

Quotients Homophones des Groupes Libres

Homophonic Quotients of Free Groups

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Ah, la recherche! Du temps perdu.

Soit G le quotient du groupe libre à 26 générateurs a, b, c, \dots, z par les relations $A = B$, pour tout couple de mots (A, B) pouvant avoir la même prononciation en anglais. (Un groupe défini d'une manière analogue a été considéré dans [Landsburg 1986].) Notre but est de déterminer la structure du groupe G .

Théorème. G est trivial.

Démonstration. (Nous nous servons sans le mentionner des faits contenus dans [Stein 1973].) La relation homophone $bye = by$ implique que e est trivial dans G (en symboles: $e = e$), après quoi les identités

$$\begin{aligned} lead = led, \quad maid = made, \quad sow = sew, \\ buy = by, \quad sow = so, \quad lye = lie \end{aligned}$$

nous donnent la trivialité des autres voyelles et demivoyelles a, i, o, u, w et y . La trivialité des générateurs h, k, n, p et b est une conséquence des formules

$$\begin{aligned} hour = our, \quad knight = night, \quad damn = dam, \\ psalter = salter, \quad plumb = plum, \end{aligned}$$

tandis que celle des générateurs s, t, l, r et m se déduit des égalités

$$\begin{aligned} bass = base, \quad butt = but, \quad tolled = told, \\ barred = bard, \quad dammed = damned \end{aligned}$$

(méthode des idempotents). Pour les générateurs

Let G be the quotient of the free group on 26 letters a, b, c, \dots, z by the relations $A = B$ whenever A and B are words having the same pronunciation in French. (A similarly defined group was considered in [Landsburg 1986].) The object of this paper is to determine the structure of G .

Theorem. G is trivial.

Proof. (We use without special mention facts that can be found in [Robert 1973].) The relation

$$soie = soi$$

shows, on canceling soi (reflexive property!), that e is trivial in G (in symbols, $e = e$). A similar argument applied to the final letters of $soit, sois$ and aux shows that t, s and x are also trivial. The triviality of r follows from the well-known fact

$$serre = sert$$

[Lam 1978], and that of c, l, d, h and n from

$$\begin{aligned} ce = se, \quad balle = bal, \quad laid = lait, \\ haut = au, \quad parlent = parle, \end{aligned}$$

after which the relations

$$\begin{aligned} allez = aller, \quad sept = cet, \\ champs = chant, \quad fard = phare \end{aligned}$$

allow one to eliminate z, p, m and f . (One could give a proof with a more topological flavor of the triviality of f using $cerf = serre$.) The triviality

d et g de G on se sert des équations

$$\textit{chased} = \textit{chaste}, \quad \textit{sign} = \textit{sine},$$

après quoi les équations

$$\begin{aligned} \textit{daze} &= \textit{days}, & \textit{cite} &= \textit{sight}, & \textit{jeans} &= \textit{genes}, \\ \textit{queue} &= \textit{cue}, & \textit{tax} &= \textit{tacks} \end{aligned}$$

nous permettent d'éliminer également z , c , j , q et x . En se servant des relations du type $\textit{ruff} = \textit{rough}$ on voit facilement que l'élément f de G est de torsion ($f^2 = e$), mais en fait il est trivial à cause de l'identité

$$\textit{phase} = \textit{faze}$$

($\textit{phrase} = \textit{frays}$ servirait également). Enfin, on peut éliminer le générateur v de G à l'aide de l'équation

$$\textit{chivvy} = \textit{chivy}$$

(ou $\textit{leitmotiv} = \textit{leitmotif}$). □

GÉNÉRALISATIONS

On peut généraliser le théorème en considérant le group G_0 défini de la même manière que G , mais avec un espace comme 27ième générateur. Alors la relation

$$\textit{be calm} = \textit{becalm}$$

montre la trivialité du nouveau générateur, et la relation

$$\textit{avowers} = \textit{of ours}$$

donne une preuve plus satisfaisante qu'avant de celle du générateur v .

Il semble en plus que l'on puisse démontrer un théorème analogue dans le cas où "anglais" est remplacé par "français" dans la définition de G .

APPLICATIONS

On a lieu de croire que le théorème ci-dessus aura des applications dans l'investigation de la conjecture $p = np$.

of the vowels a , i , y follows successively from the identities

$$\textit{an} = \textit{en}, \quad \textit{mais} = \textit{mets}, \quad \textit{bayer} = \textit{bailler},$$

after which that of g , j , b follows from

$$\textit{sang} = \textit{cent}, \quad \textit{jet} = \textit{geai}, \quad \textit{abbesse} = \textit{abaïsse}.$$

The letters k , q , u , o are more difficult:

$$\textit{khan} = \textit{quand}, \quad \textit{lacque} = \textit{lac}$$

imply that $k = qu$ and that qu is trivial. Next, $\textit{coq} = \textit{coke}$ yields the triviality of q , hence of u . Finally, $\textit{pot} = \textit{peau}$ implies that o is trivial. The triviality of the remaining letters w and v now follows from the relations

$$\textit{watt} = \textit{ouate}, \quad \textit{vaguons} = \textit{wagon}.$$

This completes the proof of the theorem. □

GENERALIZATIONS

The theorem generalizes to the enlarged group \hat{G}' obtained by adjoining to the set of generators all accented letters and $\textit{æ}$, as one sees using

$$\begin{aligned} \textit{là} &= \textit{la} \text{ (or simply } \textit{à} = \textit{a}), & \textit{guère} &= \textit{guerre}, \\ \textit{allée} &= \textit{aller}, & \textit{ôte} &= \textit{haute}, & \textit{appât} &= \textit{appas}, \\ \textit{mûr} &= \textit{mur}, & \textit{île} &= \textit{il}, & \textit{fête} &= \textit{faîte}, \\ \textit{æufs} &= \textit{eux}, & \textit{ça} &= \textit{sa}. \end{aligned}$$

One can also allow the apostrophe and the trait d'union using: $\textit{m'aime} = \textit{même}$, $\textit{voix-ci} = \textit{voici}$.

Another generalization which naturally suggests itself is that to other languages. We can prove that the theorem remains true with "French" replaced by "English" in the definition of G . The corresponding theorem for "German" has been proved very recently by Herbert Gangl, following previous pioneering work by Norbert Schappacher. On the other hand, it appears that the analogously defined group for Japanese (written in katakana) is free on 46 generators.

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