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Automorphic forms on the 5-dimensional complex ball with respect to the Picard modular group over $\mathbb{Z}[\boldsymbol{i}]$

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Abstract. We represent the 105 automorphic forms on the 5-dimensional complex ball $\mathbb{B}^5$ constructed by Matsumoto-Terasoma as the products of four linear combinations of the pull backs of theta constants under an embedding of $\mathbb{B}^5$ into the Siegel upper half space of degree 6. They were used to describe the inverse of the period map for the family of the 4-fold coverings of the complex projective line branching at eight points.

Key words: automorphic forms, theta constants.

1. Introduction

The period map for the family of the 4-fold coverings of the complex projective line $\mathbb{P}^1$ branching at eight points is studied in [MT]. Its inverse is given by 105 automorphic forms f_J on the 5-dimensional complex ball $\mathbb{B}^5$ with respect to the monodromy group of the period map, where J are (2, 2, 2, 2)-partitions $\{j_1j_2; j_3j_4; j_5j_6; j_7j_8\}$ of $\{1, \ldots, 8\}$. The automorphic form f_{J_1} for $J_1 = \langle 12; 34; 56; 78\rangle$ is given by the pull back of the product of four theta constants under the embedding $\imath$ of $\mathbb{B}^5$ into the Siegel upper half space $\mathbb{S}^6$ of degree 6 induced by the period map. The symmetric group S_8 of degree 8 transitively acts on the set of (2, 2, 2, 2)-partitions, and it is represented in the symplectic group $Sp_6(\mathbb{Q})$ (not in $Sp_6(\mathbb{Z})$) by the period map. The group S_8 represented in $Sp_6(\mathbb{Q})$ acts on f_{J_1} as the transformation of theta constants on $\mathbb{S}^6$, and this action yields all f_J from f_{J_1}. Actually, few f_J are explicitly given in terms of theta constants in [MT] since transformation formulas of theta constants for some elements of $Sp_6(\mathbb{Q})$ were not known.

In this paper, we represent all f_J as the products of four linear combinations of the pull backs of theta constants under the embedding $\imath\colon \mathbb{B}^5 \to \mathbb{S}^6$. The set of (2, 2, 2, 2)-partitions can be regarded as the quotient $G\backslash S_8$, where G is the isotropy subgroup of S_8 for the partition J_1. The group

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G is represented in $Sp_6(\mathbb{Z})$ by the period map, and the double quotient $G\backslash S_8/G$ consists of five elements: the unit σ_1, $\sigma_2 = (23)$, $\sigma_3 = (23)(45)$, $\sigma_4 = (23)(67)$ and $\sigma_5 = (23)(45)(67)$, where (jk) is the transposition of j and k. We give transformation formulas of the theta constants for the elements of $Sp_6(\mathbb{Q})$ corresponding to (23), (45) and (67), which yield the explicit forms f_{J_j} of the actions of σ_j on f_{J_1}. We can obtain all f_J by acting G represented in $Sp_6(\mathbb{Z})$ on $f_{J_2}, \ldots, f_{J_5}$. We have to study the constant $\kappa(\sigma)$ in the transformation formula of the theta constants in [I] for the elements in $Sp_6(\mathbb{Z})$ corresponding to generators of G in order to give linear relations among f_J. Some linear relations among f_J yield quadratic relations among pull backs of theta constants under $\imath$.

Another expression of fourteen linearly independent f_J in terms of liftings is given in [K].

2. Automorphic forms on $\mathbb{B}^5$

2.1. 105 polynomials

Let J be a (2, 2, 2, 2)-partition of the set $\{1, \ldots, 8\}$:

$$J=\langle j_1 j_2; j_3 j_4; j_5 j_6; j_7 j_8\rangle,$$
$$\{j_1, \ldots, j_8\}=\{1, \ldots, 8\}, \quad j_1 < j_2,\ j_3 < j_4,\ j_5 < j_6,\ j_7 < j_8.$$

The cardinality of the set $\mathcal{J}$ of (2, 2, 2, 2)-partitions of $\{1, \ldots, 8\}$ is

$$\binom{8}{2}\binom{6}{2}\binom{4}{2}/4! = \frac{8!}{2^4 \times 4!} = 105.$$

For each element of $J \in \mathcal{J}$, we define a polynomial

$$x_J = (x_{j_1} - x_{j_2})(x_{j_3} - x_{j_4})(x_{j_5} - x_{j_6})(x_{j_7} - x_{j_8})$$

of variables $x_1, \ldots, x_8$.

The symmetric group S_8 acts on $x = {}^t(x_1, \ldots, x_8) \in \mathbb{C}^8$ from the left as the transposition of coordinates, which can be represented as the left multiple $\sigma \cdot x$, where $\sigma \in S_8$ and we regard S_8 as naturally embedded in $GL_8(\mathbb{Z})$. Note that

$$(123) \cdot x = \big((12)(23)\big) \cdot x = (12) \cdot {}^t(x_1, x_3, x_2, \ldots) = {}^t(x_3, x_1, x_2, \ldots),$$
$$(132) \cdot x = \big((23)(12)\big) \cdot x = (23) \cdot {}^t(x_2, x_1, x_3, \ldots) = {}^t(x_2, x_3, x_1, \ldots),$$

where $(j_1, \ldots, j_k)$ denotes a cyclic permutation. If we regard $\sigma \in S_8$ as a map $\sigma\colon \{1, \dots, 8\} \ni k \mapsto \sigma(k) \in \{1, \dots, 8\}$, this action is equivalent to

$$\sigma \cdot {}^t(x_1, \dots, x_8) = {}^t(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(8)}).$$

The group S_8 acts on x_J from the right as its pull-back under $\sigma \in S_8$, i.e., $x_J^\sigma = \sigma^*(x_J)$. Note that $x_J^{\sigma\sigma'} = (\sigma')^*(x_J^\sigma)$. For examples,

$$x_{J_1}^{(12)(23)} = (x_3 - x_1)(x_2 - x_4)(x_5 - x_6)(x_7 - x_8) = -x_{\langle 13;24;56;78\rangle},$$
$$x_{J_1}^{(23)(12)} = (x_2 - x_3)(x_1 - x_4)(x_5 - x_6)(x_7 - x_8) = x_{\langle 23;14;56;78\rangle},$$

where J_1 is the $(2, 2, 2, 2)$-partition $\langle 12; 34; 56; 78\rangle$. We define a subgroup G of S_8 as

$$G = \{\sigma \in S_8 \mid x_{J_1}^\sigma = \pm x_{J_1}\},$$

which is generated by

$$(12),\ (34),\ (56),\ (78),\ (13)(24),\ (35)(46),\ (57)(68).$$

Note that G is a maximal subgroup of order $2^4 \cdot 4!$ and that the set $\mathcal{J}$ can be regarded as the quotient $G\backslash S_8$.

2.2. Theta constants

Let $\mathbb{S}^r$ be the Siegel upper half space of degree r, which is the set of symmetric $r \times r$ complex matrices whose imaginary parts are positive definite. The symplectic group

$$Sp_r(\mathbb{Q}) = \left\{ M \in SL_{2r}(\mathbb{Q}) \;\middle|\; {}^tM \begin{pmatrix} & -I_r \\ I_r & \end{pmatrix} M = \begin{pmatrix} & -I_r \\ I_r & \end{pmatrix} \right\}$$

acts on $\mathbb{S}^r$ as

$$M \cdot \tau = (A\tau + B)(C\tau + D)^{-1}, \quad \tau \in \mathbb{S}_r,\ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_r(\mathbb{Q}),$$

where I_r denotes the unit matrix of degree r.

Theta constants with half characteristics $a, b \in \mathbb{Z}^r$ on $\mathbb{S}^r$ are defined as

$$\vartheta_{a,b}(\tau) = \sum_{n \in \mathbb{Z}^r} \exp\Big(\pi i \Big(n + \frac{a}{2}\Big) \tau\, {}^t\Big(n + \frac{a}{2}\Big) + 2\pi i \Big(n + \frac{a}{2}\Big) \frac{{}^tb}{2}\Big),$$
$$\tau \in \mathbb{S}^r.$$

They satisfy

$$\vartheta_{a+2p,b+2q}(\tau) = \exp(\pi i a^t q)\vartheta_{a,b}(\tau)$$

for $p,\, q \in \mathbb{Z}^r$. The following transformation formula for theta constants by the action of $Sp_r(\mathbb{Z}) = Sp_r(\mathbb{Q}) \cap SL_{2r}(\mathbb{Z})$ is a basic material of this paper.

Fact 1 (Corollary in p. 176 [I]) For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_r(\mathbb{Z})$, put

$$\begin{aligned} M \cdot (a,\, b) &= (a,\, b)M^{-1} + ((C^tD)_0,\, (A^tB)_0), \\ \phi_{a,b}(M) &= -\frac{1}{8}(a^tDB^ta - 2a^tBC^tb + b^tCA^tb) \\ &\quad + \frac{1}{4}(a^tD - b^tC)^t(A^tB)_0, \end{aligned}$$

where A_0 is the row vector consisting of the diagonal components of a square matrix A. We have

$$\vartheta_{M\cdot(a,\,b)}(M\cdot\tau) = \kappa(M)\exp(2\pi i \phi_{a,b}(M))\sqrt{\det(C\tau + D)}\vartheta_{a,b}(\tau),$$

where $\kappa(M)^2$ is ± 1 or $\pm i$ depending only on M.

2.3. An embedding of $\mathbb{B}^5$ into $\mathbb{S}^6$

We put $\mathbb{B}^5 = \{y \in \mathbb{P}^5 \mid y^* H^{-1} y < 0\}$, where

$$H = \begin{pmatrix} 2 & -1+i & & & & \\ -1-i & 2 & -1+i & & & \\ & -1-i & 2 & -1+i & & \\ & & -1-i & 2 & -1+i & \\ & & & -1-i & 2 & -1+i \\ & & & & -1-i & 2 \end{pmatrix}.$$

Since the signature of H is $(5,\, 1)$, $\mathbb{B}^5$ is isomorphic to the 5-dimensional complex ball $\{(z_1,\, \dots,\, z_5) \in \mathbb{C}^5 \mid |z_1|^2 + \cdots + |z_5|^2 < 1\}$. We define discrete subgroups acting on $\mathbb{B}^5$ as follows:

$$\begin{aligned} \Gamma &= \{g \in GL_6(\mathbb{Z}[i]) \mid g^* H^{-1} g = H^{-1}\} \\ &= \{g \in GL_6(\mathbb{Z}[i]) \mid gHg^* = H\}, \\ \Gamma(1-i) &= \{g \in \Gamma \mid g \equiv I_6 \bmod (1-i)\}. \end{aligned}$$

It is shown in [MY] that the quotient group $\Gamma/\Gamma(1-i)$ is isomorphic to S_8 and that Γ is generated by seven unitary reflections of order four:

$$g_j = I_6 - (1-i)Hv_j^* v_j/(v_j H v_j^*),$$

where

$$\begin{aligned} &v_1 = (1,\,0,\,0,\,0,\,0,\,0), \quad v_2 = (0,\,1,\,0,\,0,\,0,\,0),\\ &v_3 = (0,\,0,\,1,\,0,\,0,\,0), \quad v_4 = (0,\,0,\,0,\,1,\,0,\,0),\\ &v_5 = (0,\,0,\,0,\,0,\,1,\,0), \quad v_6 = (0,\,0,\,0,\,0,\,0,\,1),\\ &v_7 = (1,\,1-i,\,-i,\,0,\,1,\,1-i). \end{aligned}$$

The reflections g_j satisfy $g_j^2 \in \Gamma(1-i)$ and the braid relations $g_j g_{j+1} g_j = g_{j+1} g_j g_{j+1}$. We put $g_{j,j+1} = g_j$ and

$$g_{jk} = (g_{k-1} \cdots g_{j+2} g_{j+1}) g_j (g_{k-1} \cdots g_{j+2} g_{j+1})^{-1}$$

for $k > j+1$. There exists a homomorphism $s\colon \Gamma \to S_8$ such that $s(g_{jk})$ is the transposition $(j,\,k)$.

The domain $\mathbb{B}^5$ is embedded into the Siegel upper half space $\mathbb{S}^6$ of degree 6 as follows:

$$\imath\colon \mathbb{B}^5 \ni y \mapsto \tau = i\left[U - 2\frac{(Ty)\,{}^t(Ty)}{{}^t(Ty)U(Ty)}\right] \in \mathbb{S}^6,$$

where $T = N_1 - iN_2$,

$$N_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0 & 0\\ 1 & 1 & 0 & -1 & 0 & 0\\ 1 & 1 & 0 & 0 & 1 & 0\\ 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 1 & 0 & 0 & 0\\ 0 & 1 & 1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0 & 0 & 0\\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix},$$

$$U = \begin{pmatrix} I_2 & & & \\ & & -1 & \\ & -1 & & \\ & & & I_2 \end{pmatrix}.$$

This embedding $\imath$ induces a homomorphism

$$\jmath\colon \Gamma \ni g \mapsto N \begin{pmatrix} \mathrm{Re}(g) & -\mathrm{Im}(g)\\ \mathrm{Im}(g) & \mathrm{Re}(g) \end{pmatrix} N^{-1} \in Sp_6(\mathbb{Q}),$$

where $N = \begin{pmatrix} N_1 & N_2 \\ UN_2 & -UN_1 \end{pmatrix}$. Note that

$$\jmath(g) = \begin{pmatrix} A & B \\ -UBU & UAU \end{pmatrix},$$

where 6×6 real matrices A, B are given by $TgT^{-1} = A + iBU$, and that $(TgT^{-1})U(TgT^{-1})^* = U$.

The maps $\imath$ and $\jmath$ satisfy

$$\imath(gy) = (A\imath(y) + B)(C\imath(y) + D)^{-1}$$

for $g \in \Gamma$ and $\jmath(g) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_6(\mathbb{Q})$.

We put $\Psi(M, \tau) = \det(C\tau + D)$ for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_6(\mathbb{Q})$ and $\tau \in \mathbb{S}^6$.

Lemma 1 *For $y \in \mathbb{B}^5$, $g \in \Gamma$ and $\jmath(g) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_6(\mathbb{Q})$, we have*

$$\Psi(\jmath(g), \imath(y)) = \det(C\imath(y) + D) = \frac{{}^t(Tgy)U(Tgy)}{\det(g)\,{}^t(Ty)U(Ty)}.$$

Proof. We have

$$\det(C\imath(y) + D) = \det(U(-BU\tau U + A)U) = \det(-BU\tau U + A)$$

where $\tau = \imath(y)$, $g \in \Gamma$ and $TgT^{-1} = A + iBU$. We put $X = -BU\tau U + A$, and $z = Ty$. Since

$$\tau U = i\left[I_6 - 2\frac{z\,{}^tzU}{{}^tzUz}\right],$$

we have $\tau Uz = -iz$, and $\tau Uw_j = iw_j$, where $w_1, \dots, w_5$ span the subspace $W = \{w \in \mathbb{C}^6 \mid {}^tzUw = 0\}$. Thus

$$\begin{aligned} Xz &= -BU(-iz) + Az = TgT^{-1}z, \\ Xw_j &= -BU(iw_j) + Aw_j = \overline{TgT^{-1}}w_j \ (j = 1, \dots, 5). \end{aligned}$$

We express Xz as the linear combination of $\overline{TgT^{-1}}z$ and $\overline{TgT^{-1}}w_j$ $(j =$

1, ..., 5):

$$Xz = c\overline{TgT^{-1}}z + \sum_{j=1}^{5} c_j \overline{TgT^{-1}} w_j.$$

Note that $\det(X)$ is equal to $c\det(\overline{TgT^{-1}}) = c\det(\bar{g}) = c/\det(g)$. Multiply ${}^t zU(\overline{TgT^{-1}})^{-1}$ from the left of this linear combination, then we have

$$ {}^t zU(\overline{TgT^{-1}})^{-1}Xz = c\,{}^t zUz$$

since ${}^t zUw_j = 0$. The left hand side of this equality is

$$\begin{aligned} {}^t zU(\overline{TgT^{-1}})^{-1}(TgT^{-1})z &= {}^t zU(U\,{}^t(TgT^{-1})U)(TgT^{-1})z \\ &= {}^t(Tgy)U(Tgy), \end{aligned}$$

since $(TgT^{-1})^*U(TgT^{-1}) = U$. $\square$

We put

$$M_j = \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix} = \begin{pmatrix} A_j & B_j \\ -UB_jU & UA_jU \end{pmatrix} = \jmath(g_j),$$

for $j = 1, \ldots, 7$, and $M_{jk} = \jmath(g_{jk})$ for $1 \le j < k \le 8$.

Fact 2 The (1, 1)-block A_j and the (1, 2)-block B_j of $M_j = \jmath(g_j)$ are given as follows:

$$A_1 = \begin{pmatrix} 0 & \\ & I_5 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & \\ & O_5 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} I_2 - \Delta & \\ & I_4 \end{pmatrix}, \quad B_2 = \begin{pmatrix} \Delta & \\ & O_4 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 1 & & & & & \\ & 0 & 1 & & & \\ & & 1 & & & \\ & -1 & 1 & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & & & & & \\ & 1 & & 1 & & \\ & & 0 & & & \\ & 1 & & 1 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} I_2 & & \\ & I_2 - \Delta & \\ & & I_2 \end{pmatrix}, \quad B_4 = \begin{pmatrix} O_2 & & \\ & \Delta & \\ & & O_2 \end{pmatrix},$$

$$A_5 = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & 1 & 1 & -1 & \\ & & 1 & & 0 & \\ & & & & & 1 \end{pmatrix}, \quad B_5 = \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 1 & 1 & \\ & & & 1 & 1 & \\ & & & & & 0 \end{pmatrix},$$

$$A_6 = \begin{pmatrix} I_4 & \\ & I_2 - \Delta \end{pmatrix}, \quad B_6 = \begin{pmatrix} O_4 & \\ & \Delta \end{pmatrix},$$

$$A_7 = \begin{pmatrix} I_5 & \\ & 0 \end{pmatrix}, \quad B_7 = \begin{pmatrix} O_5 & \\ & 1 \end{pmatrix},$$

where O_k is the zero matrix of degree k and

$$\Delta = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Remark 1 If g belongs to $\Gamma(1-i)$ then $\jmath(g)$ belongs to $Sp_6(\mathbb{Z})$. Moreover, $\jmath(g)$ belongs to $Sp_6(\mathbb{Z})$ for $g \in \Gamma$ if and only if $s(g)$ belongs to $G \subset S_8$.

We put

$$\Gamma(G) = s^{-1}(G) \subset \Gamma, \quad Sp_6(G) = \jmath(\Gamma(G)) \subset Sp_6(\mathbb{Z}).$$

2.4. 105 automorphic forms f_J

A holomorphic function f on $\mathbb{B}^5$ is called an automorphic form of weight k with respect to a finite index subgroup Γ' of Γ if f satisfies

$$f(g \cdot y) = \psi(g, y)^k f(y)$$

for any $g \in \Gamma'$, where

$$\psi(g, y) = \frac{{}^t(Tgy)U(Tgy)}{{}^t(Ty)U(Ty)}.$$

Note that $\psi(g, y) = \det(g)\Psi(\jmath(g), \imath(y))$ by Lemma 1.

We construct automorphic forms on $\mathbb{B}^5$ by using the pull-back $\vartheta_{a,b}(y)$ of theta constants $\vartheta_{a,b}(\tau)$ on $\mathbb{S}^6$ under the embedding $\imath\colon \mathbb{B}^5 \to \mathbb{S}^6$.

Fact 3 The function $\prod_{j=0}^{3} \vartheta_{m_j, m_j U}(y)$ is an automorphic form on $\mathbb{B}^5$ of weight 2 with respect to $\Gamma(1-i)$, where

$$m_0 = (0, 0, 0, 0, 0, 0), \quad m_1 = (0, 0, 1, 1, 1, 1),$$
$$m_2 = (1, 1, 0, 0, 1, 1), \quad m_3 = (1, 1, 1, 1, 0, 0).$$

We define an action of Γ on an automorphic form f on $\mathbb{B}^5$ of weight k with respect to $\Gamma(1-i)$ by

$$f^g(y) = \psi(g,\, y)^{-k} f(g \cdot y).$$

Fact 4

(1) If $mU\,{}^t m \not\equiv 0 \bmod 4$ then $\vartheta_{m,mU}(y) = 0$ for $m \in \mathbb{Z}^6$.

(2) (Corollary 4.11 in [MT]) The functions $\vartheta_{m,mU}(y)$ satisfy the quadratic relation

$$\exp\Big(\frac{\pi i}{2} v_1\,{}^t U_0\Big) \vartheta_{v_1, v_1 U}(y) \vartheta_{v_2, v_2 U}(y) + \vartheta_{0,\,\dots,\,0}(y) \vartheta_{U_0, U_0}(y) = 0,$$

where $U_0 = (1,\, 1,\, 0,\, 0,\, 1,\, 1)$, $v_2 = v_1 - U_0$, $(0,\, \dots,\, 0) \neq v_1 \in \mathbb{Z}^6$ and $v_1 U\,{}^t v_1 \equiv 0 \bmod 4$. Especially, we have

$$\vartheta_{m_0,m_0}(y) \vartheta_{m_2,m_2}(y) = \vartheta_{m_1,m_1}(y) \vartheta_{m_3,m_3}(y).$$

(3) By Fact 1, the set of theta characteristics $\{(m_0,\, m_0),\, \dots,\, (m_3,\, m_3)\}$ may change by the action of $M \in Sp_6(G)$. In spite of this situation, the function $f_1(y) = \prod_{j=0}^{3} \vartheta_{m_j, m_j U}(y)$ is invariant modulo sign under the action $g \in \Gamma(G)$ by (2).

Since $\Gamma/\Gamma(1-i)$ is isomorphic to S_8, we have 105 automorphic forms $f_J(y)$ by acting Γ on the function $f_{J_1}(y)$, where J is a $(2,\, 2,\, 2,\, 2)$-partition corresponding to an element $G\backslash S_8$ represented by $s(g) \in S_8$ for $g \in \Gamma$. The following is a main result in [MT].

Fact 5 The polynomials x_J and automorphic forms $f_J(y)$ are S_8-equivariant. The vectors $(\dots,\, x_J,\, \dots)$ and $(\dots,\, f_J(y),\, \dots)$ are proportional and the functions $f_J(y)$ satisfy the same algebraic relations those x_J satisfy.

Few $f_J(y)$ are explicitly given in terms of $\vartheta_{a,b}(y)$ in [MT], since $\jmath(g)$ does not belongs to $Sp_6(\mathbb{Z})$ for a general $g \in \Gamma$. We give representations of $f_J(y)$ in terms of $\vartheta_{a,b}(y)$ for any $J \in \mathcal{J}$ in Section 5.

3. $\kappa(M)$ for some $M \in Sp_6(\mathbb{Z})$

Lemma 2 *Let* $\kappa(M)$ *be the eighth-root of unity in* Fact 1 *for* $M \in Sp_6(G) \subset Sp_6(\mathbb{Z})$. *The map*

$$Sp_6(G) \ni M \mapsto \kappa(M)^4 \in \{\pm 1\}$$

is a homomorphism.

Proof. Note that $\phi_{a,b}(M)$ in Fact 1 becomes 0 for any $M \in Sp_6(\mathbb{Z})$ when $(a, b) = (0, \ldots, 0)$, and that the function $\Psi(M, \tau) = \det(C\tau + D)$ satisfies

$$\Psi(LM, \tau) = \Psi(L, M \cdot \tau)\Psi(M, \tau).$$

Since we have

$$\begin{aligned}
&\kappa(LM)^4\Psi(LM, \tau)^2\vartheta^4_{0,\ldots,0}(\tau) \\
&= \vartheta^4_{(LM)\cdot(0,\ldots,0)}((LM)\cdot\tau) = \vartheta^4_{L\cdot(M\cdot(0,\ldots,0))}(L\cdot(M\cdot\tau)) \\
&= \kappa(L)^4\exp(2\pi i\phi_{M\cdot(0,\ldots,0)}(L))^4\Psi(L, M\cdot\tau)^2\vartheta^4_{M\cdot(0,\ldots,0)}(M\cdot\tau) \\
&= \kappa(L)^4\kappa(M)^4\exp(2\pi i\phi_{M\cdot(0,\ldots,0)}(L))^4 \\
&\qquad\qquad\times\Psi(L, M\cdot\tau)^2\Psi(M, \tau)^2\vartheta^4_{0,\ldots,0}(\tau),
\end{aligned}$$

we show that $4\phi_{M\cdot(0,\ldots,0)}(L)$ belongs to $\mathbb{Z}$ for any $L, M \in Sp_6(G)$. Let (a, b) be $M \cdot (0, \ldots, 0)$ for $M = \begin{pmatrix} A & B \\ -UBU & UAU \end{pmatrix}$. Note that

$$b = -(UB{}^tAU)_0 \equiv (UA{}^tBU)_0 \equiv (A{}^tB)_0U \equiv aU \bmod \mathbb{Z}^6.$$

The four times of $\phi_{a,aU}(L)$ for $L = \begin{pmatrix} C & D \\ -UDU & UCU \end{pmatrix}$ is congruent to

$$\frac{1}{2}(a{}^t(UCU)D{}^ta + (aU){}^t(-UDU)C{}^t(aU)) = 0$$

modulo $\mathbb{Z}$. □

Proposition 1 *Let g be an element of $\Gamma(1-i)$. Then the constant $\kappa(M)^4$ for $M = \jmath(g) \in Sp_6(\mathbb{Z})$ is 1.*

Proof. It is shown in [MY] that the group $\Gamma(1-i)$ is generated by unitary reflections of order 2. We show that $\kappa(M)^4 = 1$ for any element $M \in Sp_6(\mathbb{Z})$ of order 2, which satisfies

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = M = M^{-1} = \begin{pmatrix} {}^tD & -{}^tB \\ -{}^tC & {}^tA \end{pmatrix}.$$

Since we have

$$\begin{aligned}
&\vartheta^2_{0,\ldots,0}(\tau) = \vartheta^2_{M^2\cdot(0,\ldots,0)}(M^2\cdot\tau) = \vartheta^2_{M\cdot(M\cdot(0,\ldots,0))}(M\cdot(M\cdot\tau)) \\
&= \kappa(M)^4\exp(4\pi i\phi_{M\cdot(0,\ldots,0)}(M))\vartheta^2_{0,\ldots,0}(\tau),
\end{aligned}$$

it is sufficient to show that $2\phi_{M\cdot(0,\,\dots,\,0)}(M) \in \mathbb{Z}$. We put $M \cdot (0, \dots, 0) = (a, b)$, which is $((C\,{}^tD)_0, (A\,{}^tB)_0)$. Note that $M^2 = I_{12}$, and

$$\begin{aligned} M \cdot (a, b) &= (a, b)M^{-1} + (a, b) \equiv (0, \dots, 0) \bmod 2\mathbb{Z}^{12}, \\ (a, b)M^{-1} &= (a\,{}^tD - b\,{}^tC, -a\,{}^tB + b\,{}^tA) \equiv (a, b) \bmod 2\mathbb{Z}^{12}. \end{aligned}$$

Since ${}^tBC - {}^tDA = B\,{}^tC - A\,{}^tD = -I_6$ by ${}^tB = -B$, ${}^tC = -C$ and ${}^tD = A$, we have

$$\begin{aligned} & 2\phi_{a,b}(M) \\ =& -\frac{1}{4}(a\,{}^tDB\,{}^ta - 2a\,{}^tBC\,{}^tb + b\,{}^tCA\,{}^tb) + \frac{1}{2}(a\,{}^tD - b\,{}^tC)\,{}^t(A\,{}^tB)_0 \\ =& \frac{1}{4}\left[(a\,{}^tD - b\,{}^tC)\,{}^t(-a\,{}^tB + b\,{}^tA) + a({}^tBC - {}^tDA)\,{}^tb + 2(a\,{}^tD - b\,{}^tC)\,{}^tb\right] \\ \equiv& \frac{1}{4}(a\,{}^tb - a\,{}^tb) + \frac{1}{2}a\,{}^tb = \frac{1}{2}a\,{}^tb \quad \bmod \mathbb{Z}. \end{aligned}$$

By the action of $M \in Sp_6(\mathbb{Z})$, the characteristic $(0, \dots, 0)$ should be transformed into an even characteristic (a, b), which satisfies $a\,{}^tb \in 2\mathbb{Z}$. $\square$

Remark 2 If $M_{jk}^2 \in Sp_6(\mathbb{Z})$ keeps the theta characteristic $(0, \dots, 0)$ invariant then the constant $\kappa(M_{jk}^2)^2$ is -1. Otherwise, $\kappa(M_{jk}^2)^2$ may take ± 1. For examples, $M_{j,j+1}^2 = M_j^2$ keeps the theta characteristic $(0, \dots, 0)$ invariant, the constant $\kappa(M_j^2)^2$ is -1. The elements M_{15}^2, M_{16}^2, M_{47}^2 and M_{48}^2 do not keep $(0, \dots, 0)$ invariant. By computational evaluations of both sides of

$$\vartheta^2_{M_{jk}^2\cdot(0,\,\dots,\,0)}(M_{jk}^2 \cdot (iI_6)) = \kappa(M_{jk}^2)^2 \Psi(M_{jk}^2, iI_6)\vartheta^2_{(0,\,\dots,\,0)}(iI_6),$$

we have $\kappa(M_{15}^2)^2 = \kappa(M_{48}^2)^2 = 1$ and $\kappa(M_{16}^2)^2 = \kappa(M_{47}^2)^2 = -1$.

Remark 3 We have $\kappa(M)^4 = (-1)^r$ for any element $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_r(\mathbb{Z})$ satisfying $M^2 = -I_{2r}$, since

$$\det(C(M \cdot \tau) + D)\det(C\tau + D) = \det(-I_r) = (-1)^r$$

for $\tau \in \mathbb{S}^r$.

Lemma 3 *The constant $\kappa(M)^2$ is i for $M = M_1$, M_3, M_5 and M_7.*

Proof. We have $M_k \cdot (0, \ldots, 0) = (0, \ldots, 0)$, $M_k \cdot \tau'_k = \tau'_k$ and $\det(C_k\tau'_k + D_k) = -i$ for $k = 1, 3, 5, 7$, where $\tau'_1 = \tau'_7 = iI_6$ and

$$\tau'_3 = i\begin{pmatrix} 1 & & & & & \\ & 3 & 2 & 2 & & \\ & 2 & 2 & 1 & & \\ & 2 & 1 & 2 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}, \quad \tau'_5 = i\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 2 & 1 & 2 & \\ & & 1 & 2 & 2 & \\ & & 2 & 2 & 3 & \\ & & & & & 1 \end{pmatrix}.$$

By Fact 1, we have

$$\vartheta_{0,\ldots,0}(\tau'_k) = \kappa(M_k)\sqrt{-i}\vartheta_{0,\ldots,0}(\tau'_k),$$

for $k = 1, 3, 5, 7$. Since $\vartheta_{0,\ldots,0}(i\tau'_k)$ are positive real numbers, $\kappa(g)\sqrt{-i}$ should be 1 for $g = M_1$, M_3, M_5 and M_7. □

Lemma 4 *The constant $\kappa(M)^2$ is -1 for $M = M_{14}M_{23}$, $M_{36}M_{45}$ and $M_{58}M_{67}$, where $M_{jk} = \jmath(g_{jk})$.*

Proof. For $M = M_{14}M_{23} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we put $g = A + iBU$. Its eigenvalues are 1 and i; the eigenspaces of 1 and i are four and two dimensional, respectively. The element $Ty = {}^t(1, 1, 1+i, 1+i, 0, 0)$ satisfies $g(Ty) = Ty$ and $y \in \mathbb{B}^5$. By Lemma 1, $\det(C\imath(y) + D) = 1/\det(T^{-1}gT) = -1$. By the property of the embedding $\imath\colon \mathbb{B}^5 \to \mathbb{S}^6$, we have $(A\imath(y)+B)(C\imath(y)+D)^{-1} = \imath(T^{-1}gTy) = \imath(y)$. Fact 1 implies that

$$\vartheta^2_{(0,\ldots,0)}(\imath(y)) = \vartheta^2_{M\cdot(0,\ldots,0)}(M \cdot \imath(y)) = \kappa(M)^2(-1)\vartheta^2_{M\cdot(0,\ldots,0)}(\imath(y)).$$

We show that $\vartheta_{0,\ldots,0}(\imath(y))$ does not vanish by computational evaluations. Since

$$\imath(y) = \begin{pmatrix} \tau' & O \\ O & iI_2 \end{pmatrix}, \quad \tau' \in \mathbb{S}^4,$$

we have

$$\vartheta_{0,\ldots,0}(\imath(y)) = \vartheta_{0,\ldots,0}(\tau')\vartheta^2_{0,0}(i).$$

The partial sum of the defining series of $\vartheta_{0,\ldots,0}(\tau')$ for $n \in \mathbb{Z}^4$ satisfying $\mathrm{Im}(n\imath(y)\,{}^tn) < 10$ is about $1.433725975 + 0.3384567922i$, and $e^{-10\pi}$ is about $0.2271101059\times10^{-13}$. $1.691709704+0.3997855731i$ and $e^{-4\pi}$ is about

$0.3487342337 \times 10^{-5}$. These evaluations imply $\vartheta_{0,\dots,0}(\imath(y)) \neq 0$. Thus we have $\kappa(M)^2 = -1$.

For $M_{36}M_{45}$ and $M_{58}M_{67}$, take Ty as

$$^t(0,\, i,\, 1+i,\, 1+i,\, 1,\, 0), \quad ^t(0,\, 0,\, 1-i,\, 1-i,\, 1,\, 1),$$

respectively. □

Lemmas 2, 3 and 4 imply the following.

Proposition 2 *For* $g \in \Gamma(G)$ *and* $y \in \mathbb{B}^5$, *we have*

$$\begin{aligned}&\det(g)^2 = \kappa(\jmath(g))^4 = \mathrm{sign}(s(g)),\\ &\kappa(\jmath(g))^4 \Psi(\jmath(g),\, \imath(y))^2 = \psi(g,\, y)^2.\end{aligned}$$

4. Transformation formulas of the theta constants for some elements of $Sp_6(\mathbb{Q})$

Proposition 3 *By the actions of* $M_2, M_4, M_6 \in Sp_6(\mathbb{Q})$, *we have*

$$\begin{aligned}&\begin{pmatrix}\vartheta_{a,b}(M_{2k}\cdot\tau)\\ \vartheta_{a+e_{[2k]},b+e_{[2k]}}(M_{2k}\cdot\tau)\end{pmatrix}\\ =&\sqrt{\det(C_{2k}\tau + D_{2k})}\exp\left(\pi i\frac{a_{2k}-a_{2k-1}}{2}\,\frac{b_{2k}-b_{2k-1}}{2}\right)\frac{1+i}{2}\\ \times&\begin{pmatrix}\exp\left(-\pi i\dfrac{a_{2k}+a_{2k-1}}{2}\right) & 1\\ 1 & -\exp\left(\pi i\dfrac{a_{2k}+a_{2k-1}}{2}\right)\end{pmatrix}\\ \times&\begin{pmatrix}\vartheta_{a',b'+e_{[2k]}}(\tau)\\ \vartheta_{a'+e_{[2k]},b'}(\tau)\end{pmatrix},\end{aligned}$$

where $(a',\, b') = -(a,\, b)M_{2k}$,

$$e_{[2]} = (1,\, 1,\, 0,\, 0,\, 0,\, 0), \quad e_{[4]} = (0,\, 0,\, 1,\, 1,\, 0,\, 0),$$
$$e_{[6]} = (0,\, 0,\, 0,\, 0,\, 1,\, 1),$$

and the branch of the square root is determined by the assignment of the value at $\tau = iI_6 \in \mathbb{S}^6$ *as*

$$\sqrt{\det(C_{2k}(iI_6) + D_{2k})} = \sqrt{-i} = \frac{1-i}{\sqrt{2}}.$$

In order to prove this proposition, we prepare the following translation formula of theta constants for $\Delta = (1/2)\left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right)$.

Lemma 5 *Theta constants on the Siegel upper half space* $\mathbb{S}^2$ *of degree* 2 *satisfy*

$$\begin{pmatrix} \vartheta_{a,b}(\tau+\Delta) \\ \vartheta_{a,b+e}(\tau+\Delta) \end{pmatrix} = \nabla(a) \begin{pmatrix} \vartheta_{a,a\Delta+b}(\tau) \\ \vartheta_{a,a\Delta+b+e}(\tau) \end{pmatrix},$$

where $e = (1, 1)$, *and*

$$\nabla(a) = \frac{c_1(a)}{2} \begin{pmatrix} 1+i & (1-i)c_2^{-1}(a) \\ (1-i)c_2(a) & 1+i \end{pmatrix},$$

$$c_1(a) = \exp\left(-\frac{\pi i a \Delta\, {}^t a}{4}\right), \quad c_2(a) = \exp\left(\frac{\pi i a\, {}^t e}{2}\right).$$

Proof. Let Λ be the lattice $\{n = (n_1, n_2) \in \mathbb{Z}^2 \mid n_1 + n_2 \in 2\mathbb{Z}\}$. Note that $[\mathbb{Z}^2 : \Lambda] = 2$ and $\mathbb{Z}^2/\Lambda = \{(0, 0), e_1 = (1, 0)\}$. We put

$$\vartheta^{\Lambda}_{a,b}(\tau) = \sum_{n\in\Lambda} \exp\left(\pi i \left(n + \frac{a}{2}\right)\tau\, {}^t\!\left(n + \frac{a}{2}\right) + 2\pi i \left(n + \frac{a}{2}\right)\frac{{}^t b}{2}\right).$$

Since $n({}^t e/2) = (n_1 + n_2)/2 \in \mathbb{Z}$ for any $n \in \Lambda$, we have

$$\vartheta^{\Lambda}_{a,b+e}(\tau) = \exp\left(\frac{\pi i a\, {}^t e}{2}\right)\vartheta^{\Lambda}_{a,b}(\tau).$$

By definition, we have

$$\vartheta_{a,b}(\tau) = \vartheta^{\Lambda}_{a,b}(\tau) + \vartheta^{\Lambda}_{a+2e_1,b}(\tau)$$

and

$$\begin{aligned} \vartheta_{a,b+e}(\tau) &= \vartheta^{\Lambda}_{a,b+e}(\tau) + \vartheta^{\Lambda}_{a+2e_1,b+e}(\tau) \\ &= \exp\left(\frac{\pi i a\, {}^t e}{2}\right)\vartheta^{\Lambda}_{a,b}(\tau) + \exp\left(\frac{\pi i (a+2e_1)\, {}^t e}{2}\right)\vartheta^{\Lambda}_{a+2e_1,b}(\tau) \\ &= c_2(a)\big(\vartheta^{\Lambda}_{a,b}(\tau) - \vartheta^{\Lambda}_{a+2e_1,b}(\tau)\big). \end{aligned}$$

Thus we have

$$\begin{pmatrix} \vartheta_{a,b}(\tau) \\ \vartheta_{a,b+e}(\tau) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ c_2(a) & -c_2(a) \end{pmatrix} \begin{pmatrix} \vartheta^{\Lambda}_{a,b}(\tau) \\ \vartheta^{\Lambda}_{a+2e_1,b}(\tau) \end{pmatrix},$$

$$\begin{pmatrix} \vartheta^{\Lambda}_{a,b}(\tau) \\ \vartheta^{\Lambda}_{a+2e_1,b}(\tau) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & c_2^{-1}(a) \\ 1 & -c_2^{-1}(a) \end{pmatrix} \begin{pmatrix} \vartheta_{a,b}(\tau) \\ \vartheta_{a,b+e}(\tau) \end{pmatrix}.$$

We express $\vartheta^\Lambda_{a,b}(\tau+\Delta)$ in terms of $\vartheta^\Lambda_{a',b'}(\tau)$. Since

$$\begin{aligned}
&\exp\left(\pi i\left(n+\frac{a}{2}\right)\Delta\,{}^t\!\left(n+\frac{a}{2}\right)\right)\\
&=\exp(\pi i n\Delta\,{}^t n)\exp\left(-\frac{\pi i a\Delta\,{}^t a}{4}\right)\exp\left(2\pi i\left(n+\frac{a}{2}\right)\Delta\frac{{}^t a}{2}\right)\\
&=\begin{cases} c_1(a)\exp\left(2\pi i\left(n+\frac{a}{2}\right)\Delta\frac{{}^t a}{2}\right) & \text{if } n\in\Lambda,\\ ic_1(a)\exp\left(2\pi i\left(n+\frac{a}{2}\right)\Delta\frac{{}^t a}{2}\right) & \text{if } n\in\Lambda+e_1,\end{cases}
\end{aligned}$$

$$\begin{aligned}
&\vartheta^\Lambda_{a,b}(\tau+\Delta)\\
&=\sum_{n\in\Lambda}\exp\left(\pi i\left(n+\frac{a}{2}\right)\tau\,{}^t\!\left(n+\frac{a}{2}\right)+2\pi i\left(n+\frac{a}{2}\right)\frac{{}^t b}{2}\right)\\
&\qquad\qquad\times\exp\left(\pi i\left(n+\frac{a}{2}\right)\Delta\,{}^t\!\left(n+\frac{a}{2}\right)\right)\\
&=c_1(a)\sum_{n\in\Lambda}\exp\left(\pi i\left(n+\frac{a}{2}\right)\tau\,{}^t\!\left(n+\frac{a}{2}\right)+2\pi i\left(n+\frac{a}{2}\right)\frac{{}^t b}{2}\right)\\
&\qquad\qquad\times\exp\left(2\pi i\left(n+\frac{a}{2}\right)\Delta\frac{{}^t a}{2}\right)\\
&=c_1(a)\vartheta^\Lambda_{a,a\Delta+b}(\tau).
\end{aligned}$$

Similarly, $\vartheta^\Lambda_{a+2e_1,b}(\tau+\Delta)=ic_1(a)\vartheta^\Lambda_{a+2e_1,a\Delta+b}(\tau)$. Thus we have

$$\begin{pmatrix}\vartheta^\Lambda_{a,b}(\tau+\Delta)\\ \vartheta^\Lambda_{a+2e_1,b}(\tau+\Delta)\end{pmatrix}=c_1(a)\begin{pmatrix}1&0\\0&i\end{pmatrix}\begin{pmatrix}\vartheta^\Lambda_{a,b+a\Delta}(\tau)\\ \vartheta^\Lambda_{a+2e_1,b+a\Delta}(\tau)\end{pmatrix}.$$

Let us compute $\vartheta_{a,b}(\tau+\Delta)$ and $\vartheta_{a,b+e}(\tau+\Delta)$:

$$\begin{aligned}
&\begin{pmatrix}\vartheta_{a,b}(\tau+\Delta)\\ \vartheta_{a,b+e}(\tau+\Delta)\end{pmatrix}=\begin{pmatrix}1&1\\c_2(a)&-c_2(a)\end{pmatrix}\begin{pmatrix}\vartheta^\Lambda_{a,b}(\tau+\Delta)\\ \vartheta^\Lambda_{a+2e_1,b}(\tau+\Delta)\end{pmatrix}\\
&=c_1(a)\begin{pmatrix}1&1\\c_2(a)&-c_2(a)\end{pmatrix}\begin{pmatrix}1&0\\0&i\end{pmatrix}\begin{pmatrix}\vartheta^\Lambda_{a,b+a\Delta}(\tau)\\ \vartheta^\Lambda_{a+2e_1,b+a\Delta}(\tau)\end{pmatrix}\\
&=\frac{c_1(a)}{2}\begin{pmatrix}1&1\\c_2(a)&-c_2(a)\end{pmatrix}\begin{pmatrix}1&0\\0&i\end{pmatrix}\begin{pmatrix}1&c_2^{-1}(a)\\1&-c_2^{-1}(a)\end{pmatrix}\begin{pmatrix}\vartheta_{a,b+a\Delta}(\tau)\\ \vartheta_{a,b+a\Delta+e}(\tau)\end{pmatrix},
\end{aligned}$$

which implies this lemma. $\square$

Proof of Proposition 3. It is sufficient to prove that

$$\begin{pmatrix}\vartheta_{a,b}(M\cdot\tau)\\ \vartheta_{a+e,b+e}(M\cdot\tau)\end{pmatrix}=\sqrt{\det(I_2-\Delta\tau-\Delta)}\exp\Big(\pi i\frac{a_2-a_1}{2}\,\frac{b_2-b_1}{2}\Big)$$
$$\times\frac{1+i}{2}\begin{pmatrix}c_2^{-1}(a) & 1\\ 1 & -c_2(a)\end{pmatrix}\begin{pmatrix}\vartheta_{a',b'+e}(\tau)\\ \vartheta_{a'+e,b'}(\tau)\end{pmatrix},$$

where $\tau\in\mathbb{S}^2$, $e=(1,1)$, $(a',b')=(a,b)M$, $c_2(a)=\exp\big(\pi i(a_1+a_2)/2\big)$,

$$M=\begin{pmatrix}I_2-\Delta & \Delta\\ -\Delta & I_2-\Delta\end{pmatrix}\in Sp_2(\mathbb{Q}),\quad \Delta=\frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix},$$

and the branch of $\sqrt{\det(I_2-\Delta\tau-\Delta)}$ is determined by the assignment of the value at $\tau=iI_2$ as $\sqrt{-i}=(1-i)/\sqrt{2}$.

Since we have

$$M=-PQPQP,\quad P=-\begin{pmatrix}I_2 & \Delta\\ O_2 & I_2\end{pmatrix},\quad Q=\begin{pmatrix}O_2 & -I_2\\ I_2 & O_2\end{pmatrix},$$

use Lemma 5 and Fact 1 for Q repeatedly. In this calculation, it is convenient to prepare formulas

$$\begin{pmatrix}\vartheta_{a,b}(\tau+\Delta)\\ \vartheta_{a,b+e}(\tau+\Delta)\\ \vartheta_{a+e,b}(\tau+\Delta)\\ \vartheta_{a+e,b+e}(\tau+\Delta)\end{pmatrix}=\Phi_P[a,b]\begin{pmatrix}\vartheta_{a,a\Delta+b}(\tau)\\ \vartheta_{a,a\Delta+b+e}(\tau)\\ \vartheta_{a+e,a\Delta+b}(\tau)\\ \vartheta_{a+e,a\Delta+b+e}(\tau)\end{pmatrix},$$
$$\begin{pmatrix}\vartheta_{a,b}(-\tau^{-1})\\ \vartheta_{a,b+e}(-\tau^{-1})\\ \vartheta_{a+e,b}(-\tau^{-1})\\ \vartheta_{a+e,b+e}(-\tau^{-1})\end{pmatrix}=\Phi_Q[a,b]\begin{pmatrix}\vartheta_{b,-a}(\tau)\\ \vartheta_{b,-a+e}(\tau)\\ \vartheta_{b+e,-a}(\tau)\\ \vartheta_{b+e,-a+e}(\tau)\end{pmatrix},$$

where

$$\Phi_P[a,b]=\begin{pmatrix}\nabla(a) & \\ & \nabla(a+e)\end{pmatrix},$$
$$\Phi_Q[a,b]=\kappa(Q)\sqrt{\det(\tau)}\exp\Big(\frac{\pi i a\,{}^t b}{2}\Big)$$

$$\times \begin{pmatrix} 1 & & & \\ & & \exp\Big(\dfrac{\pi i a^t e}{2}\Big) & \\ & \exp\Big(\dfrac{-\pi i b^t e}{2}\Big) & & \\ & & & -\exp\Big(\dfrac{\pi i (a-b)^t e}{2}\Big) \end{pmatrix}.$$

Since $(a,\, b)P = (a, a\Delta + b)$, we have

$$\begin{aligned}
&\Phi_P[a,\, b]\Phi_Q[(a,\, b)P]\Phi_P[(a,\, b)PQ]\Phi_Q[(a,\, b)PQP]\Phi_P[(a,\, b)PQPQ] \\
&= c(a,\, b)\frac{1+i}{2}\begin{pmatrix} 0 & c_2(a) & 1 & 0 \\ c_2(a) & 0 & 0 & c_2^2(a) \\ 1 & 0 & 0 & -c_2(a) \\ 0 & c_2^2(a) & -c_2(a) & 0 \end{pmatrix} = \Phi_{-M}[a,\, b],
\end{aligned}$$

where $c_2(a) = \exp(\pi i a^t e/2)$ and

$$c(a,\, b) = \kappa(Q)^2\sqrt{\det(I_2 - \Delta\tau - \Delta)}\exp\Big(\pi i \frac{a_2 - a_1}{2}\,\frac{b_2 - b_1}{2}\Big).$$

Note that the theta characteristic $(a,\, b)$ is transformed into $-(a',\, b') = -M(a,\, b)$ by $PQPQP$. We have

$$\begin{aligned}
&\begin{pmatrix} \vartheta_{a,b}(M\cdot\tau) \\ \vartheta_{a,b+e}(M\cdot\tau) \\ \vartheta_{a+e,b}(M\cdot\tau) \\ \vartheta_{a+e,b+e}(M\cdot\tau) \end{pmatrix} = \begin{pmatrix} \vartheta_{a,b}(-M\cdot\tau) \\ \vartheta_{a,b+e}(-M\cdot\tau) \\ \vartheta_{a+e,b}(-M\cdot\tau) \\ \vartheta_{a+e,b+e}(-M\cdot\tau) \end{pmatrix} \\
&= \Phi_{-M}[a,\, b]\begin{pmatrix} \vartheta_{-a',-b'}(\tau) \\ \vartheta_{-a',-b'+e}(\tau) \\ \vartheta_{-a'+e,-b'}(\tau) \\ \vartheta_{-a'+e,-b'+e}(\tau) \end{pmatrix} \\
&= c(a,\, b)\begin{pmatrix} 0 & c_2^{-1}(a) & 1 & 0 \\ c_2(a) & 0 & 0 & 1 \\ 1 & 0 & 0 & -c_2^{-1}(a) \\ 0 & 1 & -c_2(a) & 0 \end{pmatrix}\begin{pmatrix} \vartheta_{a',b'}(\tau) \\ \vartheta_{a',b'+e}(\tau) \\ \vartheta_{a'+e,b'}(\tau) \\ \vartheta_{a'+e,b'+e}(\tau) \end{pmatrix},
\end{aligned}$$

since $a'^te = (a-(a+b)\Delta)^te = a^te$, and $\vartheta_{a',b'-e}(\tau) = \exp(-\pi i a'^t e)\vartheta_{a',b'+e}(\tau)$, $\vartheta_{a'-e,b'}(\tau) = \vartheta_{a'+e,b'}(\tau)$, $\vartheta_{a'-e,b'-e}(\tau) = \exp(-\pi i a'^t e)\vartheta_{a'+e,b'+e}(\tau)$. Especially,

$$\begin{aligned}\begin{pmatrix}\vartheta_{a,b}(M\cdot\tau)\\ \vartheta_{a+e,b+e}(M\cdot\tau)\end{pmatrix} &= \kappa(Q)^2\sqrt{\det(I_2-\Delta\tau-\Delta)}\\ &\quad\times\exp\Big(\frac{\pi i}{4}(a_2-a_1)(b_2-b_1)\Big)\\ &\quad\times\frac{1+i}{2}\begin{pmatrix}c_2^{-1}(a) & 1\\ 1 & -c_2(a)\end{pmatrix}\begin{pmatrix}\vartheta_{a',b'+e}(\tau)\\ \vartheta_{a'+e,b'}(\tau)\end{pmatrix}.\end{aligned}$$

We have only to determine $\kappa(Q)^2\sqrt{\det(I_2-\Delta\tau-\Delta)}$ explicitly. The last equality for $\tau = iI_2$ and $a = b = (0,\, 0)$ becomes

$$\begin{pmatrix}\vartheta_{00,00}(iI_2)\\ \vartheta_{11,11}(iI_2)\end{pmatrix} = \kappa(Q)^2\sqrt{-i}\,\frac{1+i}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\vartheta_{00,11}(iI_2)\\ \vartheta_{11,00}(iI_2)\end{pmatrix},$$

since we have $M\cdot(iI_2) = iI_2$ and $\det(I_2-\Delta\tau-\Delta) = -i$. Note that $\vartheta_{11,11}(iI_2) = \vartheta_{1,1}(i)\vartheta_{1,1}(i) = 0$, which implies $\vartheta_{00,11}(iI_2) = \vartheta_{11,00}(iI_2)$ and

$$\vartheta_{00,00}(iI_2) = \kappa(Q)^2\sqrt{-i}(1+i)\vartheta_{11,00}(iI_2).$$

Since both of $\vartheta_{00,00}(iI_2)$ and $\vartheta_{11,00}(iI_2)$ are positive real numbers, $\kappa(Q)^2\sqrt{-i}$ should be $(1-i)/\sqrt{2}$. Thus if we choose the branch of the square root $\sqrt{\det(I_2-\Delta\tau-\Delta)}$ by the assignment of the value at $\tau = iI_2$ as $\sqrt{-i} = (1-i)/\sqrt{2}$, then $\kappa(Q)^2 = 1$. $\square$

5. Representations of $f_J(y)$

In this section, we express $f_J(y)$ in terms of theta constants for all (2, 2, 2, 2)-partitions J by acting S_8 on $f_{J_1}(y)$. In subsection 5.1, we explain an efficient method to get the representations for all forms $f_J(y)$, and in Subsections $5.2,\dots,5.5$, we list them.

5.1. An efficient method to get the representations

The following elementary lemma is a key to give 105 representations of $f_J(y)$.

Lemma 6 *The double quotient $G\backslash S_8/G$ consists of five element, which are represented by*

$$\sigma_1 = \mathrm{id},\ \sigma_2 = (23),\ \sigma_3 = (23)(45),\ \sigma_4 = (23)(67),\\ \sigma_5 = (23)(45)(67).$$

The cardinalities of the right G-orbits of $\sigma_1,\dots,\sigma_5$ are 1, 12, 32, 12 *and* 48*, respectively.*

We define an action of $\sigma \in S_8$ on an automorphic form $f(y)$ on $\mathbb{B}^5$ with respect to $\Gamma(1-i)$ by $f^\sigma(y) = f^g(y)$, where $g \in \Gamma$ satisfies $s(g) = \sigma$. We give f_J by the action $\sigma \in S_8$ on $f_{J_1} = \prod_{j=0}^4 \vartheta_{m_j}(y)$ corresponding to the polynomial x_{J_1}, where m_j are in the following table.

#	x_{J_1}	m_0 m_2	m_1 m_3
1	$x_{\langle 12;34;56;78\rangle}$	000000, 000000 110011, 110011	001111, 001111 111100, 111100

We put $f_{J_j}(y) = f_{J_1}^{\sigma_j}(y)$ for $j = 2, \ldots, 5$, which are

$$f_{J_2}(y) = \prod_{j=0}^{3} \vartheta^{g_2}_{m_j, m_j U}(y) = \prod_{j=0}^{3} \vartheta_{m_j, m_j U}\big(M_2 \cdot \imath(y)\big)/\psi(g_2, y)^2,$$

$$f_{J_3}(y) = \prod_{j=0}^{3} \vartheta^{g_2 g_4}_{m_j, m_j U}(y) = \prod_{j=0}^{3} \vartheta_{m_j, m_j U}\big((M_2 M_4) \cdot \imath(y)\big)/\psi(g_2 g_4, y)^2,$$

$$f_{J_4}(y) = \prod_{j=0}^{3} \vartheta^{g_2 g_6}_{m_j, m_j U}(y) = \prod_{j=0}^{3} \vartheta_{m_j, m_j U}\big((M_2 M_6) \cdot \imath(y)\big)/\psi(g_2 g_6, y)^2,$$

$$\begin{aligned} f_{J_5}(y) &= \prod_{j=0}^{3} \vartheta^{g_2 g_4 g_6}_{m_j, m_j U}(y) \\ &= \prod_{j=0}^{3} \vartheta_{m_j, m_j U}\big((M_2 M_4 M_6) \cdot \imath(y)\big)/\psi(g_2 g_4 g_6, y)^2. \end{aligned}$$

We express $f_{J_j}(y)$ in terms of theta constants by Proposition 4. We can express the rest one hundred $f_J(y)$ in terms of theta constants by the action of G represented in $Sp_6(\mathbb{Z})$ on $f_{J_2}(y), \ldots, f_{J_5}(y)$ by Lemma 6. In this process, we have only to apply Fact 1.

The function $f^\sigma_{J_j}(y)$ for $\sigma \in G$ is independent of the choice of $g \in \Gamma(G)$ such that $s(g) = \sigma$, but its representation in terms of theta constants $\vartheta_{a,b}(y)$ depends on the choice of g. Different representations of $f^\sigma_{J_j}(y)$ imply relations among $\vartheta_{a,b}(y)$.

At the end of this subsection, we give a lemma, which helps us to simplify the representations of $f_J(y)$.

Lemma 7 *For $y \in \mathbb{B}^5$ and a, $b \in \mathbb{Z}^6$, we have*

$$\vartheta_{(bU,aU)}(y) = \exp\Bigl(-\pi i \frac{a\,{}^t b}{2}\Bigr)\vartheta_{a,b}(y).$$

Proof. Since $\imath(y) = \imath(iI_6 y)$ and $\jmath(iI_6) = \begin{pmatrix} & U \\ -U & \end{pmatrix} = V$, Fact 1 implies

$$\begin{aligned}\vartheta_{(bU,aU)}(y) &= \vartheta_{(bU,-aU)}(y) = \vartheta_{V\cdot(a,\,b)}(V \cdot \imath(y)) \\ &= \kappa(V)\exp\Bigl(-\pi i \frac{a\,{}^t b}{2}\Bigr)\sqrt{\det(-U\imath(y))}\,\vartheta_{a,b}(y).\end{aligned}$$

Note that $\det(-U\imath(y)) = -1$ and that $\vartheta_{0,\dots,0}(y)$ is not identically zero. Thus $\kappa(V)\sqrt{\det(-U\imath(y))}$ should be 1. □

5.2. The G-orbit of $f_{J_2}(y)$

Proposition 4 *We have*

$$f_{J_2}(y) = 4\vartheta_{110000,000000}(y)\vartheta_{111111,001111}(y) \times \vartheta_{000011,110011}(y)\vartheta_{001100,111100}(y).$$

Proof. Proposition 4 implies

$$\begin{aligned}&\vartheta_{000000,000000}(M_2\cdot\tau) \\ &\qquad = \frac{1+i}{2}\sqrt{\Psi(M_2,\,\tau)}[\vartheta_{110000,000000}(\tau) + \vartheta_{000000,110000}(\tau)], \\ &\vartheta_{110000,110000}(M_2\cdot\tau) \\ &\qquad = \frac{1+i}{2}\sqrt{\Psi(M_2,\tau)}[\vartheta_{110000,000000}(\tau) - \vartheta_{000000,110000}(\tau)].\end{aligned}$$

By Fact 4 (1), $\vartheta_{110000,110000}(y)$ is identically zero on $\mathbb{B}^5$, which implies

$$\vartheta_{110000,000000}(y) = \vartheta_{000000,110000}(y).$$

Thus we have

$$\vartheta_{000000,000000}(g_2\cdot y) = (1+i)\sqrt{\Psi(M_2,\,\imath(y))}\vartheta_{110000,000000}(y).$$

Similarly, we have

$$\begin{aligned}\vartheta_{001111,001111}(g_2\cdot y) &= (1+i)\sqrt{\Psi(M_2,\,\imath(y))}\vartheta_{111111,001111}(y), \\ \vartheta_{110011,110011}(g_2\cdot y) &= (1+i)\sqrt{\Psi(M_2,\,\imath(y))}\vartheta_{000011,110011}(y), \\ \vartheta_{111100,111100}(g_2\cdot y) &= (1+i)\sqrt{\Psi(M_2,\,\imath(y))}\vartheta_{001100,111100}(y).\end{aligned}$$

Note that $\det(g_2) = i$ and that $\psi(g_2,\,y) = \det(g_2)\Psi(M_2,\,\imath(y))$. □

By the action of $\sigma_2 = (23) \in S_8$, the polynomial $x_{J_1} = x_{\langle 12;34;56;78\rangle}$ changes into $x_{J_2} = x_{\langle 13;24;56;78\rangle}$. The group

$$G_2 = \{\sigma \in G \mid x_{J_2}^{\sigma} = \pm x_{J_2}\}$$

is generated by (56), (78), (12)(34), (13)(24) and (57)(68). Since its order is 2^5, we have $[G : G_2] = 12$. We express $f_{J_2}^{\sigma}(y)$ in terms of theta constants for $\sigma \in G_2\backslash G$.

Theorem 1 *By the action of $\sigma \in G_2\backslash G$, $f_{J_2}(y) = \prod_{j=0}^{3} \vartheta_{\nu_j}(y)$ is transformed into*

$$f_{J_2}^{\sigma}(y) = 4\prod_{j=0}^{3} \vartheta_{\nu_j}^{g}(y) = 4\prod_{j=0}^{3}\big(\varepsilon_j \vartheta_{\mu_j}(y)\big) = \varepsilon 4\prod_{j=0}^{3} \vartheta_{\mu_j}(y),$$

where $\mu_0, \ldots, \mu_3$, $\varepsilon_0, \ldots, \varepsilon_3$, the sign $\varepsilon = \prod_{j=0}^{3} \varepsilon_j$ and $g \in \Gamma(G)$ such that $s(g) = \sigma$ are listed in the following table. In the table, g_{jk} such that $s(g_{jk}) = (j, k)$ are defined in the Section 2.3*,*

$$h_{12} = g_{13}g_{24}, \quad h_{13} = g_{15}g_{26}, \quad h_{14} = g_{17}g_{28},$$
$$h_{23} = g_{35}g_{46}, \quad h_{24} = g_{37}g_{48}, \quad h_{34} = g_{57}g_{68},$$

and ζ and ζ' stand for $(1+i)/\sqrt{2}$ and $(1-i)/\sqrt{2}$, respectively.

#	$g \in \Gamma(G)$	$x_{J_2}^{\sigma}$	ε;	$\varepsilon_0, \varepsilon_1$ $\varepsilon_2, \varepsilon_3$	μ_0 μ_2	μ_1 μ_3
2	id	$x_{\langle 13;24;56;78\rangle}$	$+$;	1, 1 1, 1	110000, 000000 000011, 110011	111111, 001111 001100, 111100
3	g_1	$x_{\langle 23;14;56;78\rangle}$	$+$;	1, 1 1, 1	010000, 100000 100011, 010011	011111, 101111 101100, 011100
4	h_{23}	$x_{\langle 15;26;34;78\rangle}$	$+$;	ζ, ζ ζ', ζ'	110110, 000110 000101, 110101	110101, 000101 000110, 110110
5	g_1h_{23}	$x_{\langle 25;16;34;78\rangle}$	$+$;	ζ, ζ ζ', ζ'	010110, 100110 100101, 010101	010101, 100101 100110, 010110
6	h_{24}	$x_{\langle 17;28;56;34\rangle}$	$+$;	$-i, -1$ $-i, 1$	100001, 011110 010010, 101101	101110, 010001 011101, 100010
7	g_1h_{24}	$x_{\langle 27;18;56;34\rangle}$	$+$;	$-i, -1$ $-i, 1$	000001, 111110 110010, 001101	001110, 110001 111101, 000010

8	h_{13}	$x_{\langle 35;46;12;78\rangle}$	$+;$	$1,\ i$ $1,\ -i$	$010010, 011110$ $100001, 101101$	$011101, 010001$ $101110, 100010$
9	g_1h_{13}	$x_{\langle 36;45;12;78\rangle}$	$-;$	$1,\ i$ $1,\ i$	$011000, 011000$ $101011, 101011$	$010111, 010111$ $100100, 100100$
10	h_{14}	$x_{\langle 37;48;56;12\rangle}$	$+;$	$\zeta,\ \zeta'$ $\zeta',\ \zeta$	$001110, 000001$ $111101, 110010$	$111110, 110001$ $001101, 000010$
11	g_1h_{14}	$x_{\langle 38;47;56;12\rangle}$	$+;$	$\zeta,\ \zeta'$ $\zeta',\ \zeta$	$001111, 000000$ $111100, 110011$	$111111, 110000$ $001100, 000011$
12	$h_{13}h_{24}$	$x_{\langle 57;68;12;34\rangle}$	$+;$	$1,\ 1$ $1,\ 1$	$000011, 000000$ $110000, 110011$	$001100, 001111$ $111111, 111100$
13	$g_1h_{13}h_{24}$	$x_{\langle 67;58;12;34\rangle}$	$-;$	$1,\ 1$ $1,\ -1$	$001001, 000110$ $111010, 110101$	$000110, 001001$ $110101, 111010$

5.3. The G-orbit of $f_{J_3}(y)$

Proposition 5 *We have*

$$f_{J_3} = (\vartheta^2_{111100,000000} - \vartheta^2_{110000,001100})(\vartheta^2_{000011,111111} - \vartheta^2_{001111,110011}).$$

Proof. We get f_{J_3} by $f_{J_2}^{g_4}$. Proposition 4 implies

$$\begin{aligned}
\vartheta_{110000,000000}(M_4\cdot\tau) &= c\big(\vartheta_{110000,001100}(\tau) + \vartheta_{111100,000000}(\tau)\big),\\
\vartheta_{111111,001111}(M_4\cdot\tau) &= c\big(-\vartheta_{111111,000011}(\tau) + \vartheta_{110011,001111}(\tau)\big),\\
\vartheta_{000011,110011}(M_4\cdot\tau) &= c\big(\vartheta_{000011,111111}(\tau) + \vartheta_{001111,110011}(\tau)\big),\\
\vartheta_{001100,111100}(M_4\cdot\tau) &= c\big(-\vartheta_{001100,110000}(\tau) + \vartheta_{000000,111100}(\tau)\big),
\end{aligned}$$

where $c = \big((1+i)/2\big)\sqrt{\Psi(M_4,\,\tau)}$. Recall that $\det(g_4) = i$ and that $\psi(g_4,\,y) = \det(g_4)\Psi(M_4,\,\imath(y))$, and use Lemma 7. □

By the action of $\sigma_3 = (23)(45) \in S_8$, the polynomial x_{J_1} changes into $x_{J_3} = x_{\langle 13;25;46;78\rangle}$. The group $G_3 = \{\sigma \in G \mid x_{J_3}^{\sigma} = \pm x_{J_3}\}$ is generated by (78), (13)(24)(56) and (12)(35)(46). Since its order is 12, we have $[G : G_3] = 32$. We express $f_{J_3}^{\sigma}(y)$ in terms of theta constants for $\sigma \in G_3\backslash G$.

Theorem 2 *By the action of $\sigma \in G_3\backslash G$,*

$$f_{J_3}(y) = (\vartheta_{\nu_0}(y)^2 - \vartheta_{\nu_1}(y)^2)(\vartheta_{\nu_2}(y)^2 - \vartheta_{\nu_3}(y)^2)$$

is transformed into

$$\begin{aligned}
f_{J_3}^{\sigma}(y) &= (\vartheta^g_{\nu_0}(y)^2 - \vartheta^g_{\nu_1}(y)^2)(\vartheta^g_{\nu_2}(y)^2 - \vartheta^g_{\nu_3}(y)^2)\\
&= [(\varepsilon_0\vartheta_{\mu_0}(y))^2 - (\varepsilon_1\vartheta_{\mu_1}(y))^2][(\varepsilon_2\vartheta_{\mu_2}(y))^2 - (\varepsilon_3\vartheta^g_{\mu_3}(y))^2]
\end{aligned}$$

$$= \varepsilon(\vartheta^2_{\mu_0}(y) + \varepsilon'\vartheta^2_{\mu_1}(y))(\vartheta^2_{\mu_2}(y) + \varepsilon'\vartheta^2_{\mu_3}(y)),$$

where $\mu_0, \ldots, \mu_3$, $\varepsilon_0, \ldots, \varepsilon_3$, *the signs* $\varepsilon, \varepsilon'$ *and* $g \in \Gamma(G)$ *such that* $s(g) = \sigma$ *are listed in the following table.*

#	$g \in \Gamma(g)$	$x^{\sigma}_{J_3}$	ε; $\varepsilon_0, \varepsilon_1$ ε'; $\varepsilon_2, \varepsilon_3$	μ_0 μ_2	μ_1 μ_3
14	id	$x_{\langle 13;25;46;78\rangle}$	$+$; $1, 1$ $-$; $1, 1$	111100, 000000	110000, 001100
15	g_3	$x_{\langle 14;25;36;78\rangle}$	$+$; $1, i$ $+$; $1, i$	000011, 111111	001111, 110011
16	g_1	$x_{\langle 23;15;46;78\rangle}$	$+$; $1, 1$ $-$; $1, 1$	011100, 100000	010000, 101100
17	$g_1 g_3$	$x_{\langle 24;15;36;78\rangle}$	$+$; $1, i$ $+$; $1, i$	100011, 011111	101111, 010011
18	g_5	$x_{\langle 13;26;45;78\rangle}$	$+$; $1, 1$ $-$; $1, -1$	110110, 000110	111010, 001010
19	$g_3 g_5$	$x_{\langle 14;26;35;78\rangle}$	$+$; $1, i$ $+$; $1, -i$	001001, 111001	000101, 110101
20	$g_1 g_5$	$x_{\langle 23;16;45;78\rangle}$	$+$; $1, 1$ $-$; $1, -1$	010110, 100110	011010, 101010
21	$g_1 g_3 g_5$	$x_{\langle 24;16;35;78\rangle}$	$+$; $1, i$ $+$; $1, -i$	101001, 011001	100101, 010101
22	h_{34}	$x_{\langle 13;27;48;56\rangle}$	$+$; ζ, ζ $-$; $\zeta', -\zeta'$	111110, 000001	110001, 001110
23	$g_3 h_{34}$	$x_{\langle 14;27;38;56\rangle}$	$+$; $\zeta, -\zeta'$ $+$; $\zeta', -\zeta$	000010, 111101	001101, 110010
24	$g_1 h_{34}$	$x_{\langle 23;17;48;56\rangle}$	$+$; ζ, ζ $-$; $\zeta', -\zeta'$	011110, 100001	010001, 101110
25	$g_1 g_3 h_{34}$	$x_{\langle 24;17;38;56\rangle}$	$+$; $\zeta, -\zeta'$ $+$; $\zeta', -\zeta$	100010, 011101	101101, 010010
26	$g_5 h_{34}$	$x_{\langle 13;28;47;56\rangle}$	$+$; ζ, ζ $-$; $\zeta', -\zeta'$	111111, 000000	110000, 001111
27	$g_3 g_5 h_{34}$	$x_{\langle 14;28;37;56\rangle}$	$+$; $\zeta, -\zeta'$ $+$; $\zeta', -\zeta$	000011, 111100	001100, 110011
28	$g_1 g_5 h_{34}$	$x_{\langle 23;18;47;56\rangle}$	$+$; ζ, ζ $-$; $\zeta', -\zeta'$	011111, 100000	010000, 101111
29	$g_1 g_3 g_5 h_{34}$	$x_{\langle 24;18;37;56\rangle}$	$+$; $\zeta, -\zeta'$ $+$; $\zeta', -\zeta$	100011, 011100	101100, 010011

30	h_{24}	$-x_{\langle 17;25;68;34\rangle}$	$\begin{smallmatrix}-\\+\end{smallmatrix};\begin{smallmatrix}-i,\ -1\\-1,\ -i\end{smallmatrix}$	101010, 011010 101001, 011001
31	g_3h_{24}	$-x_{\langle 18;25;67;34\rangle}$	$\begin{smallmatrix}-\\-\end{smallmatrix};\begin{smallmatrix}-i,\ -i\\-1,\ 1\end{smallmatrix}$	011010, 101010 011001, 101001
32	g_1h_{24}	$-x_{\langle 27;15;68;34\rangle}$	$\begin{smallmatrix}-\\+\end{smallmatrix};\begin{smallmatrix}-i,\ -1\\-1,\ -i\end{smallmatrix}$	001010, 111010 001001, 111001
33	$g_1g_3h_{24}$	$-x_{\langle 28;15;67;34\rangle}$	$\begin{smallmatrix}-\\-\end{smallmatrix};\begin{smallmatrix}-i,\ -i\\-1,\ 1\end{smallmatrix}$	111010, 001010 111001, 001001
34	g_5h_{24}	$-x_{\langle 17;26;58;34\rangle}$	$\begin{smallmatrix}-\\+\end{smallmatrix};\begin{smallmatrix}-i,\ -1\\-1,\ -i\end{smallmatrix}$	100000, 011100 100011, 011111
35	$g_3g_5h_{24}$	$-x_{\langle 18;26;57;34\rangle}$	$\begin{smallmatrix}-\\-\end{smallmatrix};\begin{smallmatrix}-i,\ -i\\-1,\ 1\end{smallmatrix}$	010000, 101100 010011, 101111
36	$g_1g_5h_{24}$	$-x_{\langle 27;16;58;34\rangle}$	$\begin{smallmatrix}-\\+\end{smallmatrix};\begin{smallmatrix}-i,\ -1\\-1,\ -i\end{smallmatrix}$	000000, 111100 000011, 111111
37	$g_1g_3g_5h_{24}$	$-x_{\langle 28;16;57;34\rangle}$	$\begin{smallmatrix}-\\-\end{smallmatrix};\begin{smallmatrix}-i,\ -i\\-1,\ 1\end{smallmatrix}$	110000, 001100 110011, 001111
38	h_{14}	$x_{\langle 37;58;46;12\rangle}$	$\begin{smallmatrix}+\\-\end{smallmatrix};\begin{smallmatrix}\zeta,\ -\zeta\\-\zeta',\ \zeta'\end{smallmatrix}$	000010, 000001 001110, 001101
39	g_3h_{14}	$x_{\langle 47;58;36;12\rangle}$	$\begin{smallmatrix}+\\+\end{smallmatrix};\begin{smallmatrix}\zeta,\ \zeta'\\-\zeta',\ \zeta\end{smallmatrix}$	111101, 111110 110001, 110010
40	g_1h_{14}	$x_{\langle 38;57;46;12\rangle}$	$\begin{smallmatrix}+\\-\end{smallmatrix};\begin{smallmatrix}\zeta,\ -\zeta\\-\zeta',\ \zeta'\end{smallmatrix}$	000011, 000000 001111, 001100
41	$g_1g_3h_{14}$	$x_{\langle 48;57;36;12\rangle}$	$\begin{smallmatrix}+\\+\end{smallmatrix};\begin{smallmatrix}\zeta,\ \zeta'\\-\zeta',\ \zeta\end{smallmatrix}$	111100, 111111 110000, 110011
42	g_5h_{14}	$x_{\langle 37;68;45;12\rangle}$	$\begin{smallmatrix}+\\-\end{smallmatrix};\begin{smallmatrix}\zeta,\ \zeta\\-\zeta',\ \zeta'\end{smallmatrix}$	001000, 000111 000100, 001011
43	$g_3g_5h_{14}$	$x_{\langle 47;68;35;12\rangle}$	$\begin{smallmatrix}+\\+\end{smallmatrix};\begin{smallmatrix}\zeta,\ -\zeta'\\-\zeta',\ \zeta\end{smallmatrix}$	110111, 111000 111011, 110100
44	$g_1g_5h_{14}$	$x_{\langle 38;67;45;12\rangle}$	$\begin{smallmatrix}+\\-\end{smallmatrix};\begin{smallmatrix}\zeta,\ \zeta\\-\zeta',\ \zeta'\end{smallmatrix}$	001001, 000110 000101, 001010
45	$g_1g_3g_5h_{14}$	$x_{\langle 48;67;35;12\rangle}$	$\begin{smallmatrix}+\\+\end{smallmatrix};\begin{smallmatrix}\zeta,\ -\zeta'\\-\zeta',\ \zeta\end{smallmatrix}$	110110, 111001 111010, 110101

5.4. The G-orbit of $f_{J_4}(y)$

Proposition 6 *We have*

$$f_{J_4}=(\vartheta^2_{110011,000000}-\vartheta^2_{110000,000011})(\vartheta^2_{001100,111111}-\vartheta^2_{001111,111100}).$$

Proof. We get f_{J_4} by $f_{J_2}^{g_6}$. Proposition 4 implies

$$\vartheta_{110000,000000}(M_6\cdot\tau)=c(\vartheta_{110000,000011}(\tau)+\vartheta_{110011,000000}(\tau)),$$

$$\begin{aligned}
\vartheta_{111111,001111}(M_6\cdot\tau) &= c(-\vartheta_{111111,001100}(\tau)+\vartheta_{111100,001111}(\tau)),\\
\vartheta_{000011,110011}(M_6\cdot\tau) &= c(-\vartheta_{000011,110000}(\tau)+\vartheta_{000000,110011}(\tau)),\\
\vartheta_{001100,111100}(M_6\cdot\tau) &= c(\vartheta_{001100,111111}(\tau)+\vartheta_{001111,111100}(\tau)),
\end{aligned}$$

where $c=\big((1+i)/2\big)\sqrt{\Psi(M_6,\,\tau)}$. Use Lemma 7 and the equalities $\det(g_6)=i$ and $\psi(g_6,\,y)=\det(g_6)\Psi(M_6,\,\imath(y))$. $\square$

By the action of $\sigma_4=(67)\in S_8$, the polynomial $x_{J_2}=x_{\langle 13;24;56;78\rangle}$ changes into $x_{J_4}=x_{\langle 13;24;57;68\rangle}$. The group

$$G_4=\{\sigma\in G\mid x\sigma_{J_4}=\pm x_{J_4}\}$$

is generated by (12)(34), (13)(24), (56)(78), (57)(68) and (15)(26)(37)(48). Since its order is 32, we have $[G:G_4]=12$. We express $f^{\sigma}_{J_4}(y)$ in terms of theta constants for $\sigma\in G_4\backslash G$.

Theorem 3 *For* $\sigma\in G_4\backslash G$, $f_{J_4}(y)=(\vartheta_{\nu_0}(y)^2-\vartheta_{\nu_1}(y)^2)(\vartheta_{\nu_2}(y)^2-\vartheta_{\nu_3}(y)^2)$ *is transformed into*

$$\begin{aligned}
f^{\sigma}_{J_4}(y) &= (\vartheta^g_{\nu_0}(y)^2-\vartheta^g_{\nu_1}(y)^2)(\vartheta^g_{\nu_2}(y)^2-\vartheta^g_{\nu_3}(y)^2)\\
&= [(\varepsilon_0\vartheta_{\mu_0}(y))^2-(\varepsilon_1\vartheta_{\mu_1}(y))^2][(\varepsilon_2\vartheta_{\mu_2}(y))^2-(\varepsilon_3\vartheta^g_{\mu_3}(y))^2]\\
&= (\vartheta^2_{\mu_0}(y)+\varepsilon'\vartheta^2_{\mu_1}(y))(\vartheta^2_{\mu_2}(y)+\varepsilon'\vartheta^2_{\mu_3}(y)),
\end{aligned}$$

where $\mu_0,\,\ldots,\,\mu_3$, $\varepsilon_0,\,\ldots,\,\varepsilon_3$, *the sign* ε' *and* $g\in\Gamma(G)$ *such that* $s(g)=\sigma$ *are listed in the following table.*

#	$g\in\Gamma(G)$	$x^{\sigma}_{J_4}$	ε'; $\varepsilon_0,\,\varepsilon_1$ / $\varepsilon_2,\,\varepsilon_3$	μ_0 / μ_2	μ_1 / μ_3
46	id	$x_{\langle 13;24;57;68\rangle}$	$-$; 1, 1 / 1, 1	110011, 000000 / 001100, 111111	110000, 000011 / 001111, 111100
47	h_{23}	$x_{\langle 15;26;37;48\rangle}$	$+$; -1, i / 1, i	111111, 001100 / 000000, 110011	110000, 000011 / 001111, 111100
48	h_{24}	$x_{\langle 17;28;35;46\rangle}$	$-$; 1, 1 / 1, 1	110011, 000000 / 000011, 110000	111111, 001100 / 001111, 111100
49	g_1	$x_{\langle 23;14;57;68\rangle}$	$-$; 1, 1 / 1, 1	010011, 100000 / 101100, 011111	010000, 100011 / 101111, 011100
50	g_1h_{23}	$x_{\langle 25;16;37;48\rangle}$	$+$; -1, i / 1, i	011111, 101100 / 100000, 010011	010000, 100011 / 101111, 011100
51	g_1h_{24}	$x_{\langle 27;18;35;46\rangle}$	$-$; 1, 1 / 1, 1	010011, 100000 / 100011, 010000	011111, 101100 / 101111, 011100

52	g_7	$x_{\langle 13;24;58;67\rangle}$	$-;$	$1,\ 1$ $1,\ 1$	110010, 000001 001101, 111110	110001, 000010 001110, 111101
53	g_7h_{23}	$x_{\langle 15;26;38;47\rangle}$	$+;$	$-1,\ i$ $1,\ i$	111110, 001101 000001, 110010	110001, 000010 001110, 111101
54	g_7h_{24}	$x_{\langle 17;28;45;36\rangle}$	$-;$	$1,\ -1$ $1,\ 1$	101011, 010100 011011, 100100	100111, 011000 010111, 101000
55	g_1g_7	$x_{\langle 23;14;58;67\rangle}$	$-;$	$1,\ 1$ $1,\ 1$	010010, 100001 101101, 011110	010001, 100010 101110, 011101
56	$g_1g_7h_{23}$	$x_{\langle 25;16;38;47\rangle}$	$+;$	$-1,\ i$ $1,\ i$	011110, 101101 100001, 010010	010001, 100010 101110, 011101
57	$g_1g_7h_{24}$	$x_{\langle 27;18;45;36\rangle}$	$-;$	$1,\ -1$ $1,\ 1$	001011, 110100 111011, 000100	000111, 111000 110111, 001000

5.5. The G-orbit of $f_{J_5}(y)$

Proposition 7 *We have*

$$\begin{aligned} f_{J_5} = \frac{1}{4}&(\vartheta_{111111,000000}+\vartheta_{110000,001111}+\vartheta_{001100,110011}+\vartheta_{000011,111100}) \\ &\times(\vartheta_{111111,000000}+\vartheta_{110000,001111}-\vartheta_{001100,110011}-\vartheta_{000011,111100}) \\ &\times(\vartheta_{111111,000000}-\vartheta_{110000,001111}+\vartheta_{001100,110011}-\vartheta_{000011,111100}) \\ &\times(\vartheta_{111111,000000}-\vartheta_{110000,001111}-\vartheta_{001100,110011}+\vartheta_{000011,111100}). \end{aligned}$$

Proof. We get f_{J_5} by $f_{J_4}^{g_4}$. Proposition 4 implies

$$\begin{aligned} \vartheta_{110011,000000}(M_4\cdot\tau) &= c(\vartheta_{110011,001100}(\tau)+\vartheta_{111111,000000}(\tau)), \\ \vartheta_{110000,000011}(M_4\cdot\tau) &= c(\vartheta_{110000,001111}(\tau)+\vartheta_{111100,000011}(\tau)), \\ \vartheta_{001100,111111}(M_4\cdot\tau) &= c(-\vartheta_{001100,110011}(\tau)+\vartheta_{000000,111111}(\tau)), \\ \vartheta_{001111,111100}(M_4\cdot\tau) &= c(-\vartheta_{001111,110000}(\tau)+\vartheta_{000011,111100}(\tau)), \end{aligned}$$

where $c = \big((1+i)/2\big)\sqrt{\Psi(M_4,\,\tau)}$. Use Lemma 7 and the equalities $\det(g_4) = i$ and $\psi(g_4,\,y) = \det(g_4)\Psi(M_4,\,\imath(y))$. $\square$

By the action of $\sigma_5 = (23)(45)(67) \in S_8$, the polynomial x_{J_1} changes into $x_{J_5} = x_{\langle 13;25;47;68\rangle}$. The group

$$G_5 = \{\sigma \in G \mid x_{J_5}^{\sigma} = \pm x_{J_5}\}$$

is generated by (13)(24)(57)(68), (16)(25)(38)(47) and (17)(28)(34)(56). Since its order is 8, we have $[G : G_5] = 48$. We express $f_{J_5}^{\sigma}(y)$ in terms of theta constants for $\sigma \in G_5\backslash G$.

Theorem 4 *For* $\sigma \in G_5 \backslash G$, $f^{\sigma}_{J_5}(y)$ *is*

$$\begin{aligned}
&\frac{1}{4}(\varepsilon_0\vartheta_{\mu_0}+\varepsilon_1\vartheta_{\mu_1}+\varepsilon_2\vartheta_{\mu_2}+\varepsilon_3\vartheta_{\mu_3})(\varepsilon_0\vartheta_{\mu_0}+\varepsilon_1\vartheta_{\mu_1}-\varepsilon_2\vartheta_{\mu_2}-\varepsilon_3\vartheta_{\mu_3})\\
&\times(\varepsilon_0\vartheta_{\mu_0}-\varepsilon_1\vartheta_{\mu_1}+\varepsilon_2\vartheta_{\mu_2}-\varepsilon_3\vartheta_{\mu_3})(\varepsilon_0\vartheta_{\mu_0}-\varepsilon_1\vartheta_{\mu_1}-\varepsilon_2\vartheta_{\mu_2}+\varepsilon_3\vartheta_{\mu_3})\\
&=\frac{\varepsilon}{4}(\vartheta_{\mu_0}+\varepsilon'_1\vartheta_{\mu_1}+\varepsilon'_2\vartheta_{\mu_2}+\varepsilon'_3\vartheta_{\mu_3})(\vartheta_{\mu_0}+\varepsilon'_1\vartheta_{\mu_1}-\varepsilon'_2\vartheta_{\mu_2}-\varepsilon'_3\vartheta_{\mu_3})\\
&\times(\vartheta_{\mu_0}-\varepsilon'_1\vartheta_{\mu_1}+\varepsilon'_2\vartheta_{\mu_2}-\varepsilon'_3\vartheta_{\mu_3})(\vartheta_{\mu_0}-\varepsilon'_1\vartheta_{\mu_1}-\varepsilon'_2\vartheta_{\mu_2}+\varepsilon'_3\vartheta_{\mu_3}),
\end{aligned}$$

where $\varepsilon=\varepsilon_0^4$, ε_0, $\varepsilon'_j=\varepsilon_j/\varepsilon_0$ *and* $\mu_0, \ldots, \mu_3$ *are listed in the following table.*

#	$\sigma \in G_5\backslash G$	$x^{\sigma}_{J_5}$	$\varepsilon,\ \varepsilon_0; \varepsilon'_1,\ \varepsilon'_2,\ \varepsilon'_3$	μ_0 μ_1 μ_2 μ_3
58	id	$x_{\langle 13;25;47;68\rangle}$	$+,\ 1; 1,\ 1,\ 1$	111111, 000000
59	g_3	$x_{\langle 14;25;37;68\rangle}$	$+,\ 1; i,\ i,\ 1$	110000, 001111
60	g_5	$x_{\langle 13;26;47;58\rangle}$	$+,\ 1; 1,\ i,\ i$	001100, 110011
61	g_3g_5	$x_{\langle 14;26;37;58\rangle}$	$+,\ 1; i,\ -1,\ i$	000011, 111100
62	g_1	$x_{\langle 23;15;47;68\rangle}$	$+,\ 1; 1,\ 1,\ 1$	011111, 100000
63	g_1g_3	$x_{\langle 24;15;37;68\rangle}$	$+,\ 1; i,\ i,\ 1$	010000, 101111
64	g_1g_5	$x_{\langle 23;16;47;58\rangle}$	$+,\ 1; 1,\ i,\ i$	101100, 010011
65	$g_1g_3g_5$	$x_{\langle 24;16;37;58\rangle}$	$+,\ 1; i,\ -1,\ i$	100011, 011100
66	g_7	$x_{\langle 13;25;48;67\rangle}$	$+,\ 1; 1,\ 1,\ 1$	111110, 000001
67	g_3g_7	$x_{\langle 14;25;38;67\rangle}$	$+,\ 1; i,\ i,\ 1$	110001, 001110
68	g_5g_7	$x_{\langle 13;26;48;57\rangle}$	$+,\ 1; 1,\ i,\ i$	001101, 110010
69	$g_3g_5g_7$	$x_{\langle 14;26;38;57\rangle}$	$+,\ 1; i,\ -1,\ i$	000010, 111101
70	g_1g_7	$x_{\langle 23;15;48;67\rangle}$	$+,\ 1; 1,\ 1,\ 1$	011110, 100001
71	$g_1g_3g_7$	$x_{\langle 24;15;38;67\rangle}$	$+,\ 1; i,\ i,\ 1$	010001, 101110
72	$g_1g_5g_7$	$x_{\langle 23;16;48;57\rangle}$	$+,\ 1; 1,\ i,\ i$	101101, 010010
73	$g_1g_3g_5g_7$	$x_{\langle 24;16;38;57\rangle}$	$+,\ 1; i,\ -1,\ i$	100010, 011101

74	h_{34}	$-x_{\langle 13;27;45;68\rangle}$	$-,\ \zeta; 1,\ i,\ i$	110110, 000110
75	g_3h_{34}	$-x_{\langle 14;27;35;68\rangle}$	$-,\ \zeta; i,\ -1,\ i$	111010, 001010
76	g_5h_{34}	$-x_{\langle 13;28;45;67\rangle}$	$-,\zeta; 1,\ -1,\ -1$	000101, 110101
77	$g_3g_5h_{34}$	$-x_{\langle 14;28;35;67\rangle}$	$-,\ \zeta; i,\ -i,\ -1$	001001, 111001
78	g_1h_{34}	$-x_{\langle 23;17;45;68\rangle}$	$-,\ \zeta; 1,\ i,\ i$	010110, 100110
79	$g_1g_3h_{34}$	$-x_{\langle 24;17;35;68\rangle}$	$-,\ \zeta; i,\ -1,\ i$	011010, 101010
80	$g_1g_5h_{34}$	$-x_{\langle 23;18;45;67\rangle}$	$-,\ \zeta; 1,\ -1,\ -1$	100101, 010101
81	$g_1g_3g_5h_{34}$	$-x_{\langle 24;18;35;67\rangle}$	$-,\ \zeta; i,\ -i,\ -1$	101001, 011001
82	g_7h_{34}	$-x_{\langle 13;27;46;58\rangle}$	$-,\ \zeta; 1,\ -i,\ i$	111100, 000000
83	$g_3g_7h_{34}$	$-x_{\langle 14;27;36;58\rangle}$	$-,\ \zeta; i,\ 1,\ i$	110000, 001100
84	$g_5g_7h_{34}$	$-x_{\langle 13;28;46;57\rangle}$	$-,\ \zeta; 1,\ 1,\ -1$	001111, 110011
85	$g_3g_5g_7h_{34}$	$-x_{\langle 14;28;36;57\rangle}$	$-,\ \zeta; i,\ i,\ -1$	000011, 111111
86	$g_1g_7h_{34}$	$-x_{\langle 23;17;46;58\rangle}$	$-,\ \zeta; 1,\ -i,\ i$	011100, 100000
87	$g_1g_3g_7h_{34}$	$-x_{\langle 24;17;36;58\rangle}$	$-,\ \zeta; i,\ 1,\ i$	010000, 101100
88	$g_1g_5g_7h_{34}$	$-x_{\langle 23;18;46;57\rangle}$	$-,\ \zeta; 1,\ 1,\ -1$	101111, 010011
89	$g_1g_3g_5g_7h_{34}$	$-x_{\langle 24;18;36;57\rangle}$	$-,\ \zeta; i,\ i,\ -1$	100011, 011111
90	h_{24}	$x_{\langle 17;25;38;46\rangle}$	$+,\ 1; i,\ -1,\ -i$	111000, 000100
91	g_3h_{24}	$x_{\langle 18;25;37;46\rangle}$	$+,\ 1; -1,\ -i,\ -i$	110111, 001011
92	g_5h_{24}	$x_{\langle 17;26;38;45\rangle}$	$+,\ 1; i,\ -i,\ 1$	001011, 110111
93	$g_3g_5h_{24}$	$x_{\langle 18;26;37;45\rangle}$	$+,\ 1; -1,\ 1,\ 1$	000100, 111000
94	g_1h_{24}	$x_{\langle 27;15;38;46\rangle}$	$+,\ 1; i,\ -1,\ -i$	011000, 100100
95	$g_1g_3h_{24}$	$x_{\langle 28;15;37;46\rangle}$	$+,\ 1; -1,\ -i,\ -i$	010111, 101011
96	$g_1g_5h_{24}$	$x_{\langle 27;16;38;45\rangle}$	$+,\ 1; i,\ -i,\ 1$	101011, 010111
97	$g_1g_3g_5h_{24}$	$x_{\langle 28;16;37;45\rangle}$	$+,\ 1; -1,\ 1,\ 1$	100100, 011000
98	g_7h_{24}	$x_{\langle 17;25;48;36\rangle}$	$+,\ 1; i,\ -1,\ i$	100000, 010000
99	$g_3g_7h_{24}$	$x_{\langle 18;25;47;36\rangle}$	$+,\ 1; 1,\ -i,\ i$	101111, 011111
100	$g_5g_7h_{24}$	$x_{\langle 17;26;48;35\rangle}$	$+,\ 1; ,\ -i,\ -1$	010011, 100011
101	$g_3g_5g_7h_{24}$	$x_{\langle 18;26;47;35\rangle}$	$+,\ 1; -1,\ 1,\ -1$	011100, 101100
102	$g_1g_7h_{24}$	$x_{\langle 27;15;48;36\rangle}$	$+,\ 1; i,\ -1,\ i$	000000, 110000
103	$g_1g_3g_7h_{24}$	$x_{\langle 28;15;47;36\rangle}$	$+,\ 1; -1,\ -i,\ i$	001111, 111111
104	$g_1g_5g_7h_{24}$	$x_{\langle 27;1;48;35\rangle}$	$+,\ 1; i,\ -i,\ -1$	110011, 000011
105	$g_1g_3g_5g_7h_{24}$	$x_{\langle 28;16;47;35\rangle}$	$+,\ 1; -1,\ 1,\ -1$	111100, 001100

6. Some relations among $\vartheta_{a,b}(y)$ on $\mathbb{B}^5$

6.1. Quadratic relations among $\vartheta_{a,b}(y)$ on $\mathbb{B}^5$

Proposition 8 *The functions $\vartheta_{a,b}(y)$ on $\mathbb{B}^5$ satisfy the following quadratic relations*:

(1) *For $\mu_0, \ldots, \mu_3$ and ε in* Theorem 1*,*

$$\vartheta_{\mu_0}(y)\vartheta_{\mu_2}(y) = \varepsilon\vartheta_{\mu_1}(y)\vartheta_{\mu_3}(y);$$

(2) *For $\mu_0, \ldots, \mu_3$ and $\varepsilon_0, \ldots, \varepsilon_3$ in* Theorem 2*,*

$$\varepsilon_0\varepsilon_3\vartheta_{\mu_0}(y)\vartheta_{\mu_3}(y) = -\varepsilon_1\varepsilon_2\vartheta_{\mu_1}(y)\vartheta_{\mu_2}(y);$$

(3) *For $\mu_0, \ldots, \mu_3$ and ε' in* Theorem 3*,*

$$\vartheta^2_{\mu_0}(y) + \varepsilon'\vartheta^2_{\mu_1}(y) = \vartheta^2_{\mu_2}(y) + \varepsilon'\vartheta^2_{\mu_3}(y);$$

(4) *For $\mu_0, \ldots, \mu_3$ and $\varepsilon'_1, \varepsilon'_2, \varepsilon'_3$ in* Theorem 4*,*

$$\vartheta_{\mu_0}(y)\vartheta_{\mu_2}(y) = \varepsilon'_1{\varepsilon'_2}^{-1}\varepsilon'_3\vartheta_{\mu_1}(y)\vartheta_{\mu_3}(y).$$

Proof. (1) Act g_2 on the equality $\vartheta_{m_0}(y)\vartheta_{m_2}(y) = \vartheta_{m_1}(y)\vartheta_{m_3}(y)$ in Fact 4 (2). Then we have

$$\vartheta_{\mu_0}(y)\vartheta_{\mu_2}(y) = \vartheta_{\mu_1}(y)\vartheta_{\mu_3}(y)$$

for μ_j in Proposition 4. Act $\sigma \in G_2\backslash G$ on this equality.

(2) Act g_4 on the equality $\vartheta_{\mu_0}(y)\vartheta_{\mu_2}(y) = \vartheta_{\mu_1}(y)\vartheta_{\mu_3}(y)$ for μ_j in #2 of the table in Theorem 1. Then we have

$$\begin{aligned}(\vartheta_{\nu_0}(y) + \vartheta_{\nu_1}(y))(\vartheta_{\nu_2}(y) + \vartheta_{\nu_3}(y))&\\ = (\vartheta_{\nu_0}(y) - \vartheta_{\nu_1}(y))(\vartheta_{\nu_2}(y) - \vartheta_{\nu_3}(y)),&\end{aligned}$$

where ν_j are in #14 of the table in Theorem 2. This equality is equivalent to $\vartheta_{\nu_0}(y)\vartheta_{\nu_3}(y) = -\vartheta_{\nu_1}(y)\vartheta_{\nu_2}(y)$. Act $\sigma \in G_3\backslash G$ on this equality.

(3) Act g_6 on the equality $\vartheta_{\mu_0}(y)\vartheta_{\mu_2}(y) = \vartheta_{\mu_1}(y)\vartheta_{\mu_3}(y)$ for μ_j in #2 of the table in Theorem 1. Then we have

$$\vartheta^2_{\nu_0}(y) - \vartheta^2_{\nu_1}(y) = \vartheta^2_{\nu_2}(y) - \vartheta^2_{\nu_3}(y),$$

where ν_j are in #46 of the table in Theorem 3. Act $\sigma \in G_4\backslash G$ on this equality.

(4) Act g_4 on the equality

$$\vartheta^2_{\mu_0}(y) - \vartheta^2_{\mu_1}(y) = \vartheta^2_{\mu_2}(y) - \vartheta^2_{\mu_3}(y),$$

for μ_j in #46 of the table in Theorem 3. Then we have

$$\begin{aligned}&\big(\vartheta_{\nu_0}(y) + \vartheta_{\nu_1}(y) + \vartheta_{\nu_2}(y) + \vartheta_{\nu_3}(y)\big)\\ &\qquad\qquad \times\big(\vartheta_{\nu_0}(y) - \vartheta_{\nu_1}(y) + \vartheta_{\nu_2}(y) - \vartheta_{\nu_3}(y)\big)\\ &= \big(\vartheta_{\nu_0}(y) + \vartheta_{\nu_1}(y) - \vartheta_{\nu_2}(y) - \vartheta_{\nu_3}(y)\big)\\ &\qquad\qquad \times\big(\vartheta_{\nu_0}(y) - \vartheta_{\nu_1}(y) - \vartheta_{\nu_2}(y) + \vartheta_{\nu_3}(y)\big),\end{aligned}$$

where ν_j are in #58 of the table in Theorem 4. This equality is equivalent to $\vartheta_{\nu_0}(y)\vartheta_{\nu_2}(y) = \vartheta_{\nu_1}(y)\vartheta_{\nu_3}(y)$. Act $\sigma \in G_5\backslash G$ on this equality. □

6.2. Relations among f_J corresponding to those among x_J

The polynomials x_J satisfy the relations

$$x_{\langle j_1j_2;j_3j_4;j_5j_6;j_7j_8\rangle} - x_{\langle j_1j_3;j_2j_4;j_5j_6;j_7j_8\rangle} + x_{\langle j_1j_4;j_2j_3;j_5j_6;j_7j_8\rangle} = 0,$$

where $j_1 < j_2 < j_3 < j_4$. Thus $f_J(y)$ satisfy the same relations. For examples,

$$f_{\langle 12;34;56;78\rangle}(y) - f_{\langle 13;24;56;78\rangle}(y) + f_{\langle 14;23;56;78\rangle}(y) = 0,$$

and

$$f_{\langle 25;68;14;37\rangle}(y) - f_{\langle 26;58;14;37\rangle}(y) + f_{\langle 28;56;14;37\rangle}(y) = 0.$$

By #27, #59 and #61 in the tables in Theorems 1 and 4, the left hand side of the second equality becomes

$$2(\vartheta_{\mu_0}(y)\vartheta_{\mu_2}(y) - \vartheta_{\mu_1}(y)\vartheta_{\mu_3}(y))^2,$$

where

$$\begin{pmatrix}\mu_0\\ \mu_1\\ \mu_2\\ \mu_3\end{pmatrix} = \begin{pmatrix}111111, 000000\\ 110000, 001111\\ 001100, 110011\\ 000011, 111100\end{pmatrix}.$$

Note that

$$\vartheta_{\mu_0}(y)\vartheta_{\mu_2}(y) - \vartheta_{\mu_1}(y)\vartheta_{\mu_3}(y) = 0$$

by Proposition 8.

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