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Characterization of connection coefficients for hypergeometric systems

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Introduction

In the paper [4] it was shown that a connection problem for the hypergeometric system of linear differential equations

$$(0.1)\qquad (t-B)\frac{dX}{dt} = AX,$$

where X is an n-dimensional column vector, $B=\mathrm{diag}\,[\lambda_1, \lambda_2,\dots, \lambda_n]$ and $A \in M_n(\boldsymbol{C})$, can be solved by the global analysis of the system of linear difference equations

$$(0.2)\qquad (B-\lambda)(z+1)G(z+1) = (z-A)G(z),$$

which determines coefficients of power series solutions of (0.1). The method of [4] was effectively applied to solve the connection problem for a system of linear differential equations corresponding to a one-dimensional section of Appell's $F_3(\alpha, \alpha', \beta, \beta', \gamma; x, y)$ in [8]. In this paper, dealing with the complete solution of a connection problem for (0.1) with A which is diagonalizable and has only *two* distinct eigenvalues, we shall clear up the relation between solutions of (0.1) and (0.2), and the structure of connection coefficients in more detail.

In Section 1 we shall be concerned with power series solutions of (0.1) near singularities. In Section 2 we study the system (0.2). In Section 3 we analyze Barnes-integral representations of solutions of (0.1) and characterize the connection coefficients between solutions of (0.1) near a finite singularity and near the infinity. It will be shown that these coefficients are given by solutions of an $(n-1)$-dimensional hypergeometric system obtained from (0.1). In the last section, §4, we deal with some examples.

As for other investigations related to this paper, we refer the reader to [1], [3], [6] and [7].

Hereafter we assume that the diagonal elements λ_j $(j=1,\dots, n)$ of B are all distinct and $A=[a_{jk}]$ is similar to

$$\mathrm{diag}\,[\overbrace{\mu_1,\dots, \mu_1}^{n_1}, \overbrace{\mu_2,\dots, \mu_2}^{n_2}] \quad (n_1+n_2=n).$$

Denoting a_{jj} by ν_j $(j=1,\dots, n)$, we furthermore assume the following:

[A_0] No three λ_j lie on a line.
[A_1] None of the quantities

$$\nu_j \quad (j=1,\dots, n), \quad \mu_1 - \mu_2$$

is an integer.
This implies that there appear no logarithmic solutions.
[A_2] None of the quantities

$$\mu_k \quad (k=1, 2), \quad \mu_k - \nu_j \quad (k=1, 2; j=1,\dots, n)$$

is an integer.
This condition is related to the reducibility of (0.1) (see [1] and [6]).

§1. Power series solutions of (0.1)

Since (0.1) is a Fuchsian system with regular singularities $t=\lambda_j$ $(j=1,\dots, n)$ and ∞, (0.1) has convergent power series solutions at each singularity. Let $t=\lambda$ be one of the finite singularities. For brevity of the notation we may assume that $\lambda_n=\lambda$. This can always be done by interchanging the components in (0.1) by the linear transformation $X=PY$ with a permutation matrix P. Furthermore, without loss of generality, we may assume that

$$\arg(\lambda_1-\lambda) < \arg(\lambda_2-\lambda) < \cdots < \arg(\lambda_{n-1}-\lambda) < \arg(\lambda_1-\lambda) + 2\pi$$

by the assumption [A_0].

Near $t=\lambda$ $(\lambda=\lambda_n)$, there exists a non-holomorphic solution of (0.1) of the form

$$\hat{X}(t) = (t-\lambda)^{\nu} \sum_{m=0}^{\infty} \hat{G}(m)(t-\lambda)^m \quad (|t-\lambda|<R), \tag{1.1}$$

where $\nu=\nu_n$ and $R=\min\{|\lambda_j-\lambda|; j=1,\dots, n-1\}$. The coefficient vectors $\hat{G}(m)$ $(m\geqslant 0)$ are determined uniquely up to a constant factor by the system of linear difference equations

$$\begin{cases} (B-\lambda)(m+1+\nu)\hat{G}(m+1) = (m+\nu-A)\hat{G}(m) \quad (m\geqslant 0) \\ (B-\lambda)\nu\hat{G}(0) = 0. \end{cases} \tag{1.2}$$

Besides there exist $n-1$ holomorphic solutions of (0.1) of the form

$$X(t) = \sum_{m=0}^{\infty} G(m)(t-\lambda)^m,$$

where the coefficient vectors are characterized by the system of linear difference equations replacing ν by 0 in (1.2), i.e., (0.2).

Near $t=\infty$, there exist n linearly independent solutions of (0.1) of the form

$$(1.3)\quad Y^{kl}(t) = (t-\lambda)^{\mu_k} \textstyle\sum_{r=0}^{\infty} H^{kl}(r)(t-\lambda)^{-r} \quad (|t-\lambda|>R';\ k=1,\ 2;\ l=1,\dots,\ n_k),$$

where $R'=\max\{|\lambda_j-\lambda|;\ j=1,\dots,\ n-1\}$. The coefficient vectors $H^{kl}(r)$ $(r\geqslant 1)$ are determined by the system of linear difference equations

$$(1.4)\qquad (r-\mu_k+A)H^{kl}(r) = (B-\lambda)(r-1-\mu_k)H^{kl}(r-1) \quad (r\geqslant 1)$$

subject to the initial condition

$$(1.5)\qquad (A-\mu_k)H^{kl}(0) = 0.$$

For each k $(=1,\ 2)$, since rank $(A-\mu_k)\doteq n-n_k$, we can choose $H^{kl}(0)$ $(l=1,\dots,\ n_k)$ as n_k linearly independent eigenvectors of A corresponding to μ_k.

§ 2. Solutions of (0.2)

In this section, for Barnes-integral representation of $\hat{X}(t)$ and $X(t)$, we consider the system of linear difference equations (0.2), where z is a complex variable and $\lambda=\lambda_n$.

We first show the following lemma.

LEMMA.

$$(z-A)^{-1} = \frac{1}{(z-\mu_1)(z-\mu_2)}(z-\mu_1-\mu_2+A).$$

PROOF. Since A is similar to diag $[\mu_1,\dots,\ \mu_1,\ \mu_2,\dots,\ \mu_2]$, we have

$$(2.1)\qquad (\mu_1-A)(\mu_2-A) = 0.$$

Then we have

$$(z-\mu_1-\mu_2+A)(z-A) = (z-\mu_1)(z-\mu_2)I,$$

which proves the Lemma. □

Using this formula, we can rewrite (0.2) as

$$(2.2)\qquad \frac{z+1}{(z-\mu_1)(z-\mu_2)}(z-\mu_1-\mu_2+A)(B-\lambda)G(z+1) = G(z).$$

Then, putting

$$(2.3)\qquad G(z) = \frac{\Gamma(z-\mu_1)\Gamma(z-\mu_2)}{\Gamma(z+1)}\bar{G}(z),$$

we can transform (2.2) into

$$(z-\mu_1-\mu_2+A)(B-\lambda)\bar{G}(z+1) = \bar{G}(z),$$

which is rewritten in the form

$$(2.4) \qquad (z-\mu_1-\mu_2+\tilde{B}\tilde{A}_{11}\tilde{B}^{-1})\tilde{G}(z+1) = \tilde{B}\tilde{G}(z)$$

and

$$(2.5) \qquad \tilde{A}_{21}\tilde{B}^{-1}\tilde{G}(z+1) = \tilde{g}(z),$$

where

$$\tilde{A}_{11} = \begin{bmatrix} a_{11} \cdots a_{1n-1} \\ \cdots\cdots \\ a_{n-11} \cdots a_{n-1n-1} \end{bmatrix}, \quad \tilde{A}_{12} = \begin{bmatrix} a_{1n} \\ \vdots \\ a_{n-1n} \end{bmatrix},$$

$$\tilde{A}_{21} = [a_{n1} \cdots a_{nn-1}], \qquad \tilde{A}_{22} = [a_{nn}],$$

$$\tilde{B} = \mathrm{diag}\,[\tilde{\lambda}_1, \dots, \tilde{\lambda}_{n-1}], \quad \tilde{\lambda}_j = (\lambda_j - \lambda)^{-1} \quad (j=1,\dots, n-1),$$

and $\tilde{G}(z)$ and $\tilde{g}(z)$ are the vectors which consist of the first $n-1$ components and the last one of $\bar{G}(z)$, respectively. Namely, as the system of difference equations, (2.4) is an essential part. We observe that the dimension of (2.4) is equal to $n-1$.

2.1. Principal solutions in the right half plane

By the Mellin-transformation

$$\tilde{G}(z) = \frac{1}{\Gamma(z-\rho)} \int t^{z-\rho-1}\tilde{\Psi}(t)dt,$$

ρ being a complex parameter, (2.4) is transformed into an $(n-1)$-dimensional hypergeometric system

$$(2.6) \qquad (t-\tilde{B})\frac{d\tilde{\Psi}}{dt} = (\rho-1-\mu_1-\mu_2+\tilde{B}\tilde{A}_{11}\tilde{B}^{-1})\tilde{\Psi}.$$

In order to define the arguments of $t-\tilde{\lambda}_j$, we put

$$\mathscr{D} = \boldsymbol{C} \setminus \cup_{j=1}^{n-1} \{s\tilde{\lambda}_j;\ s \geqslant 1\}.$$

Then, for $t \in \mathscr{D}$, we take the value between $\arg\tilde{\lambda}_j - 2\pi$ and $\arg\tilde{\lambda}_j$ ($\arg\tilde{\lambda}_j = -\arg(\lambda_j - \lambda)$) for $\arg(t-\tilde{\lambda}_j)$ $(j=1,\dots, n-1)$. Let $\tilde{\Psi}_j(\rho;\, t)$ $(j=1,\dots, n-1)$ be solutions of (2.6) in $\mathscr{D}$ which are developed as

$$\tilde{\Psi}_j(\rho;\, t)$$
$$= (e^{\pi i}(t-\tilde{\lambda}_j))^{\rho-1-\mu_1-\mu_2+\nu_j} \sum_{m=0}^{\infty} \frac{1}{\Gamma(\rho-\mu_1-\mu_2+\nu_j+m)} \tilde{C}_j(m)(t-\tilde{\lambda}_j)^m$$

near $t=\tilde{\lambda}_j$, where $\tilde{C}_j(m)$ $(m \geqslant 0)$ are the vectors determined uniquely by the systems of linear difference equations

$$\begin{cases} (\tilde{B}-\tilde{\lambda}_j)\tilde{C}_j(m+1) = (m+\nu_j-\tilde{B}\tilde{A}_{11}\tilde{B}^{-1})\tilde{C}_j(m) \quad (m \geqslant 0) \\ \tilde{C}_j(0) = \tilde{e}_j, \end{cases}$$

$\tilde{e}_j$ being the j-th unit $(n-1)$-vector $(j=1,\dots,n-1)$. Using these $\tilde{\Psi}_j(\rho;\, t)$, we define

$$\tilde{G}_j(z) = \frac{1}{\Gamma(z-\rho)} \int_0^{\tilde{\lambda}_j} t^{z-\rho-1}\, \tilde{\Psi}_j(\rho;\, t)dt \quad (j=1,\dots,n-1) \tag{2.7}$$

for z satisfying $\mathrm{Re}(z-\rho)>0$, where the path of integration is the straight line from 0 to $\tilde{\lambda}_j$, $\arg t=\arg \tilde{\lambda}_j$ and the parameter ρ is selected as $\mathrm{Re}(\rho-1-\mu_1-\mu_2+\nu_j)>0$ (for every $j=1,\dots,n-1$). We here observe that, under the assumption $[\mathrm{A}_0]$, $\arg \tilde{\lambda}_i \neq \arg \tilde{\lambda}_j \pmod{2\pi}$ $(i\neq j)$. Concerning these $\tilde{G}_j(z)$, we obtain the following

PROPOSITION. (i) *$\tilde{G}_j(z)$ is holomorphic and is a solution of* (2.4) *in* $\mathrm{Re}(z-\rho)>0$, *therefore an entire solution* $(j=1,\dots,n-1)$.

(ii) $\tilde{G}_j(z)=\tilde{\Psi}_j(z;\, 0)$ *for every* $z\in \boldsymbol{C}$. *Therefore* $\tilde{G}_j(z)$ *does not depend on* ρ $(j=1,\dots,n-1)$.

(iii) *As* $z\to\infty$, $|\arg z|<\pi/2+\varepsilon$ *for sufficiently small* $\varepsilon>0$,

$$\tilde{G}_j(z) \sim \tilde{F}_j(z),$$

where $\tilde{F}_j(z)$ is a formal solution of (2.4) *of the form*

$$\begin{cases} \tilde{F}_j(z) = \Gamma(z)^{-1}\, \tilde{\lambda}_j^{z-1-\mu_1-\mu_2+\nu_j} z^{\mu_1+\mu_2-\nu_j} \sum_{r=0}^{\infty} \tilde{f}_j(r) z^{-r}, \\ \tilde{f}_j(0) = \tilde{e}_j \quad (j=1,\dots,n-1). \end{cases}$$

PROOF. (i) It is trivial that $\tilde{G}_j(z)$ is holomorphic and is a solution of (2.4) in $\mathrm{Re}(z-\rho)>0$. Then $\tilde{G}_j(z)$ is analytically continued into the left half plane by the equation (2.4).

(ii) For $\mathrm{Re}(z-\rho)>0$ and $t\in\mathscr{D}$, we put

$$\tilde{G}_j(z;\, t) = \frac{1}{\Gamma(z-\rho)} \int_t^{\tilde{\lambda}_j} (\tau-t)^{z-\rho-1}\, \tilde{\Psi}_j(\rho;\, \tau)d\tau,$$

where the path of integration is a curve in $\mathscr{D}$ from t to $\tilde{\lambda}_j$ and $\arg(\tau-t)$ is taken continuously along the path of integration as $\arg(\tilde{\lambda}_j-t)=\arg(t-\tilde{\lambda}_j)+\pi$ at the endpoint $\tilde{\lambda}_j$. For $|t-\tilde{\lambda}_j|$ sufficiently small, we have

$$\begin{aligned} &\tilde{G}_j(z;\, t) \\ &= (e^{\pi i}(t-\tilde{\lambda}_j))^{z-1-\mu_1-\mu_2+\nu_j} \sum_{m=0}^{\infty} \frac{1}{\Gamma(z-\mu_1-\mu_2+\nu_j+m)}\, \tilde{C}_j(m)\, (t-\tilde{\lambda}_j)^m \\ &= \tilde{\Psi}_j(z;\, t) \end{aligned}$$

by termwise integration. Since both $\tilde{G}_j(z;\, t)$ and $\tilde{\Psi}_j(z;\, t)$ are holomorphic in $\mathscr{D}$, we obtain $\tilde{G}_j(z;\, t) = \tilde{\Psi}_j(z;\, t)$ for $t \in \mathscr{D}$ and $\mathrm{Re}(z-\rho) > 0$, in particular

$$\tilde{G}_j(z;\, 0) = \tilde{G}_j(z) = \tilde{\Psi}_j(z;\, 0).$$

Moreover, since both $\tilde{G}_j(z)$ and $\tilde{\Psi}_j(z;\, 0)$ are holomorphic in $\boldsymbol{C}$, we obtain the above formula for every $z \in \boldsymbol{C}$.

(iii) This asymptotic expansion for $|\arg z| < \pi/2$ is immediately seen by applying Watson's lemma (e.g. [5; p. 4]) to the integral representation of $\tilde{G}_j(z)$ with $t = \tilde{\lambda}_j e^{-\tau}$, i.e.,

$$\tilde{G}_j(z) = \frac{\tilde{\lambda}_j^{z-\rho}}{\Gamma(z-\rho)} \int_0^\infty e^{-(z-\rho)\tau} \tilde{\Psi}_j(\rho;\, \tilde{\lambda}_j e^{-\tau}) d\tau$$

and by using Stirling's formula. Then, putting $\tau = e^{i\theta}\zeta$ for $|\theta|$ sufficiently small, we have the asymptotic expansion in the enlarged half plane as stated above (cf. [5; p. 6 Lemma 2]). $\square$

REMARK. (i) As to the holomorphy of $\tilde{\Psi}_j(z;\, 0)$ in $z \in \boldsymbol{C}$, see [7].

(ii) According to the general theory of difference equations, the fundamental set of solutions of (2.4) characterized by the asymptotic behavior in (iii) is uniquely determined.

Now we shall denote by $G_j(z)$ the solution of (0.2) constructed by $\tilde{G}_j(z)$ with (2.5) and (2.3) ($j = 1,\dots, n-1$), which will be used for Barnes-integral representation of the holomorphic solutions of (0.1) near $t = \lambda$ in §3.

2.2. Solution having zeros in the left half plane

For the solution of (1.2) we next consider a solution of (2.4) which has zeros in the left half plane. We first calculate Casorati's determinant of $\tilde{G}_j(z)$ ($j = 1,\dots, n-1$). Since from the Lemma we have

$$\begin{bmatrix} \dfrac{1}{(z-\mu_1)(z-\mu_2)}(z-\mu_1-\mu_2+\tilde{A}_{11}) & \dfrac{1}{(z-\mu_1)(z-\mu_2)}\tilde{A}_{12} \\ 0 & 1 \end{bmatrix}(z-A)$$
$$= \begin{bmatrix} I_{n-1} & 0 \\ -\tilde{A}_{21} & z-\tilde{A}_{22} \end{bmatrix},$$

I_{n-1} being the $(n-1)$-dimensional identity matrix, we then obtain

$$\det(z-\mu_1-\mu_2+\tilde{B}\tilde{A}_{11}\tilde{B}^{-1}) = (z-\nu)(z-\mu_1)^{n_2-1}(z-\mu_2)^{n_1-1}.$$

Therefore we have

$$\det[\tilde{G}_j(z)] = \frac{\prod_{j=1}^{n-1} \tilde{\lambda}_j^z}{\Gamma(z-\nu)\Gamma(z-\mu_1)^{n_2-1}\Gamma(z-\mu_2)^{n_1-1}}\, p(z)\,,$$

where $[\tilde{G}_j(z)]$ denotes $[\tilde{G}_1(z), \tilde{G}_2(z),\dots, \tilde{G}_{n-1}(z)]$ for short and $p(z)$ is a periodic function of period 1. From Proposition (iii) and Stirling's formula it is easy to see that

$$p(z) \sim \prod_{j=1}^{n-1} \tilde{\lambda}_j^{-1-\mu_1-\mu_2+\nu_j}$$

as $z\to\infty$, $|\arg z|<\pi/2+\varepsilon$. Hence $p(z)$ is indeed a constant equal to the right hand side of this formula. We consequently obtain

$$\det[\tilde{G}_j(z)] = \frac{\prod_{j=1}^{n-1} \tilde{\lambda}_j^{z-1-\mu_1-\mu_2+\nu_j}}{\Gamma(z-\nu)\Gamma(z-\mu_1)^{n_2-1}\Gamma(z-\mu_2)^{n_1-1}}\,.$$

Observing from the (1, 2)-block of (2.1) that

(2.8) $$(\nu-\mu_1-\mu_2+\tilde{B}\tilde{A}_{11}\tilde{B}^{-1})\tilde{B}\tilde{A}_{12} = 0,$$

we can define the constants γ_j $(j=1,\dots, n-1)$ as the solution of the system of linear equations

$$\sum_{j=1}^{n-1} \gamma_j\tilde{G}_j(\nu+1) = -\frac{\Gamma(\nu+1)}{\Gamma(\nu+1-\mu_1)\,\Gamma(\nu+1-\mu_2)}\,\tilde{B}\tilde{A}_{12}.$$

In fact, since $\det[\tilde{G}_j(\nu+1)]\neq 0$, these constants are uniquely determined. Using these γ_j, we define a solution $\tilde{G}_0(z)$ of (2.4) by

$$\tilde{G}_0(z) = \sum_{j=1}^{n-1} \gamma_j\tilde{G}_j(z)\,.$$

Then it is easy to see that $\tilde{G}_0(\nu-r)=0$ for $r=0, 1, 2,\dots$.

Now we shall denote by $G_0(z)$ the solution of (0.2) constructed by $\tilde{G}_0(z)$ with (2.5) and (2.3), and put

$$\hat{G}(z) = G_0(z+\nu)\,.$$

Then it is immediately seen that $\hat{G}(-r)=0$ for $r=1, 2,\dots$. Observing that

$$(\nu-A)G_0(\nu) = (B-\lambda)(\nu+1)G_0(\nu+1) = (\nu-A)e_n,$$

e_n being the n-th unit n-vector, and $\det(\nu-A)\neq 0$ by the assumption [A_2], we obtain

$$\hat{G}(0) = G_0(\nu) = e_n\,.$$

Hence $\hat{G}(z)$ is a (non-trivial) solution of (1.2), which will be used for Barnes-integral representation of the non-holomorphic solution of (0.1) near $t=\lambda$ in §3.

The solution $\tilde{G}_0(z)$ of (2.4) will be considered again at the last part of §3.

§ 3. Barnes-integral representation

We consider a Barnes-integral

$$X_j(t) = -\frac{1}{2\pi i}\int_C G_j(z)\frac{\pi e^{-\pi i z}}{\sin \pi z}(t-\lambda)^z dz \quad (j=1,\dots, n-1),$$

where the path of integration C is a Barnes-contour running along the straight line $z=-ia$ from $+\infty-ia$ to $0-ia$, a curve from $0-ia$ to $0+ia$ and the straight line $z=ia$ from $0+ia$ to $+\infty+ia$ such that the points $z=m$ $(m=0, 1, 2,\dots)$ lie to the right of C and the points $z=\mu_k-r$ $(r=0, 1, 2,\dots;\ k=1, 2)$ lie to the left of C. The constant a is taken as $a>\max\{|\mathrm{Im}\,\mu_k|;\ k=1, 2\}$. In view of the asymptotic behavior of $G_j(z)$, the above integral is absolutely convergent for $|t-\lambda|<|\lambda_j-\lambda|$ $(=1/|\tilde{\lambda}_j|)$ and equal to the sum of residues at $z=m$ $(m=0, 1, 2,\dots)$, i.e.,

$$X_j(t) = \textstyle\sum_{m=0}^{\infty} G_j(m)(t-\lambda)^m \quad (|t-\lambda|<|\lambda_j-\lambda|),$$

which is a holomorphic solution of (0.1) near $t=\lambda$.

Now let ξ be an arbitrary negative number not equal to $\mathrm{Re}(\mu_k-r)$ $(r=0, 1, 2,\dots;\ k=1, 2)$. We take the positive integer N_k $(k=1, 2)$ such that

$$-(N_k+1) < \xi - \mathrm{Re}\,\mu_k < -N_k \quad (k=1, 2).$$

Replacing the path C by the rectilinear contour L_ξ which runs first from $+\infty-ia$ to $\xi-ia$, next from $\xi-ia$ to $\xi+ia$ and finally from $\xi+ia$ to $+\infty+ia$, we obtain

$$\begin{aligned} X_j(t) &= -\frac{1}{2\pi i}\int_{L_\xi} G_j(z)\frac{\pi e^{-\pi i z}}{\sin \pi z}(t-\lambda)^z dz \\ &\quad -\sum \mathrm{Res}\left[G_j(z)\frac{\pi e^{-\pi i z}}{\sin \pi z}(t-\lambda)^z\right], \end{aligned}$$

where the summation covers all poles in the domain encircled by L_ξ and the curve from $-ia$ to ia of C. Since $G_j(-r)=0$ $(r=1, 2,\dots)$ by (2.3), $z=-r$ $(r=1, 2,\dots)$ are no longer poles. Then, by (2.3), the integrand has simple poles only at $z=\mu_k-r$ $(r=0, 1,\dots, N_k;\ k=1, 2)$ in that domain. Hence we obtain

$$\sum \mathrm{Res}\left[G_j(z)\frac{\pi e^{-\pi i z}}{\sin \pi z}(t-\lambda)^z\right] = \textstyle\sum_{k=1}^{2}\sum_{r=0}^{N_k}\dfrac{\pi e^{-\pi i \mu_k}}{\sin \pi\mu_k} H_j^k(r)(t-\lambda)^{-r+\mu_k},$$

where

$$H_j^k(r) = \lim_{z\to\mu_k-r}[(z-\mu_k+r)G_j(z)] \quad (r=0, 1, 2,\dots;\ k=1, 2),$$

which is a solution of the system of linear difference equations (1.5) and (1.4).

In fact, since $G_j(z)$ is holomorphic at $z=\mu_k+1$, we have

$$\begin{aligned}(A-\mu_k)H_j^k(0) &= \lim_{z\to\mu_k}[-(z-A)(z-\mu_k)G_j(z)]\\ &= \lim_{z\to\mu_k}[-(B-\lambda)(z+1)(z-\mu_k)G_j(z+1)] = 0.\end{aligned}$$

For $r\geqslant 1$, we have

$$\begin{aligned}(r-\mu_k+A)H_j^k(r) &= \lim_{z\to\mu_k-r}[-(z-A)(z-\mu_k+r)G_j(z)]\\ &= \lim_{z\to\mu_k-r}[-(B-\lambda)(z+1)(z-\mu_k+r)G_j(z+1)]\\ &= (B-\lambda)(r-1-\mu_k)\lim_{z+1\to\mu_k-(r-1)}[(z+1-\mu_k+r-1)G_j(z+1)]\\ &= (B-\lambda)(r-1-\mu_k)H_j^k(r-1).\end{aligned}$$

Hence there exist the constants T_j^{kl} $(k=1,\ 2;\ l=1,\dots,\ n_k)$ such that

$$-\frac{\pi e^{-\pi i\mu_k}}{\sin\pi\mu_k}H_j^k(r) = \sum_{l=1}^{n_k} T_j^{kl}H^{kl}(r) \quad (r=0,\ 1,\ 2,\dots;\ k=1,\ 2).$$

Actually these constants are uniquely determined by the system of linear equations

$$\sum_{l=1}^{n_k} T_j^{kl}H^{kl}(0) = -\frac{\pi e^{-\pi i\mu_k}}{\sin\pi\mu_k}H_j^k(0)$$

($H^{kl}(0)$ $(k=1,\ 2;\ l=1,\dots,\ n_k)$ are given in advance). This, in turn, is equivalent to

$$\text{(3.1)}\quad \sum_{l=1}^{n_k} T_j^{kl}\tilde{H}^{kl}(0) = -\frac{\pi e^{-\pi i\mu_k}}{\sin\pi\mu_k}\cdot\frac{\Gamma(2\mu_k-\mu_1-\mu_2)}{\Gamma(\mu_k+1)}\cdot\tilde{\Psi}_j(\mu_k;\,0) \quad (k=1,\ 2),$$

where $\tilde{H}^{kl}(0)$ denotes the vector which consists of the first $n-1$ components of $H^{kl}(0)$, since for an eigenvector H of A corresponding to μ_k we have

$$\tilde{h} = (\mu_k-\nu)^{-1}\tilde{A}_{21}\tilde{H},$$

where $\tilde{H}$ and $\tilde{h}$ denote the vectors which consist of the first $n-1$ components and the last one of H, respectively. We therefore obtain

$$\begin{aligned}X_j(t) &= \sum_{k=1}^{2}\sum_{l=1}^{n_k} T_j^{kl}\{(t-\lambda)^{\mu_k}\sum_{r=0}^{N_k} H^{kl}(r)(t-\lambda)^{-r}\}\\ &\quad -\frac{1}{2\pi i}\int_{L_\xi} G_j(z)\frac{\pi e^{-\pi iz}}{\sin\pi z}(t-\lambda)^z dz \quad (|t-\lambda|<|\lambda_j-\lambda|).\end{aligned}$$

We here apply the results of B. L. J. Braaksma [2; pp. 271–278] to the last term. The integral

$$-\frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} G_j(z)\frac{\pi e^{-\pi iz}}{\sin\pi z}(t-\lambda)^z\,dz$$

is the analytic continuation of the above integral for t which, in view of the asymptotic behavior of $G_j(z)$, lies in the sector

$$S'_j = \{t \in \boldsymbol{C};\ \arg(\lambda_j - \lambda) + \varepsilon' \leqslant \arg(t-\lambda) \leqslant \arg(\lambda_j - \lambda) + 2\pi - \varepsilon'\} \quad (\varepsilon' > 0).$$

Moreover we have

$$\left\| -\frac{1}{2\pi i} \int_{\xi - i\infty}^{\xi + i\infty} G_j(z) \frac{\pi e^{-\pi i z}}{\sin \pi z} (t-\lambda)^z dz \right\| < K|t-\lambda|^{\xi}$$

as $t \to \infty$, $t \in S'_j$, where K is a positive constant independent of t (but depending on ξ). Hence $X_j(t)$ is analytically continued into $\boldsymbol{C} \backslash \{\lambda + s(\lambda_j - \lambda);\ s \geqslant 1\}$ and has the asymptotic expansion

$$X_j(t) \sim \textstyle\sum_{k=1}^{2} \sum_{l=1}^{n_k} T_j^{kl} Y^{kl}(t)$$

as $t \to \infty$, $t \in S_j = \{t \in \boldsymbol{C};\ \arg(\lambda_j - \lambda) < \arg(t-\lambda) < \arg(\lambda_j - \lambda) + 2\pi\}$. Since $t = \infty$ is a regular singularity of (0.1) and $Y^{kl}(t)$ $(k=1, 2;\ l=1,\dots, n_k)$ are convergent, we therefore obtain

$$X_j(t) = \textstyle\sum_{k=1}^{2} \sum_{l=1}^{n_k} T_j^{kl} Y^{kl}(t)$$

for $t \in S_j$. Consequently the connection coefficients between $X_j(t)$ and $Y^{kl}(t)$ $(k=1, 2;\ l=1,\dots, n_k)$ are given by $\tilde{\Psi}_j(\mu_k;\ 0)$ $(k=1, 2)$.

As to the non-holomorphic solution of (0.1) near $t=\lambda$, we analyze the Barnes-integral

$$\hat{X}(t) = -\frac{1}{2\pi i} \int_{C'} \hat{G}(z) \frac{\pi e^{-\pi i z}}{\sin \pi z} (t-\lambda)^{z+\nu}\, dz,$$

where the path of integration C' is a suitable Barnes-contour defined in a way similar to the above consideration. Then, for $|t-\lambda| < R$, we have

$$\hat{X}(t) = \textstyle\sum_{m=0}^{\infty} \hat{G}(m)(t-\lambda)^{m+\nu}$$

and

$$\hat{X}(t) = \textstyle\sum_{j=1}^{n-1} \gamma_j \Big\{ \sum_{k=1}^{2} \sum_{l=1}^{n_k} e^{\pi i \nu} \dfrac{\sin \pi \mu_k}{\sin \pi(\mu_k - \nu)} T_j^{kl} \sum_{r=0}^{N'_k} H^{kl}(r)(t-\lambda)^{-r+\mu_k}$$
$$- \frac{1}{2\pi i} \int_{L'_{\xi'}} G_j(z+\nu) \frac{\pi e^{-\pi i z}}{\sin \pi z} (t-\lambda)^{z+\nu}\, dz \Big\},$$

where $L'_{\xi'}$ is a suitable rectilinear contour and N'_k $(k=1, 2)$ are suitable positive integers. Hence the above $\hat{X}(t)$ is the non-holomorphic solution of (0.1) near $t=\lambda$ and is analytically continued into the sector

$$\textstyle\bigcap_{j=1}^{n-1} S_j = \{t \in \boldsymbol{C};\ \arg(\lambda_{n-1} - \lambda) < \arg(t-\lambda) < \arg(\lambda_1 - \lambda) + 2\pi\},$$

and we have

$$\hat{X}(t) = \sum_{k=1}^{2} \sum_{l=1}^{n_k} \left\{ e^{\pi i \nu} \frac{\sin \pi\mu_k}{\sin \pi(\mu_k - \nu)} \sum_{j=1}^{n-1} \gamma_j T_j^{kl} \right\} Y^{kl}(t) \tag{3.2}$$

for $t \in \bigcap_{j=1}^{n-1} S_j$. In order to obtain an analytic continuation of $\hat{X}(t)$ into another sector, we anew take

$$\hat{G}(z) = \sum_{j=1}^{n-1} \gamma_j e^{2\pi i \delta_j z} G_j(z+\nu),$$

where δ_j $(j=1,\dots, n-1)$ are suitable integers, instead of

$$\hat{G}(z) = \sum_{j=1}^{n-1} \gamma_j G_j(z+\nu).$$

Then we obtain the analytic continuation of $\hat{X}(t)$ into $\bigcap_{j=1}^{n-1} S_j(\delta_j)$, where

$$S_j(\delta_j) = \{t \in \boldsymbol{C};\ \arg(\lambda_j - \lambda) - 2\pi\delta_j < \arg(t-\lambda) < \arg(\lambda_j - \lambda) - 2\pi\delta_j + 2\pi\} \quad (j=1,\dots, n-1). \tag{3.3}$$

Now, in order to characterize γ_j and the coefficients in (3.2), we shall investigate the solution $\tilde{G}_0(z)$ of (2.4). Let $\tilde{\Phi}(\rho;\, t)$ be a solution of (2.6) near $t=\infty$ corresponding to the exponent $-(\rho-1-\nu)$. Then it is easy to see that $\tilde{\Phi}(\rho;\, t)$ has an expansion of the form

$$\tilde{\Phi}(\rho;\, t) = (e^{\pi i} t)^{\rho-1-\nu} \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(\rho-\nu-r)} \tilde{K}(r) t^{-r}$$

near $t=\infty$, where $\tilde{K}(r)$ $(r \geqslant 1)$ are the vectors determined uniquely by the system of linear difference equations

$$(r+1+\nu-\mu_1-\mu_2+\tilde{B}\tilde{A}_{11}\tilde{B}^{-1})\tilde{K}(r+1) = \tilde{B}\tilde{K}(r) \tag{3.4}$$

subject to the initial condition

$$(\nu-\mu_1-\mu_2+\tilde{B}\tilde{A}_{11}\tilde{B}^{-1})\tilde{K}(0) = 0.$$

Observing (2.8), we here define

$$\tilde{K}(0) = -\frac{\Gamma(\nu+1)}{\Gamma(\nu+1-\mu_1)\,\Gamma(\nu+1-\mu_2)} \tilde{B}\tilde{A}_{12} \quad (=\tilde{G}_0(\nu+1)).$$

Then, since (3.4) is equivalent to (2.4), we have

$$\tilde{K}(r) = \tilde{G}_0(r+1+\nu) \quad (r=0,\, 1,\, 2,\dots).$$

We therefore consider a Barnes-integral

$$\hat{\Phi}(\rho;\, t) = -\frac{1}{2\pi i} \int_{C''} \tilde{G}_0(z+1+\nu)\, \Gamma(z-\rho+1+\nu) \frac{\pi e^{-\pi i z}}{\sin \pi z} t^{\rho-1-\nu-z}\, dz,$$

where

$$\tilde{G}_0(z+1+\nu) = \sum_{j=1}^{n-1} \gamma_j e^{2\pi i \delta_j z} \tilde{G}_j(z+1+\nu) \quad (\delta_j \in \boldsymbol{Z}),$$

the path of integration C'' is a suitable Barnes-contour and we temporarily assume that $\rho \not\equiv \nu \pmod 1$. By an analysis similar to the above we obtain

$$\begin{aligned}\hat{\Phi}(\rho;\, t) &= \sum_{r=0}^{\infty} \tilde{G}_0(r+1+\nu)\Gamma(r-\rho+1+\nu)t^{\rho-1-\nu-r} \\ &= \frac{\pi}{\sin \pi(\rho-\nu)} \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(\rho-\nu-r)} \tilde{K}(r) t^{\rho-1-\nu-r} \\ &= -\frac{\pi e^{-\pi i(\rho-\nu)}}{\sin \pi(\rho-\nu)} \tilde{\Phi}(\rho;\, t)\end{aligned}$$

for $|t| > \max\{|\tilde{\lambda}_j|;\, j=1,\dots, n-1\}$ (observe that $e^{2\pi i \delta_j r}=1$ for $r \in \boldsymbol{Z}$). Moreover we obtain

$$\begin{aligned}\hat{\Phi}(\rho;\, t) = &-\frac{\pi e^{-\pi i(\rho-\nu)}}{\sin \pi(\rho-\nu)} \sum_{m=0}^{N''} \frac{(-1)^m}{m!} \tilde{G}_0(\rho-m)t^m \\ &-\frac{1}{2\pi i}\int_{L''_{\xi''}} \tilde{G}_0(z+1+\nu)\, \Gamma(z-\rho+1+\nu) \frac{\pi e^{-\pi i z}}{\sin \pi z} t^{\rho-1-\nu-z}\, dz,\end{aligned}$$

where $L''_{\xi''}$ is a suitable rectilinear contour and N'' is a suitable positive integer. Since

$$\frac{\partial}{\partial t} \tilde{\Psi}_j(\rho;\, t) = -\tilde{\Psi}_j(\rho-1;\, t),$$

we have

$$\begin{aligned}&\sum_{m=0}^{N''} \frac{(-1)^m}{m!} \tilde{G}_0(\rho-m)t^m \\ &= \sum_{j=1}^{n-1} \gamma_j e^{2\pi i \delta_j(\rho-\nu)} \sum_{m=0}^{N''} \frac{1}{m!} \cdot \frac{\partial^m}{\partial t^m} \tilde{\Psi}_j(\rho;\, t)\Big|_{t=0} \cdot t^m.\end{aligned}$$

Then, observing the asymptotic behavior of $\tilde{G}_j(z)$ and that the last summation in the above is the sum of the first $N''+1$ terms of Taylor's series of $\tilde{\Psi}_j(\rho;\, t)$, we see that $\tilde{\Phi}(\rho;\, t)$ is analytically continued into the sector $\bigcap_{j=1}^{n-1} \tilde{S}_j(\delta_j)$, where

$$\tilde{S}_j(\delta_j) = \{t \in \mathscr{D};\, \arg \tilde{\lambda}_j + 2\pi\delta_j - 2\pi < \arg t < \arg \tilde{\lambda}_j + 2\pi\delta_j\} \quad (j=1,\dots, n-1),$$

and we have

$$\tilde{\Phi}(\rho;\, t) = \sum_{j=1}^{n-1} \gamma_j e^{2\pi i \delta_j(\rho-\nu)} \tilde{\Psi}_j(\rho;\, t) \tag{3.5}$$

for $t \in \bigcap_{j=1}^{n-1} \tilde{S}_j(\delta_j)$. Consequently $\gamma_j e^{2\pi i \delta_j(\rho-\nu)}$ $(j=1,\dots, n-1)$ are the connection coefficients between $\tilde{\Phi}(\rho;\, t)$ and $\tilde{\Psi}_j(\rho;\, t)$ $(j=1,\dots, n-1)$ in the sector $\bigcap_{j=1}^{n-1} \tilde{S}_j(\delta_j)$. Since we can drop the assumption that $\rho \not\equiv \nu \pmod 1$ in (3.5) by the holomorphy of $\tilde{\Phi}(\rho;\, t)$ and $\tilde{\Psi}_j(\rho;\, t)$ in $\rho \in \boldsymbol{C}$, we therefore obtain

$$\tilde{G}_0(z)\ (=\textstyle\sum_{j=1}^{n-1}\gamma_j e^{2\pi i\delta_j(z-\nu)}\tilde{G}_j(z)) = \tilde{\Phi}(z;0)$$

for every $z\in\boldsymbol{C}$, where $\tilde{\Phi}(z;\cdot)$ means the analytic continuation through the sector $\bigcap_{j=1}^{n-1}\tilde{S}_j(\delta_j)$. Hence the coefficients

$$(3.6)\qquad \hat{T}^{kl}(\delta_1,\dots,\delta_{n-1}) = e^{\pi i\nu}\frac{\sin\pi\mu_k}{\sin\pi(\mu_k-\nu)}\textstyle\sum_{j=1}^{n-1}\gamma_j e^{2\pi i\delta_j(\mu_k-\nu)}T_j^{kl}$$
$$(k=1,2;\ l=1,\dots,n_k),$$

which correspond to the analytic continuation of $\hat{X}(t)$ into the sector $\bigcap_{j=1}^{n-1}S_j(\delta_j)$, are characterized as the unique solutions of the systems of linear equations

$$(3.7)\qquad \textstyle\sum_{l=1}^{n_k}\hat{T}^{kl}(\delta_1,\dots,\delta_{n-1})\tilde{H}^{kl}(0)$$
$$= -\frac{\pi e^{-\pi i(\mu_k-\nu)}}{\sin\pi(\mu_k-\nu)}\cdot\frac{\Gamma(2\mu_k-\mu_1-\mu_2)}{\Gamma(\mu_k+1)}\cdot\tilde{\Phi}(\mu_k;0)\quad(k=1,2),$$

where $\tilde{\Phi}(\mu_k;\cdot)$ means the analytic continuation through the sector $\bigcap_{j=1}^{n-1}\tilde{S}_j(\delta_j)$. Consequently the connection coefficients between $\hat{X}(t)$ and $Y^{kl}(t)$ ($k=1,2$; $l=1,\dots,n_k$) are given by $\tilde{\Phi}(\mu_k;0)$ ($k=1,2$), of which the sector of analytic continuation corresponds to that of $\hat{X}(t)$.

We summarize all results derived above in the following

THEOREM. *Let* $Y^{kl}(t)$ ($k=1,2$; $l=1,\dots,n_k$) *be solutions of* (0.1) *of the form* (1.3) *near* $t=\infty$, *and let* $S_j(\delta_j)$ ($j=1,\dots,n-1$) *be sectors defined by* (3.3), *where the* δ_j *are integers.*

(i) *For each* j ($=1,\dots,n-1$), *the holomorphic solution* $X_j(t)$ *of* (0.1) *near* $t=\lambda$ *which is characterized by* $X_j(\lambda)=G_j(0)$, *where* $G_j(0)$ *is constructed by* (2.7) *with* (2.4), (2.5) *and* (2.3), *is holomorphic in* $\boldsymbol{C}\setminus\{\lambda+s(\lambda_j-\lambda);\ s\geqslant 1\}$ *and*

$$X_j(t) = \textstyle\sum_{k=1}^{2}\sum_{l=1}^{n_k}e^{2\pi i\delta_j\mu_k}T_j^{kl}Y^{kl}(t)\quad(t\in S_j(\delta_j))$$

holds, where the constants T_j^{kl} ($k=1,2$; $l=1,\dots,n_k$) *are determined by* (3.1).

(ii) *The non-holomorphic solution* $\hat{X}(t)$ *of* (0.1) *near* $t=\lambda$ *of the form* (1.1) *with* $\hat{G}(0)=e_n$ *is analytically continued into a sector* $\bigcap_{j=1}^{n-1}S_j(\delta_j)$ *and*

$$\hat{X}(t) = \textstyle\sum_{k=1}^{2}\sum_{l=1}^{n_k}\hat{T}^{kl}(\delta_1,\dots,\delta_{n-1})Y^{kl}(t)\quad(t\in\bigcap_{j=1}^{n-1}S_j(\delta_j))$$

holds, where the constants $\hat{T}^{kl}(\delta_1,\dots,\delta_{n-1})$ ($k=1,2$; $l=1,\dots,n_k$) *are determined by* (3.7) (*or* (3.6)).

(iii) $X_j(t)$ ($j=1,\dots,n-1$) *and* $\hat{X}(t)$ *form a fundamental set of solutions of* (0.1).

REMARK. (i) About the integers δ_j ($j=1,\dots,n-1$) in (ii) we actually choose them as $\bigcap_{j=1}^{n-1}S_j(\delta_j)\neq\emptyset$.

(ii) The fact in (iii) is immediately seen by $\det[\tilde{G}_j(0)] \neq 0$.

REMARK. The system (2.6) with $\rho=\mu_k$, whose solutions give the connection coefficients as above, is indeed reducible to an n_k-dimensional system of linear differential equations (which is no longer a hypergeometric system in general). This fact is easily seen by $\operatorname{rank}(\mu_k-\mu_1-\mu_2+A)=n_k$ $(k=1, 2)$.

§4. Examples

EXAMPLE 1 (1-dimensional section of Appell's F_3).

$$B = \operatorname{diag}[\lambda_0, \lambda_0, \lambda_1, \lambda_2],$$

$$A = [a_{jk}] \sim \operatorname{diag}[\mu_1, \mu_1, \mu_2, \mu_2] \quad \text{with} \quad a_{12} = a_{21} = 0.$$

See [8] in detail.

EXAMPLE 2 (Jordan-Pochhammer system).

$$(t-B)\frac{dX}{dt} = \left(\rho + \begin{bmatrix} a_1 & a_1 \cdots a_1 \\ & \cdots\cdots \\ a_n & a_n \cdots a_n \end{bmatrix}\right) X, \tag{4.1}$$

where $B=\operatorname{diag}[\lambda_1,\dots, \lambda_n]$. In this case the $(n-1)$-dimensional hypergeometric system corresponding to (2.6) becomes again a Jordan-Pochhammer system. We here describe connection formulas only for non-holomorphic solutions near finite singularities: Assume the assumptions corresponding to $[\mathrm{A}_0 \sim \mathrm{A}_2]$. Let $P_n^k\left(\rho;\, \begin{matrix}\lambda_1,\dots, \lambda_n \\ a_1,\dots, a_n\end{matrix}\middle|\, t\right)$ $(k=1,\dots, n)$ be solutions of (4.1) characterized by

$$\lim_{t\to\lambda_k} (t-\lambda_k)^{-(\rho+a_k)} P_n^k\left(\rho;\, \begin{matrix}\lambda_1,\dots, \lambda_n \\ a_1,\dots, a_n\end{matrix}\middle|\, t\right) = e_k \quad (k=1,\dots, n),$$

where e_k denotes the k-th unit n-vector. Besides let $\hat{Q}_n\left(\rho;\, \begin{matrix}\lambda_1,\dots, \lambda_n \\ a_1,\dots, a_n\end{matrix}\middle|\, t\right)$ and $Q_n^{k,j}\left(\rho;\, \begin{matrix}\lambda_1,\dots, \lambda_n \\ a_1,\dots, a_n\end{matrix}\middle|\, t\right)$ $(k, j=1,\dots, n, j\neq k)$ be solutions of (4.1) characterized by

$$\lim_{t\to\infty} t^{-(\rho+\alpha)} \hat{Q}_n\left(\rho;\, \begin{matrix}\lambda_1,\dots, \lambda_n \\ a_1,\dots, a_n\end{matrix}\middle|\, t\right) = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix},$$

where $\alpha=a_1+\cdots+a_n$, and

$$\lim_{t\to\infty} t^{-\rho} Q_n^{k,j}\left(\rho;\, \begin{matrix}\lambda_1,\dots, \lambda_n \\ a_1,\dots, a_n\end{matrix}\middle|\, t\right) = e_j - e_k \quad (k, j=1,\dots, n, j\neq k),$$

respectively. Then we have

$$P_n^k\left(\rho;\, \begin{matrix}\lambda_1,\dots, \lambda_n \\ a_1,\dots, a_n\end{matrix}\middle|\, t\right)$$

$$= \frac{\Gamma(\alpha)\,\Gamma(\rho+a_k+1)}{\Gamma(\rho+\alpha+1)\,\Gamma(a_k+1)} \prod_{j\neq k} (-\tilde{\lambda}_j^k)^{a_j} \hat{Q}_n\left(\rho;\, \begin{matrix}\lambda_1,\dots,\lambda_n\\ a_1,\dots,a_n\end{matrix}\middle|\, t\right)$$

$$- \frac{\Gamma(-\alpha)\,\Gamma(\rho+a_k+1)}{\Gamma(\rho+1)\,\Gamma(a_k+1-\alpha)} [Q_n^{k,1}(t),\dots, Q_n^{k,k-1}(t),\, Q_n^{k,k+1}(t),\dots, Q_n^{k,n}(t)]$$

$$\times\, \tilde{B}_k \hat{Q}_{n-1}\left(-\alpha-1;\, \begin{matrix}\tilde{\lambda}_1^k,\dots,\tilde{\lambda}_{k-1}^k,\, \tilde{\lambda}_{k+1}^k,\dots,\tilde{\lambda}_n^k\\ a_1,\dots,a_{k-1},\, a_{k+1},\dots,a_n\end{matrix}\middle|\, 0\right),$$

where $Q_n^{k,j}(t)$ denotes $Q_n^{k,j}\left(\rho;\, \begin{matrix}\lambda_1,\dots,\lambda_n\\ a_1,\dots,a_n\end{matrix}\middle|\, t\right)$ for short, $\tilde{\lambda}_j^k = (\lambda_j-\lambda_k)^{-1} (j\neq k)$ and

$$\tilde{B}_k = \mathrm{diag}\,[\tilde{\lambda}_1^k,\dots,\tilde{\lambda}_{k-1}^k,\, \tilde{\lambda}_{k+1}^k,\dots,\tilde{\lambda}_n^k] \quad (k=1,\dots,n).$$

The paths of analytic continuation in these formulas are taken as in §3.

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