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On the Lefschetz module

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Abstract. Let G be a finite group. We define a Lefschets module $L(G, \Pi)$ which consists of equivalent classes of all Π-maps and prove that it is isomorphic to the Burnside module $\Omega(G, \Pi)$.

Key words: G-complex, G-map, G-poset, Lefschets module.

1. Introduction and statement of results

The purpose of this paper is to define a Lefschets module and to show that it is isomorphic to a Burnside module. R. Oliver and T. Petrie introduced the Burnside module $\Omega(G, \Pi)$ to solve a topological problem [10], where G is a finite group and Π is a partially ordered set with a G-action. This notion is a generalization of the Burnside ring $\Omega(G)$. The study of this direction is done by T. Yoshida [13], in which he presented an antecednt such that the free abelian group $\Omega(G, \mathfrak{X})$ has a ring structure. On the other hand, E. Laitinen and W. Lück defined the Lefschetz ring $L(G)$ [8]. It is well-known that the Burnside ring is isomorphic to the Lefschetz ring. In this paper, we study the Lefschets module $L(G, \Pi)$ which is a group version of the Lefschetz ring.

Our main theorem is the following.

Theorem 1.1 *Let* (Π, ρ) *be a* G*-poset. Then a map*

$$\overline{\varphi}\colon \Omega(G, \Pi) \longrightarrow L(G, \Pi)$$

given by $[(G_\alpha/\rho(\alpha))^+] \mapsto [\mathrm{id}_{(G_\alpha/\rho(\alpha))^+}]$ *is an group isomorphism, where* $\mathrm{id}_{(G_\alpha/\rho(\alpha))^+}\colon (G_\alpha/\rho(\alpha))^+ \to (G_\alpha/\rho(\alpha))^+$ *is the identity map on* $(G_\alpha/\rho(\alpha))^+$.

The proof of Theorem 1.1 is carried out in Section 4. We set

$$\begin{aligned} &S((G), \alpha) \\ &\quad = \{K \in S(G) \mid (K/\rho(\alpha)) \in \Phi(G_\alpha/\rho(\alpha)) \text{ and } K/\rho(\alpha) \text{ is cyclic}\}, \end{aligned}$$

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where $S(G)$ is the set of all subgroups of G, G_α denotes an isotropy subgroup of G at α ($\alpha \in \Pi$) and $\Phi(G)$ is the conjugacy class set of G. Applying the above theorem, we have a Burnside relation for the Lefschetz module.

Corollary 1.2 *Let α be an element of Π. Given a Π-map $f\colon X \to X$, one has*

$$\sum_{K \in S((G),\alpha)} \frac{|G_\alpha/\rho(\alpha)|}{|N_{G_\alpha/\rho(\alpha)}(K/\rho(\alpha))|} \cdot \phi(|K/\rho(\alpha)|) \cdot \overline{\Lambda}(f_\alpha^K) \equiv 0 \mod |G_\alpha/\rho(\alpha)|.$$

where $(|G_\alpha/\rho(\alpha)|)/(|N_{G_\alpha/\rho(\alpha)}(K/\rho(\alpha))|)$ is the order of $(G_\alpha/\rho(\alpha))/(N_{G_\alpha/\rho(\alpha)}(K/\rho(\alpha)))$ and $\phi(|K/\rho(\alpha)|)$ is the number of generators of the cyclic group $K/\rho(\alpha)$.

2. Notations and Preliminaries

Notations G always denotes a finite group. The set of all subgroups of G is denoted by $S(G)$. We regard $S(G)$ as a G-set via the action $(g, H) \mapsto gHg^{-1}$($g \in G$ and $H \in S(G)$) and as a partially ordered set via

$$H \leq K \text{ if and only if } H \supseteq K \quad (H, K \in S(G)).$$

By a G-complex we will mean a CW-complex X together with an action of G on X which permutes the cells. Thus we have for each $g \in G$ a homeomorphism $x \mapsto gx$ of X such that the image $g\sigma$ of any cell σ of X is again a cell. For example, if X is a simplicial complex on which G acts simplicially, then X is a G-complex.

Preliminaries. 1. A G-poset (=a partially ordered set with a G-action). Suppose that Π is a partially ordered set and G acts on it preserving the partial order. For any $\alpha \in \Pi$, we set

$$\Pi_\alpha = \{\beta \in \Pi \mid \beta \geq \alpha\}, \text{ and } G_\alpha = \{g \in G \mid g\alpha = \alpha\}.$$

In particular, G_α is called an *isotropy subgroup* of G at α. Let $\rho\colon \Pi \to S(G)$ be an order preserving G-map. A pair (Π, ρ) is called a *G-poset* if it is satisfying the following condition: for any $\alpha \in \Pi$,

$$\rho(\alpha) \lhd G_\alpha \quad \text{and} \quad \rho\colon \Pi_\alpha \to S(G)_{\rho(\alpha)} \quad \text{is injective.}$$

Note that $S(G)_{\rho(\alpha)} = S(\rho(\alpha)) \subset S(G_\alpha)$ and $G_\alpha \subset G_{\rho(\alpha)} = N_G(\rho(\alpha))$, the normalizer of $\rho(\alpha)$ in G. As example of a G-poset, consider $(S(G), \mathrm{id})$.

2. Burnside modules. References: Oliver-Petrie [10], Fujita [5]. Let a pair (Π, ρ) be a *G-poset*. A finite G-complex X with a base point $*$ is called a Π-*complex* if it is equipped with a specified set $\{X_\alpha \mid \alpha \in \Pi\}$ of subcomplexes X_α of X, satisfying the following four conditions:
(i) $* \in X_\alpha$,
(ii) $gX_\alpha = X_{g\alpha}$ for $g \in G$, $\alpha \in \Pi$,
(iii) $X_\alpha \subseteqq X_\beta$ if $\alpha \leqq \beta$ in Π, and
(iv) for any $H \in S(G)$,

$$X^H = \bigvee_{\alpha \in \Pi \text{ with } \rho(\alpha) = H} X_\alpha \quad \text{(the wedge sum of } X_\alpha\text{'s).}$$

On some examples of Π-complexes and its basic properties, see [4] for details. Let $\mathcal{F}$ denote the family of all Π-complexes and define the equivalence relation $\sim$ on $\mathcal{F}$ by

$$Z \sim W \text{ if and only if } \chi(Z_\alpha) = \chi(W_\alpha) \quad \text{for all } \alpha \in \Pi \quad (Z,\, W \in \mathcal{F})$$

where $\chi(Z_\alpha)$ is the Euler characteristic of Z_α.

The set $\Omega(G, \Pi) = \mathcal{F}/{\sim}$ is an abelian group via

$$[Z] + [W] = [Z \vee W] \quad (Z,\, W \in \mathcal{F}).$$

The unit element is the equivalence class of a point. We call $\Omega(G, \Pi)$ the *Burnside module associated with a G-poset* Π.

Let α be any element of Π and X a Π-complex. Construct a new space X' by attaching α-cells $G/\rho(\alpha) \times D^i$'s to X. Each attachment map

$$\varphi\colon G/\rho(\alpha) \times S^{i-1} \to X$$

is defined such that $\varphi(g\rho(\alpha) \times S^{i-1}) \subset X_{g\alpha}$. The space X' is equipped with a Π-complex structure:

$$(X')_\beta = X_\beta \cup \Big(\bigcup \{g\rho(\alpha) \times D^i \mid g\alpha \le \beta,\, g \in G\}\Big) \quad \text{for} \quad \beta \in \Pi.$$

Any Π-complex is constructed from one point by attaching α-cells for $\alpha \in \Pi$.

Proposition 2.1 ([10, Proposition 1.5]) *One has*

$$\Omega(G,\Pi) \cong \bigoplus_{\alpha\in\mathcal{A}} \mathbb{Z}.$$

Any finite Π*-complex* X *is equivalent in* $\Omega(G,\Pi)$ *to a sum of the form* $\sum_{\alpha\in\mathcal{A}} a_\alpha[(G/\rho(\alpha))^+]$*, and the map* $[X]\to\{a_\alpha\}_{\alpha\in\mathcal{A}}$ *defines the group isomorphism.*

3. Definitions

For a G-complex X and a self-map $f\colon X\to X$, we define

$$\Lambda(f) = \sum_{i=0}^{\infty}(-1)^i \operatorname{trace}[f_*\colon H_i(X;\mathbb{Q})\to H_i(X;\mathbb{Q})],$$

which is called the *Lefschetz number* of f. Remark that each homology group is a vector space over $\mathbb{Q}$; moreover, $f\colon X\to X$ is continuous, then $f_*\colon H_i(X;\mathbb{Q})\to H_i(X;\mathbb{Q})$ can be seen to be a linear transformation, and so the trace of f_* is now the usual trace of linear algebra. If a self-map f is an identity map, $\Lambda(f)$ is equal to the Euler characteristic of X. Here we set $\overline{\Lambda}(f)=\Lambda(f)-1$, which is called the *reduced Lefschetz number* of f. If $\{*\}\in X$ is a base point, and two base point preserving maps are homotopic, each reduced Lefschetz number coincides. Let A be a base pointed G-subcomplex of X. Then X/A is naturally equipped with a G-complex structure. Let f_A be the restriction of f to A and $f_{X/A}$ the quotient map $X/A\to X/A$. If the following diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f_A"] & X \arrow[r] \arrow[d, "f"] & X/A \arrow[d, "f_{X/A}"] \\ A \arrow[r] & X \arrow[r] & X/A \end{tikzcd}$$

commutes, then we have $\overline{\Lambda}(f)=\overline{\Lambda}(f_A)+\overline{\Lambda}(f_{X/A})$. Moreover let $\sum f$ be the suspension map of f. An easy computation shows that $\overline{\Lambda}(\sum f)=-\overline{\Lambda}(f)$.

Let X, Y be Π-complexes. The map f is called a Π-*map* if a map $f\colon X\to Y$ be a base point preserving G-map such that

$$f(X_\alpha)\subset Y_\alpha \quad \text{for all } \alpha\in\Pi.$$

We denote by f_α the restriction of f to X_α. For self Π-maps $f\colon X\to X$,

$g\colon Y \to Y$, we define a equivalence relation by

$$\overline{\Lambda}(f_\alpha) = \overline{\Lambda}(g_\alpha) \quad \text{for all } \alpha \in \Pi.$$

Let $\mathcal{F}_{\text{map}}$ denote the set of all self Π-maps. Then we let $L(G, \Pi)$ the quotient set of $\mathcal{F}_{\text{map}}$ by the equivalence relation. The quotient set is an abelian group via

$$[f] + [g] = [f \vee g] \quad (f, g \in \mathcal{F}_{\text{map}}),$$

where a map $f \vee g$ is the standard wedge map. We call $L(G, \Pi)$ the *Lefschetz module associated with a G-poset* Π.

4. Proof of the main theorem

We need the following lemma to prove Theorem 1.1.

Lemma 4.1 *Let X be a Π-complex with a subcomplex A and $f\colon X \to X$ be a Π-map with $f(A) \subset A$. Then for the commutative diagram*

$$\begin{array}{ccccc} A & \longrightarrow & X & \longrightarrow & X/A \\ \downarrow{\scriptstyle f_A} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_{X/A}} \\ A & \longrightarrow & X & \longrightarrow & X/A, \end{array}$$

$[f] = [f_A] + [f_{X/A}] \in L(G, \Pi)$, *where f_A is the restriction of f to A and $f_{X/A}$ is the quotient map $X/A \to X/A$.*

Proof. Let α be any element of Π. Consider the following cellular chain complex

$$0 \longrightarrow \overline{C_*}(A_\alpha) \longrightarrow \overline{C_*}(X_\alpha) \longrightarrow \overline{C_*}(X_\alpha/A_\alpha) \longrightarrow 0.$$

Since each term of this chain complex is a vevtor space over $\mathbb{Q}$, it splits, therefore $\overline{C_*}(X_\alpha) \cong \overline{C_*}(A_\alpha) \oplus \overline{C_*}(X_\alpha/A_\alpha)$. To calculate trace$(f_\alpha)_\sharp$, consider the following diagram:

$$\begin{array}{ccccc} \overline{C_*}(X_\alpha) & \cong & \overline{C_*}(A_\alpha) & \oplus & \overline{C_*}(X_\alpha/A_\alpha) \\ \downarrow{\scriptstyle (f_\alpha)_\sharp} & & \downarrow{\scriptstyle (f_{A_\alpha})_\sharp} & & \downarrow{\scriptstyle (f_{X_\alpha/A_\alpha})_\sharp} \\ \overline{C_*}(X_\alpha) & \cong & \overline{C_*}(A_\alpha) & \oplus & \overline{C_*}(X_\alpha/A_\alpha). \end{array}$$

Let (a, b) be an element of $\overline{C_*}(A_\alpha) \oplus \overline{C_*}(X_\alpha/A_\alpha)$ and the image of (a, b) by

$(f_\alpha)_\sharp$ be $(f_1(a, b), f_2(a, b))$. Then

$$\begin{pmatrix} f_1(a, b) \\ f_2(a, b) \end{pmatrix} = \begin{pmatrix} (f_{A_\alpha})_\sharp & * \\ 0 & (f_{X_\alpha/A_\alpha})_\sharp \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

Hence we have $\mathrm{trace}(f_\alpha)_\sharp = \mathrm{trace}(f_{A_\alpha})_\sharp + \mathrm{trace}(f_{X_\alpha/A_\alpha})_\sharp$. By [12, Lemma 9.18], $\overline{\Lambda}(f_\alpha) = \overline{\Lambda}(f_{A_\alpha}) + \overline{\Lambda}(f_{X_\alpha/A_\alpha})$. Thus we get the assertion. □

Proof of Theorem 1.1. It is obvious that $\overline{\varphi}$ is a group homomorphism. Moreover since

$$\overline{\chi}((G_\alpha/\rho(\alpha))^+) = \chi(G_\alpha/\rho(\alpha)) = \Lambda(\mathrm{id}_{G_\alpha/\rho(\alpha)}) = \overline{\Lambda}(\mathrm{id}_{(G_\alpha/\rho(\alpha))^+}),$$

we easily verify that $\overline{\varphi}$ is injective. We shall show the surjectivity of $\overline{\varphi}$. For any $[f\colon X \to X] \in L(G, \Pi)$, we want to show $\mathrm{Im}\,\overline{\varphi} \ni [f]$. We proceed by induction on the number of G-cell. The Π-complex X is composed with the cell structure:

$$X = X_0 \cup_{\varphi_0} \left(\coprod_{i_0 \in I_0} (G/\rho(\alpha) \times D^0)_{i_0} \right) \cup \cdots \cup_{\varphi_n} \left(\coprod_{i_n \in I_n} (G/\rho(\alpha) \times D^n)_{i_n} \right),$$

where X_0 is a subcomplex of X. Moreover we may suppose that $\rho(\alpha)$ is a minimum isotropy subgroup of $X \backslash \{*\}$ and $X_0 \backslash \{*\}$ contains no cells the isotropy type of which is $(\rho(\alpha))$. It then follows that the Π-map $f\colon X \to X$ satisfies $f(X_0) \subset X_0$. Let f_0 be the restriction of f to X_0 and f' be the quotient map $X/X_0 \to X/X_0$. By Lemma 2.1, we have $[f] = [f_0] + [f'] \in L(G, \Pi)$. In the case of $X_0 \backslash \{*\} \neq \emptyset$, the assertion is already done by induction. As for the case where $X_0 \backslash \{*\} = \emptyset$, namely, $X = \{*\} \cup \{$the cell's of its isotropy type $(\rho(\alpha))\}$, X has the following cell structure:

$$X = X^{n-1} \cup_{\varphi_n} \left(\coprod_{i_n \in I_n} (G/\rho(\alpha) \times D^n)_{i_n} \right),$$

where X^{n-1} is the $(n-1)$-skeleton of X. By considering the map on the homotopic level, we may assume that the Π-map f is a cellular map, so that, $f(X^{n-1}) \subset X^{n-1}$ (This is done by the cellular approximation theorem). If $X^{n-1} \neq \{*\}$, by Lemma 4.1 and induction, we have $[f] \in \mathrm{Im}\,\overline{\varphi}$. Finally we consider the case of $X^{n-1} = \{*\}$. (Remark that $[f] = [f_{X^{n-1}}] +$

$[f_{X/X^{n-1}}] \in L(G, \Pi)$ in this case, but we can not prove the surjectivity of $\overline{\varphi}$ by induction.) Then X is expressible as a wegde sum of some suspensions:

$$X = \bigvee_{i\in I} \left((G/\rho(\alpha))^+ \wedge S^n\right)_i = \bigvee_{i\in I} \left(\sum\nolimits^n (G/\rho(\alpha))^+\right)_i,$$

where $\sum^n$ is the n-th suspension operator. Next we compute the chain complex of X.

Claim 4.2

$$\overline{C_n}(X) = \bigoplus_{i\in I} (\mathbb{Q}[G/\rho(\alpha)])_i$$

Proof. We now compute:

$$\begin{aligned}
\overline{C_n}(X) &= \overline{C_n}\left(\bigvee_{i\in I}\left(\sum\nolimits^n (G/\rho(\alpha))^+\right)\right) \\
&= \bigoplus_{i\in I} \overline{C_0}((G/\rho(\alpha))^+) \\
&= \bigoplus_{i\in I} \overline{H_0}\left((G/\rho(\alpha))^+; \mathbb{Q}\right) \\
&= \bigoplus_{i\in I} (\mathbb{Q}[G/\rho(\alpha)])_i,
\end{aligned}$$

where each $(\mathbb{Q}[G/\rho(\alpha)])_i$ is the copy of $\mathbb{Q}[G/\rho(\alpha)]$. □

Let $f_\sharp$ be a self-chain map on the cellular chain complex $\overline{C_*}(X)$, where $f_{i\sharp}\colon \overline{C_i}(X) \to \overline{C_i}(X)$ is the i-th term of the chain map $f_\sharp$. Note that each $\overline{C_i}(X)$ is a finite-dimensional vector space over $\mathbb{Q}$ and the map $f_{i\sharp}$ is a linear transformation then a choice of basis of $\overline{C_i}(X)$ associates a square matrix A to $f_{i\sharp}$. Let m be the order of the index set I. Let f_{ij} be a linear transformation from $(\mathbb{Q}[G/\rho(\alpha)])_i$ to $(\mathbb{Q}[G/\rho(\alpha)])_j$. Then there exists the following diagram:

$$\begin{array}{ccccc}
\overline{C_n}(X) & \cong & (\mathbb{Q}[G/\rho(\alpha)])_1 \oplus \cdots & & \cdots \oplus (\mathbb{Q}[G/\rho(\alpha)])_m \\
\downarrow{\scriptstyle f_{n\sharp}} & & \downarrow{\scriptstyle f_{11}} & & \downarrow{\scriptstyle f_{mm}} \\
\overline{C_n}(X) & \cong & (\mathbb{Q}[G/\rho(\alpha)])_1 \oplus \cdots & & \cdots \oplus (\mathbb{Q}[G/\rho(\alpha)])_m.
\end{array}$$

If $\{x_{i_1}, \ldots, x_{i_n}\}$ is a basis of $(\mathbb{Q}[G/\rho(\alpha)])_i$, one extends it to a basis of

$\overline{C_n}(X)$. The matrix A of $f_{n\sharp}$ with respect to the extended basis is

$$\begin{bmatrix} A_{11} & * & * & * \\ & A_{22} & * & * \\ & & \ddots & * \\ 0 & & & A_{nn} \end{bmatrix},$$

where A_{ii} is the matrix of f_{ii} with respect to $\{x_{i_1}, \ldots, x_{i_n}\}$. Hence we have that $\mathrm{trace}(f_{n\sharp}) = \sum_{i=1}^{m} \mathrm{trace}(f_{ii})$, and so $\overline{\Lambda}(f) = (-1)^n \sum_{i=1}^{m} \mathrm{trace}(f_{ii})$. For each $i = 1, \ldots, m$, we denote by A_i the copy of the subset $\bigvee_{g\in G}(g\rho(\alpha) \wedge S^n)$ of X. A new map g_i is the composition:

$$A_i \xrightarrow{i} X \xrightarrow{f} X \xrightarrow{q_i} A_i,$$

which is a Π-map. In this diagram, $i\colon A_i \to X$ denotes an inclusion map and $q_i\colon X \to A_i$ is the i-th term projection map. It then follows that $g_{i\sharp} = f_{ii}\colon \overline{C_n}(A_i) \to \overline{C_n}(A_i)$. Therefore $\overline{\Lambda}(f) = (-1)^n \sum_{i=1}^{m} \mathrm{trace}(g_{i\sharp})$. By restricting f to X_α, we have $\overline{\Lambda}(f_\alpha) = \sum_{i=1}^{m} \overline{\Lambda}((g_i)_\alpha)$, and so $[f] = \sum_{i=1}^{m}[g_i]$. We show that each $[g_i] \in \overline{\varphi}(\Omega(G, \Pi))$. We also denote by f the Π-map $\sum^n(G/\rho(\alpha))^+ \to \sum^n(G/\rho(\alpha))^+$ without confusion. The desired map f_i is the composition:

$$g_i\rho(\alpha)/\rho(\alpha) \xrightarrow{j} \sum\nolimits^n (G/\rho(\alpha))^+ \xrightarrow{f} \sum\nolimits^n (G/\rho(\alpha))^+ \xrightarrow{q_i} g_i\rho(\alpha)/\rho(\alpha),$$

which is not a Π-map. In this diagram, $j\colon g_i\rho(\alpha)/\rho(\alpha) \to \sum^n(G/\rho(\alpha))^+$ denotes an inclusion map and $q_i\colon \sum^n(G/\rho(\alpha))^+ \to g_i\rho(\alpha)/\rho(\alpha)$ is the i-th term projection map. Let g_1 be the unit element of G. Now consider the following diagram:

$$\begin{array}{ccc} g_1\rho(\alpha)/\rho(\alpha) \times S^n & \xrightarrow{f_1} & g_1\rho(\alpha)/\rho(\alpha) \times S^n \\ {\scriptstyle g_i}\big\downarrow & & \big\downarrow{\scriptstyle g_i} \\ g_i\rho(\alpha)/\rho(\alpha) \times S^n & \xrightarrow[f_i]{} & g_i\rho(\alpha)/\rho(\alpha) \times S^n. \end{array}$$

Here the symbol f_i denotes a map from $g_i\rho(\alpha)/\rho(\alpha) \times S^n$ to itself and g_i is the left translation by g_i. This diagram is obviously commutative. Moreover g_i is a homeomorphism. Hence $\overline{\Lambda}(f_i) = \overline{\Lambda}(f_1)$. Let β be any element of Π.

Since $\sum^n (G/\rho(\alpha))^+_\beta = \vee\{g_i\rho(\alpha)/\rho(\alpha) \times S^n \mid g_i\alpha \le \beta\}$, we see that

$$\begin{aligned}\overline{\Lambda}(f_\beta) &= \sum_i \overline{\Lambda}(f_i) \quad \text{(the index i satisfies $g_i\alpha \le \beta$)} \\ &= \sum_i \overline{\Lambda}(f_1) \\ &= \overline{\Lambda}(f_1)|(G/\rho(\alpha))_\beta| \\ &= \overline{\Lambda}(f_1)\overline{\chi}((G/\rho(\alpha))^+_\beta),\end{aligned}$$

and hence that $[f] \in \operatorname{Im}\overline{\varphi}$. This concludes the proof. □

We have a Burnside relation for the Lefschetz module. We set

$$\begin{aligned}&S((G),\alpha) \\ &\quad = \{K \in S(G) \mid (K/\rho(\alpha)) \in \Phi(G_\alpha/\rho(\alpha)) \text{ and } K/\rho(\alpha) \text{ is cyclic}\}.\end{aligned}$$

From Theorem 1.1, we have the following corollary (see [5, Theorem 1.6]).

Corollary 1.2 *Let α be an element of Π. Given a Π-map $f\colon X \to X$, one has*

$$\sum_{K\in S((G),\alpha)} \frac{|G_\alpha/\rho(\alpha)|}{|N_{G_\alpha/\rho(\alpha)}(K/\rho(\alpha))|} \cdot \phi(|K/\rho(\alpha)|) \cdot \overline{\Lambda}(f^K_\alpha) \equiv 0 \mod |G_\alpha/\rho(\alpha)|.$$

where $|G_\alpha/\rho(\alpha)|/|N_{G_\alpha/\rho(\alpha)}(K/\rho(\alpha))|$ is the order of $G_\alpha/\rho(\alpha)/N_{G_\alpha/\rho(\alpha)}(K/\rho(\alpha))$ and $\phi(|K/\rho(\alpha)|)$ is the number of generators of the cyclic group $K/\rho(\alpha)$.

Proof. From Theorem 1.1, the group $L(G,\Pi)$ is generated by the isomorphism classes of the identity maps of the form $[\mathrm{id}_{(G_\alpha/\rho(\alpha))^+}]$. It is sufficient to prove for $\mathrm{id}_X\colon X \to X$ for a Π-complex X. Then clearly $\overline{\chi}(X^K_\alpha) = \overline{\Lambda}(\mathrm{id}_{X^K_\alpha})$, so that we have the desired result. □

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