sup_{B_n(\rho, R|\rho|^{1/s})} u + \delta \sup_{B_n(\zeta_\rho, r)} u$$
$$\forall \zeta \in B_n(\rho, R_1|\rho|^{1/s}),\ (\tau, \zeta) \in \omega. \tag{2.20}$$

By assumption

$$|\tau| \leq \lambda(1 + |\zeta|)^{p_o} - 1 \quad \forall (\tau, \zeta) \in V$$

for some $\lambda > 1$, and hence from (2.11) we obtain, for $|\rho| \geq R^{\frac{s}{s-1}}$:

$$\begin{aligned}
\sup_{B_n(\rho, R|\rho|^{1/s})} u &\leq \sup_{B_n(\rho, R|\rho|^{1/s})} \{\alpha_u \log(1 + |\tau|) + \alpha_u |\zeta|^{1/s} + A|\operatorname{Im}\zeta| + c_u\} \\
&\leq \sup_{B_n(\rho, R|\rho|^{1/s})} \{\alpha_u \log\lambda + \alpha_u p_o \log(1 + |\zeta|) \\
&\qquad\qquad + \alpha_u |\zeta|^{1/s} + A|\operatorname{Im}\zeta| + c_u\} \\
&\leq \alpha_u(p_o s + 1)(|\rho| + R|\rho|^{1/s})^{1/s} + AR|\rho|^{1/s} + \alpha_u \log\lambda + c_u \\
&\leq [2^{1/s}\alpha_u(p_o s + 1) + AR]|\rho|^{1/s} + \alpha_u \log\lambda + c_u, \quad (\tau, \zeta) \in \omega,
\end{aligned} \tag{2.21}$$

and, by $(H')^s$:

$$\begin{aligned}
\sup_{B_n(\zeta_\rho,r)} u &\le \sup_{B_n(\zeta_\rho,r)} \{\alpha\log(1+|\tau|)+\alpha|\zeta|^{1/s}+\alpha\kappa_\Gamma(\tau)+A|\operatorname{Im}\zeta|\} \\
&\le \sup_{B_n(\zeta_\rho,r)} \{\alpha\log\lambda+\alpha p_o s|\zeta|^{1/s}+\alpha|\zeta|^{1/s}+\alpha c_1|\zeta|^{1/s}+A|\operatorname{Im}\zeta|\} \\
&\le \alpha(p_o s+1+c_1)(|\rho|+R|\rho|^{1/s})^{1/s}+AR|\rho|^{1/s}+\alpha\log\lambda \\
&\le [2^{1/s}\alpha(p_o s+1+c_1)+AR]|\rho|^{1/s}+\alpha\log\lambda, \quad (\tau,\zeta)\in\omega.
\end{aligned} \tag{2.22}$$

For $\zeta\in B_n(\rho,R_1|\rho|^{1/s})$ with $|\rho|\le R^{\frac{s}{s-1}}$, form (2.11) and (2.2):

$$\begin{aligned}
&u(\tau(\zeta),\zeta) \\
&\quad\le \sup_{\substack{\zeta\in B_n(\rho,R_1|\rho|^{1/s})\\ |\rho|\le R^{\frac{s}{s-1}}}} \{\alpha\log\lambda+\alpha(p_o s+1)|\zeta|^{1/s}+\alpha\kappa_\Gamma(\tau)+A|\operatorname{Im}\zeta|\} \\
&\quad\le \sup_{|\rho|\le R^{\frac{s}{s-1}}} \{\alpha\log\lambda+\alpha(p_o s+1)(|\rho|+R_1|\rho|^{1/s})^{1/s} \\
&\qquad\qquad + \alpha k(1+|\rho|+R_1|\rho|^{1/s})^{p_o}+AR_1|\rho|^{1/s}\} \\
&\quad\le C(R), \quad (\tau,\zeta)\in\omega,
\end{aligned} \tag{2.23}$$

for some positive constant $C(R)$ depending on R.

From (2.20), (2.21), (2.22) and (2.23) we finally obtain, for $\zeta\in B_n(\rho,R_1|\rho|^{1/s})$, $\rho\in\mathbb{R}^n$:

$$\begin{aligned}
u(\tau(\zeta),\zeta) \le (1-\delta)&\{[2^{1/s}\alpha_u(p_o s+1)+AR]|\rho|^{1/s}+\alpha_u\log\lambda+c_u\} \\
&+\delta\{[2^{1/s}\alpha(p_o s+1+c_1)+AR]|\rho|^{1/s}+\alpha\log\lambda\} \\
&+C(R).
\end{aligned}$$

This holds, in particular, for all $\theta=(\tau,\zeta)\in V$ with $\pi_n(\theta)=\rho\in\mathbb{R}^n$, and therefore the function $\varphi(\eta)=\sup_{\theta\in V\cap\pi_n^{-1}(\eta)}u(\theta)$ satisfies:

$$\begin{aligned}
\varphi(\eta) \le \{&(1-\delta)[2^{1/s}\alpha_u(p_o s+1)+AR] \\
&+\delta[2^{1/s}\alpha(p_o s+1+c_1)+AR]\}|\eta|^{1/s} \\
&+(1-\delta)(\alpha_u\log\lambda+c_u)+\delta\alpha\log\lambda+C(R) \quad \forall\eta\in\mathbb{R}^n
\end{aligned}$$

and, by the second inequality of (2.11):

$$\varphi(\eta)\le\alpha_u(p_o s+1)|\eta|^{1/s}+A|\operatorname{Im}\eta|+\alpha_u\log\lambda+c_u \quad \forall\eta\in\mathbb{C}^n.$$

It follows, by the classical Phragmén-Lindelöf principle (Proposition 1.7), that

$$\varphi(\eta) \leq A|\operatorname{Im}\eta| + \ell_u|\eta|^{1/s} + \sigma_u + \sigma \quad \forall \eta \in \mathbb{C}^n,$$

where

$$\begin{aligned}
\ell_u &= \frac{2n}{\cos\frac{\pi}{2s}}\{(1-\delta)[2^{1/s}\alpha_u(p_o s+1)+AR] \\
&\qquad\qquad + \delta[2^{1/s}\alpha(p_o s+1+c_1)+AR]\} \\
\sigma_u &= (1-\delta)(\alpha_u \log\lambda + c_u) \\
\sigma &= \delta\alpha\log\lambda + C(R).
\end{aligned}$$

Then $v(\theta) = u(\theta) - \sigma$ satisfies:

$$\begin{aligned}
v(\theta) \leq A|\operatorname{Im}\zeta| &+ \frac{2n}{\cos\frac{\pi}{2s}}\{(1-\delta)[2^{1/s}\alpha_u(p_o s+1)+AR] \\
&+ \delta[2^{1/s}\alpha(p_o s+1+c_1)+AR]\}|\zeta|^{1/s} \\
&+ (1-\delta)(\alpha_u\log\lambda + c_u) \quad \forall\theta = (\tau,\zeta)\in V.
\end{aligned}$$

After ℓ steps we obtain that

$$v(\theta) \leq A|\operatorname{Im}\zeta| + M_\ell|\zeta|^{1/s} + \lambda_\ell \quad \forall\theta = (\tau,\zeta)\in V,$$

with

$$\begin{cases}
M_0 = \alpha_u(p_o s+1) \\
M_\ell = \dfrac{2n}{\cos\frac{\pi}{2s}}\{(1-\delta)[2^{1/s}M_{\ell-1}+AR] \\
\qquad\qquad + \delta[2^{1/s}\alpha(p_o s+1+c_1)+AR]\}
\end{cases} \tag{2.24}$$

$$\begin{cases}
\lambda_0 = \alpha_u\log\lambda + c_u \\
\lambda_\ell = (1-\delta)\lambda_{\ell-1}.
\end{cases} \tag{2.25}$$

It is now possible to take R sufficiently large to choose then $\delta = \delta(R) \in (0,1)$ such that

$$\frac{2^{1+\frac{1}{s}}n}{\cos\frac{\pi}{2s}}(1-\delta) < 1, \quad \text{i.e.} \quad \delta > 1 - \frac{\cos\frac{\pi}{2s}}{2^{1+\frac{1}{s}}n}.$$

For such δ the sequence $\{M_\ell\}_{\ell\in\mathbb{N}}$ converges and, by (2.24),

$$\lim_{\ell\to+\infty} M_\ell = M = \frac{2n}{\cos\frac{\pi}{2s}}\{(1-\delta)2^{1/s}M + (1-\delta)AR \\ + \delta[2^{1/s}\alpha(p_o s + 1 + c_1) + AR]\},$$

that is

$$M = \frac{(1-\delta)AR + \delta[2^{1/s}\alpha(p_o s + 1 + c_1) + AR]}{1 - \dfrac{2^{1+\frac{1}{s}}n}{\cos\frac{\pi}{2s}}(1-\delta)} < +\infty.$$

Moreover, from (2.25) we have that $\lim_{\ell\to+\infty}\lambda_\ell = 0$.

For $\ell \to +\infty$ we finally obtain that

$$u(\theta) \le A|\operatorname{Im}\zeta| + M|\zeta|^{1/s} + \sigma \quad \forall\theta = (\tau,\zeta)\in V,$$

and hence (2.12), with $\beta = M$ and $c = \sigma$. □

By Remark 2.23, from Theorems 2.7 and 2.24 we have in particular:

Corollary 2.25 *Let $\mathcal{M}$ be a unitary finitely generated $\mathcal{P}$-module, for which $\Sigma \simeq \mathbb{R}^{N-1}_x \subset \mathbb{R}^N$ is formally non-characteristic and quasi-free. Let K be a compact convex subset of Σ with a non-empty interior in $\mathbb{R}^{N-1}_x$ and Γ a closed cone of $\mathbb{R}_t$ with non-empty interior and vertex in 0.*

Then, for $1 < s \in \mathbb{Q}$, condition $(H)^s$ is necessary and sufficient in order that the pair $(K, \Gamma\times K)$ is of evolution for $\mathcal{M}$ in the class $\tilde{\gamma}^{(s)}$.

From Lemma 2.2 and Theorem 2.24 we immediately obtain:

Theorem 2.26 *Let $\mathcal{M}$ be a unitary finitely generated $\mathcal{P}$-module, for which $\Sigma \simeq \mathbb{R}^n_x$ is formally non-characteristic and quasi-free, with reduced order p_o. Let K be a compact convex subset of Σ with a non-empty interior in $\mathbb{R}^n_x$ and Γ a closed convex cone of $\mathbb{R}^k_t$ with non-empty interior and vertex in 0.*

Then, if condition $(H')^s$ is satisfied for some $s > 1$ on all $V(\wp)$ for $\wp \in \operatorname{Ass}(\mathcal{M})$, it follows that the pair $(K, \Gamma\times K)$ is of evolution for $\mathcal{M}$ in the classes $\gamma^{(r,s')}$ for all real $r, s' > 1$ with $1 < s' \le s$ and $r \ge p_o s'$.

By Remark 2.23:

Corollary 2.27 *Let $\mathcal{M}$ be a unitary finitely generated $\mathcal{P}$-module, for which $\Sigma \simeq \mathbb{R}^{N-1}_x \subset \mathbb{R}^N$ is formally non-characteristic and quasi-free, with* *reduced order* p_o*. Let* K *be a compact convex subset of* Σ *with a non-empty interior in* $\mathbb{R}_x^{N-1}$ *and* Γ *a closed cone of* $\mathbb{R}_t$ *with non-empty interior and vertex in* 0*.*

Then, for $1 < s \in \mathbb{Q}$*, condition* $(H)^s$ *is necessary and sufficient in order that the pair* $(K, \Gamma \times K)$ *is of evolution for* $\mathcal{M}$ *in the class* $\gamma^{(r,s)}$ *for* $r > 1$ *with* $r \geq p_o s$*.*

Example 2.28 Let us consider the heat operator $\partial_t - \Delta_x$ in $\mathbb{R}_t^+ \times \mathbb{R}_x^n$. The associated affine algebraic variety is given by

$$V = \left\{ (\tau, \zeta) \in \mathbb{C} \times \mathbb{C}^n : \tau = -i \sum_{j=1}^{n} \zeta_j^2 \right\},$$

and the reduced order of the system is $p_o = 2$.

Consider $\Gamma = \{t \in \mathbb{R} : t \geq 0\} = \mathbb{R}_t^+$. For $(\tau, \zeta) \in V$ with $\zeta = \xi \in \mathbb{R}^n$ we have that

$$\kappa_\Gamma(\tau) = (\operatorname{Im} \tau)^+ = 0 \quad \forall (\tau, \zeta) \in V \text{ with } \zeta = \xi \in \mathbb{R}^n.$$

This means that condition $(H)^s$ with $k = 1$ is satisfied for all $s > 1$. Therefore, every pair of the form $(K, \mathbb{R}_t^+ \times K)$, for a compact convex subset K of $\mathbb{R}_x^n$, is of evolution for the heat operator in the classes $\tilde{\gamma}^{(s)}$ and $\gamma^{(r,s)}$ for all $s > 1$ and $r \geq 2s$.

Example 2.29 Let us consider the wave operator $\partial_{tt} - \Delta_x$ in $\mathbb{R}_t \times \mathbb{R}_x^n$, with associated affine algebraic variety

$$V = \left\{ (\tau, \zeta) \in \mathbb{C} \times \mathbb{C}^n : \tau^2 = \sum_{j=1}^{n} \zeta_j^2 \right\},$$

and reduced order $p_o = 1$. If $(\tau, \zeta) \in V$ and $\zeta = \xi \in \mathbb{R}^n$, we have that $\operatorname{Im} \tau = 0$.

Then condition $(H)^s$ with $k = 1$ is satisfied for $\Gamma = \mathbb{R}_t$ and $s > 1$, and hence for every compact convex subset K of $\mathbb{R}_x^n$ the pair $(K, \mathbb{R}_t \times K)$ is of evolution for the wave operator in the classes $\tilde{\gamma}^{(s)}$ and $\gamma^{(r,s)}$ for all $s > 1$ and $r \geq s$.

We shall see by the following example that condition $(H')^s$ is sufficient, but in general not necessary, for evolution.

Example 2.30 Let us consider the following operator in $\mathbb{R}^2_t \times \mathbb{R}_x$:

$$\begin{cases} \dfrac{\partial}{\partial t_1} - i\dfrac{\partial^2}{\partial x^2} \\ \dfrac{\partial}{\partial t_2} + i\dfrac{\partial^2}{\partial x^2}. \end{cases}$$

The associated affine algebraic variety is given by

$$V = \{(\tau_1, \tau_2, \zeta) \in \mathbb{C}^3 : \tau_1 = \zeta^2,\ \tau_2 = -\zeta^2\}$$

with reduced order $p_o = 2$. Consider $\Gamma = \{t \in \mathbb{R}^2 : t_1 \geq 0,\ t_2 \geq 0\}$. By Propositions 2.10 and 2.13, for every compact convex subset K of $\mathbb{R}_x$ the pair $(K, \Gamma \times K)$ is of evolution for the given system in the classes $\tilde{\gamma}^{(s)}$ and $\gamma^{(r,s)}$ with $s > 1$ and $r \geq 2s$, since

$$\kappa_\Gamma(\tau) = (\operatorname{Im}\tau_1)^+ + (\operatorname{Im}\tau_2)^+ = 2|\operatorname{Re}\zeta| \cdot |\operatorname{Im}\zeta| = 0$$
$$\forall (\tau, \zeta) \in V \ \text{ with } \ \zeta = \xi \in \mathbb{R}.$$

However, condition $(H')^s$ is not satisfied for any $s > 1$.

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