The Propositional Theory of Closure

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CONTENTS

We study the simplest fragment of topological theory: those statements that can be expressed using one set variable, interior and closure operators, and inclusion. We introduce a formal system that is simple enough to be implemented on a computer and exhaustively studied and yet rich enough to be sound and complete for the fragment of theory under consideration. This fragment is rich enough to capture concepts such as regular open sets, extremal disconnectedness, partition topologies, and the nodec property.

1. INTRODUCTION

It is well known that the structure of a topological space may be characterized by either its closure operator $\mathrm{cl}(\cdot)$ or its interior operator $\mathrm{int}(\cdot)$. Moreover, any operator $\mathrm{cl}(\cdot) : \mathbb{P}(X) \to \mathbb{P}(X)$ such that for all $A, B \subseteq X$,

1. $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$,
2. $A \subseteq \mathrm{cl}(A)$,
3. $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$,
4. $\mathrm{cl}(\varnothing) = \varnothing$,

is the closure operator for a topology on X, with a similar characterization for interior operators (see, for example, [Engelking 89, Proposition 1.2.7]).

In this paper we explore the strength of a somewhat restricted theory, in which we are allowed only one set variable and a closure and an interior operator, and a relation symbol for $\subseteq$. We shall see that even this restricted language enables us to discuss familiar topological concepts such as the notion of regular open sets and that of extremally disconnected spaces.

We will explore formal systems in this restricted language, using "production systems" in which we identify a set S of well-formed formulas, a subset A of S of axioms, and a set of rules of inference; for $\Gamma \subseteq S$ and $\varphi \in S$ we write $\Gamma \vdash \varphi$ if there is a finite sequence $(\psi_1, \psi_2, \ldots, \psi_n)$

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with $\psi_n = \varphi$ such that each ψ_i is in Γ, is in A, or follows from earlier terms in the sequence by a rule of inference. Such a sequence is called a *derivation*. We will exploit the equivalence of such systems with approaches using consequence operators $\mathrm{Con} : \mathbb{P}(S) \to \mathbb{P}(S)$, where $\mathrm{Con}(\Gamma) = \{\varphi \in S \mid \Gamma \vdash \varphi\}$ is the smallest superset of Γ that contains A and is closed under the rules of inference. We will also use the fact that if $\Gamma, \Omega \subseteq S$ and $\varphi \in S$ with $\Omega \vdash \varphi$ and $\Gamma \vdash \psi$ for all $\psi \in \Omega$, then $\Gamma \vdash \varphi$.

We will describe a production system $\mathfrak{Cl}$ that is intended to capture the theory of our restricted language. We will examine this system, showing it to be sound and adequate. Because the system is so simple, it is amenable to implementation and testing on a computer: many of our conjectures were tested using scripts written in the language Perl, described in the last section.

2. THE PROPOSITIONAL LANGUAGE $\mathfrak{Cl}$

We now introduce the language of the proposed theory:

Definition 2.1. A *word* is a term that can be built from a single variable A and the two unary operations cl and int.

Examples of words are $\mathsf{cl\,int\,A}$ and $\mathsf{cl\,cl\,int\,cl\,int\,int\,A}$.

The semantic intention, of course, is that A represent a subset of a topological space and that cl and int represent the closure and interior operators $\mathrm{cl}(\cdot)$ and $\mathrm{int}(\cdot)$ in that space.

Given a word w and a subset A of a topological space X, let w_A denote the corresponding subset of X: more formally, we have

1. $\mathsf{A}_A = A$;
2. $(\mathsf{cl}\,w)_A = \mathrm{cl}(w_A)$;
3. $(\mathsf{int}\,w)_A = \mathrm{int}(w_A)$.

Using four properties that are true in any topological space, it is well known that there are only seven "different" words [Kuratowski 22].

Proposition 2.2. *Let X be a topological space, and let $A \subseteq X$. Then we have*

1. $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$*;*
2. $\mathrm{int}(\mathrm{int}(A)) = \mathrm{int}(A)$*;*
3. $\mathrm{cl}(\mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))) = \mathrm{cl}(\mathrm{int}(A))$*;*
4. $\mathrm{int}(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))) = \mathrm{int}(\mathrm{cl}(A))$*;*

From this it follows that we may restrict ourselves to the following seven "reduced" words:

$$\mathsf{A},\ \mathsf{cl\,A},\ \mathsf{int\,A},\ \mathsf{int\,cl\,A},\ \mathsf{cl\,int\,A},\ \mathsf{cl\,int\,cl\,A},\ \mathsf{int\,cl\,int\,A};$$

that is, for every word w there is a reduced word w' such that for every X and every $A \subseteq X$, we have $w_A = (w')_A$. We denote the above words by v_0 to v_6 respectively, and we denote $\{v_0, \ldots, v_6\}$ by W.

Definition 2.3. A *property* is a string of symbols of the form $w < v$, where w and v are reduced words. We denote the set of properties by $\mathcal{P}$.

Examples of properties include $\mathsf{A} < \mathsf{cl\,int\,A}$ and $\mathsf{int\,A} < \mathsf{cl\,A}$. We denote the property $v_i < v_j$ by (ij); for example, the property $\mathsf{A} < \mathsf{cl\,int\,A}$ will be denoted by (04).

Let A be a subset of a space X, and let P be the property $w < v$. We say that A has property P if $w_A \subseteq v_A$.

Definition 2.4. We define the operations $c, i : W \to W$ as follows:

w	$c(w)$	$i(w)$
A	cl A	int A
cl A	cl A	int cl A
int A	cl int A	int A
int cl A	cl int cl A	int cl A
cl int A	cl int A	int cl int A
cl int cl A	cl int cl A	int cl A
int cl int A	cl int A	int cl int A

Syntactically, $c(w)$ is obtained by prepending cl to w and then canceling according to the rules implied by Proposition 2.2, and $i(w)$ is obtained by prepending int and canceling. The semantic effect of the above operations is embodied in the following result.

Proposition 2.5. *Let $w \in W$, and let $A \subseteq X$ for some topological space X. Then we have*

1. $c(w)_A = \mathrm{cl}(w_A)$*;*
2. $i(w)_A = \mathrm{int}(w_A)$.

We may now introduce the system $\mathfrak{Cl}$. This system has as its well-formed formulas the properties $\mathcal{P}$. It has the following axioms:

A1 $\mathsf{A} < \mathsf{A}$.

A2 $w < c(w)$, for each $w \in W$.

A3 $i(w) < w$, for each $w \in W$.

It has the following rules of inference:

Cl1 From $w < v$ and $v < u$ infer $w < u$.

Cl2 From $w < v$ infer $i(w) < i(v)$.

Cl3 From $w < v$ infer $c(w) < c(v)$.

Definition 2.6. Let $P_1, P_2, \ldots, P_n, Q \in \mathcal{P}$. We write $P_1, P_2, \ldots, P_n \vdash_{\mathfrak{Cl}} Q$ if there is a derivation $R_1, R_2, \ldots, R_k$, where $R_k = Q$ and each R_i is an axiom, is P_j for some j, or follows from one or more of the earlier R_j's by one of the rules of inference. We write $P_1, P_2, \ldots, P_n \models Q$ if for every subset A of a topological space X, if A has property P_i for $1 \le i \le n$, then A has property Q.

As usual, our goal is to show that the syntactic notion and the semantic notion are equivalent.

Proposition 2.7. (Soundness of $\mathfrak{Cl}$.) *Let*

$$P_1, P_2, \ldots, P_n, Q \in \mathcal{P}.$$

If $P_1, P_2, \ldots, P_n \vdash_{\mathfrak{Cl}} Q$, then $P_1, P_2, \ldots, P_n \models Q$.

Proof: This follows easily by induction, since all the axioms and rules of inference are sound. □

The proof that $\mathfrak{Cl}$ is adequate will be given in Section 4. The system $\mathfrak{Cl}$ is implemented in the script `full_cons`, described in Section 7.1. When this script is invoked with input a list of properties $P_1, P_2, \ldots, P_n$, its output is a list of all properties Q such that $P_1, P_2, \ldots, P_n \vdash_{\mathfrak{Cl}} Q$.

3. ELEMENTARY PROPERTIES AND THE SYSTEM $\mathfrak{Cl}\_$

Although the set $\mathcal{P}$ of properties has only 49 elements, it is still too large to allow practical computations considering all subsets. We will identify a set $\mathcal{E}$ of "elementary" properties such that every property is equivalent to a (possibly empty) conjunction of elementary properties. First, we should define this notion of equivalence.

Definition 3.1. Let $P_1, P_2, \ldots, P_n, Q \in \mathcal{P}$. We write $Q \Leftrightarrow P_1 \wedge P_2 \wedge \cdots \wedge P_n$ (or $Q \Leftrightarrow \bigwedge_{i=1}^n P_i$) if

- for each i, $Q \vdash_{\mathfrak{Cl}} P_i$; and
- $P_1, P_2, \ldots, P_n \vdash_{\mathfrak{Cl}} Q$.

Note that it might have been more natural to define equivalence in terms of $\models$ rather than $\vdash_{\mathfrak{Cl}}$: when we have proved Theorem 4.5, we will see that the two notions of equivalence are equivalent. Note also that $Q \Leftrightarrow P_1 \wedge P_2 \wedge \cdots \wedge P_n$ if and only if $\mathrm{Con}_{\mathfrak{Cl}}(\{Q\}) = \mathrm{Con}_{\mathfrak{Cl}}(\{P_1, P_2, \ldots, P_n\})$.

We define $\mathcal{A} = \mathrm{Con}_{\mathfrak{Cl}}(\varnothing)$. The properties in $\mathcal{A}$ may be identified using the script `full_cons` with no properties input: if we do so, we find that there are 23 such "tautological" properties. As a result, the script `proper_cons` was written: this script is identical to `full_cons` except that the 23 tautological properties are suppressed from the output. For $P \in \mathcal{P}$, we define the set of proper consequences of P to be

$$\begin{aligned} \mathrm{Con}'_{\mathfrak{Cl}}(P) &= \mathrm{Con}_{\mathfrak{Cl}}(P) \smallsetminus \mathcal{A} \\ &= \{\, Q \in \mathcal{P} \mid P \vdash_{\mathfrak{Cl}} Q \text{ and } Q \text{ is not tautological} \,\}. \end{aligned}$$

Our first attempt at a definition of "elementary" is the following notion.

Definition 3.2. A property P is *absolutely elementary* if there is no subset $S = \{Q_1, Q_2, \ldots, Q_n\} \subseteq \mathcal{P} \smallsetminus \{P\}$ such that $P \Leftrightarrow \bigwedge_{i=1}^n Q_i$.

However, this fails to take into account pairs like (05) and (15), for which we have $(05) \Leftrightarrow (15)$, so neither property is absolutely elementary. However, $\mathrm{Con}'_{\mathfrak{Cl}}((05)) = \mathrm{Con}'_{\mathfrak{Cl}}((15)) = \{(05), (15)\}$, so any set of elementary properties must include one or the other of these two.

Definition 3.3. We order $\mathcal{P}$ lexicographically, so that $(ij) \preccurlyeq (i'j')$ if $i < i'$ or $i = i'$ and $j \le j'$. Then a property P is *canonical* if $P \preccurlyeq Q$ for every Q with $P \Leftrightarrow Q$.

To determine the canonical properties, we use the script `canonical`, described in Section 7.2. From this script we find that there are 21 canonical properties: we denote the set of canonical properties by $\mathcal{C}$. We have

$$\begin{aligned} \mathcal{C} = \{&(00), (02), (03), (04), (05), (06), (10), (12), (13), \\ &(16), (30), (34), (40), (42), (43), (46), (50), (52), \\ &(53), (56), (60)\}. \end{aligned}$$

We can now give a more suitable definition of *elementary*.

Definition 3.4. A property P is *elementary* if $P \in \mathcal{C}$ and there is no subset $S = \{Q_1, Q_2, \ldots, Q_n\} \subseteq \mathcal{C} \smallsetminus \{P\}$ such that $P \Leftrightarrow \bigwedge_{i=1}^n Q_i$.

The elementary properties $\mathcal{E}$ can be found using the script `find_elementaries` described in Section 7.4. They are

(02) $\mathsf{A} < \mathsf{int}\,\mathsf{A}$

(03) $\mathsf{A} < \mathsf{int}\,\mathsf{cl}\,\mathsf{A}$

(05) $\mathsf{A} < \mathsf{cl}\,\mathsf{int}\,\mathsf{cl}\,\mathsf{A}$

(10) $\mathsf{cl}\,\mathsf{A} < \mathsf{A}$

(34) $\mathsf{int}\,\mathsf{cl}\,\mathsf{A} < \mathsf{cl}\,\mathsf{int}\,\mathsf{A}$

(40) $\mathsf{cl}\,\mathsf{int}\,\mathsf{A} < \mathsf{A}$

(43) $\mathsf{cl}\,\mathsf{int}\,\mathsf{A} < \mathsf{int}\,\mathsf{cl}\,\mathsf{A}$

(46) $\mathsf{cl}\,\mathsf{int}\,\mathsf{A} < \mathsf{int}\,\mathsf{cl}\,\mathsf{int}\,\mathsf{A}$

(53) $\mathsf{cl}\,\mathsf{int}\,\mathsf{cl}\,\mathsf{A} < \mathsf{int}\,\mathsf{cl}\,\mathsf{A}$

(60) $\mathsf{int}\,\mathsf{cl}\,\mathsf{int}\,\mathsf{A} < \mathsf{A}$

Using the script `check_elementary` with this list of elementary properties, we can verify that each property is indeed (either tautological or) equivalent to a conjunction of properties in $\mathcal{E}$.

We note that many of these elementary properties have been studied in the literature; see, for example, [Cao et al. 01] and [Isomichi 71].

Definition 3.5. Let X be a topological space. A subset A of X is called:

- *regular open* if $A = \mathrm{int}(\mathrm{cl}(A))$;
- *regular closed* if $A = \mathrm{cl}(\mathrm{int}(A))$;
- *α-closed* if $\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))) \subseteq A$;
- *semiclosed* (or *supercondensed*) if $\mathrm{int}(\mathrm{cl}(A)) \subseteq A$;
- *preclosed* if $\mathrm{cl}(\mathrm{int}(A)) \subseteq A$;
- *β-closed* if $\mathrm{int}(\mathrm{cl}(\mathrm{int}(A))) \subseteq A$;
- *α-open* if $X \smallsetminus A$ is α-closed, or equivalently, if $A \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$;
- *semiopen* (or *subcondensed*) if $X \smallsetminus A$ is semiclosed, or equivalently, if $A \subseteq \mathrm{cl}(\mathrm{int}(A))$;
- *preopen* if $X \smallsetminus A$ is preclosed, or equivalently, if $A \subseteq \mathrm{int}(\mathrm{cl}(A))$;
- *β-open* if $X \smallsetminus A$ is β-closed, or equivalently, if $A \subseteq \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$.

Thus the elementary property (02) is "open," (10) is "closed," (03) is "preopen," (40) is "preclosed," (05) is "β-open," and (60) is "β-closed." The remaining properties in Definition 3.5 are conjunctions of elementary properties and are identified in Table 1.

For $S \subseteq \mathcal{P}$, we define the "elementary consequences" of S to be $ec(S) = \mathrm{Con}_{\mathfrak{Cl}}(S) \cap \mathcal{E}$.

We will now introduce the system $\mathfrak{Cl}_{-}$, which is essentially $\mathfrak{Cl}$ restricted to $\mathcal{E}$. The system has $\mathcal{E}$ as its set of well-formed formulas. It has no axioms, and has as its rules of inference the following:

Clm1 From (02) infer (03), i.e., from $\mathsf{A} < \mathsf{int}\,\mathsf{A}$ infer $\mathsf{A} < \mathsf{int}\,\mathsf{cl}\,\mathsf{A}$.

Clm2 From (03) infer (05), i.e., from $\mathsf{A} < \mathsf{int}\,\mathsf{cl}\,\mathsf{A}$ infer $\mathsf{A} < \mathsf{cl}\,\mathsf{int}\,\mathsf{cl}\,\mathsf{A}$.

Clm3 From (02) infer (34), i.e., from $\mathsf{A} < \mathsf{int}\,\mathsf{A}$ infer $\mathsf{int}\,\mathsf{cl}\,\mathsf{A} < \mathsf{cl}\,\mathsf{int}\,\mathsf{A}$.

Clm4 From (10) infer (34), i.e., from $\mathsf{cl}\,\mathsf{A} < \mathsf{A}$ infer $\mathsf{int}\,\mathsf{cl}\,\mathsf{A} < \mathsf{cl}\,\mathsf{int}\,\mathsf{A}$.

Clm5 From (10) infer (40), i.e., from $\mathsf{cl}\,\mathsf{A} < \mathsf{A}$ infer $\mathsf{cl}\,\mathsf{int}\,\mathsf{A} < \mathsf{A}$.

Clm6 From (40) infer (60), i.e., from $\mathsf{cl}\,\mathsf{int}\,\mathsf{A} < \mathsf{A}$ infer $\mathsf{int}\,\mathsf{cl}\,\mathsf{int}\,\mathsf{A} < \mathsf{A}$.

Clm7 From (53) infer (43), i.e., from $\mathsf{cl}\,\mathsf{int}\,\mathsf{cl}\,\mathsf{A} < \mathsf{int}\,\mathsf{cl}\,\mathsf{A}$ infer $\mathsf{cl}\,\mathsf{int}\,\mathsf{A} < \mathsf{int}\,\mathsf{cl}\,\mathsf{A}$.

Clm8 From (46) infer (43), i.e., from $\mathsf{cl}\,\mathsf{int}\,\mathsf{A} < \mathsf{int}\,\mathsf{cl}\,\mathsf{int}\,\mathsf{A}$ infer $\mathsf{cl}\,\mathsf{int}\,\mathsf{A} < \mathsf{int}\,\mathsf{cl}\,\mathsf{A}$.

Clm9 From (34) and (43) infer (46), i.e., from $\mathsf{int}\,\mathsf{cl}\,\mathsf{A} < \mathsf{cl}\,\mathsf{int}\,\mathsf{A}$ and $\mathsf{cl}\,\mathsf{int}\,\mathsf{A} < \mathsf{int}\,\mathsf{cl}\,\mathsf{A}$ infer $\mathsf{cl}\,\mathsf{int}\,\mathsf{A} < \mathsf{int}\,\mathsf{cl}\,\mathsf{int}\,\mathsf{A}$.

Clm10 From (34) and (43) infer (53), i.e., from $\mathsf{int}\,\mathsf{cl}\,\mathsf{A} < \mathsf{cl}\,\mathsf{int}\,\mathsf{A}$ and $\mathsf{cl}\,\mathsf{int}\,\mathsf{A} < \mathsf{int}\,\mathsf{cl}\,\mathsf{A}$ infer $\mathsf{cl}\,\mathsf{int}\,\mathsf{cl}\,\mathsf{A} < \mathsf{int}\,\mathsf{cl}\,\mathsf{A}$.

Clm11 From (05) and (53) infer (03), i.e., from $\mathsf{A} < \mathsf{cl}\,\mathsf{int}\,\mathsf{cl}\,\mathsf{A}$ and $\mathsf{cl}\,\mathsf{int}\,\mathsf{cl}\,\mathsf{A} < \mathsf{int}\,\mathsf{cl}\,\mathsf{A}$ infer $\mathsf{A} < \mathsf{int}\,\mathsf{cl}\,\mathsf{A}$.

Clm12 From (46) and (60) infer (40), i.e., from $\mathsf{cl}\,\mathsf{int}\,\mathsf{A} < \mathsf{int}\,\mathsf{cl}\,\mathsf{int}\,\mathsf{A}$ and $\mathsf{int}\,\mathsf{cl}\,\mathsf{int}\,\mathsf{A} < \mathsf{A}$ infer $\mathsf{cl}\,\mathsf{int}\,\mathsf{A} < \mathsf{A}$.

Clm13 From (03) and (40) infer (43), i.e., from $\mathsf{A} < \mathsf{int}\,\mathsf{cl}\,\mathsf{A}$ and $\mathsf{cl}\,\mathsf{int}\,\mathsf{A} < \mathsf{A}$ infer $\mathsf{cl}\,\mathsf{int}\,\mathsf{A} < \mathsf{int}\,\mathsf{cl}\,\mathsf{A}$.

Clm14 From (03) and (34) and (60) infer (02), i.e., from $\mathsf{A} < \mathsf{int}\,\mathsf{cl}\,\mathsf{A}$ and $\mathsf{int}\,\mathsf{cl}\,\mathsf{A} < \mathsf{cl}\,\mathsf{int}\,\mathsf{A}$ and $\mathsf{int}\,\mathsf{cl}\,\mathsf{int}\,\mathsf{A} < \mathsf{A}$ infer $\mathsf{A} < \mathsf{int}\,\mathsf{A}$.

Clm15 From (05) and (34) and (40) infer (10), i.e., from $\mathsf{A} < \mathsf{cl\,int\,cl\,A}$ and $\mathsf{int\,cl\,A} < \mathsf{cl\,int\,A}$ and $\mathsf{cl\,int\,A} < \mathsf{A}$ infer $\mathsf{cl\,A} < \mathsf{A}$.

Proposition 3.6. *For all $P_1, P_2, \ldots, P_n, Q \in \mathcal{E}$ we have $P_1, P_2, \ldots, P_n \vdash_{\mathfrak{Cl}} Q$ if and only if $P_1, P_2, \ldots, P_n \vdash_{\mathfrak{Cl}_-} Q$.*

Proof: For each subset $S = \{P_1, P_2, \ldots, P_n\}$ of $\mathcal{E}$ we can find all the consequences of S in $\mathfrak{Cl}$ and in $\mathfrak{Cl}_-$. The result we are proving is that $ec(S) = \mathrm{Con}_{\mathfrak{Cl}}(S) \cap \mathcal{E} = \mathrm{Con}_{\mathfrak{Cl}_-}(S)$ for every $S \subseteq \mathcal{E}$.

The script `cl-minus` described in Section 7.5 calculates $ec(S)$ and $\mathrm{Con}_{\mathfrak{Cl}_-}(S)$ for each $S \subseteq E$ and verifies that they are equal, as required. □

In fact, the list **Clm1–Clm15** was found using this script. The list of rules of inference that we initially conjectured to be sufficient had 19 rules, but was not strong enough to derive all consequences. With the use of this script, we were able to find the appropriate strengthening of two of the rules in our original list, and also to discover that four of the rules in our original list could be derived from the remaining rules.

4. TYPES AND THE ADEQUACY OF $\mathfrak{Cl}$

The goal of this section is to show that $\mathfrak{Cl}$ is adequate, in other words, that if $P_1, P_2, \ldots, P_n \models Q$, then $P_1, P_2, \ldots, P_n \vdash_{\mathfrak{Cl}} Q$. In fact, we will show the contrapositive: if $P_1, P_2, \ldots, P_n \nvdash_{\mathfrak{Cl}} Q$, then $P_1, P_2, \ldots, P_n \not\models Q$. Furthermore, by the results in Section 3 we may restrict our attention to the case that $P_1, P_2, \ldots, P_n, Q \in \mathcal{E}$ and $P_1, P_2, \ldots, P_n \nvdash_{\mathfrak{Cl}_-} Q$.

Definition 4.1. Let A be a subset of a topological space X. The *type* of A, $\mathrm{type}(A)$, is the set of $P \in \mathcal{E}$ such that A has property P.

A *type* is a subset T of $\mathcal{E}$ such that $T = \mathrm{type}(A)$ for some subset A of some topological space X.

Lemma 4.2. *If S is a type, then S is closed under $\vdash_{\mathfrak{Cl}_-}$ (in other words, if $Q \in \mathcal{E}$ and $S \vdash_{\mathfrak{Cl}_-} Q$, then $Q \in S$).*

Proof: This follows from the fact that all 15 rules of inference are sound. □

Lemma 4.3. *There are exactly 49 subsets of $\mathcal{E}$ that are closed under $\vdash_{\mathfrak{Cl}_-}$.*

Proof: For each of the 1024 subsets S of $\mathcal{E}$ it is a simple matter to determine whether S is closed under each of the 15 rules of inference. The script `types` described in Section 7.6 does this. Its output is a list of 49 subsets, which are given in Table 1. In what follows, we will refer to these sets as Type 0 to Type 48. □

We will now show that these 49 potential types are indeed all types. To do this we will give examples of topological spaces X_i and subsets S_i for $0 \le i \le 48$ such that S_i has type i. The spaces X_i will be finite. We will actually define the topology on X_i by specifying a preorder on X_i, in other words, a relation that is reflexive and transitive (but not necessarily antisymmetric). For each such preorder $\le$ there is an associated topology given by declaring that $U \subseteq X_i$ is open if for every $x \in U$ and every y with $x \le y$ we have $y \in U$. Equivalently, for each topology we may define a preorder $\le$ by declaring that

$$x \le y \text{ if } x \in \mathrm{cl}(\{y\}).$$

This preorder is called the *specialization order* for the topology.

Example 4.4. For each i with $0 \le i \le 48$ we will construct a preorder on the ten-element set $X_i = \{a, b, \ldots, j\}$ and find a subset S_i of X_i that has Type i in the corresponding topology. These examples were found using the script `find_types` described in Section 7.6.

The examples are listed in Table 2. For example, the row "Type 16 | $\{d, f\}$ | di, gf, id" means that the subset $\{d, f\}$ of X_{16}, which has topology with specialization order generated by $d < i$, $g < f$, $i < d$, has Type 16 ($\{(03), (05), (43), (46), (53)\}$).

Theorem 4.5. (Adequacy of $\mathfrak{Cl}$.) *For any $P_1, P_2, \ldots, P_n$, $Q \in \mathcal{P}$, if $P_1, P_2, \ldots, P_n \models Q$, then $P_1, P_2, \ldots, P_n \vdash_{\mathfrak{Cl}} Q$.*

Proof: Suppose that $P_1, P_2, \ldots, P_n \nvdash_{\mathfrak{Cl}} Q$. For each i, choose $R_{i,1}, \ldots, R_{i,k_i} \in \mathcal{E}$ with $P_i \Leftrightarrow \bigwedge_{j=1}^{k_i} R_{i,j}$. Choose $S_1, \ldots, S_k \in \mathcal{E}$ with $Q \Leftrightarrow \bigwedge_{j=1}^{k} S_j$. Put $\Gamma = \{\, R_{i,j} \mid 1 \le i \le n, 1 \le j \le k_i \,\}$.

We claim that there is some j_0 such that $\Gamma \nvdash_{\mathfrak{Cl}_-} S_{j_0}$. If this were not true, then we would have $\Gamma \vdash_{\mathfrak{Cl}} S_j$ for all j (by Proposition 3.6). Hence, since $S_1, \ldots, S_k \vdash_{\mathfrak{Cl}} Q$, we would have $\Gamma \vdash_{\mathfrak{Cl}} Q$. We also have $P_i \vdash_{\mathfrak{Cl}} R_{i,j}$ for all i, j, so $P_1, \ldots, P_n \vdash_{\mathfrak{Cl}} R$ for all $R \in \Gamma$. Thus $P_1, \ldots, P_n \vdash_{\mathfrak{Cl}} Q$, contradicting our assumption.

Type 0	$\varnothing$	
Type 1	$\{(05)\}$	β-open
Type 2	$\{(03),(05)\}$	preopen
Type 3	$\{(34)\}$	
Type 4	$\{(05),(34)\}$	semi-open
Type 5	$\{(03),(05),(34)\}$	α-open
Type 6	$\{(02),(03),(05),(34)\}$	open
Type 7	$\{(43)\}$	
Type 8	$\{(05),(43)\}$	
Type 9	$\{(03),(05),(43)\}$	
Type 10	$\{(43),(46)\}$	
Type 11	$\{(05),(43),(46)\}$	
Type 12	$\{(03),(05),(43),(46)\}$	
Type 13	$\{(43),(53)\}$	
Type 14	$\{(03),(05),(43),(53)\}$	
Type 15	$\{(43),(46),(53)\}$	
Type 16	$\{(03),(05),(43),(46),(53)\}$	
Type 17	$\{(34),(43),(46),(53)\}$	
Type 18	$\{(03),(05),(34),(43),(46),(53)\}$	
Type 19	$\{(02),(03),(05),(34),(43),(46),(53)\}$	
Type 20	$\{(60)\}$	β-closed
Type 21	$\{(05),(60)\}$	
Type 22	$\{(03),(05),(60)\}$	
Type 23	$\{(34),(60)\}$	semiclosed
Type 24	$\{(05),(34),(60)\}$	condensed
Type 25	$\{(02),(03),(05),(34),(60)\}$	regular open
Type 26	$\{(40),(60)\}$	preclosed
Type 27	$\{(05),(40),(60)\}$	
Type 28	$\{(34),(40),(60)\}$	α-closed
Type 29	$\{(10),(34),(40),(60)\}$	closed
Type 30	$\{(05),(10),(34),(40),(60)\}$	regular closed
Type 31	$\{(43),(60)\}$	
Type 32	$\{(05),(43),(60)\}$	
Type 33	$\{(03),(05),(43),(60)\}$	
Type 34	$\{(40),(43),(60)\}$	
Type 35	$\{(05),(40),(43),(60)\}$	
Type 36	$\{(03),(05),(40),(43),(60)\}$	
Type 37	$\{(40),(43),(46),(60)\}$	
Type 38	$\{(05),(40),(43),(46),(60)\}$	
Type 39	$\{(03),(05),(40),(43),(46),(60)\}$	
Type 40	$\{(43),(53),(60)\}$	
Type 41	$\{(03),(05),(43),(53),(60)\}$	
Type 42	$\{(40),(43),(53),(60)\}$	
Type 43	$\{(03),(05),(40),(43),(53),(60)\}$	
Type 44	$\{(40),(43),(46),(53),(60)\}$	
Type 45	$\{(03),(05),(40),(43),(46),(53),(60)\}$	
Type 46	$\{(34),(40),(43),(46),(53),(60)\}$	
Type 47	$\{(10),(34),(40),(43),(46),(53),(60)\}$	
Type 48	$\{(02),(03),(05),(10),(34),(40),(43),(46),(53),(60)\}$	

TABLE 1. The 49 types of subsets.

Now let Π be the smallest subset of $\mathcal{E}$ that contains Γ and is closed under $\vdash_{\mathfrak{Cl}_-}$. Then $S_{j_0} \notin \Pi$. By Lemmas 4.2 and 4.3 and Example 4.4, Π is a type, so there is a subset A of a space X such that $\text{type}(A) = \Pi$. Then A has property $R_{i,j}$ for all i and j, and so A has property P_i for each i. On the other hand, since A does not have property S_{j_0}, A does not have property Q. Thus $P_1, P_2, \ldots, P_n \not\models Q$. $\square$

5. UNIVERSALS AND PROPERTIES OF SPACES

Up to this point we have considered properties of a given subset of a topological space X. We turn now to the question of properties of the space X that can be characterized in our system. In the first instance we consider "universals," which are properties of spaces that may be defined by insisting that all subsets have certain proper-

Type	Subset	Order
Type 0	$\{a, g, i\}$	$ah, bd, bj, ch, gf, ha, ji$
Type 1	$\{a, b, i\}$	$ah, bd, bj, ch, gf, ha, ji$
Type 2	$\{a, i\}$	$ah, bd, bj, ch, gf, ha, ji$
Type 3	$\{a, d\}$	ba, di, fd, fh, ja, jh
Type 4	$\{f, i\}$	ba, di, fd, fh, ja, jh
Type 5	$\{g, h\}$	ac, ah, ba, ci, dh, gd
Type 6	$\{a\}$	ba, di, fd, fh, ja, jh
Type 7	$\{b, e, i\}$	$af, bh, cb, ce, db, eg, fh, fi, hb, ji$
Type 8	$\{a, b, g\}$	$af, bh, cb, ce, db, eg, fh, fi, hb, ji$
Type 9	$\{b, g\}$	$af, bh, cb, ce, db, eg, fh, fi, hb, ji$
Type 10	$\{b, c, e\}$	$bh, dc, ei, fb, fj, hb, ij$
Type 11	$\{c, f, g\}$	ce, ch, dg, fh, hf
Type 12	$\{f, g\}$	ce, ch, dg, fh, hf
Type 13	$\{b, d, j\}$	$bh, dc, ei, fb, fj, hb, ij$
Type 14	$\{b, j\}$	$bh, dc, ei, fb, fj, hb, ij$
Type 15	$\{a, b, c\}$	ae, bj, ea, ic
Type 16	$\{d, f\}$	di, gf, id
Type 17	$\{d, f\}$	fe, hd, ji
Type 18	$\{d, h\}$	ae, ag, bd, hb
Type 19	$\{d\}$	fe, hd, ji
Type 20	$\{c, e, j\}$	be, bf, ch, hc, jd
Type 21	$\{b, c, g\}$	$bh, cd, ch, hb, ic, jc, jg$
Type 22	$\{b, d\}$	$bh, cd, ch, hb, ic, jc, jg$
Type 23	$\{b, h\}$	ba, di, fd, fh, ja, jh
Type 24	$\{f, h\}$	ba, di, fd, fh, ja, jh

Type	Subset	Order
Type 25	$\{h\}$	ba, di, fd, fh, ja, jh
Type 26	$\{b, c, e, j\}$	be, bf, ch, hc, jd
Type 27	$\{b, g, j\}$	$bh, cd, ch, hb, ic, jc, jg$
Type 28	$\{a, b, e\}$	ae, ag, bd, hb
Type 29	$\{a, e, h\}$	ae, ag, bd, hb
Type 30	$\{a, e\}$	ae, ag, bd, hb
Type 31	$\{a, b, f\}$	$bh, cd, ci, da, dc, di, eh, ej, fi, if, jf, jh$
Type 32	$\{a, e, g\}$	$ah, ca, cg, ea, ej, fa, hf$
Type 33	$\{a, g\}$	$ah, ca, cg, ea, ej, fa, hf$
Type 34	$\{b, c, e, h\}$	$ac, bc, be, cj, dh, dj, fc, fd, gc, gi, hi, jc$
Type 35	$\{a, c, e, g\}$	$ah, ca, cg, ea, ej, fa, hf$
Type 36	$\{a, c, g\}$	$ah, ca, cg, ea, ej, fa, hf$
Type 37	$\{d, f\}$	ce, ch, dg, fh, hf
Type 38	$\{b, c\}$	$bh, cd, ch, hb, ic, jc, jg$
Type 39	$\{b\}$	$bh, cd, ch, hb, ic, jc, jg$
Type 40	$\{d, e, f\}$	ce, ch, dg, fh, hf
Type 41	$\{e, f\}$	ce, ch, dg, fh, hf
Type 42	$\{c, d, e, f\}$	ce, ch, dg, fh, hf
Type 43	$\{c, e, f\}$	ce, ch, dg, fh, hf
Type 44	$\{d, g\}$	di, gf, id
Type 45	$\{d\}$	di, gf, id
Type 46	$\{d\}$	ba, di, fd, fh, ja, jh
Type 47	$\{f\}$	fe, hd, ji
Type 48	$\varnothing$	fe, hd, ji

TABLE 2. Subsets having all 49 types.

ties. In other words, these are spaces that may be characterized by a statement of the form $(\forall \mathsf{A}) \bigwedge_{i=1}^{n} P_i$, where for each P_i, $P_i \in \mathcal{E}$.

We first ask how many such universals there are. For example, $(\forall \mathsf{A})(\mathsf{cl\,A} < \mathsf{A})$ is equivalent to $(\forall \mathsf{A})(\mathsf{A} < \mathsf{int\,A})$: insisting that every subset be closed is the same as insisting that every subset be open.

Definition 5.1. Let X be a topological space. The *universal* of X, $u(X)$, is the set of $P \in \mathcal{E}$ such that every subset of X has property P, or equivalently, $u(X) = \bigcap_{A \subseteq X} \mathrm{type}(A)$.

A *universal* is a subset S of $\mathcal{E}$ such that $S = u(X)$ for some topological space X.

To find the universals, we will extend $\mathfrak{Cl}$ to a system $\mathfrak{U}$: whereas $S \vdash_{\mathfrak{Cl}} P$ was intended to mean, "if A has all the properties in S, then A has property P," $S \vdash_{\mathfrak{U}} P$ is intended to mean, "if every subset of X has all the properties in S, then every subset of X has property P."

We first introduce three further operations on W.

Definition 5.2. We define the operations $d, C, I : W \to W$ as follows:

w	$d(w)$	$C(w)$	$I(w)$
A	A	cl A	int A
cl A	int A	cl A	cl int A
int A	cl A	int cl A	int A
int cl A	cl int A	int cl A	int cl int A
cl int A	int cl A	cl int cl A	cl int A
cl int cl A	int cl int A	cl int cl A	cl int A
int cl int A	cl int cl A	int cl A	int cl int A

Syntactically, $d(w)$ is obtained by replacing each cl with int and vice versa, $C(w)$ is obtained by replacing A by $\mathsf{cl\,A}$ in w and then canceling according to the rules implied by Proposition 2.2, and $I(w)$ is obtained by replacing A by $\mathsf{int\,A}$ and canceling.

The semantic effect of the above operations is embodied in the following result.

Proposition 5.3. *Let $w \in W$, and let $A \subseteq X$ for some topological space X. Then we have*

1. $d(w)_A = (w_{A'})'$, *where* $B' = X \setminus B$;
2. $C(w)_A = w_{\mathrm{cl}(A)}$;
3. $I(w)_A = w_{\mathrm{int}(A)}$.

	$U \cap \mathcal{E}_{\mathcal{U}}$	U
quasiuniversal 0	$\varnothing$	$\varnothing$
quasiuniversal 1	$\{(34)\}$	$\{(34)\}$
quasiuniversal 2	$\{(43)\}$	$\{(43),(46),(53)\}$
quasiuniversal 3	$\{(34),(43)\}$	$\{(34),(43),(46),(53)\}$
quasiuniversal 4	$\{(05)\}$	$\{(05),(60)\}$
quasiuniversal 5	$\{(05),(34)\}$	$\{(05),(34),(60)\}$
quasiuniversal 6	$\{(05),(43)\}$	$\{(03),(05),(40),(43),(46),(53),(60)\}$
quasiuniversal 7	$\{(05),(34),(43)\}$	$\{(02),(03),(05),(10),(34),(40),(43),(46),(53),(60)\}$

TABLE 3. The quasiuniversals.

Proposition 5.4. *Let X be a topological space, and let $A \subseteq X$. Let u and v be words. Then*

- *A has property $C(u) < C(v)$ if and only if $\mathrm{cl}(A)$ has property $u < v$;*
- *A has property $I(u) < I(v)$ if and only if $\mathrm{int}(A)$ has property $u < v$;*
- *A has property $d(v) < d(u)$ if and only if A' has property $u < v$.*

Proof: This follows immediately from Proposition 5.3. □

If U is the universal of X, then for every $A \subseteq X$, we must have that $\mathrm{cl}(A)$, $\mathrm{int}(A)$, and A' have all the properties in U. Thus the system $\mathfrak{U}$ should also have rules of inference such as "from $u < v$ infer $C(u) < C(v)$," "from $u < v$ infer $I(u) < I(v)$," and "from $u < v$ infer $d(v) < d(u)$." Let $\mathfrak{U}_q$ be the system obtained from $\mathfrak{Cl}$ by adding these rules of inference.

Again, we can simplify our discussion by identifying those elementary properties that are "universal-elementary," in other words, to identify a set $\mathcal{F}$ such that for every $P \in \mathcal{E}$ there is an $S \subseteq \mathcal{F}$ with $S \vdash_{\mathfrak{U}_q} P$ and $P \vdash_{\mathfrak{U}_q} Q$ for every $Q \in S$.

Again, we first find the properties that are "universal-canonical," that is, such that $P \preccurlyeq Q$ for every Q with $\mathrm{Con}_{\mathfrak{U}_q}(P) = \mathrm{Con}_{\mathfrak{U}_q}(Q)$. With a modified version of the script `canonical`, we find that there are eight universal-canonical properties, namely

$$\mathcal{C}_{\mathfrak{U}} = \{(00),(02),(03),(04),(05),(34),(43),(56)\}.$$

We then make the following definition:

Definition 5.5. A property P is *universal-elementary* if $P \in \mathcal{C}_{\mathfrak{U}}$ and

$$P \notin \mathrm{Con}_{\mathfrak{U}_q}((\mathrm{Con}_{\mathfrak{U}_q}(\{P\}) \cap \mathcal{C}_{\mathfrak{U}}) \smallsetminus \{P\}).$$

Again, we may find these using a modified version of the script `find_elementaries`. The universal-elementary properties are (05), (34), and (43). Again, a modified version of `check_elementary` confirms that each property is equivalent in $\mathfrak{U}_q$ to a conjunction of properties from $\mathcal{E}_{\mathcal{U}} = \{(05),(34),(43)\}$, so we may use these as our universal-elementary properties.

Definition 5.6. A subset U of $\mathcal{E}$ is a *quasiuniversal* if it is the intersection with $\mathcal{E}$ of a set that is closed under the rules of inference of $\mathfrak{U}_q$, or equivalently if $U = \mathcal{E} \cap \mathrm{Con}_{\mathfrak{U}_q}(U)$.

Clearly, if U and V are quasiuniversals, then $U = V$ if and only if $U \cap \mathcal{E}_{\mathcal{U}} = V \cap \mathcal{E}_{\mathcal{U}}$. Thus there are at most $2^{|\mathcal{E}_{\mathcal{U}}|} = 8$ quasiuniversals. We may readily confirm that all of these sets are indeed quasiuniversals, using a modified version of the script `types` called `quasi-universals`. They are listed in Table 3.

The script `find_types` was modified to give the script `find_universals` described in Section 7.7. This script was used to find examples of quasiuniversals 0, 1, 2, 3, 6, and 7, but no examples of quasiuniversals 4 and 5. This is because of the following result.

Lemma 5.7. *Let X be a topological space in which every subset is β-open (i.e., has property (05), that is, $\mathsf{A} < \mathsf{cl\,int\,cl\,A}$). Then every open set in X is closed.*

Proof: Let U be an open set in X, and put $A = \partial U = \mathrm{cl}(U) \smallsetminus U$. Then A is β-open, so $A \subseteq \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$. But $\mathrm{int}(\mathrm{cl}(A)) = \varnothing$, so $A \subseteq \mathrm{cl}(\varnothing) = \varnothing$. Thus $\partial U = \varnothing$ for every open set U; in other words, every open set is closed. □

Corollary 5.8. *Let S be a universal with $(05) \in S$. Then $(43) \in S$.*

	$U \cap \mathcal{E}_{\mathcal{U}}$	U	Order
universal 0	$\varnothing$	$\varnothing$	$ae, af, be, cf, eb, ha, hc$
universal 1	$\{(34)\}$	$\{(34)\}$	ea, ef
universal 2	$\{(43)\}$	$\{(43), (46), (53)\}$	be, dh, fd, hd
universal 3	$\{(34), (43)\}$	$\{(34), (43), (46), (53)\}$	be
universal 4	$\{(05), (43)\}$	$\{(03), (05), (40), (43), (46), (53), (60)\}$	$ac, af, ag, ah, cd, ce, da, ed, fg, ge, hc, hd$
universal 5	$\{(05), (34), (43)\}$	$\{(02), (03), (05), (10), (34), (40), (43), (46), (53), (60)\}$	

TABLE 4. The universals.

Proof: By the previous lemma, we know that every open set is closed; in other words, we have $\mathrm{cl}(\mathrm{int}(A)) \subseteq \mathrm{int}(A)$ for each $A \subseteq X$, so every subset A of X satisfies $\mathsf{cl\,int\,A} < \mathsf{int\,A}$, or (42). Since $ec(\{(42)\}) = \{(40), (43), (46), (60)\}$, we have $(43) \in S$, as required. □

In light of Corollary 5.8, we obtain the system $\mathfrak{U}$ from $\mathfrak{U}_q$ by adding the rule of inference "from (05) infer (43)." Adding this rule of inference to `quasi-universals` to get the script `universals`, we find that there are exactly six subsets of $\mathcal{P}$ that are closed under the rules of inference of $\mathfrak{U}$, namely those sets whose intersections with $\mathcal{E}$ are quasiuniversals 0, 1, 2, 3, 6, and 7.

Theorem 5.9. *There are precisely six universals.*

Proof: Clearly, every universal is the intersection with $\mathcal{E}$ of a subset of $\mathcal{P}$ that contains the axioms and is closed under the rules of inference of $\mathfrak{U}$. Thus there are at most six universals.

In Table 4 we list examples of topological spaces with each of these six universals. In each case, the underlying set X is $\{a, b, \ldots, h\}$, and the topology is that with specialization order generated by the given edges. For example, the row "universal 1 | $\{(34)\}$ | ea, ef" means that the topology on X with specialization order generated by $e < a$, $e < f$ has universal $u(X) = \{(34)\}$.

□

Note that universal 5 is the property that the topology is discrete, which is why the example has no generators in its specialization order. The other universals have also been studied in the literature.

Definition 5.10. Let X be a topological space.

1. X is *extremally disconnected* if the closure of every open set is open.
2. X is *irresolvable* if it cannot be written as a union of two disjoint dense subsets. It is *strongly irresolvable* (or open hereditarily irresolvable) if every nonempty open subset is irresolvable.
3. X is a *partition topology* if there is an equivalence relation on X such that a subset is open if and only if it is a union of equivalence classes, or equivalently if every closed set is open.

Theorem 5.11. *Let X be a topological space.*

1. *X is extremally disconnected if and only if it satisfies universal* 2.
2. *X is strongly irresolvable if and only if it satisfies universal* 1.
3. *X is a partition topology if and only if it satisfies universal* 4.

Proof:
(1) Note that X is extremally disconnected if and only if the closure of $\mathrm{int}(A)$ is open for each $A \subseteq X$, in other words, if and only if every subset of X satisfies $\mathsf{cl\,int\,A} < \mathsf{int\,cl\,int\,A}$, or (46). Thus X is extremally disconnected if and only if $\mathrm{Con}_{\mathfrak{U}}((46)) \subseteq u(X)$. Since (46) and (43) are equivalent in $\mathfrak{U}$, a space is extremally disconnected if and only if it satisfies universal 2.

(2) We first observe that if X is strongly irresolvable and A is dense in an open set U, then $\mathrm{int}(A)$ is dense in U. Otherwise, there would be a nonempty open $V \subseteq U \smallsetminus \mathrm{int}(A)$, and V would be the union of the sets $V \smallsetminus A$ and $V \cap A$, each of which is dense in V.

So suppose that X is strongly irresolvable. Let $A \subseteq X$. We must show that $\mathrm{int}(\mathrm{cl}(A)) \subseteq \mathrm{cl}(\mathrm{int}(A))$. So let $B = \mathrm{int}(\mathrm{cl}(A)) \smallsetminus \mathrm{cl}(\mathrm{int}(A))$. Note that B is open and $A \cap B$ is dense in B. Thus by the above observation

$U = \mathrm{int}(A \cap B)$ is dense in B. But $U \subseteq \mathrm{int}(A)$ so $B \subseteq \mathrm{cl}(\mathrm{int}(A))$, so $B = \varnothing$, as required.

Conversely, suppose X is not strongly irresolvable. Then there is a nonempty open set U with disjoint subsets A and B that are both dense in U. On the one hand, we have $U \subseteq \mathrm{int}(\mathrm{cl}(A))$. On the other hand, $\mathrm{int}(A)$ is open in U and misses B, so it is empty. So $\mathrm{int}(\mathrm{cl}(A)) \nsubseteq \mathrm{cl}(\mathrm{int}(A))$, so A does not satisfy (34), so X does not have universal 1.

(3) Suppose first that X is a partition topology. Let $A \subseteq X$. Then $\mathrm{cl}(A)$ is the union of all the equivalence classes meeting A, so $\mathrm{cl}(A)$ is open, so we have $A \subseteq \mathrm{cl}(A) \subseteq \mathrm{int}(\mathrm{cl}(A))$. Thus A satisfies (03), as required.

Conversely, suppose $(03) \in u(X)$. If A is closed, then $A \subseteq \mathrm{int}(\mathrm{cl}(A)) = \mathrm{int}(A)$, so A is open. Thus every open set is closed, so X is a partition topology. □

Another way to view a universal U is to consider the extension $\mathfrak{Cl}[U]$ of $\mathfrak{Cl}$ obtained by adding all the properties in U as axioms.

Theorem 5.12. *For each universal U, the system $\mathfrak{Cl}[U]$ is sound and adequate, in the sense that if $P_1, P_2, \ldots, P_n, Q \in \mathcal{P}$, then $P_1, P_2, \ldots, P_n \vdash_{\mathfrak{Cl}[U]} Q$ if and only if for every space X with $u(X) \supseteq U$ and every subset A of X that satisfies $P_1, P_2, \ldots, P_n$, A satisfies Q.*

Proof: Soundness again follows from the fact that all the axioms and rules of inference are sound.

For adequacy, we will again consider types. First, consider the subsets of $\mathcal{E}$ that are the intersection with $\mathcal{E}$ of a subset of $\mathcal{P}$ that contains the axioms of $\mathfrak{Cl}[U]$ and is closed under the rules of inference of $\mathfrak{Cl}[U]$. These are precisely the types that contain U. There are forty-nine types containing universal 0, sixteen types containing universal 1, ten types containing universal 2, six types containing universal 3, two types containing universal 4, and one type containing universal 5. For each of these universals U, examples have been found of spaces with universal U and with each of the appropriate types of subset. □

6. FURTHER DIRECTIONS

Our system $\mathfrak{Cl}$ and its extensions discussed above are adequate for properties of spaces of the form "all subsets have property P." We might extend this idea to properties of spaces of the form "each subset with property P has property Q."

Recall that a subset A of a topological space X is *nowhere dense* if $\mathrm{int}(\mathrm{cl}(A)) = \varnothing$. Van Douwen introduced the class of *nodec* spaces: those spaces in which every nowhere dense set is closed [van Douwen 93]. This class is important, for example, in the study of submaximal spaces. Although superficially not being of the form mentioned above (since the definition involves a "constant" $\varnothing$), Cao, Greenwood, and Reilly have shown that a space is nodec if and only if every α-closed subset is closed [Cao et al. 01, Theorem 2.10], in other words, if every subset with property (50) has property (10). Since we have

$$(50) \Leftrightarrow (34) \wedge (40) \wedge (60),$$

we might abbreviate this as $(\forall \mathsf{A})((34) \wedge (40) \wedge (60) \Rightarrow (10))$.

If we inspect the list of types in Table 1, we see that only Types 28 and 46 violate this: thus a space is nodec if and only if it has no subsets of Type 28 or 46. Of course, the dual property is also true: all α-open sets are open, so Types 5 and 18 are similarly excluded. Once again, a modified version of `find_types` has found examples of all 45 types of subsets of nodec spaces.

Again, we can think of the nodec property as being an extension of $\mathfrak{Cl}$ or $\mathfrak{Cl}_{-}$, this time obtained by adding the new rules of inference "from (34) and (40) and (60) infer (10)" and "from (03) and (05) and (34) infer (02)," and the same argument as before shows that this system is sound and adequate for nodec spaces.

Another direction for future research is to consider a slightly richer language, for example, adding constant symbols for $\varnothing$ and X, or a binary operation symbol for $\setminus$. Note that if we introduce both a constant symbol for X and a binary operation symbol for $\setminus$, we will in effect have complements, unions, and interiors. In general, these three operations can generate infinitely many distinct sets [Bowron and Rabinowitz 97].

7. PROGRAMS

The programs used to develop the results presented in this paper were all written in the scripting language Perl. They are described briefly in the following subsections. The scripts themselves are available for download at http://www.math.auckland.ac.nz/~mcintyre/ptc.

The following conventions are used:

- Properties are represented by two-digit strings such as `02`.
- The operations c, i, d, C, and I on W are represented by arrays `@c`, `@i`, etc. For example, `$c[2] == 4`,

because $c(\mathsf{int\,A}) = \mathsf{cl\,int\,A}$, and we number $\mathsf{int\,A}$ as 2 and $\mathsf{cl\,int\,A}$ as 4.

- When finding $\mathrm{Con}(S)$, we usually use an associative array (or hash) `%hold`, with `$hold{uv} == 1` if $(uv) \in \mathrm{Con}(S)$, and `$hold{uv}` undefined otherwise. This means that we can extract $\mathrm{Con}(S)$ using `keys %hold`.

7.1 The Script `full_cons`

The main part of the script is the subroutine `find_consequences`, which calculates $\mathrm{Con}_{\mathfrak{Cl}}(S)$. This first sets `$hold{P} = 1` for each axiom P and each $P \in S$. It then enters a loop: in each iteration, we apply each of the rules of inference to each property known to hold. We check the number of properties known to hold at the start and end of each loop. When these values are equal, we exit the loop.

7.2 The Script `canonical`

The script `canonical` finds the canonical properties. Recall that these are defined as those properties P such that $P \preccurlyeq Q$ whenever $P \Leftrightarrow Q$, and note that $P \Leftrightarrow Q$ if and only if $\mathrm{Con}_{\mathfrak{Cl}}(P) = \mathrm{Con}_{\mathfrak{Cl}}(Q)$. The script works as follows. We consider the properties in lexicographical order. For each property P, $S = \mathrm{Con}_{\mathfrak{Cl}}(\{P\})$ is found using the subroutine `find_consequences` described above. We then check the associative array `found_canonical` to see whether `found_canonical{S}` is defined. If not, then we declare P to be canonical and set `found_canonical{S}` to be true.

7.3 The Script `find_elementaries`

The script `find_elementaries` first uses the above method to identify the canonical properties. It then runs through all the canonical properties P: for each P it finds the consequences of P, finds those consequences that are canonical and distinct from P, finds the consequences of this list, and checks whether P is in this latter set of consequences. If not, P is elementary.

7.4 The Script `check_elementary`

This script takes as input a list E of properties and tests whether each property is equivalent to a conjunction of properties in E. For each P, it first uses `find_consequences` to find $\mathrm{Con}_{\mathfrak{Cl}}(\{P\})$, then intersects this with E using the subroutine `get_elementaries`. It then finds the consequences of this set, and checks that the original property P is in this list.

7.5 The Script `cl-minus`

This script takes each $S \subseteq \mathcal{E}$, finds $\mathrm{Con}_{\mathfrak{Cl}}(S)$ using `find_consequences`, and intersects it with E. It also finds the consequences of S in the restricted system $\mathfrak{Cl}_{-}$, using the subroutine `restricted_consequences`. This latter subroutine is similar in structure to `find_consequences`. However, note that the rules of inference are now concrete instances rather than rule schemata.

7.6 The Scripts `types` and `find_types`

The script `types` finds the types. It successively considers each subset of the list of elementary properties, and tests whether that subset is closed under each of the rules of inference of $\mathfrak{Cl}_{-}$.

The script `find_types` first performs the same code as `types` in order to identify by number the 49 types. It then generates a number of topologies on a given set, finds each subset of the set, determines the type of that subset, and (if it is a type not previously encountered or has fewer elements than the example previously found) stores the example in an associative array indexed by the type.

The underlying set for the space is stored as the array `@points`. The topologies are generated as follows. First, a list of all pairs xy with $x \neq y$ is formed, and stored as `@edges`. Then, for each edge, a random number is generated between 0 and 100: if this random number exceeds a certain threshold (the "edge probability"), the edge is added to a list of generators. The specialization order for the topology is then the reflexive, transitive closure of the set of generators. This process is repeated a number of times. The edge probability is increased, with a certain number of topologies generated for each edge probability. The range of probabilities, step size, and number of topologies generated are all parameters that can be set by the user.

The method used to determine the type of each subset is as follows. First, for each point x, the smallest neigborhood of $m(x)$ is found. This is $\{\, y \mid x \leq y \,\}$. Then, given a set A, we can find $\mathrm{int}(A)$ as $\{\, y \mid m(y) \subseteq A \,\}$ and $\mathrm{cl}(A)$ as $\{\, y \mid m(y) \cap A \neq \varnothing \,\}$. Now for each set A, the sets w_A for each $w \in W$ are found, and for each elementary property $w < v$, we test whether $w_A \subseteq v_A$.

7.7 The Scripts `quasi-universals`, `universals`, and `find_universals`

The script `quasi-universals` is a modification of the script `types`.

Conceptually it is obtained by adding all the rules of inference of the form "from $w < v$ infer $C(w) < C(v)$," and similarly for the operators I and d. However, in most cases $C(w) < C(v)$ and $I(w) < I(v)$ are not elementary even when $w < v$ is. Thus in fact, the extra rules are all rules of the form "from $w < v$ infer $w' < v'$," where $w' < v' \in \mathrm{Con}_{\mathfrak{Cl}}(C(w) < C(v)) \cap \mathcal{E}$, and similarly for I and d. The script `universals` is obtained from `quasi-universals` by adding the rule "from (05) infer (43)" from Corollary 5.8.

The script `find_universals` is a modified version of the script `find_types`. A number of topologies are generated in the same way. Once again, the type of each subset is found, giving a list of types found in this topology. Then each elementary property is tested to check whether it occurs in each type found in this topology.

In this way, the universal for this topology is found. The script also records which types of subset have been seen in a space with each given universal: as mentioned in Section 5, this may be used to establish the adequacy of the systems $\mathfrak{Cl}[U]$.

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