Local probabilities for random walks with negative drift conditioned to stay nonnegative*

Denis Denisov[†]   Vladimir Vatutin[‡]   Vitali Wachtel[§]

Abstract

Let $\{S_n, n \geq 0\}$ with $S_0 = 0$ be a random walk with negative drift and let $\tau_x = \min\{k > 0 : S_k < -x\}$, $x \geq 0$. Assuming that the distribution of the i.i.d. increments of the random walk is absolutely continuous with subexponential density we describe the asymptotic behavior, as $n \to \infty$, of the probabilities $\mathbf{P}(\tau_x = n)$ and $\mathbf{P}(S_n \in [y, y + \Delta), \tau_x > n)$ for fixed x and various ranges of y. The case of lattice distribution of increments is considered as well.

1 Introduction

Let $\{S_n, n \geq 0\}$ be a random walk with $S_0 = 0$ and $S_n = X_1 + X_2 + \ldots + X_n$ for all $n \geq 1$, where $X_1, X_2, \ldots$ are independent copies of a random variable X. For each $x \geq 0$ let τ_x denote the first passage time to $(-\infty, -x)$, that is,

$$\tau_x = \min\{k > 0 : S_k < -x\}.$$

The main purpose of the present note is to investigate the asymptotic behaviour, as $n \to \infty$, of the probabilities $\mathbf{P}(S_n \in [y, y + \Delta), \tau_x > n)$ and $\mathbf{P}(S_n \in [y, y + \Delta), \tau_x = n + 1)$ for random walks with negative drift:

$$\mathbf{E}[X] = -a < 0.$$

The driftless case attracted a lot of attention in the last decade and is well studied in the literature, see [8, 9, 12, 13, 21, 22].

[†] School of Mathematics, University of Manchester, Oxford Road, Manchester M13 9PL, UK
  E-mail: denis.denisov@manchester.ac.uk
[‡] Steklov Mathematical Institute RAS, Gubkin street 8, 19991 Moscow, Russia
  E-mail: vatutin@mi.ras.ru
[§] Mathematical Institute, University of Munich, Theresienstrasse 39, D–80333 Munich, Germany.
  E-mail: wachtel@mathematik.uni-muenchen.de

Local probabilities for conditioned random walks

The study of the random walks with negative drift conditioned to stay nonnegative was apparently initiated by Iglehart [17]. He has proved that if

$$\mathbf{E}\left[Xe^{pX}\right] = 0 \text{ for some } p > 0 \tag{1.1}$$

and $\mathbf{E}\left[X^2e^{pX}\right] < \infty$, then the sequence $\mathcal{L}\{S_n|\tau_0 > n\}$ converges weakly to a distribution on $\mathbb{R}_+$. Since no scaling is needed here, one have also an information on local probabilities $\mathbf{P}(S_n \in [y, y+\Delta), \tau_0 > n)$ for fixed y. An explicit expression for the limit of the conditional probabilities $\mathbf{P}(S_n \in [y, y+\Delta)|\tau_0 > n)$ can be found in Theorem 1.3 by Keener [19].

Much less is known for the case when (1.1) is not valid. If the variance of X is finite and the tail $\mathbf{P}(X > x)$ varies regularly with index $-\beta < -2$, then, as $n \to \infty$,

$$\mathbf{P}(S_n \geq un|\tau_0 > n) \to (1 + u/a)^{-\beta}, \quad u \geq 0. \tag{1.2}$$

This is a particular case of a conditional functional limit theorem proved by Durrett [14]. In contrast to Iglehart's situation, for regularly varying tails one can not derive asymptotics for local probabilities from the integral limit theorem.

We are going to consider conditional local probabilities of the random walks having heavy-tailed increments. More precisely, we shall work with the following classes of functions and distributions.

We say that a function $f : \mathbb{R} \to \mathbb{R}_+$ is *(asymptotically) locally constant* and write $f \in \mathcal{L}$ if

$$\lim_{x \to \infty} \frac{f(x+h)}{f(x)} = 1$$

for any $h > 0$. Further, see [18], Definition 3 and [2], Appendix B, we say that a function $f : \mathbb{R} \to \mathbb{R}_+$ belongs to the class $\mathcal{S}d$ *of subexponential densities* if $f \in \mathcal{L}$ and

$$\lim_{x \to \infty} \frac{\int_0^{x/2} f(y)f(x-y)dy}{f(x)} = \int_0^{\infty} f(y)dy < \infty.$$

A positive, measurable function f defined in a neighborhood of infinity is called $O-$*regularly varying* if

$$0 < \liminf_{x \to \infty} \frac{f(xy)}{f(x)} \leq \limsup_{x \to \infty} \frac{f(xy)}{f(x)} < \infty.$$

Recall, finally, that $f : \mathbb{R} \to \mathbb{R}_+$ is called *almost decreasing* (see Section 2.2 of [6]) if $f(x) \geq c \sup_{y \geq x} f(y)$ for some positive constant c.

We assume in the sequel that the distribution of X is either absolutely continuous or is supported by the integers $\mathbb{Z}$ (and not by a sublattice thereof). Let $b(x)$ denote the Lebesgue density of X in the absolute continuous case or the mass function in the lattice case.

Theorem 1.1. *Assume that $\mathbf{E}\left[|X|^\kappa\right] < \infty$ for some $1 < \kappa \leq 2$, $b(x)$ is almost decreasing, $x^\kappa b(x)$ either belongs to $\mathcal{S}d$ or is O-regularly varying, and*

$$\lim_{x \to \infty} \sup_{0 \leq t \leq x^{1/\kappa}} \left| \frac{b(x-t)}{b(x)} - 1 \right| = 0. \tag{1.3}$$

Then, for all $x \geq 0$, $y \geq -x$ and each $\Delta > 0$ (in the lattice case all these variables should be integer),

$$\lim_{n \to \infty} \frac{\mathbf{P}\left(S_n \in [y, y+\Delta), \tau_x > n\right)}{b(an)} = \mathbf{E}\left[\tau_x\right] \int_{y-x}^{y-x+\Delta} \mathbf{P}\left(\max_{j \geq 1} S_j < z\right) dz.$$

All the conditions of this theorem are taken from [2], and they are sufficient for the relation

$$\mathbf{P}(S_n \in [y, y + \Delta)) \sim \Delta n b(an + y) \text{ uniformly in } y \geq -(a - \varepsilon)n$$

to be valid for every $\varepsilon > 0$, see Corollary 2.1 in [2]. This asymptotics for unconditioned probabilities is one of the most important ingredients for the proof.

Remark 1.2. *Under much stronger conditions Theorem 1.1 was proved in [4].*

Theorem 1.3. *Assume that the conditions of Theorem 1.1 are fulfilled. Suppose additionally that, as $x \to \infty$,*

$$\mathbf{P}(X \geq x) = O(xb(x)). \tag{1.4}$$

Then, for every fixed $x \geq 0$,

$$\mathbf{P}\left(\tau_x = n\right) \sim a \mathbf{E}\left[\tau_x\right] b(an). \tag{1.5}$$

The starting point in the proof of Theorem 1.1 is the Wiener-Hopf factorization. It seems, however, that this method does not work in the case when $y = y_n \to \infty$. In order to analyze this situation we use a probabilistic approach which requires stronger restrictions on the jump distribution.

We consider the algebraic decay of the tail of X.

Theorem 1.4. *Assume that $\mathbf{E}\left[|X|^\kappa\right] < \infty$ for some $1 < \kappa \leq 2$, $b(x)$ is regularly varying with index $-\beta < -2$. Then, for every sequence $y_n \to \infty$ as $n \to \infty$ and any fixed x and $\Delta > 0$,*

$$\sup_{y \geq y_n} \left| \frac{\mathbf{P}(S_n \in [y, y + \Delta), \tau_x > n)}{b(an + y)} - \Delta \mathbf{E}\left[\tau_x\right] \right| \to 0 \quad \text{as } n \to \infty. \tag{1.6}$$

This theorem is a local counterpart of Durrett's result mentioned earlier.

The method we use to prove Theorem 1.4 works also for bounded values of y, but it requires stronger, compared to Theorem 1.1, conditions on the function $b(x)$.

2 Proof of Theorem 1.1

Since the proofs in absolutely continuous and lattice cases are almost identical, we consider here only the first possibility.

We start with a series of auxiliary statements.

The first result is Corollary 2.1 from [2].

Lemma 2.1. *Under the conditions of Theorem 1.1, for any fixed $\varepsilon > 0$ and $\Delta > 0$,*

$$\lim_{n \to \infty} \sup_{y \geq -(a - \varepsilon)n} \left| \frac{\mathbf{P}\left(S_n \in [y, y + \Delta)\right)}{n b(na + y)} - 1 \right| = 0. \tag{2.1}$$

The next lemma can be found in Embrechts and Hawkes [15] or Asmussen et al [1].

Lemma 2.2. *Let $\{\beta_n, n \geq 0\}$ be a subexponential sequence with $\sum_{k=0}^{\infty} \beta_k < \infty$.*

1. *If $\delta_n \sim d\beta_n$, $\eta_n \sim e\beta_n$, then $\sum_{i=0}^{n} \delta_i \eta_{n-i} \sim c\beta_n$ with $c := d\sum_{k=0}^{\infty} \eta_k + e\sum_{k=0}^{\infty} \delta_k$ as $n \to \infty$.*
2. *If $\sum_{k=0}^{\infty} \alpha_k t^k = \exp\left(\sum_{k=0}^{\infty} \beta_k t^k\right)$ for $|t| < 1$, then $\alpha_n \sim c\beta_n$ with $c := \sum_{k=0}^{\infty} \alpha_k$ as $n \to \infty$.*

The first statement of Lemma 2.2 follows from Proposition 3 of [1]. The second statement of the Lemma follows from Theorem 1 of [15] or Theorem 7 of [1]. To apply the results from [1] one should take there $\Delta = (0, 1]$ and notice that for lattice random variables subexponentiality of probability mass function is equivalent to Definition 2 of [1] with $\Delta = (0, 1]$.

Lemma 2.3. *Put $Z(x) = |\log b(x)|$. If the condition (1.3) is valid, then there exists a constant $c \in (0, \infty)$ such that $Z(x) \leq cx^{1-1/\kappa}$ for all sufficiently large x.*

Proof. By (1.3) $b(x) \leq 2b(x - tx^{1/\kappa})$ for $t \leq 1$. Now note that we can pick a sequence C_k such that

$$x - C_k x^{1/\kappa} - (x - C_k x^{1/\kappa})^{1/\kappa} \geq x - C_{k+1} x^{1/\kappa}. \tag{2.2}$$

Indeed, observe that

$$x - C_k x^{1/\kappa} - (x - C_k x^{1/\kappa})^{1/\kappa} = x - C_k x^{1/\kappa} - x^{1/\kappa}(1 - C_k x^{1/\kappa - 1})^{1/\kappa}$$
$$\geq x - C_k x^{1/\kappa} - 0.5^{1/\kappa} x^{1/\kappa}$$

if $C_k x^{1/\kappa - 1} \leq 0.5$. Clearly one can take $C_k = (k-1)0.5^{1/\kappa}$. Let $k(x)$ be the maximal integer such that $C_k x^{1/\kappa - 1} \leq 0.5$. It is not difficult to see that $k(x) \sim 0.5x^{1-1/\kappa}$ as $x \to \infty$. Then, using (2.2) we can iteratively use (1.3) to conclude that

$$b(x) \leq 2b(x - C_1 x^{1/\kappa}) \leq 2^2 b(x - C_1 x^{1/\kappa}) \leq \ldots \leq 2^{k(x)} b(x - C_{k(x)} x^{1/\kappa}).$$

This implies $b(x) \leq 2^{k(x)} b(0.5x)$. Applying the latter inequality iteratively we see that, for a fixed x_0

$$b(x) \leq 2^{\sum_{i=1}^{\log_2 x} k(x 2^{-i})} \sup_{y \leq x_0} b(y).$$

Taking logarithms from both sides gives $\log b(x) \leq cx^{1-1/\kappa}$. Inequality $\log b(x) \geq -cx^{1-1/\kappa}$ can be proved similarly. □

The next statement immediately follows from Theorem 2.2 of [3].

Lemma 2.4. *If $\mathbf{P}(S_n > y)/\mathbf{P}(S_n > 0) \to 1$ for every $y > 0$ and $\mathbf{P}(S_n > 0)/n$ is a subexponential sequence then, as $n \to \infty$,*

$$\mathbf{P}(\tau_x > n) \sim \mathbf{E}[\tau_x] \frac{\mathbf{P}(S_n \geq 0)}{n}. \tag{2.3}$$

Moreover, under the conditions of Theorem 1.1,

$$\mathbf{P}(\tau_x > n) \sim \mathbf{E}[\tau_x] \mathbf{P}(X \geq na).$$

Indeed, the conditions of Theorem 1.1 of the present paper correspond to the conditions of Theorem 2.1 of [3]. Additional conditions of Lemma 2.4 correspond to the conditions of Theorem 1.2 of [3] for $\alpha = \gamma = 0$.

We define

$$L_n := \min_{0 \leq k \leq n} S_k, \quad M_n := \max_{1 \leq k \leq n} S_k$$

and

$$T_n = \min\{0 \leq k \leq n : S_k = L_n\}, \tag{2.4}$$

and specify two renewal functions

$$u(x) = 1 + \sum_{k=1}^{\infty} \mathbf{P}(-S_k \leq x, M_k < 0), \ x \geq 0,$$
$$v(x) = 1 + \sum_{k=1}^{\infty} \mathbf{P}(S_k < x, L_k \geq 0), \ x \geq 0.$$

Lemma 2.5. *Assume that all the conditions of Theorem 1.1 are fulfilled. Then, for any* $\lambda > 0$, *as* $n \to \infty$,

$$\mathbf{E}\left[e^{\lambda S_n}; T_n = n\right] = \mathbf{E}\left[e^{\lambda S_n}; M_n < 0\right] \sim K_1(\lambda)b(an), \tag{2.5}$$

$$\mathbf{E}\left[e^{-\lambda S_n}; \tau_0 > n\right] = \mathbf{E}\left[e^{-\lambda S_n}; L_n \geq 0\right] \sim K_2(\lambda)b(an), \tag{2.6}$$

where

$$
\begin{aligned}
K_1(\lambda) &= \frac{1}{\lambda} \exp\left\{\sum_{n=1}^{\infty} \frac{1}{n} \mathbf{E}\left[e^{\lambda S_n}; S_n < 0\right]\right\} \\
&= \frac{1}{\lambda}\left(1 + \sum_{n=1}^{\infty} \mathbf{E}\left[e^{\lambda S_n}; M_n < 0\right]\right) = \int_0^{\infty} e^{-\lambda x} u(x)\,dx
\end{aligned} \tag{2.7}
$$

and

$$
\begin{aligned}
K_2(\lambda) &= \frac{1}{\lambda} \exp\left\{\sum_{n=1}^{\infty} \frac{1}{n} \mathbf{E}\left[e^{-\lambda S_n}; S_n \geq 0\right]\right\} \\
&= \frac{1}{\lambda}\left(1 + \sum_{n=1}^{\infty} \mathbf{E}\left[e^{-\lambda S_n}; L_n \geq 0\right]\right) = \int_0^{\infty} e^{-\lambda z} v(z)\,dz.
\end{aligned} \tag{2.8}
$$

Proof. We first check the validity of (2.5). Since the random walks $\{S_k : k = 0, 1, \ldots, n\}$ and $\{S'_k := S_n - S_{n-k} : k = 0, 1, \ldots, n\}$ have the same law and the event $\{T_n = n\}$ for $\{S_k\}$ corresponds to the event $\{M_n < 0\}$ for $\{S'_k\}$, the equality in (2.5) follows from the mentioned duality. To go further we set $Z(x) = |\log b(x)|$ and evaluate the quantity

$$\mathbf{E}\left[e^{\lambda S_n}; S_n < 0\right] = \mathbf{E}\left[e^{\lambda S_n}; -2\lambda^{-1} Z(an) \leq S_n < 0\right] + O\left(b^2(an)\right). \tag{2.9}$$

Clearly, for any $h > 0$,

$$
\sum_{0 \leq k \leq 2\lambda^{-1}h^{-1}Z(an)} e^{-\lambda(k+1)h} \mathbf{P}\left(-(k+1)h \leq S_n \leq -kh\right)
$$
$$
\leq \mathbf{E}\left[e^{\lambda S_n}; -2\lambda^{-1} Z(an) \leq S_n < 0\right]
$$
$$
\leq \sum_{0 \leq k \leq 2\lambda^{-1}h^{-1}Z(an)} e^{-\lambda kh} \mathbf{P}\left(-(k+1)h \leq S_n \leq -kh\right).
$$

Note that according to Lemma 2.3, $Z(x) \leq cx^{1-1/\kappa}$ for sufficiently large x. With this in view we have by Lemma 2.1,

$$
\lim_{n \to \infty} \sup_{0 \leq k \leq 2\lambda^{-1}h^{-1}Z(na)} \left|\frac{\mathbf{P}\left(-(k+1)h \leq S_n < -kh\right)}{nhb(an)} - 1\right| = 0.
$$

Thus,

$$
\sum_{0 \leq k \leq 2\lambda^{-1}h^{-1}Z(an)} e^{-\lambda kh} \mathbf{P}\left(-(k+1)h \leq S_n \leq -kh\right)
$$
$$
= (1 + o(1))\, nhb(an) \sum_{0 \leq k \leq 2\lambda^{-1}h^{-1}Z(na)} e^{-\lambda kh}
$$
$$
= nb(an)(1 + o(1))h \times \sum_{k=0}^{\infty} e^{-\lambda kh}
$$
$$
= nb(an)(1 + o(1))\frac{h}{1 - e^{-\lambda h}}.
$$

By similar arguments we get

$$\sum_{0 \leq k \leq 2\lambda^{-1}h^{-1}Z(an)} e^{-\lambda(k+1)h} \mathbf{P}\left(-(k+1)h \leq S_n < -kh\right)$$

$$= (1 + o(1)) \, nhb(an) \sum_{0 \leq k \leq 2\lambda^{-1}h^{-1}Z(na)} e^{-\lambda(k+1)h}$$

$$= nb(an)(1 + o(1))h \times \sum_{k=0}^{\infty} e^{-\lambda(k+1)h}$$

$$= nb(an)(1 + o(1))\frac{he^{-\lambda h}}{1 - e^{-\lambda h}}.$$

Now

$$\frac{he^{-\lambda h}}{1 - e^{-\lambda h}} \quad \leq \quad \liminf_{n \to \infty} \frac{\mathbf{E}\left[e^{\lambda S_n}; -2\lambda^{-1}Z(an) \leq S_n < 0\right]}{nb(an)}$$

$$\leq \quad \limsup_{n \to \infty} \frac{\mathbf{E}\left[e^{\lambda S_n}; -2\lambda^{-1}Z(an) \leq S_n < 0\right]}{nb(an)} \leq \frac{h}{1 - e^{-\lambda h}}$$

and letting $h \downarrow 0$ we get that, as $n \to \infty$,

$$\mathbf{E}\left[e^{\lambda S_n}; -2\lambda^{-1}Z(an) \leq S_n < 0\right] \sim \lambda^{-1}nb(an).$$

Combining this with (2.9) and the fact that $b(n) = o(n)$ due to the existence of the first moment, we conclude that, as $n \to \infty$

$$\mathbf{E}\left[e^{\lambda S_n}; S_n < 0\right] \sim \lambda^{-1}nb(an). \tag{2.10}$$

We know by the Baxter identity that

$$1 + \sum_{n=1}^{\infty} t^n \mathbf{E}\left[e^{\lambda S_n}; M_n < 0\right] = \exp\left\{\sum_{n=1}^{\infty} \frac{t^n}{n} \mathbf{E}\left[e^{\lambda S_n}; S_n < 0\right]\right\},$$

see, for example, Chapter XVIII.3 in [16] or Chapter 8.9 in [6]. From (2.10), Theorem 1.4.3 in [7] and (ii) of Lemma 2.2 we deduce

$$\mathbf{E}\left[e^{\lambda S_n}; M_n < 0\right] \sim K_1(\lambda)b(an),$$

where $K_1(\lambda)$ is given by (2.7). This proves the equivalence in (2.5).

The proof of (2.6) follows the same line by using the Baxter identity

$$1 + \sum_{n=1}^{\infty} t^n \mathbf{E}\left[e^{-\lambda S_n}; L_n \geq 0\right] = \exp\left\{\sum_{n=1}^{\infty} \frac{t^n}{n} \mathbf{E}\left[e^{-\lambda S_n}; S_n \geq 0\right]\right\}. \tag{2.11}$$

$\square$

Lemma 2.6. *Under the conditions of Theorem 1.1, as* $n \to \infty$,

$$\mathbf{P}\left(S_n \in [y, y + \Delta), L_n \geq 0\right) \sim b(an) \int_y^{y+\Delta} v(z) \, dz \tag{2.12}$$

and

$$\mathbf{P}\left(-S_n \in [y, y + \Delta), M_n < 0\right) \sim b(an) \int_y^{y+\Delta} u(z) \, dz. \tag{2.13}$$

Proof. Lemma 2.5, the extended continuity theorem for Laplace transforms (see [16], Ch.XIII.1, Theorem 2) and the boundness of $u(x)$ and $v(x)$ on each finite interval of the nonnegative semi-axis lead to

$$\mathbf{P}\left(S_n \in [y, y+\Delta), L_n \geq 0\right) \sim b(an) \int_y^{y+\Delta} v(z)\, dz$$

and

$$\mathbf{P}\left(-S_n \in [y, y+\Delta), M_n < 0\right) \sim b(an) \int_y^{y+\Delta} u(z)\, dz.$$

□

The next lemma is a crucial step in proving Theorem 1.1.

Lemma 2.7. *Under the conditions of Theorem 1.1, for $x \geq 0$ and $\theta > 0$,*

$$\mathbf{E}[e^{-\theta S_n}, L_n \geq -x] \sim b(an)u(x)e^{-\theta x} \int_0^\infty e^{-\theta z} v(z)\, dz , \tag{2.14}$$

and

$$\mathbf{E}[e^{\theta S_n}, M_n < x] \sim b(an)v(x)e^{\theta x} \int_0^\infty e^{-\theta z} u(z)\, dz. \tag{2.15}$$

Proof. The same as earlier, Lemma 2.5 and the extended continuity theorem for Laplace transforms imply, as $n \to \infty$,

$$\frac{\mathbf{E}[e^{\theta S_n}; M_n < 0, S_n > -x]}{b(an)} \to \int_0^x e^{-\theta z} u(z)\, dz , \tag{2.16}$$

$$\frac{\mathbf{E}[e^{\theta S_n}; L_n \geq 0, S_n < x]}{b(an)} \to \int_0^x e^{\theta z} v(z)\, dz , \tag{2.17}$$

which for finite $x \geq 0$ are valid for every $\theta \in \mathbb{R}^+$, since the limit measures involved here have densities with respect to the Lebesgue measure.

Next we fix some $x > 0$. By the total probability formula we may write

$$\mathbf{E}[e^{\theta S_n}; M_n < x]$$
$$= \sum_{i=0}^{n-1} \mathbf{E}[e^{\theta S_n}; S_0 \leq S_i, \ldots, S_{i-1} \leq S_i, S_i < x , S_i > S_{i+1}, \ldots, S_i > S_n]$$
$$+ \mathbf{E}[e^{\theta S_n}; S_0 \leq S_n, \ldots, S_{n-1} \leq S_n, S_n < x]. \tag{2.18}$$

Now we can apply the duality arguments. Since the random walks $\{S_k : k = 0, 1, \ldots, n\}$ and $\{S_k' := S_n - S_{n-k} : k = 0, 1, \ldots, n\}$ have the same law, the measures $\mathbf{P}\{S_0 \leq S_i, \ldots, S_{i-1} \leq S_i, S_i \in dy\}$ and $\mathbf{P}\{S_1 \geq 0, \ldots, S_i \geq 0, S_i \in dy\}$ are equal. Moreover, by the Markov property, $\mathbf{P}\{S_i > S_{i+1}, \ldots, S_i > S_n, S_n \in dz | S_i = y\} = \mathbf{P}\{S_1 < 0, \ldots, S_{n-i} < 0, S_{n-i} \in dz | S_0 = y\}$. Hence we can continue (2.18) to obtain

$$\mathbf{E}[e^{\theta S_n}; M_n < x] = \sum_{i=0}^n \mathbf{E}[e^{\theta S_i}; L_i \geq 0, S_i < x] \cdot \mathbf{E}[e^{\theta S_{n-i}}; M_{n-i} < 0] .$$

This formula combined with (2.16), (2.17) and the equations (note that $v(z)$ is left con-

tinuous for $z > 0$ and that $v(0) = v(0-) = 1$)

$$1 + \sum_{k=1}^{\infty} \mathbf{E}[e^{\theta S_k}; L_k \geq 0, S_k < x]$$

$$= 1 + \int_{(0,x)} e^{\theta z}\, dv(z) \;=\; e^{\theta x} v(x) - \theta \int_0^x e^{\theta z} v(z)\, dz \;,$$

$$1 + \sum_{k=1}^{\infty} \mathbf{E}[e^{\theta S_k}; M_k < 0] \;=\; \theta \int_0^{\infty} e^{-\theta z} u(z)\, dz$$

imply by means of Lemma 2.2 i) for $\theta > 0$ and $x > 0$

$$\frac{\mathbf{E}[e^{\theta S_n}; M_n < x]}{b(an)} \;\rightarrow\; v(x) e^{\theta x} \int_0^{\infty} e^{-\theta z} u(z)\, dz \;.$$

The second statement can be proved using similar arguments. $\qquad\square$

Proof of Theorem 1.1. By the same arguments that have been used to deduce (2.12) and (2.13) from (2.5) and (2.6), we infer from (2.14) that

$$\frac{\mathbf{P}(S_n \in [y, y+\Delta), L_n \geq -x)}{b(an)} \sim u(x) \int_y^{y+\Delta} v(z-x) dz. \qquad (2.19)$$

It remains to rewrite u and v in terms of $\max_{j \geq 1} S_j$ and $\mathbf{E}[\tau_x]$. Applying the duality, we get

$$v(z) = 1 + \sum_{k=1}^{\infty} \mathbf{P}(S_k < z, L_k \geq 0)$$

$$= 1 + \sum_{k=1}^{\infty} \mathbf{P}\Big(S_k < z, k \text{ is a (weak ascending) ladder epoch}\Big).$$

Define $\tau^+ = \min\{k \geq 1 : S_k \geq 0\}$. From the factorization identity, see e.g. Section XVIII.3 of [16],

$$1 - s = (1 - \mathbf{E}[s^{\tau_0}]) \left(1 - \mathbf{E}[s^{\tau^+}; \tau^+ < \infty]\right)$$

we infer that

$$\mathbf{P}(\tau^+ = \infty) = 1/\mathbf{E}[\tau_0].$$

Then

$$\frac{v(z)}{\mathbf{E}[\tau_0]} = \mathbf{P}(\tau^+ = \infty) \left(1 + \sum_{k=1}^{\infty} \mathbf{P}\Big(S_k < z, k \text{ is a (weak ascending) ladder epoch}\Big)\right)$$

$$= \mathbf{P}(\tau^+ = \infty) + \sum_{k=1}^{\infty} \mathbf{P}\Big(S_k < z, k \text{ is the last (weak ascending) ladder epoch}\Big)$$

$$= \mathbf{P}\left(\max_{j \geq 1} S_j < z\right). \qquad (2.20)$$

Define

$$\sigma(x) := \min\{k \geq 1 : \chi_1 + \ldots + \chi_k > x\},$$

where χ_i are independent copies of the first strict descending ladder height. Then, by the Wald identity,

$$\mathbf{E}\sigma(x) = \frac{\mathbf{E}[\tau_x]}{\mathbf{E}[\tau_0]}.$$

Furthermore,

$$\mathbf{E}\sigma(x) = \sum_{k=0}^{\infty} \mathbf{P}(\sigma(x) > k) = 1 + \sum_{k=1}^{\infty} \mathbf{P}(\chi_1 + \ldots + \chi_k \leq x)$$

$$= 1 + \sum_{l=1}^{\infty} \mathbf{P}(S_l \geq -x, l \text{ is a strict descending ladder epoch}).$$

By the duality, for each $l \geq 1$,

$$\mathbf{P}(S_l \geq -x, l \text{ is a strict descending ladder epoch}) = \mathbf{P}(S_l \geq -x, M_l < 0)$$

and, recalling the definition of $u(x)$, we finally get

$$u(x) = \frac{\mathbf{E}[\tau_x]}{\mathbf{E}[\tau_0]}. \tag{2.21}$$

Combining (2.19)–(2.21) completes the proof. $\qquad\square$

3 Local limit theorem for the first exit moment from the positive semi-axis

3.1 Proof of (1.5) for $x = 0$

We write in this subsection τ for τ_0. Setting $\lambda = 0$ in (2.11) and differentiating the result with respect to t, one can easily get

$$\mathbf{P}(\tau > n) = \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{P}(\tau > k) \mathbf{P}(S_{n-k} > 0).$$

Hence it follows that

$$\mathbf{P}(\tau = n) = \mathbf{P}(\tau > n - 1) - \mathbf{P}(\tau > n)$$

$$= \frac{1}{n-1} \sum_{k=0}^{n-2} \mathbf{P}(\tau > k) \mathbf{P}(S_{n-1-k} > 0) - \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{P}(\tau > k) \mathbf{P}(S_{n-k} > 0)$$

$$= \left(\frac{1}{n-1} - \frac{1}{n} \right) (n-1) \mathbf{P}(\tau > n - 1)$$

$$\quad + \frac{1}{n} \left(\sum_{k=0}^{n-2} \mathbf{P}(\tau > k) \mathbf{P}(S_{n-1-k} > 0) - \sum_{k=0}^{n-1} \mathbf{P}(\tau > k) \mathbf{P}(S_{n-k} > 0) \right)$$

$$= \frac{1}{n} \left(\mathbf{P}(\tau > n - 1) - \mathbf{P}(\tau > n - 1) \mathbf{P}(S_1 > 0) \right)$$

$$\quad + \frac{1}{n} \sum_{k=0}^{n-2} \mathbf{P}(\tau > k) (\mathbf{P}(S_{n-1-k} > 0) - \mathbf{P}(S_{n-k} > 0)).$$

As a result,

$$\mathbf{P}(\tau = n) = \frac{1}{n} \mathbf{P}(\tau > n - 1) \mathbf{P}(S_1 \leq 0) \tag{3.1}$$

$$\quad + \frac{1}{n} \sum_{k=0}^{n-2} \mathbf{P}(\tau > k) \left(\mathbf{P}(S_{n-1-k} > 0) - \mathbf{P}(S_{n-k} > 0) \right).$$

By Lemma 2.4,

$$\mathbf{P}(\tau > n) \sim \mathbf{E}[\tau] \, \mathbf{P}(X \geq na), \quad n \to \infty.$$

Therefore, for any fixed integer A, as $n \to \infty$,

$$\frac{1}{n}\mathbf{P}(\tau > n-1)\mathbf{P}(S_1 \le 0) + \frac{1}{n}\sum_{k=n-A}^{n-2}\mathbf{P}(\tau > k)\left(\mathbf{P}(S_{n-1-k} > 0) - \mathbf{P}(S_{n-k} > 0)\right)$$

$$\sim \mathbf{E}[\tau]\frac{\mathbf{P}(X \ge na)}{n}\left(1 - \mathbf{P}(S_1 > 0) + \sum_{k=n-A}^{n-2}\left(\mathbf{P}(S_{n-1-k} > 0) - \mathbf{P}(S_{n-k} > 0)\right)\right)$$

$$\sim \mathbf{E}[\tau]\frac{\mathbf{P}(X \ge na)}{n}(1 - \mathbf{P}(S_A > 0)).$$

Since the random walk under consideration has a negative drift, we can select for any fixed $\varepsilon > 0$ a sufficiently large A to meet the inequality $\mathbf{P}(S_A > 0) \le \varepsilon$. In fact, we can assume that $k \le n - A(n) \to \infty$. As a result,

$$\frac{1}{n}\mathbf{P}(\tau > n-1)\mathbf{P}(S_1 \le 0) + \frac{1}{n}\sum_{k=n-A(n)}^{n-2}\mathbf{P}(\tau > k)\left(\mathbf{P}(S_{n-1-k} > 0) - \mathbf{P}(S_{n-k} > 0)\right)$$

$$\sim \mathbf{E}[\tau]\frac{\mathbf{P}(X \ge na)}{n}(1 + o(1)). \tag{3.2}$$

Now we analyze the difference

$$\mathbf{P}(S_{i-1} > 0) - \mathbf{P}(S_i > 0) = \mathbf{P}(S_{i-1} > 0, S_i \le 0) - \mathbf{P}(S_{i-1} \le 0, S_i > 0).$$

Applying Lemma 2.1, we obtain

$$\mathbf{P}(S_{i-1} > 0, S_i \le 0) = \int_{-\infty}^{0}\mathbf{P}(X_i \in dy)\mathbf{P}(S_{i-1} \in (0, -y])$$

$$\sim (i-1)\int_{-\infty}^{0}\mathbf{P}(X_i \in dy)\int_{0}^{-y}b((i-1)a + z)dz$$

$$= (i-1)\int_{0}^{\infty}dz\, b((i-1)a + z)\mathbf{P}(X_i \le -z).$$

Since $b(x)$ is almost decreasing, we have

$$\int_{A}^{\infty}dz\, b((i-1)a + z)\mathbf{P}(X_i \le -z) \le Cb((i-1)a)\int_{A}^{\infty}\mathbf{P}(X_i \le -z)dz.$$

Using long-tailedness, we deduce that, as $i \to \infty$,

$$\int_{0}^{A}dz\, b((i-1)a + z)\mathbf{P}(X_i \le -z) \sim b((i-1)a)\int_{0}^{A}\mathbf{P}(X_i \le -z)dz.$$

Hence, letting $A \to \infty$, we conclude that

$$\mathbf{P}(S_{i-1} > 0, S_i \le 0) \sim (i-1)b((i-1)a)\mathbf{E}[X^-], \tag{3.3}$$

where $X^- = \max(0, -X)$. Next, for any $\varepsilon \in (0, a)$ we have

$$\mathbf{P}(S_{i-1} \le 0, S_i > 0) = \int_{0}^{\infty}\mathbf{P}(X_i \in dy)\mathbf{P}(S_{i-1} \in (-y, 0])$$

$$= \int_{0}^{(a-\varepsilon)i}\mathbf{P}(X_i \in dy)\mathbf{P}(S_{i-1} \in (-y, 0])$$

$$+ \int_{(a-\varepsilon)i}^{\infty}\mathbf{P}(X_i \in dy)\mathbf{P}(S_{i-1} \in (-y, 0]). \tag{3.4}$$

Repeating the arguments used to derive (3.3), we obtain, as $i \to \infty$,

$$\int_0^{(a-\varepsilon)i} \mathbf{P}(X_i \in dy)\mathbf{P}(S_{i-1} \in (-y, 0]) \sim (i-1)b((i-1)a)\mathbf{E}\left[X^+\right]. \qquad (3.5)$$

We split the second integral in (3.4) into three parts

$$\begin{aligned}
\int_{(a-\varepsilon)i}^{\infty} \mathbf{P}(X_i \in dy)\mathbf{P}(S_{i-1} \in (-y, 0]) \;=\; & \int_{ai+Ai^{1/\kappa}}^{\infty} \mathbf{P}(X_i \in dy)\mathbf{P}(S_{i-1} \in (-y, 0]) \\
& + \int_{ai-Ai^{1/\kappa}}^{ai+Ai^{1/\kappa}} \mathbf{P}(X_i \in dy)\mathbf{P}(S_{i-1} \in (-y, 0]) \\
& + \int_{(a-\varepsilon)i}^{ai-Ai^{1/\kappa}} \mathbf{P}(X_i \in dy)\mathbf{P}(S_{i-1} \in (-y, 0]).
\end{aligned}$$

By the insensitivity assumption (1.3), the second integral admits the estimate

$$\int_{ai-Ai^{1/\kappa}}^{ai+Ai^{1/\kappa}} \mathbf{P}(X_i \in dy)\mathbf{P}(S_{i-1} \in (-y, 0]) \le Cb(ai)2Ai^{1/\kappa} = o((i-1)b((i-1)a)),$$

while for the first we have

$$\lim_{A \to \infty} \lim_{i \to \infty} \frac{1}{\mathbf{P}(X \ge (i-1)a)} \int_{ai+Ai^{1/\kappa}}^{\infty} \mathbf{P}(X_i \in dy)\mathbf{P}(S_{i-1} \in (-y, 0]) = 1.$$

The evaluation of the third integral requires more delicate arguments based on a number of results we borrow from [2]. First we note that according to Lemma 6.2 of [2], the sequence $h_i := i^{1/\kappa}$ is a truncation sequence, see formula (4) in [2] for more detail. Hence we may apply Lemma 2.5 of the mentioned article to conclude that, as $i \to \infty$,

$$\begin{aligned}
\mathbf{P}(S_i > x) = \; & \mathbf{P}(S_i > x, \max_{k \le i} X_k \le h_i) \\
& + i(1+o(1))\mathbf{P}(S_i > x, X_1 > h_i, \max_{2 \le k \le i} X_k \le h_i)
\end{aligned}$$

uniformly in x. Consequently,

$$\begin{aligned}
& \int_{(a-\varepsilon)i}^{ai-Ai^{1/\kappa}} \mathbf{P}(X_i \in dy)\mathbf{P}(S_{i-1} \in (-y, 0]) \\
& \le \int_{(a-\varepsilon)i}^{ai-Ai^{1/\kappa}} \mathbf{P}(X_i \in dy)\mathbf{P}(S_{i-1} > -y, \max_{k<i} X_k \le h_i) \\
& + i(1+o(1)) \int_{(a-\varepsilon)i}^{ai-Ai^{1/\kappa}} \mathbf{P}(X_i \in dy)\mathbf{P}(S_{i-1} > -y, X_1 > h_i, \max_{2 \le k < i} X_k \le h_i).
\end{aligned}$$

Applying Lemma 7.1 from [2] to the centered random walk $S_n + na$, we obtain

$$\mathbf{P}(S_{i-1} > -y, \max_{k<i} X_k \le h_i) \le C\exp\left\{-\frac{(ai-y)}{h_i}\right\}.$$

Furthermore, using (1.3), one can get, for all sufficiently large i,

$$\frac{b(y)}{b(ai)} \le C \prod_{k \le \frac{(ai-y)}{h_i}} \frac{b(y+kh_i))}{b(y+(k+1)h_i)} \le C\exp\left\{\varepsilon\frac{(ai-y)}{h_i}\right\}.$$

These bounds imply

$$\frac{1}{b(ai)} \int_{(a-\varepsilon)i}^{ai-Ai^{1/\kappa}} \mathbf{P}(X_i \in dy)\mathbf{P}(S_{i-1} > -y, \max_{k<i} X_k \le h_i)$$

$$\le C \int_{(a-\varepsilon)i}^{ai-Ai^{1/\kappa}} \exp\left\{-(1-\varepsilon)\frac{(ai-y)}{h_i}\right\} dy = O(h_i).$$

It is easy to see that

$$\int_{(a-\varepsilon)i}^{ai-Ai^{1/\kappa}} \mathbf{P}(X_i \in dy)\mathbf{P}(S_{i-1} > -y, X_1 > h_i, \max_{2 \le k < i} X_k \le h_i)$$

$$\le \mathbf{P}(S_i > 0, X_1 > (a-\varepsilon)i, X_2 > h_i, \max_{3 \le k \le i} X_k \le h_i).$$

To bound this probability we apply estimates from [2]. Applying the first display on page 1958 of [2] we obtain,

$$\mathbf{P}(S_i > 0, X_1 > (a-\varepsilon)i, X_2 > h_i, \max_{3 \le k \le i} X_k \le h_i)$$

$$\le o(1/i)\mathbf{P}(X_2 + \cdots + X_i > 0, X_2 > h_i, \max_{3 \le k \le i} X_k \le h_i).$$

Applying the third display on page 1958 of [2] gives

$$\mathbf{P}(S_i > 0, X_1 > (a-\varepsilon)i, X_2 > h_i, \max_{3 \le k \le i} X_k \le h_i)$$

$$= (1/\beta)o(1/i)\mathbf{P}(S_i > 0, X_1 > h_i, \max_{2 \le k \le i} X_k \le h_i) = o(\mathbf{P}(S_i > 0)/i),$$

where $\beta = 2^{-1}\mathbf{P}(X_1 + x > 0) > 0$. Noting that (2.1) yields $\mathbf{P}(S_i > 0) \sim i\mathbf{P}(X_1 \ge ai)$, we get

$$\int_{(a-\varepsilon)i}^{ai-Ai^{1/\kappa}} \mathbf{P}(X_i \in dy)\mathbf{P}(S_{i-1} > -y, X_1 > h_i, \max_{2 \le k < i} X_k \le h_i) = o(\mathbf{P}(X_1 \ge ai)).$$

As a result,

$$\mathbf{P}(S_{i-1} \le 0, S_i > 0) \sim (i-1)b((i-1)a)\mathbf{E}\left[X^+\right] + \mathbf{P}(X_1 \ge ai). \tag{3.6}$$

Combining (3.3) and (3.6), we deduce, as $i \to \infty$,

$$\mathbf{P}(S_{i-1} > 0) - \mathbf{P}(S_i > 0) \sim aib(ia) - \mathbf{P}(X \ge ia).$$

Then,

$$\frac{1}{n}\sum_{k=0}^{n-A(n)} \mathbf{P}(\tau > k)\left(\mathbf{P}(S_{n-1-k} > 0) - \mathbf{P}(S_{n-k} > 0)\right)$$

$$\sim \frac{1}{n}\sum_{k=0}^{n-A(n)} \mathbf{P}(\tau > k)\left(a(n-k)b((n-k)a) - \mathbf{P}(X \ge (n-k-1)a)\right).$$

For the second term we have

$$
-\frac{1}{n} \sum_{k=0}^{n-A(n)} \mathbf{P}(\tau > k) \mathbf{P}\left(X \geq (n-k-1)a\right)
$$

$$
\sim -\frac{\mathbf{P}\left(X \geq na\right)}{n} \sum_{k=0}^{A(n)} \mathbf{P}(\tau > k) + \frac{1}{n} \sum_{k=A(n)}^{n-A(n)} \mathbf{P}(\tau > k) \mathbf{P}\left(X \geq (n-k-1)a\right)
$$

$$
\sim -\frac{\mathbf{E}\tau \mathbf{P}\left(X \geq na\right)}{n} + \frac{1}{n} \sum_{k=A(n)}^{n-A(n)} \mathbf{P}\left(X \geq ka\right) \mathbf{P}\left(X \geq (n-k-1)a\right)
$$

$$
\sim -\frac{\mathbf{E}\tau \mathbf{P}\left(X \geq na\right)}{n}.
$$

In the second equivalence we used the second assertion of Lemma 2.4. In the third equivalence we used the subexponentiality of the tail of X. This term will be canceled with (3.2).

Finally, choosing $A(n) = h_n = n^{1/\kappa}$ (here and in what follows we agree to consider $n^{1/\kappa}$ as $\left[n^{1/\kappa}\right]$, i.e, as a positive integer number) and taking into account (1.3), we get

$$
\mathbf{P}(\tau = n) = \frac{1}{n} \sum_{k=0}^{n-A(n)} \mathbf{P}(\tau > k) a(n-k) b((n-k)a) + o\left(\frac{\mathbf{P}\left(X \geq an\right)}{n}\right)
$$

$$
= ab(na) \sum_{k=0}^{A(n)} \mathbf{P}(\tau > k) + \frac{\mathbf{E}\left[\tau\right]}{n} \sum_{k=A(n)}^{n-A(n)} \mathbf{P}\left(X \geq ka\right) a(n-k) b((n-k)a)
$$

$$
+ o\left(\frac{\mathbf{P}\left(X \geq an\right)}{n}\right).
$$

Using again the estimate (4) from [2], we obtain

$$
\frac{\mathbf{E}\left[\tau\right]}{n} \sum_{k=A(n)}^{n-A(n)} \mathbf{P}\left(X \geq ka\right) a(n-k) b((n-k)a) \leq a\mathbf{E}\left[\tau\right] \sum_{k=A(n)}^{n-A(n)} \mathbf{P}\left(X \geq ka\right) b((n-k)a)
$$

$$
= o(\mathbf{P}(X \geq an)/n).
$$

Recalling the condition (1.4), we get the desired result.

3.2 Proof of (1.5) for $x > 0$

By the total probability formula

$$
\mathbf{P}(T_n = n) = \mathbf{P}(T_n = n, S_n \geq -x) + \mathbf{P}(\tau_x = n).
$$

Then decomposing the event $\{T_n = n, S_n \geq -x\} = \cup_{i=0}^{n-1}\{\tau_0 = n - i, T_n = n, S_n \geq -x\}$ we obtain

$$\mathbf{P}(\tau_x = n) = \sum_{i=0}^{n-1} \mathbf{P}(\tau_0 = n - i)\mathbf{P}(S_i \geq -x; T_i = i) - \mathbf{P}(S_n \geq -x; T_n = n)$$

$$= \sum_{i=0}^{n^{1/\kappa}} \mathbf{P}(\tau_0 = n - i)\mathbf{P}(S_i \geq -x; T_i = i)$$

$$+ \sum_{i=n^{1/\kappa}+1}^{n-n^{1/\kappa}} \mathbf{P}(\tau_0 = n - i)\mathbf{P}(S_i \geq -x; T_i = i)$$

$$+ \sum_{i=n-n^{1/\kappa}+1}^{n} \mathbf{P}(\tau_0 = n - i)\mathbf{P}(S_i \geq -x; T_i = i) - \mathbf{P}(S_n \geq -x; T_n = n).$$

Applying (1.5) with $x = 0$, we get

$$\sum_{i=0}^{n^{1/\kappa}} \mathbf{P}(\tau_0 = n - i)\mathbf{P}(S_i \geq -x; T_i = i)$$

$$\sim \left(\sum_{i=0}^{\infty} \mathbf{P}(S_i \geq -x; T_i = i)\right) a\mathbf{E}\left[\tau_0\right] b(an) = au(x)\mathbf{E}\left[\tau_0\right] b(an) = a\mathbf{E}\left[\tau_x\right] b(an).$$

Here we use the following duality arguments in the first equality

$$\mathbf{P}(S_i \geq -x; T_i = i) = \mathbf{P}(S_i \geq -x; S_1 > S_i, \ldots, S_{i-1} > S_i)$$
$$= \mathbf{P}(S_i \geq -x; 0 > S_{i-1}, \ldots, 0 > S_1)$$

and (2.21) in the second equality. Recalling (2.13) and taking into account our insensitivity condition (1.3), we conclude that

$$\sum_{i=n-n^{1/\kappa}+1}^{n} \mathbf{P}(\tau_0 = n - i)\mathbf{P}(S_i \geq -x; T_i = i) - \mathbf{P}(S_n \geq -x; T_n = n) = o(b(an)).$$

Finally, since $b(x)$ is subexponential, there exists a positive constant $C(x)$ such that

$$\sum_{i=n^{1/\kappa}+1}^{n-n^{1/\kappa}} \mathbf{P}(\tau_0 = n - i)\mathbf{P}(S_i \geq -x; T_i = i) \sim C(x) \sum_{i=n^{1/\kappa}+1}^{n-n^{1/\kappa}} b(ai)b((n-i)a) = o(b(an)).$$

This completes the proof of (1.5).

4 Proof of Theorem 1.4

For diversity, we give a proof in the lattice case and assume that, as $n \to \infty$,

$$\mathbf{P}(X = n) \sim l(n)n^{-\beta}, \beta > 2.$$

Fix an $\varepsilon \in (0, 1/2)$ and introduce

$$\eta := \min\{k \geq 1 : X_k \geq \varepsilon(an + y)\}.$$

Then for $y > -x$

$$\mathbf{P}(S_n = y, \tau_x > n) = \mathbf{P}(S_n = y, \eta > n, \tau_x > n) + \sum_{k=1}^{n} \mathbf{P}(S_n = y, \eta = k, \tau_x > n).$$

By the Markov property,

$$\mathbf{P}(S_n = y, \eta = k, \tau_x > n) = \sum_{z=-x+1}^{\infty} \mathbf{P}(S_{k-1} = z, \eta > k - 1, \tau_x > k - 1)$$

$$\times \sum_{w \geq \varepsilon(an+y)} \mathbf{P}(X_k = w)\mathbf{P}(S_{n-k} = y, \tau_x > n - k | S_0 = z + w). \qquad (4.1)$$

Note that

$$\mathbf{P}(S_n = y, \eta = k, \tau_x > n)$$

$$\leq \sup_{w \geq \varepsilon(an+y)} \mathbf{P}(X = w) \sum_{z=-x+1}^{\infty} \mathbf{P}(S_{k-1} = z, \tau_x > k - 1) \sum_{w \geq \varepsilon(an+y)} \mathbf{P}(S_{n-k} = y - z - w)$$

$$\leq C\varepsilon^{-\beta} \mathbf{P}(X = an + y)\mathbf{P}(\tau_x > k - 1) \qquad (4.2)$$

uniformly in k and y.

We next investigate the behavior of $\mathbf{P}(S_n = y, \eta = k, \tau_x > n)$ for every fixed k. Define

$$A(y) = \{w : |w - an - y| \leq \varepsilon(an + y)\},$$
$$B(y) = \{w : w \geq \varepsilon(an + y) \text{ and } w \notin A(y)\}.$$

It is not difficult to see that, for every fixed z and all sufficiently large n,

$$\sum_{w \in B(y)} \mathbf{P}(X_k = w)\mathbf{P}(S_{n-k} = y, \tau_x > n - k | S_0 = z + w)$$

$$\leq C\varepsilon^{-\beta}\mathbf{P}(X = an + y) \sum_{w \in B(y)} \mathbf{P}(S_{n-k} = y - w - z)$$

$$\leq C\varepsilon^{-\beta}\mathbf{P}(X = an + y)\mathbf{P}(|S_{n-k} - a(n - k)| > \varepsilon n).$$

Applying the law of large numbers, we conclude that, uniformly in y,

$$\sum_{w \in B(y)} \mathbf{P}(X_k = w)\mathbf{P}(S_{n-k} = y, \tau_x > n - k | S_0 = z + w) = o(\mathbf{P}(X = an + y)). \qquad (4.3)$$

Since $\mathbf{P}(X = n)$ is regularly varying with index $-\beta$, we have

$$\limsup_{n \to \infty} \sup_{w \in A(y)} \left| \frac{\mathbf{P}(X = w)}{\mathbf{P}(X = an + y)} - 1 \right| \leq \varepsilon^{\beta}. \qquad (4.4)$$

Fix some sequence $y_n \to \infty$. Inverting the time ($\widetilde{S}_i := \widetilde{X}_1 + \ldots + \widetilde{X}_i$, where $\widetilde{X}_j := -X_{n-k-j+1}$ for $j = 1, 2, \ldots, n - k$), we obtain, for $y \geq y_n$,

$$\sum_{w \in A(y)} \mathbf{P}(S_{n-k} = y, \tau_x > n - k | S_0 = z + w)$$

$$= \sum_{w \in A(y)} \mathbf{P}(\widetilde{S}_{n-k} = y - w - z, \min_{j \leq n-k} \widetilde{S}_j \geq -x - y)$$

$$\geq \sum_{w \in A(y)} \mathbf{P}(\widetilde{S}_{n-k} = y - w - z) - \mathbf{P}(\min_{j \leq n-k} \widetilde{S}_j < -y_n).$$

Since $\widetilde{S}_j$ has a positive drift, $\mathbf{P}(\min_{j \leq n-k} \widetilde{S}_j < -y_n) \to 0$ as $n \to \infty$. Hence, recalling the definition of $A(y)$ and using the law of large numbers, we see that

$$\sum_{w \in A(y)} \mathbf{P}(S_{n-k} = y, \tau_x > n - k | S_0 = z + w) \to 1$$

uniformly in $y \geq y_n$. Combining this relation with (4.3) and (4.4), we obtain

$$\limsup_{n \to \infty} \left| \frac{\mathbf{P}(S_n = y, \eta = k, \tau_x > n)}{\mathbf{P}(X = an + y)} - \mathbf{P}(\tau_x > k - 1) \right| \leq \varepsilon^\beta.$$

From this pointwise convergence and (4.2) we infer that

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \left| \sum_{k=1}^{n} \frac{\mathbf{P}(S_n = y, \eta = k, \tau_x > n)}{\mathbf{P}(X = an + y)} - \mathbf{E}\left[\tau_x\right] \right| = 0 \qquad (4.5)$$

uniformly in $y \geq y_n$. Thus, it remains to consider $\mathbf{P}(S_n = y, \eta > n, \tau_x > n)$. Here it suffices to apply one of the Fuk-Nagaev inequalities, see Theorem 1.2 [20] and its proof,

$$\mathbf{P}(S_n = y, \eta > n, \tau_x > n) \leq \mathbf{P}(S_n \geq y, \eta > n) \leq \left(\frac{e^2 n \mathbf{E}\left[|X|^\kappa\right]}{\varepsilon^{\kappa-1}(an + y)^\kappa} \right)^{1/2\varepsilon}.$$

Choosing ε sufficiently small, we conclude that

$$\mathbf{P}(S_n = y, \eta > n, \tau_x > n) = o(\mathbf{P}(X = an + y)). \qquad (4.6)$$

Combining (4.5) and (4.6), we obtain (1.6).

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