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Quandle cocycle invariants of pretzel links

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ABSTRACT. In this paper, we give a formula for the quandle cocycle invariants associated with the Fox shadow p-coloring of all pretzel links.

1. Introduction

A quandle cocycle invariant of a link L is defined when a quandle 3-cocycle is fixed. In this paper, we consider a quandle cocycle invariant $\Psi_p(L)$ associated with the 3-cocycle θ_p of the dihedral quandle of order p founded by Mochizuki [7], where p is an odd integer. This invariant takes value in a Laurent polynomial ring $\mathbf{Z}[T, T^{-1}]/(T^p - 1)$. It is calculated when L is a torus knot and p is an odd prime [2], when L is a 2-bridge knot and p is an odd prime [6], and when L is a 3-braid knot and $p = 3$ [10]. The purpose of this paper is to calculate the invariants $\Psi_p(L)$ for all pretzel links. (Asami [1] calculates quandle cocycle invariants of alternating odd pretzel knots. However his result (Proposition 4 of [1]) is not correct.)

For integers $m_1, \dots, m_n$, we denote by $D(m_1, \dots, m_n)$ the diagram shown in Fig. 1, where m_i indicates an m_i half twist on the i-th column for each i. We call such a diagram a *pretzel link diagram*. A link represented by $D(m_1, \dots, m_n)$ is called a *pretzel link* and denoted by $P(m_1, \dots, m_n)$.

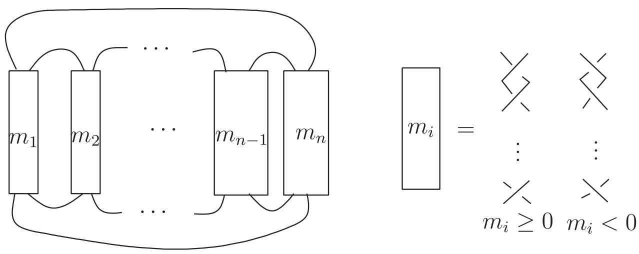

Fig. 1

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Let p be an odd prime integer. We denote $\{i \in \{1, \dots, n\} \mid m_i \equiv 0 \pmod{p}\}$ by $I_p = I_p(m_1, \dots, m_n)$. The *p-fundamental sequence* of $(m_1, \dots, m_n)$ is the sequence obtained from $(m_1, \dots, m_n)$ by dropping all elements that are not multiples of p, and is denoted by $F_p = F_p(m_1, \dots, m_n)$. For example, $F_3(4, -6, 5, 3, -2, 0, 1) = (-6, 3, 0)$. For an element x in $\mathbf{Z}_p$ (or $\mathbf{Z}_{p^2}$, resp.), we denote by $\bar{x}$ (or x^{-1}, resp.) the multiplicative inverse element of x if there exists. Put $\overline{W} = \overline{m_1} + \overline{m_2} + \cdots + \overline{m_n} \in \mathbf{Z}_p$ and $\hat{W} = m_1^{-1} + \cdots + m_n^{-1} \in \mathbf{Z}_{p^2}$ when $I_p = \phi$.

Theorem 1.1. *Let $L = P(m_1, \dots, m_n)$ be a pretzel link and p be an odd prime integer. Let $(M_1, \dots, M_l)$ be the p-fundamental sequence of $(m_1, \dots, m_n)$, where $l = |I_p|$.*

(i) *If $l \geq 2$, then*

$$\Psi_p(L) = p^2 \sum_{s_1 + \cdots + s_l \equiv 0} T^{-\{(M_1/p)s_1^2 + \cdots + (M_l/p)s_l^2\}},$$

where $s_1, \dots, s_l$ run over $0, 1, \dots, p-1$ such that $s_1 + \cdots + s_l \equiv 0 \pmod{p}$.

(ii) *If $l = 0$ and $\overline{W} \equiv 0 \pmod{p}$ (i.e., $\hat{W}$ is divisible by p), then*

$$\Psi_p(L) = p^2 \sum_{s=0}^{p-1} T^{(\hat{W}/p)s^2}.$$

(iii) *If l and $\overline{W}$ are not in the case of* (i) *nor* (ii)*, then*

$$\Psi_p(L) = p^2.$$

By Theorem 1.1, we see that the invariant $\Psi_p(L)$ of a pretzel link $L = P(m_1, \dots, m_n)$ does not depend on the order of the sequence $(m_1, \dots, m_n)$, and that if $l \geq 2$, then $\Psi_p(L)$ depends only on the p-fundamental sequence of $(m_1, \dots, m_n)$.

For a surface link S in $\mathbf{R}^4$, we denoted by $\Phi_p(S)$ the quandle cocycle invariant of S (defined in [3]) associated with the 3-cocycle θ_p. As a corollary of Theorem 1.1, we have quandle cocycle invariants of twist-spin of pretzel links.

Corollary 1.2. *Let $L = P(m_1, \dots, m_n)$ be a pretzel link and p be an odd prime integer. Let r be an even integer and let $\tau^r L$ be an r-twist-spin of L. Let $(M_1, \dots, M_l)$ be the p-fundamental sequence of $(m_1, \dots, m_n)$, where $l = |I_p|$.*

(i) *If $l \geq 2$, then*

$$\Phi_p(\tau^r L) = p \sum_{s_1 + \cdots + s_l \equiv 0} T^{-r\{(M_1/p)s_1^2 + \cdots + (M_l/p)s_l^2\}},$$

where $s_1, \dots, s_l$ run over $0, 1, \dots, p-1$ such that $s_1 + \cdots + s_l \equiv 0 \pmod{p}$.

(ii) *If* $l = 0$ *and* $\overline{W} \equiv 0 \pmod{p}$ *(i.e.,* $\hat{W}$ *is divisible by* p*), then*

$$\Phi_p(\tau^r L) = p \sum_{s=0}^{p-1} T^{r(\hat{W}/p)s^2}.$$

(iii) *If* l *and* $\overline{W}$ *are not in the case of* (i) *nor* (ii)*, then*

$$\Phi_p(\tau^r L) = p.$$

When r is an odd integer, any diagram of $\tau^r L$ has only p trivial p-colorings, and hence $\Phi_p(\tau^r L) = p$ (cf. [9]).

Throughout this paper, by $x \equiv y$ and $x \equiv_2 y$, we mean $x \equiv y \pmod{p}$ and $x \equiv y \pmod{p^2}$, respectively. Furthermore, for a color a of an arc or a region, we fix a representative element in $\mathbf{Z}$, and it may be also denoted by the same symbol a. Some elements in $\mathbf{Z}_{p^2}$ may be defined by using such representative elements. For example, $\delta \equiv_2 a' - a$ in Lemma 2.1 is an element in $\mathbf{Z}_{p^2}$ such that $\delta \equiv_2 z' - z$, where z and z' are fixed representative elements in $\mathbf{Z}$ of a and a', respectively. In §2, we review quandle cocycle invariants. We prove Theorem 1.1 in §3, and Corollary 1.2 in §4.

2. Quandle cocycle invariants

A quandle cocycle invariant of a link associated with a 3-cocycle f of a finite quadle Q, which is based on [3, 8], is defined in [4]. When we calculate it, we need to take account of signs of crossing points. However, when Q is the dihedral quandle of order p and f is Mochizuki's 3-cocycle θ_p, we do not need to be careful about this as seen below (cf. [6, 11]).

Let D be a diagram of a link (or a tangle) L, and $\Sigma(D)$ the set of arcs of D. A map $C : \Sigma(D) \to \mathbf{Z}_p$ is a *p-coloring* of D if $C(\mu_1) + C(\mu_2) \equiv 2C(\nu)$ at each crossing x, where μ_1 and μ_2 are under-arcs separated by an over-arc ν. A *shadow p-coloring of D extending C* is a map $\tilde{C} : \tilde{\Sigma}(D) \to \mathbf{Z}_p$, where $\tilde{\Sigma}(D)$ is the union of $\Sigma(D)$ and the set of regions separated by the underlying immersed curve of D, satisfying the following conditions: (i) $\tilde{C}$ restricted to $\Sigma(D)$ coincides with C, and (ii) if λ_1 and λ_2 are regions separated by an arc μ, then $\tilde{C}(\lambda_1) + \tilde{C}(\lambda_2) \equiv 2\tilde{C}(\mu)$. The set of p-colorings (or shadow p-colorings) is denoted by $Col_p(D)$ (or $\widetilde{Col_p}(D)$). A p-coloring of D is *trivial* if the image of the p-coloring consists of a single element; otherwise *non-trivial*. A shadow p-coloring $\tilde{C}$ of D is *trivial* if $\tilde{C}$ restricted to $\Sigma(D)$ is a trivial coloring; otherwise *non-trivial*.

Let $\tilde{C}$ be a shadow p-coloring, and at a crossing point x, let a, b, c, R and $R' \in \mathbf{Z}_p$ be the colors of three arcs and two regions as in Fig. 2. We note that

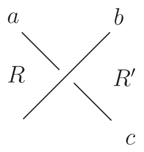

Fig. 2

$(R-a)\dfrac{a^p+c^p-2b^p}{p} \equiv (R'-c)\dfrac{a^p+c^p-2b^p}{p} \in \mathbf{Z}_p$. Thus, we can define the Boltzmann weight at x by

$$W_p(x;\tilde{C}) = (R-a)\frac{a^p+c^p-2b^p}{p} \in \mathbf{Z}_p.$$

We put $W_p(\tilde{C}) = \sum_x W_p(x;\tilde{C})$, where x runs over all crossings of the diagram D. Consider the state-sum

$$\Psi_p(D) = \sum_{\tilde{C} \in \widetilde{Col}_p(D)} T^{W_p(\tilde{C})} \in \mathbf{Z}[T, T^{-1}]/(T^p - 1).$$

The state-sum $\Psi_p(D)$ is invariant under Reidemeister moves and does not depend on a choice of the orientation of L, and hence we may denote it by $\Psi_p(L)$ (cf. [6, 11]). This is equal to the quandle cocycle invariant associated with Mochizuki's 3-cocycle θ_p, in the sense of [4].

We need the following lemma and corollary for later calculations.

Lemma 2.1 ([11]). *Let D be a tangle diagram and let R, a and a' be the colors of the region and two arcs in D as in Fig. 3 by a shadow p-coloring $\tilde{C}$, where m in Fig. 3 indicates m half twist. Then,*

$$\begin{aligned} W_p(\tilde{C}) \equiv\ & \frac{(R-a)a^p + (a-R-\delta)a'^p}{p} \\ & + \frac{(a-R+m\delta)(m\delta+a)^p + (R-a-(m-1)\delta)(m\delta+a')^p}{p}, \end{aligned}$$

where $\delta \equiv_2 a' - a$.

Corollary 2.2. *In Lemma 2.1, if m is divisible by p, then*

$$W_p(\tilde{C}) \equiv \frac{-m\delta^2}{p}.$$

Proof. Since m is divisible by p, $\dfrac{(m\delta+x)^p}{p} \equiv \dfrac{x^p}{p}$ for any $x \in \mathbf{Z}_p$. Therefore, by Lemma 2.1,

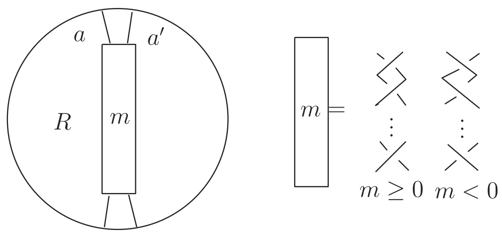

Fig. 3

$$W_p(\tilde{C}) \equiv \frac{(R-a)a^p + (a-R-\delta)a'^p}{p} + \frac{(a-R+m\delta)a^p + (R-a-(m-1)\delta)a'^p}{p}$$
$$\equiv \frac{m\delta(a^p - a'^p)}{p}.$$

Since m is divisible by p,

$$W_p(\tilde{C}) \equiv \frac{m\delta(a-a')}{p} = \frac{-m\delta^2}{p}. \qquad \square$$

3. Pretzel links

Let $L = P(m_1, \dots, m_n)$ be a pretzel link and $D = D(m_1, \dots, m_n)$ be a pretzel link diagram. Let μ_i, ν_i and λ_i be the top-, bottom-left arcs and the left-side region of the i-th column for each i. (Put $\mu_{n+1} = \mu_1$, $\nu_{n+1} = \nu_1$ and $\lambda_{n+1} = \lambda_1$.) See Fig. 4. Let $(M_1, \dots, M_l)$ be the p-fundamental sequence of $(m_1, \dots, m_n)$.

Lemma 3.1. *Let $D = D(m_1, \dots, m_n)$ be a pretzel link diagram such that $I_p \neq \phi$.*

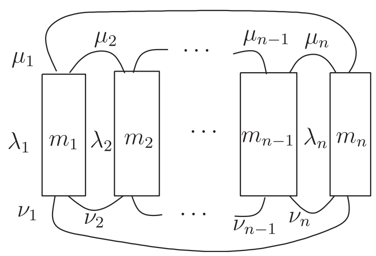

Fig. 4

(i) *Any p-coloring C of D satisfies a condition that* $C(\mu_i) \equiv C(\mu_{i+1})$ *if* $i \notin I_p$.

(ii) *For* $a_1, \dots, a_l \in \mathbf{Z}_p$, *there is a unique p-coloring* $C = C(a_1, \dots, a_l)$ *of D such that* $C(\mu_{\tau_i}) \equiv a_i$, *where* $\{\tau_1, \dots, \tau_l\} = I_p$, $\tau_1 < \dots < \tau_l$.

(iii) *Any p-coloring of D is represented as* $C(a_1, \dots, a_l)$ *for some* $a_1, \dots, a_l \in \mathbf{Z}_p$.

(iv) *If* $|I_p| = 1$, $Col_p(D)$ *consists of trivial colorings.*

Proof. Put $\mu_{n+i} = \mu_i$ and $\nu_{n+i} = \nu_i$ for each i. We remark that $C(\nu_i) \equiv m_i(C(\mu_{i+1}) - C(\mu_i)) + C(\mu_i)$ and $C(\nu_{i+1}) \equiv m_i(C(\mu_{i+1}) - C(\mu_i)) + C(\mu_{i+1})$ for each i. Hence we have $C(\nu_i) - C(\mu_i) \equiv C(\nu_{i+1}) - C(\mu_{i+1})$ for each i. Therefore

$$C(\nu_1) - C(\mu_1) \equiv C(\nu_2) - C(\mu_2) \equiv \dots \equiv C(\nu_n) - C(\mu_n).$$

Since $I_p \neq \phi$, there exists an integer $i_0 \in I_p$ with $m_{i_0} \equiv 0$. Since $C(\nu_{i_0}) \equiv m_{i_0}(C(\mu_{i_0+1}) - C(\mu_{i_0})) + C(\mu_{i_0})$, we have $C(\nu_{i_0}) \equiv C(\mu_{i_0})$. Since $C(\nu_i) - (\mu_i) \equiv C(\nu_{i_0}) - C(\mu_{i_0})$ for each i, we have $C(\nu_i) \equiv C(\mu_i)$ for any i. Thus, if $i \notin I_p$, then $C(\mu_i) \equiv C(\mu_{i+1})$ by $C(\nu_i) \equiv m_i(C(\mu_{i+1}) - C(\mu_i)) + C(\mu_i)$ and $C(\nu_i) \equiv C(\mu_i)$. This completes the proof of (i) and (iv) and induces that any p-coloring is deteremined by the colors of $\mu_{\tau_1}, \dots, \mu_{\tau_l}$. For $a_1, \dots, a_l \in \mathbf{Z}_p$, we can construct a unique p-coloring $C = C(a_1, \dots, a_l)$ of D such that $C(\mu_{\tau_i}) \equiv a_i$ for each i. Concretely, there is a p-coloring C satisfying the following conditions; see Fig. 5:

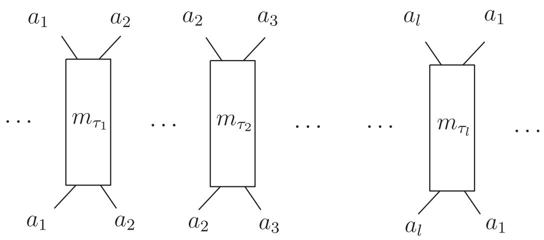

Fig. 5

(a) $C(\mu_{\tau_i}) \equiv C(\nu_{\tau_i}) \equiv a_i$ for each i.

(b) If $\tau_i < j < \tau_{i+1}$ $(j \notin I_p)$, then $C(x) \equiv a_{i+1}$ for any arc x in the i-th column.

Thus, we have (ii). The statement (iii) follows from the fact that any p-coloring C' can be represented as $C(C'(\mu_{\tau_1}), \dots, C'(\mu_{\tau_l}))$. □

When $I_p \neq \phi$, by Lemma 3.1 (ii), for any $a_1, \dots, a_l$ and $R \in \mathbf{Z}_p$, we see that there is a unique shadow p-coloring $\tilde{C} = \tilde{C}(a_1, \dots, a_l, R)$ such that $\tilde{C}|_{\Sigma(D)} = C$

and $\tilde{C}(\lambda_1) = R$ for the unbounded region λ_1. Any shadow p-coloring is represented as $\tilde{C}(a_1, \dots, a_l, R)$ by Lemma 3.1 (iii).

PROPOSITION 3.2. *Let $D = D(m_1, \dots, m_n)$ be a pretzel link diagram such that $|I_p| \geq 2$ and $(M_1, \dots, M_l)$ be the p-fundamental sequence of $(m_1, \dots, m_n)$. Let $\tilde{C} = \tilde{C}(a_1, \dots, a_l, R)$. Put $s_i \equiv_2 a_{i+1} - a_i \in \mathbf{Z}_{p^2}$ for $i = 1, \dots, l-1$ and $s_l \equiv_2 a_1 - a_l$. Then,*

$$W_p(\tilde{C}) \equiv -\frac{M_1 s_1^2 + \cdots + M_l s_l^2}{p}.$$

PROOF. Let X_i be the set of crossing points of the i-th column of D for each i. By Lemma 3.1 (i), if $i \notin I_p$, then $\tilde{C}(\mu_i) \equiv \tilde{C}(\mu_{i+1})$, so $\sum_{x \in X_i} W_p(x; \tilde{C}) \equiv 0$. Thus, by Corollary 2.2, $W_p(\tilde{C}) \equiv \sum_{i=1}^{n} \sum_{x \in X_i} W_p(x; \tilde{C}) \equiv \sum_{i \in I_p} \sum_{x \in X_i} W_p(x; \tilde{C}) \equiv -\sum_{i=1}^{l} \frac{M_i (a_{i+1} - a_i)^2}{p} \equiv -\sum_{i=1}^{l} \frac{M_i s_i^2}{p}$. □

LEMMA 3.3. *Let $D = D(m_1, \dots, m_n)$ be a pretzel link diagram such that $I_p = \phi$. Then,*

(i) *there is a non-trivial p-coloring of D if and only if $\overline{W} \equiv 0$. If $\overline{W} \equiv 0$, then*

(ii) *for $a_1, b_1 \in \mathbf{Z}_p$, there is a unique p-coloring $C = C(a_1, b_1)$ such that $C(\mu_1) = a_1$ and $C(v_1) = b_1$,*

(iii) *any p-coloring of D is represented by $C(a_1, b_1)$ for some a_1 and b_1.*

PROOF. Since $C(v_i) \equiv m_i(C(\mu_{i+1}) - C(\mu_i)) + C(\mu_i)$ and $C(v_{i+1}) \equiv m_i(C(\mu_{i+1}) - C(\mu_i)) + C(\mu_{i+1})$ for each i, we have $C(\mu_2) \equiv \overline{m_1}(C(v_1) - C(\mu_1)) + C(\mu_1)$ and $C(v_2) \equiv \overline{m_1}(C(v_1) - C(\mu_1)) + C(v_1)$. In the same way, we have $C(\mu_i) \equiv C(\mu_1) + \left(\sum_{j=1}^{i-1} \overline{m_j}\right)(C(v_1) - C(\mu_1))$ and $C(v_i) \equiv C(v_1) + \left(\sum_{j=1}^{i-1} \overline{m_j}\right)(C(v_1) - C(\mu_1))$ for each i. In particular, $C(\mu_1) \equiv C(\mu_{n+1}) \equiv \overline{W}(C(v_1) - C(\mu_1)) + C(\mu_1)$ and $C(v_1) \equiv C(v_{n+1}) \equiv \overline{W}(C(v_1) - C(\mu_1)) + C(v_1)$. Thus, there is a p-coloring C if and only if $\overline{W} \equiv 0$ or $C(\mu_1) \equiv C(v_1)$. When $C(\mu_1) \equiv C(v_1)$, C is trivial. When $\overline{W} \equiv 0$ and $C(\mu_1) \not\equiv C(v_1)$, C is non-trivial. Thus, we have (i). For $a_1, b_1 \in \mathbf{Z}_p$, we can construct a unique p-coloring $C = C(a_1, b_1)$ of D such that $C(\mu_1) \equiv a_1$ and $C(v_1) \equiv b_1$. Concretely, there is a p-coloring C such that $C(\mu_i) \equiv a_1 + \left(\sum_{j=1}^{i-1} \overline{m_j}\right)(b_1 - a_1)$ and $C(v_i) \equiv b_1 + \left(\sum_{j=1}^{i-1} \overline{m_j}\right)(b_1 - a_1)$. This completes the proof of (ii). The statement (iii) follows from the fact that any p-coloring C' can be represented as $C(C'(\mu_1), C'(v_1))$. □

When $I_p = \phi$ and $\overline{W} \equiv 0$, by Lemma 3.3 (ii), for any a_1, b_1 and $R \in \mathbf{Z}_p$, we see that there is a unique shadow p-coloring $\tilde{C} = \tilde{C}(a_1, b_1, R)$ such that $\tilde{C}|_{\Sigma(D)} = C$ and $\tilde{C}(\lambda_1) = R$ for the unbounded region λ_1. Any shadow p-coloring is represented as $\tilde{C}(a_1, b_1, R)$ by Lemma 3.3 (iii). Proposition 3.4 follows from Lemmas 3.5 and 3.6.

Proposition 3.4. *Let $D = D(m_1, \dots, m_n)$ be a pretzel link diagram such that $I_p = \phi$ and $\overline{W} \equiv 0$. Let $\tilde{C} = \tilde{C}(a_1, b_1, R)$. Then,*

$$W_p(\tilde{C}) \equiv \frac{\hat{W} c_0^2}{p},$$

where $c_0 \equiv_2 b_1 - a_1$.

For a shadow p-coloring $\tilde{C} = \tilde{C}(a_1, b_1, R)$, let $a_2, \dots, a_n$ be the elements in $\mathbf{Z}_p$ such that $\tilde{C}(\mu_i) = a_i$. Put $a_{n+1} = a_1$. Let A_i and B_i be the elememts in $\mathbf{Z}_{p^2}$ such that

$$A_i \equiv_2 (R_i - a_i)a_i^p + (a_i - R_i - \delta_i)a_{i+1}^p$$

and

$$B_i \equiv_2 (a_i - R_i + m_i\delta_i)(m_i\delta_i + a_i)^p + (R_i - a_i - (m_i - 1)\delta_i)(m_i\delta_i + a_{i+1})^p,$$

where

$$\delta_i \equiv_2 a_{i+1} - a_i \quad \text{and} \quad R_i \equiv_2 R + 2\sum_{j=1}^{i-1} \delta_j,$$

for each i. It is seen that $c_0 \equiv m_1\delta_1 \equiv \cdots \equiv m_n\delta_n$ and $\tilde{C}(\lambda_i) \equiv R_i$, so we will assume that for each i, R_i is given by a representative element in $\mathbf{Z}$ for $\tilde{C}(\lambda_i)$. By Lemma 2.1,

$$W_p(\tilde{C}) \equiv \frac{1}{p}\sum_{i=1}^{n}(A_i + B_i).$$

Lemma 3.5.

$$\sum_{i=1}^{n} A_i \equiv_2 0.$$

Proof. Since $\delta_{i-1} \equiv_2 a_i - a_{i-1}$ and $R_i - R_{i-1} \equiv_2 2\delta_{i-1}$ for $i = 2, 3, \dots, n$, we have $(R_i - a_i) + (a_{i-1} - R_{i-1} - \delta_{i-1}) \equiv_2 0$ for $i = 2, 3, \dots, n$. Since $\delta_n \equiv_2 a_1 - a_n$ and $R_n \equiv_2 R_1 + 2\sum_{j=1}^{n-1} \delta_j$, we have $(R_1 - a_1) + (a_n - R_n - \delta_n) \equiv_2$

$-2\sum_{j=1}^{n}\delta_j \equiv_2 0$. Therefore, $\sum_{i=1}^{n} A_i \equiv_2 (R_1 - a_1)a_1^p + \sum_{i=2}^{n} a_i^p\{(R_i - a_i) + (a_{i-1} - R_{i-1} - \delta_{i-1})\} + (a_n - R_n - \delta_n)a_1^p \equiv_2 0$. $\square$

Lemma 3.6.

$$\sum_{i=1}^{n} B_i \equiv_2 \hat{W}c_0^2.$$

Proof. Put $m_0 = m_n$ and $\delta_0 = \delta_n$. Let η_i be the element in $\mathbf{Z}_{p^2}$ such that $\eta_i \equiv_2 m_i\delta_i - m_{i-1}\delta_{i-1}$ for each i. By $c_0 \equiv m_1\delta_1 \equiv \cdots \equiv m_n\delta_n$, η_i is divisible by p. This induces $(m_1\delta_1 + x)^p \equiv_2 \cdots \equiv_2 (m_n\delta_n + x)^p$ for any $x \in \mathbf{Z}_{p^2}$. Thus, since $R_i - R_{i-1} \equiv_2 2\delta_{i-1}$ and η_i is divisible by p,

$$\begin{aligned}
\sum_{i=1}^{n} B_i &\equiv_2 \sum_{i=1}^{n}(m_i\delta_i - m_{i-1}\delta_{i-1})(m_i\delta_i + a_i)^p \\
&\equiv_2 \sum_{i=1}^{n}\eta_i(c_0 + a_i)^p \\
&\equiv_2 \sum_{i=1}^{n}\eta_i(c_0 + a_i) \\
&\equiv_2 \sum_{i=1}^{n}(-m_i\delta_i^2).
\end{aligned}$$

Let ξ_i be the element in $\mathbf{Z}_{p^2}$ such that $\xi_i = c_0 - m_i\delta_i$ for each i. Then, $\delta_i \equiv_2 (c_0 - \xi_i)m_i^{-1}$. Since $\delta_n = -(\delta_1 + \delta_2 + \cdots + \delta_{n-1})$,

$$\begin{aligned}
\sum_{i=1}^{n} B_i \equiv_2 {}& \sum_{i=1}^{n-1} -(m_i + m_n)\delta_i^2 - 2m_n\sum_{i=2}^{n-1}\sum_{j=1}^{i-1}\delta_i\delta_j \\
\equiv_2 {}& \sum_{i=1}^{n-1} -(m_i + m_n)(m_i^{-1})^2(c_0^2 - 2c_0\xi_i) \\
& - 2m_n\sum_{i=2}^{n-1}\sum_{j=1}^{i-1} m_i^{-1}m_j^{-1}(c_0^2 - (\xi_i + \xi_j)c_0) \\
\equiv_2 {}& -c_0^2(m_1^{-1} + \cdots + m_{n-1}^{-1} + m_n(m_1^{-1} + \cdots + m_{n-1}^{-1})^2) \\
& + 2c_0\sum_{i=1}^{n-1}\xi_i\left(m_i^{-1} + m_n(m_i^{-1})^2 + m_nm_i^{-1}\sum_{1\le j\le n, j\ne i} m_j^{-1}\right)
\end{aligned}$$

$$\equiv_2 -c_0^2(m_1^{-1} + \cdots + m_{n-1}^{-1})m_n\hat{W} + 2c_0\sum_{i=1}^{n-1}\xi_i m_i^{-1}m_n\hat{W}$$

$$\equiv_2 -c_0^2(\hat{W} - m_n^{-1})m_n\hat{W} + 2c_0\sum_{i=1}^{n-1}\xi_i m_i^{-1}m_n\hat{W}.$$

Since ξ_i and $\hat{W}$ are divisible by p,

$$\sum_{i=1}^{n} B_i \equiv_2 -c_0^2(m_n^{-1})m_n\hat{W} \equiv_2 c_0^2\hat{W}. \qquad \square$$

Proof of Theorem 1.1. (i) and (ii) follows from Propositions 3.2 and 3.4, respectively. For the remaining cases, all colorings are trivial by Lemmas 3.1 (iv) and 3.3 (i). Thus, we have (iii). $\square$

4. Twist-spin of pretzel links

Proof of Corollary 1.2. Asami and Satoh [2] defined a quandle cocycle invariant $\Psi_p^*(L)$ of a link L with a base point and proved that if r is even, $\Phi_p(\tau^r L) = \Psi_p^*(L)|_{T\to T^r}$. It is shown in [12] that $\Psi_p(L) = p\Psi_p^*(L)$, and hence $\Psi_p^*(L) = p^{-1}\Psi_p(L)$. Thus, Corollary 1.2 is obtained from Theorem 1.1. $\square$

Remark 4.1. For an n-component link L, there are n r-twist-spins of L. They need not to be equivalent to each other, but quandle cocycle invariants of them are the same by Corollary 1.2. For example, for a 2-component pretzel link $L = P(2, 3, 6m)$, there are two 2-twist-spins of L. One is a union of 2-twist-spin of a trefoil and an unknotted torus, and the other is a union of an unknotted 2-sphere and a knotted torus. Thus, they are not equivalent.

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