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Closed ideals of Lie algebras

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0. Introduction

In 1960 Goldie [7] showed how to develop a structure theory for semi-prime rings with maximum condition in terms of what he called *closed ideals*. An alternative and slightly different definition was given by Lesieur and Croisot [9] and used by Divinsky [6]. The aim of this paper is to define analogous concepts for Lie algebras, and to establish their basic properties.

In §1 we introduce two analogous notions for Lie algebras, which we call *closed ideals* and **closed ideals* to distinguish them. They are defined in terms of the *closure* $cl(I)$ and **closure* $cl^*(I)$ of an ideal I, see Definitions 1.1 and 1.2. We show that $cl(I) \subseteq cl^*(I)$ and that the two closures coincide for semisimple Lie algebras (defined below). The basic properties of closed ideals are established in §2, where we show in particular that the closure of an ideal need not be an ideal—indeed it need not even be a vector subspace. Analogous questions for cl^* are investigated in §3; in contrast, the *closure of an ideal is always an ideal.

In §4 we study semisimple algebras. The main result is that the following four concepts are equivalent for semisimple algebras: centralizer ideal, complement ideal, closed ideal, and *closed ideal. In §5 we discuss, for arbitrary Lie algebras, relations between centralizer ideals, complement ideals, closed ideals, *closed ideals, and ideals with no proper essential extension, where the latter concept is analogous to one defined for rings in Behrens [5] and Goodearl [8]. The main result is that *closed ideals are always closed; closed ideals are always complement ideals; and being a complement ideal is equivalent to having no proper essential extension. Moreover, no other implications between these properties are valid. We also show that the sum of two complement (respectively closed) ideals need not be a complement (respectively closed) ideal. Finally, in §6, we investigate various chain conditions on closed and *closed ideals, extending work in Aldosray and Stewart [3] and answering part of Question 1.7 of that paper. In particular we show that the ascending chain condition for complement ideals is equivalent to the descending chain condition for complement ideals.

All Lie algebras considered are of finite or infinite dimension over a field **k** of arbitrary characteristic, unless otherwise specified. Most notation used is standard, and may be found in Amayo and Stewart [4], Aldosray [1], or Aldosray and Stewart [2, 3]. We write $I \lhd L$ if I is an ideal of L, and $I \leq L$ if I is a subalgebra. The centralizer of I in L is written $C_L(I)$, and $\zeta_1(L)$ is the centre of L. The subalgebra generated by a subset $X \subseteq L$ is denoted $\langle X \rangle$, and the ideal generated by a subset $X \subseteq L$ is denoted $\langle X \rangle^L$. If $X = \{x\}$ is a singleton we write $\langle x \rangle^L$ in place of $\langle \{x\} \rangle^L$.

An ideal $E \lhd L$ is *essential* if E intersects nontrivially with every nonzero ideal of L. The *singular ideal* of L is

$$Z(L) = \{x \in L : [x, E] = 0 \text{ for some essential ideal } E \text{ of } L\}.$$

L is *semisimple* if it has no nonzero abelian ideals: by Aldosray and Stewart [3] Lemma 2.2 this is equivalent to $Z(L)$ being zero.

Any other notation is defined as it is needed. The end (or absence) of a proof is signalled by a box $\square$.

1. Alternative definitions of closed ideals

In the ring-theoretic literature, several different definitions of the closure of an ideal, and of closed ideals, may be found. We wish to investigate their analogues for Lie algebras. In this section we introduce the main concepts to be studied in this paper.

The first definition is a direct analogue of that originally introduced by Lesieur and Croisot [9], and used by Divinsky [6].

DEFINITION 1.1. The *closure* in L of an ideal $I \lhd L$ is

$$cl_L(I) = \{x \in L | \langle y \rangle^L \cap I \neq 0 \text{ for all } 0 \neq y \in \langle x \rangle^L\}.$$

When L is clear from the context we may write just $cl(I)$.

The second definition is analogous to that used by Goldie [7]:

DEFINITION 1.2. The **closure* in L of an ideal $I \lhd L$ is

$$cl_L^*(I) = \{x \in L | [x, E] \subseteq I \text{ for some essential ideal } E \text{ of } L\}.$$

When L is clear from the context we may write just $cl^*(I)$.

We also require a module version:

DEFINITION 1.3. Let M be an L-module, and S a submodule of M. The **closure* of S in M is

$$cl_M^*(S) = \{x \in M | xE \subseteq S \text{ for some essential ideal } E \text{ of } L\}.$$

It is easy to show that

$$cl^*(I) = \{x \in L | [\langle x \rangle^L, E] \subseteq I \text{ for some essential ideal } E \text{ of } L\}.$$

The two notions of closure are almost, but *not* exactly, the same. The relation between them is given in the next result.

THEOREM 1.4. *For every Lie algebra L, and every ideal I of L, $cl(I) \subseteq cl^*(I)$. If L is semisimple then for every ideal I of L, $cl(I) = cl^*(I)$.*

PROOF. Let $x \in cl(I)$. Let $E = \{z \in L: [\langle x \rangle^L, z] \subseteq I\}$. We claim that E is essential in L. This is true provided, given $0 \neq J \lhd L$, we can find $z \in J$, $z \neq 0$, such that $[\langle x \rangle^L, z] \subseteq I$. If $J \cap \langle x \rangle^L = 0$ then $[\langle x \rangle^L, J] = 0 \subseteq I$, hence any $0 \neq z \in J$ suffices. If $J \cap \langle x \rangle^L \neq 0$ then pick $0 \neq y \in J \cap \langle x \rangle^L$. Since $x \in cl(I)$ we have $\langle y \rangle^L \cap I \neq 0$, so $[\langle x \rangle^L, z] \subseteq I$ for $z \in \langle y \rangle^L \cap I \subseteq J$. Therefore E is essential, and by definition $x \in cl^*(I)$. Therefore $cl(I) \subseteq cl^*(I)$.

Suppose that L is semisimple and $x \in cl^*(I) \backslash cl(I)$. Then there exists an essential ideal E of L such that $[\langle x \rangle^L, E] \subseteq I$, and there exists $0 \neq y \in \langle x \rangle^L$ such that $\langle y \rangle^L \cap I = 0$. Then $(\langle y \rangle^L \cap E)^2 \subseteq [\langle y \rangle^L, \langle x \rangle^L \cap E] \subseteq \langle y \rangle^L \cap I = 0$. Since L is semisimple $\langle y \rangle^L \cap E = 0$, a contradiction. □

EXAMPLE 1.5. If L is not semisimple then $cl(I)$ can be strictly smaller than $cl^*(I)$.

Let A be abelian with $B \leq A$, $B \neq A$. It is easy to check that $cl_A(B) = B$ but $cl_A^*(B) = A$. □

DEFINITION 1.6. An ideal $I \lhd L$ is *closed* in L if and only if $cl_L(I) = I$.

DEFINITION 1.7. An ideal $I \lhd L$ is **closed* in L if and only if $cl_L^*(I) = I$.

By Example 1.5, closed ideals need not be *closed; but *closed ideals are always closed by Theorem 1.4. The following characterisation of closed ideals is immediate from Definitions 1.1 and 1.6:

PROPOSITION 1.8. *An ideal $I \lhd L$ is closed if and only if, for all $x \notin I$, there exists $0 \neq y \in \langle x \rangle^L$ such that $\langle y \rangle^L \cap I = 0$.* □

COROLLARY 1.9. *An ideal $I \lhd L$ is closed and essential in L if and only if $I = L$.* □

2. Closed ideals

In this section we establish the basic properties of closed ideals, in the sense of Definition 1.1.

LEMMA 2.1. *Let I, J be ideals of L. Then*

(a) *$I \supseteq J$ implies that $cl(I) \supseteq cl(J)$.*

(b) *Let I, $J \lhd L$. Then $cl(I \cap J) = cl(I) \cap cl(J)$.*

(c) *The intersection of any set of closed ideals is always closed.*
(d) *If K is an ideal of L and $K \cap cl(I) \neq 0$ then $K \cap I \neq 0$.*
(e) $cl(cl(I)) = cl(I)$.
(f) *Let $I \subseteq J \lhd L$, with J closed in L. Then J/I is closed in L/I.*

PROOF. (a) This follows from the definition.

(b) From (a) it follows that $cl(I \cap J) \subseteq cl(I) \cap cl(J)$. Now suppose that $x \in cl(I) \cap cl(J)$. Let $0 \neq y \in \langle x \rangle^L$. Since $x \in cl(I)$ we have $\langle y \rangle^L \cap I \neq 0$. Let $0 \neq z \in \langle y \rangle^L \cap I$. Since $\langle y \rangle^L \cap I \subseteq \langle x \rangle^L$ and $x \in cl(J)$ we have $\langle z \rangle^L \cap J \neq 0$. Therefore $\langle y \rangle^L \cap I \cap J \neq 0$, so $x \in cl(I \cap J)$. Therefore $cl(I \cap J) \supseteq cl(I) \cap cl(J)$ and we are done.

(c) Let $\{I_\alpha\}$ be a family of closed ideals of L, and let $I = \bigcap I_\alpha$. Clearly $I \lhd L$. If $b \notin I$ then there exist some α such that $b \notin I_\alpha$. Since I_α is closed, there exists $0 \neq a \in \langle b \rangle^L$ such that $I_\alpha \cap \langle a \rangle^L = 0$. Then $I \cap \langle a \rangle^L = 0$, so I is closed.

(d) If $K \cap cl(I) \neq 0$ then there exists $0 \neq k \in K \cap cl(I)$. By the definition of $cl(I)$, we have $\langle k \rangle^L \cap I \neq 0$. Therefore $K \cap I \neq 0$.

(e) By (a) we have $cl(I) \subseteq cl(cl(I))$. Suppose that $x \in cl(cl(I))$. Let $0 \neq y \in \langle x \rangle^L$. Then $\langle y \rangle^L \cap cl(I) \neq 0$. By part (d) $\langle y \rangle^L \cap I \neq 0$. Therefore $x \in cl(I)$. So $cl(cl(I)) \subseteq cl(I)$ and we are done.

(f) Let bars denote images modulo I. Suppose $\bar{x} \in \bar{L}$ is such that $\langle \bar{y} \rangle^{\bar{L}} \cap \bar{J} \neq \bar{0}$ for all $\bar{0} \neq \bar{y} \in \langle \bar{x} \rangle^{\bar{L}}$. Then $\langle y \rangle^L \cap J \neq 0$ for all $y \in \langle x \rangle^L \backslash I$. On the other hand, if $y \in I$ then $\langle y \rangle^L \cap J \neq 0$ since $J \supseteq I$. Therefore $x \in cl(J) = J$ so $\bar{x} \in \bar{J}$. $\square$

We do not know whether a statement similar to part (b) holds for arbitrary intersections. Note also that in Lemma 2.1 we have not stated that $cl(I) \lhd L$. It is important to realise that this is not the case. Indeed a stronger statement holds:

EXAMPLE 2.2. If $I \lhd L$ then $cl(I)$ need not be a vector subspace of L. Let L have a basis

$$\{x, y, z, w, \delta, \varepsilon\} \tag{1}$$

over a field $\mathbf{k}$, where

$$[x, y] = z = -[y, x]$$

$$[x, \delta] = w = -[\delta, x]$$

$$[y, \varepsilon] = w = -[\varepsilon, y]$$

and all other products are zero. Then all triple products are zero, so the Jacobi identity holds and L is nilpotent of class 2.

Let $I = \langle w, \delta, \varepsilon \rangle \lhd L$. We show that $cl(I)$ is not a vector subspace. The following notation is useful: if $u \in L$ then write

$$u \approx x$$

if the x-coordinate (relative to the basis (1)) of u is nonzero, and similarly if x is replaced by any other basis element.

If $a \approx x$ or $a \approx y$, then $z \in \langle a \rangle^L$, and $\langle z \rangle^L = \langle z \rangle$, so $\langle z \rangle^L \cap I = 0$. Therefore by Definition 1.1, $a \notin cl(I)$. Thus $cl(I) \subseteq \langle z, w, \delta, \varepsilon \rangle = U$, say. Let $u \in U$. If $u \approx \delta$ or $u \approx \varepsilon$ then $\langle u \rangle^L = \langle u, w \rangle$. Moreover, every element $v \in \langle u \rangle^L \backslash \langle w \rangle$ satisfies $v \approx \delta$ or $v \approx \varepsilon$, and $\langle v \rangle^L = \langle u, w \rangle$. Therefore $\langle v \rangle^L \cap I \ni w$, so $\langle v \rangle^L \cap I \neq 0$. But if $0 \neq v \in \langle w \rangle$, then $\langle v \rangle^L = \langle w \rangle$, and $\langle v \rangle^L \cap I \neq 0$. Therefore $u \in cl(I)$ provided $u \approx \delta$ or $u \approx \varepsilon$. Otherwise, $u \in \langle z, w \rangle$, so $\langle u \rangle^L = \langle u \rangle$. Now $u \in cl(I)$ if and only if $u \in \langle w \rangle$. We conclude that

$$cl(I) = (\langle z, w, \delta, \varepsilon \rangle \backslash \langle z, w \rangle) \cup \langle w \rangle$$

which is not a vector space. □

This example sheds some light on possible modifications of the notion of closure that would make the closure of an ideal always an ideal. Here are two possibilities:

(a) $cl'(I) = \bigcap \{J \mid I \subseteq J \lhd L,\ J \text{ closed in } L\}$.

(b) $cl''(I) = \langle cl(I) \rangle^L$.

Let us compute these for the ideal I of L in Example 2.2. Clearly $cl''(I) = \langle z, w, \delta, \varepsilon \rangle = \langle z \rangle + I$. Suppose that J is a closed ideal that contains I. Since $I \subseteq J$, $cl(I) \subseteq cl(J) = J \lhd L$. Hence $cl''(I) \subseteq J$. But now J contains $\zeta_1(L) = \langle w, z \rangle$ which is essential, so $J = L$ by Corollary 1.9. Therefore $cl'(I) = L$. In particular, $cl'(I) \neq cl''(I)$.

3. *Closed ideals

Next we establish the basic properties of cl^*, and of *closed ideals, using Definitions 1.2 and 1.7.

LEMMA 3.1. *Let I, J be ideals of L. Then*

(a) *If $I \lhd L$ then $I \subseteq cl^*(I)$.*

(b) *If $I \lhd L$ then $cl^*(I) \lhd L$.*

(c) *If $I \supseteq J$ then $cl^*(I) \supseteq cl^*(J)$.*

(d) *Let $I, J \lhd L$. Then $cl^*(I \cap J) = cl^*(I) \cap cl^*(J)$.*

(e) *The intersection of two *closed ideals is *closed.*

(f) *If $I \subseteq J \lhd L$, $I \lhd L$, and J is *closed in L, then J/I is *closed in L/I.*

PROOF. (a) Use $E = L$ in Definition 1.2.

(b) Let x_1, $x_2 \in cl^*(I)$, $y \in L$, and λ, $\mu \in \mathbf{k}$. Let E_i $(i = 1, 2)$ be essential ideals of L such that $[x_i, E_i] \subseteq I$ $(i = 1, 2)$. Then

$$[\lambda x_1 + \mu x_2, E_1 \cap E_2] \subseteq [x_1, E_1] + [x_2, E_2] \subseteq I .$$

$$[[x_1, y], E_1] \subseteq [[x_1, E_1], y] + [x_1, [y, E_1]] \subseteq [I, y] + [x_1, E] \subseteq I .$$

Hence $cl^*(I) \lhd L$.

(c) This follows from Definition 1.2.

(d) By part (c) we have $cl^*(I \cap J) \subseteq cl^*(I) \cap cl^*(J)$. Suppose that $x \in cl^*(I) \cap cl^*(J)$. Then there exist essential ideals E and F such that $[x, E] \subseteq I$ and $[x, F] \subseteq J$. Then $E \cap F$ is essential, and $[x, E \cap F] \subseteq I \cap J$. So $x \in cl^*(I \cap J)$.

(e) This follows from (d).

(f) Let $x + I \in cl^*(J/I)$. Then there is an essential ideal $\bar{E}$ of L/I such that $[x + I, \bar{E}] \subseteq J/I$. Let $\bar{E} = E/I$ where $I \subseteq E \lhd L$. Then E is essential in L by Aldosray and Stewart [3] Lemma 2.3; and $[x, E] \subseteq J$. Thus $x \in cl^*(J) = J$. Therefore $x + I \in J/I$ as required. □

REMARKS 3.2. 1. The analogue of part (e) of Lemma 2.1 does not hold for cl^*: see Example 3.3 below.

2. Part (d) of Lemma 2.1 does not hold for cl^*: see Example 5.11. □

EXAMPLE 3.3. $cl^*(cl^*(I))$ need not equal $cl^*(I)$.

Let L have basis a_1, a_2, b_1, b_2, where $[a_i, b_i] = a_i$ and all other products are zero. It is easy to see that E is essential in L if and only if $E \supseteq A = \langle a_1, a_2 \rangle$. Hence for any ideal I we have $cl^*(I) = \{x \in L\colon [x, A] \subseteq I\}$. Therefore $cl^*(\langle a_1 \rangle) = \langle a_1, a_2, b_1 \rangle$. However, $cl^*(cl^*(\langle a_1 \rangle) = L$.

In view of this example, we cannot conclude that $cl^*(I)$ is always *closed. We should perhaps therefore define the **closure $cl^{**}(I)$ to be the smallest *closed ideal containing I. If $cl^*(I) = I$ then $cl^{**}(I) = I$; moreover $cl^{**}(cl^{**}(I)) = I$. Clearly

$$cl^{**}(I) = \bigcup\nolimits_{\alpha:\text{ordinal}} cl^{*(\alpha)}(I)$$

where $cl^{*(\alpha+1)}(I) = cl^*(cl^{*(\alpha)}(I))$. We do not consider $cl^{**}(I)$ further here. □

4. Semisimple algebras

In this section we specialise to the case of semisimple algebras, where many of the concepts under discussion become equivalent. We begin by recalling the appropriate definitions: see also Aldosray and Stewart [2, 3]. An ideal $C \lhd L$ is a *centralizer ideal* if there exists an ideal $I \lhd L$ such that $C = C_L(I)$. An ideal $K \lhd L$ is a *complement ideal* if there exists an ideal

$I \lhd L$ such that $K \cap I = 0$ and if $J \supseteq K$, $J \lhd L$, and $J \cap I = 0$, then $J = K$. That is, K is maximal subject to $K \cap I = 0$. We say that K is a *complement to* I.

The main result is:

THEOREM 4.1. *Let L be semisimple. Then the following are equivalent*:
(a) *I is a centralizer ideal of L.*
(b) *I is a complement ideal of L.*
(c) *I is a closed ideal of L.*
(d) *I is a *closed ideal of L.*

We first prove:

THEOREM 4.2. (a) *Let I be an ideal of L with $Z(I) = 0$. Then every complement in I is a *closed submodule of I.*

(b) *Let I be an ideal of L. Then every *closed ideal of L contained in I is a complement in I.*

PROOF. (a) Suppose that K is a complement (to X, say) in I. Let $x \in cl^*(K) \cap X$. Then there exists an essential ideal E such that $xE \subseteq K$. Therefore $xE \subseteq K \cap X = 0$. Therefore $x = 0$ since $Z(I) = 0$, so $cl^*(K) \cap X = 0$. Since K is a complement, $K = cl^*(K)$, so K is *closed.

(b) This follows from Theorem 1.4, and Corollary 5.3 and Proposition 5.4 below. □

We can now give the:

PROOF OF THEOREM 4.1. Using Theorems 1.4 and 4.2, and Lemma 2.3 of [2], it remains to show that if L is semisimple and I is a centralizer ideal in L, then I is closed and *closed. Since $cl(I) = cl^*(I)$ when L is semisimple, it suffices to consider *closure. Let $I = C_L(K)$ where $K \lhd L$. Suppose I is not *closed, and let $x \in cl(I) \cap K$. By definition $[\langle x\rangle^L, E] \subseteq I$ for some essential ideal E of L. Then $[\langle x\rangle^L, E]^2 = 0$, which implies that $[\langle x\rangle^L, E] = 0$ by semisimplicity. Therefore $cl^*(I) \cup K = 0$, so $cl^*(I) \subseteq C_L(K) = I$. Hence $x \in Z(L)$, so $x = 0$. □

We note a further result:

PROPOSITION 4.3. *If L is semisimple and $I \lhd L$, then $C_L(I) = C_L(cl(I))$.*

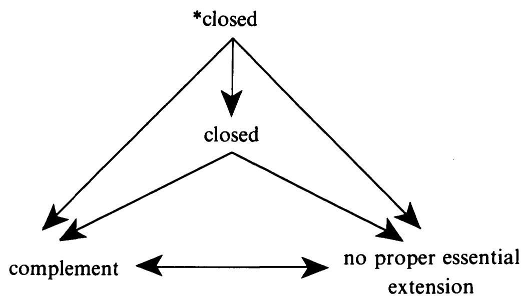
PROOF. We have $I \subseteq cl(I)$, so $C_L(I) \supseteq C_L(cl(I))$. Let $x \in C_L(I)$. Then $I \subseteq C_L(\langle x\rangle^L)$, so that $cl(I) \subseteq cl(C_L(\langle x\rangle^L))$ by Lemma 2.1a. But $C_L(\langle x\rangle^L)$ is a closed ideal by Theorem 4.1, hence $cl(C_L(\langle x\rangle^L)) = C_L(\langle x\rangle^L)$. Therefore $[\langle x\rangle^L, cl(I)] = 0$ and $x \in C_L(cl(I))$. Hence $C_L(I) = c_L(cl(I))$. □

5. Relation to centralizers and complements

In this section we discuss relations between closed ideals, *closed ideals, centralizer ideals, complement ideals, and essential extensions, where the latter is defined by analogy with ring theory, see Behrens [5] and Goodearl [8], as follows:

DEFINITION 5.1. Let $I \lhd L$. Then J is an *essential extension* of I if $J \lhd L$, $J \supseteq I$, and whenever $K \lhd L$, $K \cap J \neq 0$, then $K \cap I \neq 0$. It is a *proper* essential extension if $J \neq I$.

The aim of this section is to prove that the implications shown by the arrows in the following diagram are valid, and that no others are.



We first prove the implications shown, noting that *closed implies closed by Theorem 1.4.

PROPOSITION 5.2. *If J is an essential extension of I, then $J \subseteq cl(I)$.*

PROOF. If $0 \neq x \in J$ then $\langle x\rangle^L \subseteq J$. Let $0 \neq y \in \langle x\rangle^L$. Then $\langle y\rangle^L \subseteq J$. So $\langle y\rangle^L \cap I \neq 0$ by Definition 5.1. Thus $x \in cl(I)$. □

COROLLARY 5.3. *If I is closed then I has no proper essential extension.* □

PROPOSITION 5.4. *If I is an ideal having no proper essential extension, then I is a complement ideal.*

PROOF. If $I = L$ there is nothing to prove. If $I \neq L$ then L is not an essential extension of I, so there exists $0 \neq X \lhd L$, $I \cap X = 0$. Choose $K \lhd L$ maximal with respect to $K \supseteq X$, $I \cap K = 0$. Then K is a complement to I. Let $J \lhd L$ be maximal such that $J \supseteq I$, $J \cap K = 0$. We claim that J is an essential extension of I. For a contradiction, suppose that $0 \neq J' \subseteq J$, $J' \lhd L$, and $J' \cap I = 0$. Then $I + J' \subseteq J$ and $J' \cap I = 0$. Then $I + J' \subseteq J$ and $I + J' =$

$I \oplus J'$. Now $(I \oplus J') \cap K = 0$, so $(I \oplus J') + K$ is direct, and we may write it as $(I \oplus J') \oplus K$. So $I \cap (J' \oplus K) = 0$, whence $J' \subseteq K$ by maximality. But this is a contradiction. Thus J is an essential extension of I, so $J = I$. Therefore I is a complement ideal. □

PROPOSITION 5.5. *If I is a complement ideal then I has no proper essential extension.*

PROOF. Let I be a complement ideal, so that $I \cap K = 0$ for $K \lhd L$ maximal with this property. Suppose that I' is a proper extension of I. Then $I' \cap K \neq 0$. Therefore $K \cap I \neq 0$, a contradiction. □

This completes the proofs of the implications. That no others exist is shown by the following sequence of examples.

EXAMPLES 5.6. If an ideal I has no proper essential extension, then I need not be closed.

Let L be as in Example 2.2, and let $I = \langle w, \delta, \varepsilon \rangle$. We have proved in Example 2.2 that I is not closed. We claim that I has no proper essential extension. Let $I < J \lhd L$. If $u \in J \backslash I$ then either $u \approx x$, $u \approx y$, or $u \in I + \langle z \rangle$ and $u \approx z$. In the third case, $J \supseteq I + \langle z \rangle$. Suppose $u \approx x$. Then $J \ni [u, y] = \alpha z + \beta w$ where $\alpha \neq 0$. Therefore $z \in J$. Similarly if $u \approx y$ we have $z \in J$. We conclude that $J \supseteq I + \langle z \rangle$. But $\langle z \rangle \lhd L$ and $\langle z \rangle \cap I = 0$, so J is not an essential extension. □

EXAMPLE 5.7. Complement ideals need not be closed.
This follows from Example 5.6 and Proposition 5.4. □

It follows that complement ideals need not be *closed, since *closed implies closed.

EXAMPLE 5.8. If I is a closed ideal of L and $I \subseteq J \lhd L$, it need not follow that I is closed in J.

Let L be as in Example 2.2, and let $Z = \langle z \rangle$. First we claim that Z is closed in L. To prove this, let $a \notin Z$. Then $a \approx x, y, \delta, \varepsilon$, or $a \in \langle z, w \rangle$. If any of the former hold, then $w \in \langle a \rangle^L$. Now $\langle w \rangle^L = \langle w \rangle$, so $\langle w \rangle^L \cap Z = 0$. Therefore $a \notin cl(Z)$. It follows that $cl(Z) \subseteq \langle z, w \rangle = U$, say. If $u \in U$ then $\langle u \rangle^L = \langle u \rangle$. Therefore $u \in cl(Z)$ if and only if $u \in Z$. So Z is closed.

Let $B = \langle x, y, z, w \rangle \lhd L$. We claim that Z is not closed in B, and indeed that $x \in cl_B(Z)$. To prove this, observe that $\langle x \rangle^B = \langle x \rangle + \langle z \rangle$ and that the only ideals of B lying inside $\langle x \rangle + \langle z \rangle$ are $\langle x \rangle + \langle z \rangle$, $\langle z \rangle$, and 0. Thus if $0 \neq y \in \langle x \rangle + \langle z \rangle$, we have $\langle y \rangle^B \ni z$, so $\langle y \rangle^B \cap Z \neq 0$. □

To end this section we show that three plausible conjectures are false.

EXAMPLE 5.9. The sum of complement ideals need not be a complement ideal.

Let L have basis $\{x, y, z, u, v\}$ where

$$[x, y] = z = -[y, x]$$

$$[x, u] = u = -[u, x]$$

$$[y, v] = v = -[v, y]$$

and all other products are zero. The Jacobi identity follows easily. Let $V = \langle u, v\rangle \lhd L$, $Z = \langle z\rangle \lhd L$. We claim both V and Z are complement ideals. Clearly $V \cap Z = 0$.

Suppose that $V' \supseteq V$, $V' \cap Z = 0$. If $a \in V'$ and $a \approx x$ then without loss of generality $a = x + \lambda v + b$ for $\lambda \in k$, $b \in \langle y, z, u\rangle$. Then V' contains $[a, y] = z - \lambda v$, so V' contains z. Therefore $a \not\approx x$. Similarly $a \not\approx y$. Thus $a \in V + Z$, so $a \in V$, and V is a complement ideal.

Now let $Z' \supseteq Z$, $V \cap Z' = 0$. Let $a \in Z' \backslash Z$. Then $a \approx x$ or $a \approx y$. If $a \approx x$ then $[a, u] = \lambda u$, $\lambda \neq 0$, a contradiction. Similarly $a \approx y$, a contradiction, so Z is a complement ideal.

On the other hand, $Z + V$ is essential and proper in L, so is not a complement ideal. □

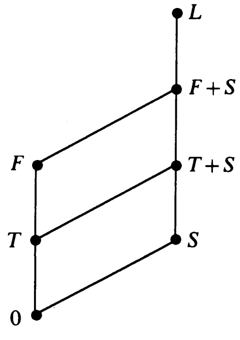
EXAMPLE 5.10. Sums of closed ideals need not be closed.

Take L as in example 5.9. It can easily be shown that Z and V are closed. But $Z + V$ is essential, so is not closed. □

We do not know whether sums of *closed ideals are always *closed.

EXAMPLE 5.11. Part (d) of Theorem 2.1 is false for cl^*.

Let $L = \mathscr{L}(V)$, the Lie algebra of all linear maps $V \to V$ where $\dim_k V = \aleph_0$. By Stewart [10], see also Amayo and Stewart [4] p. 173, the lattice of ideals of L is



where F is the algebra of linear maps of finite rank, T those of trace zero, and S the scalar multiples of the identity. The essential ideals are $T + S$, $F + S$, and L. It follows easily that $cl^*(T) = L$, and then $S \cap T = 0$ but $S \cap cl^*(T) \neq 0$.

6. Chain conditions

We define the classes Max-CL and Max-CL* of Lie algebras with the ascending chain conditions on closed and *closed ideals respectively.

PROPOSITION 6.1. (a) *If* $I \supseteq L^2$ *then* $cl^*(I) = L$.
(b) *If* $\zeta_1(L)$ *is essential in* L *then* $cl^*(I) = L$ *for all* $I \lhd L$ *and* $L \in$ Max-CL*.
(c) *If* L *is hypercentral then* $cl^*(I) = L$ *for all* $I \lhd L$ *and* $L \in$ Max-CL*.

PROOF. (a) Take $E = L$ in the definition of $cl^*(I)$.
(b) Take $E = \zeta_1(L)$ in the definition of $cl^*(I)$.
(c) If L is hypercentral then $\zeta_1(L)$ is essential: now use part (b). □

Theorem 4.1 implies that

$$\mathcal{S} \cap \text{Max-c} = \mathcal{S} \cap \text{Max-ci} = \mathcal{S} \cap \text{Max-cl} = = \mathcal{S} \cap \text{Max-cl}^*$$

where $\mathcal{S}$ is the class of semisimple Lie algebras. Here Max-C is the class of Lie algebras with the maximal condition on centralizer ideals, and Max-CI is the class of Lie algebras with the maximal condition on complement ideals: see Aldosray and Stewart [3].

The next theorem answers Question 1.7 of Aldosray and Stewart [3]:

THEOREM 6.2. *Let* $I \lhd L$ *and suppose that* L/I *has no infinite direct sum of ideals, and* I *contains no infinite direct sum of ideals. Then* L *contains no infinite direct sum of ideals.*

PROOF. This is a direct consequence of Proposition 3.13c of Goodearl [8] applied to L, considered as a module over its universal enveloping algebra. □

Since Lemma 2.2 of [2] shows that L has no infinite direct sum of ideals if and only if $L \in$ Max-CI, we have:

COROLLARY 6.3. Max-CI *is* E-*closed.* □

LEMMA 6.4. (a) *Let* I *be a complement ideal to* J *in* L. *Then* $I \oplus J$ *is essential in* L.

(b) *Let* I *be a complement ideal in* L, *and suppose that* $I \subseteq K$ *where* K *is an essential ideal of* L. *Then* K/I *is an essential ideal of* L/I.

PROOF. (a) If there exists a nonzero ideal P of L such that $(I \oplus J) \cap P = 0$ then $L = (I \oplus P) \oplus J$ contradicting I being a complement ideal. Therefore $I \cap P \neq 0$, so I is essential.

(b) If K/I is not essential in L/I then there exists an ideal P of L such that $I \subsetneqq P$ and $P \cap K = 0$. Since I is a complement ideal of L there exists an ideal J of L such that I is maximal with respect to $I \cap J = 0$. We claim $P \cap J = 0$. If not, then since K is essential, $K \cap (P \cap J) \neq 0$, so $(K \cap P) \cap J \neq 0$, so $I \cap J \neq 0$, which is a contradiction. But $P \cap J = 0$ and $I \subsetneqq P$ contradicts maximality of I with respect to $I \cap J = 0$. □

THEOREM 6.5. Max-CI = Min-CI.

PROOF. Suppose $L \in$ Max-CI and let $I_0 \supseteq I_1 \supseteq \dots$ be a descending chain of ideals. Inductively choose complement ideals K_i to I_i such that $K_i \subseteq K_{i+1}$. (This can be done using a Zorn's lemma argument, but taking into account the ascending chain condition. Inductively choose K_{i+1} maximal subject to $K_{i+1} \supseteq K_i$ and $K_{i+1} \cap I_{i+1} = 0$.) By Max-CI the chain stops, say $K_{i+1} = K_i$. By the modular law, $(I_{i+1} \oplus K_i) \cap I_i = I_{i+1}$, so that

$$\frac{I_{i+1} \oplus K_{i+1}}{I_{i+1}} \cap \frac{I_i}{I_{i+1}} = 0\,.$$

But K_{i+1} is a complement to I_{i+1}, so that $I_{i+1} \oplus K_{i+1}$ is essential in L by Lemma 6.4a. Since I_{i+1} is a complement ideal in L and $I_{i+1} \subseteq I_{I=1} \oplus K_{i+1}$, it follows that $(I_{i+1} \oplus K_{i+1})/I_{i+1}$ is essential in L/I_{i+1} by Lemma 6.4b. Therefore $I_i/I_{i+1} = 0$ so that $I_i = I_{i+1}$. Therefore the chain of ideals stops.

The converse is similar. □

EXAMPLE 6.6. Max-CL* does not imply Max-CL or Max-CI.

Let L be infinite-dimensional abelian. Then $L \in$ Max-CL* but $L \notin$ Max-CL and $L \notin$ Max-CI. □

EXAMPLE 6.7. Max-CL is not Q-closed.

Let $A = \mathbf{k}[x_1, x_2, \dots]$ be the polynomial algebra in countably many commuting indeterminates, considered as an abelian Lie algebra. Define derivations δ_i such that $\delta_i(f) = x_i f$ for $f \in A$. The δ_i commute. Let $D = \langle \delta_i \mid i = 1, 2, \dots \rangle$ and $L = A + D$. Then $L \in$ Max-CI by Aldosray and Stewart [3] Example 1.2, so $L \in$ Max-CL by Corollary 5.3 and Proposition 5.4. But L/A is infinite-dimensional abelian. However, abelian algebras with Max-CL are obviously finite-dimensional.

PROPOSITION 6.8. *If* $L/I \in$ Max-CL (*respectively* Max-CL*) *for all nonzero closed* (*respectively* **closed*) *ideals* I *of* L, *then* $L \in$ Max-CL (*respectively* Max-CL*).

PROOF. Let $0 \neq I_0 \subseteq I_1 \subseteq \dots \subseteq I_n \subseteq \dots$ be an ascending chain of closed (respectively *closed) ideals of L. Then $I_1/I_0 \subseteq I_2/I_0 \subseteq \dots$ is an ascending chain of closed (respectively *closed) ideals of L/I_0, by Lemma 2. If (respectively Lemma 3.1f). But $L/I_0 \in$ Max-CL (respectively Max-CL*) so the chain stops.

We end with two open questions:

QUESTIONS 6.9. 1. Is Max-CL E-closed?
2. Is Max-CL* E-closed?

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