Small deviations for time-changed Brownian motions and applications to second-order chaos

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Dedicated to the memory of Wenbo Li

Abstract

We prove strong small deviations results for Brownian motion under independent time-changes satisfying their own asymptotic criteria. We then apply these results to certain stochastic integrals which are elements of second-order homogeneous chaos.

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1 Introduction

In this paper, we study small deviations for some time-changed Brownian motions, for the purpose of applications to certain elements of Wiener chaos. Large deviation estimates for Wiener chaos are well-studied (see for example [15]), largely due to the work of Borell (see for example [5] and [6]). However, small deviations in this setting are much less understood and are of interest for their myriad interactions with other concentration, comparison, and correlation inequalities as well as various limit laws for stochastic processes; see for example the surveys [17] and [19]. The present work gives strong small deviations results for certain elements of second-order homogeneous chaos. In particular, let $(\mathcal{W}, \mathcal{H}, \mu)$ be an abstract Wiener space, $\{W_t\}_{t \geq 0}$ denote Brownian motion on $\mathcal{W}$, and $\omega : \mathcal{W} \times \mathcal{W} \to \mathbb{R}$ be a continuous bilinear antisymmetric map . We will study processes $\{Z(t)\}_{t \geq 0}$ of the form

$$Z(t) := \int_0^t \omega(W_s, dW_s). \tag{1.1}$$

(A precise definition is given in Section 3.) In particular, we show that Z is equal in distribution to a Brownian motion running on an independent random clock for which small deviation probabilities are known, and thus the small deviations behavior of Z follows. From these results one may infer, for example, a functional law of iterated logarithm and hence a Chung-type law of iterated logarithm for Z. To the authors' knowledge, these are the first results for small deviations of elements of Wiener chaos in the infinite-dimensional context beyond the first-order Gaussian case.

1.1 Statement of main results

We first discuss the general small deviations result for time-changed Brownian motion we will be using. We will assume that the random clocks satisfy the following.

Assumption 1.1. *Suppose $\{C(t)\}_{t \geq 0}$ is a continuous non-negative non-decreasing process such that $C(0) = 0$ and there exist $\alpha > 0$, $\beta \in \mathbb{R}$, and a non-decreasing function $K : (0, \infty) \to (0, \infty)$ such that for any $m \in \mathbb{N}$, $\{d_i\}_{i=1}^m \subset (0, \infty)$ a decreasing sequence, and $0 = t_0 < t_1 < \cdots < t_m$,*

$$\lim_{\varepsilon \downarrow 0} \varepsilon^\alpha |\log \varepsilon|^\beta \log \mathbb{P}\left(\sum_{i=1}^m d_i \Delta_i C \leq \varepsilon\right) = -\left(\sum_{i=1}^m (d_i^\alpha K(t_{i-1}, t_i))^{1/(1+\alpha)}\right)^{(1+\alpha)} \tag{1.2}$$

where $\Delta_i C = C_{t_i} - C_{t_{i-1}}$.

By the exponential Tauberian theorem (see Theorem 2.1), equation (1.2) is equivalent to

$$\lim_{\lambda \to \infty} \lambda^{-\alpha/(1+\alpha)} (\log \lambda)^{\beta/(1+\alpha)} \log \mathbb{E}\left[\exp\left(-\lambda \sum_{i=1}^m d_i \Delta_i C\right)\right]$$
$$= -(1+\alpha)^{1+\beta/(1+\alpha)} \alpha^{-\alpha/(1+\alpha)} \sum_{i=1}^m (d_i^\alpha K(t_{i-1}, t_i))^{1/(1+\alpha)}. \tag{1.3}$$

Also via the exponential Tauberian theorem, equation (1.2) clearly holds when C is a subordinator satisfying

$$\lim_{\varepsilon \downarrow 0} \varepsilon^\alpha |\log \varepsilon|^\beta \log \mathbb{P}(C(t) \leq \varepsilon) = -K(t)$$

for any $t > 0$, although the additional requirement of continuity makes this example trivial (since in this case $C(t) = ct$ a.s. for some $c \geq 0$). More generally, (1.2) holds if C has independent increments which satisfy

$$\lim_{\varepsilon \downarrow 0} \varepsilon^\alpha |\log \varepsilon|^\beta \log \mathbb{P}(C(t) - C(s) \leq \varepsilon) = -K(s, t)$$

for all $0 \leq s < t$. However, it is not necessary for C to have independent or stationary increments for Assumption 1.1 to hold. One important source of examples for the present paper is the following theorem from [16] for weighted L^p norms of a Brownian motion.

Theorem 1.2. *Let $p \in [1, \infty)$ and $\rho : [0, \infty) \to [0, \infty]$ be a Lebesgue measurable function satisfying*

(i) ρ *is bounded or non-increasing on $[0, a]$ for some $a > 0$;*

(ii) $t^{(2+p)/p}\rho(t)$ *is bounded or non-decreasing on $[A, \infty)$ for some $A < \infty$;*

(iii) ρ *is bounded on $[a, A]$; and*

(iv) $\rho^{2p/(p+2)}$ *is Riemann integrable on $[0, \infty)$.*

Then

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{2/p} \log \mathbb{P}\left(\int_0^\infty \rho(s)^p |B(s)|^p \, ds \leq \varepsilon\right) = -\kappa_p \left(\int_0^\infty \rho(s)^{2p/(2+p)} \, ds\right)^{(2+p)/p},$$

where $\kappa_p = 2^{2/p} p \left(\frac{\lambda_1(p)}{2+p}\right)^{(2+p)/2}$ for

$$\lambda_1(p) = \inf_{\substack{\phi \in L^2(-\infty, \infty) \\ \|\phi\| = 1}} \left\{\int_{-\infty}^\infty |x|^p \phi^2(x) \, dx + \frac{1}{2} \int_{-\infty}^\infty (\phi'(x))^2 \, dx\right\}.$$

For example, if $\tilde{\rho}$ is any non-negative continuous function on $[0, \infty)$ and

$$C(t) = \int_0^t \tilde{\rho}(s)^p |B(s)|^p \, ds,$$

then

$$\sum_{i=1}^m d_i \Delta_i C = \sum_{i=1}^m d_i \int_{t_{i-1}}^{t_i} \tilde{\rho}(s)^p |B(s)|^p \, ds$$

and applying Theorem 1.2 with $\rho(s) = \sum_{i=1}^m d_i^{1/p} 1_{(t_{i-1}, t_i]}(s) \tilde{\rho}(s)$ gives (1.2) with $\alpha = 2/p$, $\beta = 0$, and

$$K(t_{i-1}, t_i) = \left(\int_{t_{i-1}}^{t_i} \tilde{\rho}(s)^{2p/(p+2)} \, ds\right)^{(2+p)/p}.$$

A particularly relevant example to our later applications is the simplest case where $p = 2$ and $\tilde{\rho} \equiv 1$, for which

$$C(t) = \int_0^t B(s)^2 \, ds, \tag{1.4}$$

$\kappa_2 = 1/8$, and $K(t_{i-1}, t_i) = (\Delta_i t)^2$ where $\Delta_i t := t_i - t_{i-1}$.

See Section 6 of [17] for more results related to Theorem 1.2. Additionally, Chapter 7.3 of [20] contains these results under weaker assumptions. Known small deviations for weighted L^p norms of other stochastic processes provide other interesting examples. For example, in [21] a result analogous to Theorem 1.2 is proved for fractional Brownian motions. See this and related references for further examples.

Now working under Assumption 1.1, one may prove the following.

Theorem 1.3. *Suppose that $\{Z(t)\}_{t\geq 0}$ is a stochastic process given by $Z(t) = B(C(t))$, where C is as in Assumption 1.1 and B is a standard real-valued Brownian motion independent of C. Let $M(t) := \sup_{0\leq s\leq t} |Z(s)|$. Then, for any $m \in \mathbb{N}$, $0 = t_0 < t_1 < \cdots < t_m < \infty$, and $0 \leq a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_m < b_m$,*

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{2\alpha/(1+\alpha)} |\log \varepsilon|^{\beta/(1+\alpha)} \log \mathbb{P}\left(\bigcap_{i=1}^{m} \{a_i \varepsilon \leq M(t_i) \leq b_i \varepsilon\}\right)$$

$$= -2^{-\beta/(1+\alpha)}(1+\alpha)^{1+\beta/(1+\alpha)} \left(\frac{\pi^2}{8\alpha}\right)^{\alpha/(1+\alpha)} \sum_{i=1}^{m} \left(\frac{K(t_{i-1}, t_i)}{b_i^{2\alpha}}\right)^{1/(1+\alpha)}.$$

Such estimates have been previously studied for processes $\{Z_t\}_{t\geq 0}$ that are symmetric α-stable processes [8], fractional Brownian motions [12], certain stochastic integrals [13], m-fold integrated Brownian motions [30], and integrated α-stable processes [31]. In particular, the stochastic integrals studied in [13] are essentially finite-dimensional versions of the class of stochastic integral processes we study, and the proof that we give for Theorem 1.3 follows the outline of the proof of small ball estimates in that reference.

We apply Theorem 1.3 to stochastic integrals of the form (1.1) as follows.

Theorem 1.4. *Let $\{Z(t)\}_{t\geq 0}$ be as in equation (1.1). Then $\{Z(t)\} \stackrel{d}{=} \{B(C(t))\}$ for B a standard real-valued Brownian motion and*

$$C(t) = \sum_{k=1}^{\infty} \|\omega(e_k, \cdot)\|_{\mathcal{H}}^2 \int_0^t (W_s^k)^2 \, ds$$

where $\{e_k\}_{k=1}^{\infty}$ is any orthonormal basis of $\mathcal{H}$ contained in $\mathcal{H}_* := \{h \in \mathcal{H} : \langle h, \cdot \rangle$ extends to a continuous linear functional on $\mathcal{W}\}$ and $\{W^k\}_{k=1}^{\infty}$ are independent standard Brownian motions which are also independent of B. If we further suppose that $\|\omega(e_k, \cdot)\|_{\mathcal{H}} = O(k^{-r})$ for $r > 1$, then, for any $m \in \mathbb{N}$, $0 = t_0 < t_1 < \cdots < t_m$, and $\{d_i\}_{i=1}^{m} \subset (0, \infty)$ a decreasing sequence,

$$\lim_{\varepsilon \downarrow 0} \varepsilon \log \mathbb{P}\left(\sum_{i=1}^{m} d_i^2 \Delta_i C \leq \varepsilon\right) = -\frac{1}{2} \|\omega\|_1^2 \left(\sum_{i=1}^{m} d_i \Delta_i t\right)^2,$$

where $\Delta_i t := t_i - t_{i-1}$ and

$$\|\omega\|_1 := \sum_{k=1}^{\infty} \|\omega(e_k, \cdot)\|_{\mathcal{H}} < \infty.$$

Thus, for any $0 \leq a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_m < b_m$,

$$\lim_{\varepsilon \downarrow 0} \varepsilon \log P\left(\bigcap_{i=1}^{m} \{a_i \varepsilon \leq \sup_{0 \leq s \leq t_i} |Z_s| \leq b_i \varepsilon\}\right) = -\frac{\pi}{4} \|\omega\|_1 \sum_{i=1}^{m} \frac{\Delta_i t}{b_i}.$$

Remark 1.5. *Note that in the above theorem, and in the sequel, we make the standard identification between $\mathcal{H}^*$ and $\mathcal{H}$ via Riesz representation. That is, for a linear functional φ on $\mathcal{H}$, we write*

$$\|\varphi\|_{\mathcal{H}}^2 = \sum_{j=1}^{\infty} |\langle \varphi, e_j \rangle_{\mathcal{H}}|^2 = \sum_{j=1}^{\infty} |\varphi(e_j)|^2.$$

In particular, for fixed $h \in \mathcal{H}$, we write $\|\omega(h, \cdot)\|_{\mathcal{H}}^2 := \|\omega(h, \cdot)|_{\mathcal{H}}\|_{\mathcal{H}}^2 = \sum_{j=1}^{\infty} |\omega(h, e_j)|^2$.

Applications of such estimates include using the small deviations in Theorem 1.4 to prove a Chung-type law of iterated logarithm as well as a functional law of iterated logarithm for the process Z. We record these results in Theorem 3.17 and 3.18.

1.2 Discussion

First-order small deviation estimates of the standard form

$$\log \mathbb{P} \left(\sup_{0 \leq s \leq t} |Z(s)| \leq \varepsilon \right)$$

were studied in [25] for processes $Z(t) = \int_0^t \omega(W_s, dW_s)$ with W an n-dimensional Brownian motion and $\omega : \mathbb{R}^n \to \mathbb{R}$ given by $\omega(x, y) = Ax \cdot y$ for A a skew-symmetric $n \times n$ matrix. These estimates were then applied to prove an analogue of the classical limit result of Chung. (This was done earlier in [29] in the case $n = 2$ and $A = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}$, that is, for Z the stochastic Lévy area.) In [13], the authors improved these results by proving stronger asymptotic results like those in Theorem 1.3 for the same Z as in [25] and applying these results to prove a functional law of iterated logarithm.

In the present paper, the proof of the small ball estimates established in Theorem 1.3 is a direct generalization of the techniques of [13]. However, Theorem 1.3 is sufficiently general to be of independent interest for other potential applications. Thus for that purpose, as well as for clarity and self-containment, we include the proof here. It is also clear from the proofs that, given only the asymptotic order for C, one could infer the asymptotic order for Z instead.

We also mention the reference [2], in which the authors study general iterated processes of the form $X \circ Y$ where X is a two-sided self-similar process and Y is a continuous process independent of X. Since X is two-sided, it is not required that Y satisfy any monotonicity or positivity criteria. In this general setting, under the assumption that the first-order ($m = 1$) asymptotics are known for X and Y, the authors are able to prove a first-order small ball estimate (Theorem 4 of [2]). Theorem 1.3 is stated in the restricted setting that X is a Brownian motion; however, the proof carries through for first-order estimates for processes X satisfying more general assumptions (as in [2]). See Proposition 2.9 for more details.

The organization of the paper is as follows. In Section 2 we give the proof of Theorem 1.3. In Section 3, we apply Theorem 1.3 to prove small ball estimates for the relevant collection of stochastic integrals. In Section 3, we define precisely the processes of interest, and in Theorem 3.10 we prove that these processes have a representation as Brownian motions on an independent random clock. In Subsection 3.2, we determine the small ball asymptotics of the clock. Thus we are able to apply Theorem 1.3, and we additionally record a Chung-type law of iterated logarithm and functional law of iterated logarithm that follow from these estimates.

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2 Small deviation estimates

In this section, we prove separately the upper and lower bounds of Theorem 1.3. The outline of the proof here follows Section 4 of [13]. First, we record a standard relation between asymptotics of the Laplace transform and small ball estimate of a positive random variable in the form of the exponential Tauberian theorem (see for example Theorem 4.12.9 in [3]). We give a special case of that theorem here.

Theorem 2.1. *Suppose that X is a positive random variable. There exist $\alpha > 0$, $\beta \in \mathbb{R}$, and $K \in (0, \infty)$ such that*

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{\alpha} |\log \varepsilon|^{\beta} \log \mathbb{P}(X \leq \varepsilon) = -K$$

if and only if

$$\lim_{\lambda \to \infty} \lambda^{-\alpha/(1+\alpha)} (\log \lambda)^{\beta/(1+\alpha)} \log \mathbb{E}[e^{-\lambda X}] = -(1 + \alpha)^{1+\beta/(1+\alpha)} (\alpha^{-\alpha} K)^{1/(1+\alpha)}.$$

We will use this theorem repeatedly in the sequel along with the standard fact that, for any $\varepsilon > 0$,

$$\frac{2}{\pi} e^{-\frac{\pi^2}{8\varepsilon^2}} \leq \mathbb{P}\left(\sup_{0 \leq s \leq 1} |B(s)| \leq \varepsilon \right) \leq \frac{4}{\pi} e^{-\frac{\pi^2}{8\varepsilon^2}}, \tag{2.1}$$

see for example [9]. Now the upper bound of Theorem 1.3 follows almost immediately from this and the upper bound for the random clock C via conditioning.

Notation 2.2. *For C as in Theorem 1.3, we let $P_C(\cdot) = \mathbb{P}(\cdot \mid C)$.*

Proposition 2.3. *Under the hypotheses of Theorem 1.3, we have that*

$$\limsup_{\varepsilon \downarrow 0} \varepsilon^{2\alpha/(1+\alpha)} |\log \varepsilon|^{\beta/(1+\alpha)} \log \mathbb{P}\left(\bigcap_{i=1}^{m} \{a_i \varepsilon \leq M(t_i) \leq b_i \varepsilon\} \right)$$

$$\leq -2^{-\beta/(1+\alpha)} (1 + \alpha)^{1+\beta/(1+\alpha)} \left(\frac{\pi^2}{8\alpha} \right)^{\alpha/(1+\alpha)} \sum_{i=1}^{m} \left(\frac{K(t_{i-1}, t_i)}{b_i^{2\alpha}} \right)^{1/(1+\alpha)}.$$

Proof. We will show that

$$\mathbb{P}\left(\bigcap_{i=1}^{m} \{a_i \varepsilon \leq M(t_i) \leq b_i \varepsilon\} \right) \leq \left(\frac{4}{\pi} \right)^{m} \mathbb{E}\left[\exp\left(-\frac{\pi^2}{8\varepsilon^2} \sum_{i=1}^{m} \frac{\Delta_i C}{b_i^2} \right) \right]. \tag{2.2}$$

Then applying equation (1.3) with $d_i = 1/b_i^2$ finishes the proof. So first we define

$$A_i := \left\{ \sup_{t_{i-1} \leq s \leq t_i} |Z(s)| \leq b_i \varepsilon \right\}.$$

Then we have that

$$P_C\left(\bigcap_{i=1}^{m} \{a_i \varepsilon \leq M(t_i) \leq b_i \varepsilon\} \right) \leq P_C\left(\bigcap_{i=1}^{m} A_i \right),$$

and for $\mu_{C,t_{m-1}}(\cdot) = P_C(Z(t_{m-1}) \in \cdot)$

$$P_C\left(\bigcap_{i=1}^{m} A_i \right)$$

$$= \int_{\mathbb{R}} P_C\left(\bigcap_{i=1}^{m-1} A_i, \sup_{t_{m-1} \leq s \leq t_m} |Z(s) - Z(t_{m-1}) + x| \leq b_m \varepsilon \,\middle|\, Z(t_{m-1}) = x \right) d\mu_{C,t_{m-1}}(x)$$

$$= \int_{\mathbb{R}} P_C\left(\bigcap_{i=1}^{m-1} A_i \,\middle|\, Z(t_{m-1}) = x \right)$$

$$\times P_C\left(\sup_{t_{m-1} \leq s \leq t_m} |Z(s) - Z(t_{m-1}) + x| \leq b_m \varepsilon \right) d\mu_{C,t_{m-1}}(x)$$

since $\displaystyle\sup_{t_{m-1}\leq s\leq t_m}|Z(s) - Z(t_{m-1}) + x|$ is P_C independent of $Z(t_{m-1})$ and $\displaystyle\bigcap_{i=1}^{m-1} A_i$ by the P_C independent increments of Z.

Since $\{Z(t)\}_{t\geq 0}$ is a P_C Gaussian centered process, we have by Anderson's inequality (see, for example Theorem 1.8.5 of [4]) that

$$P_C\left(\sup_{t_{m-1}\leq s\leq t_m}|Z(s) - Z(t_{m-1}) + x|\leq b_m\varepsilon\right)$$

$$\leq P_C\left(\sup_{t_{m-1}\leq s\leq t_m}|Z(s) - Z(t_{m-1})|\leq b_m\varepsilon\right)$$

$$= P_C\left(\sup_{0\leq s\leq 1}|B(s)|\leq \frac{b_m\varepsilon}{\sqrt{\Delta_m C}}\right),$$

by the monotonicity and continuity of C and the stationary and scaling properties of Brownian motion. Thus

$$P_C\left(\bigcap_{i=1}^{m} A_i\right)\leq P_C\left(\bigcap_{i=1}^{m-1} A_i\right)P_C\left(\sup_{0\leq s\leq 1}|B(s)|\leq \frac{b_m\varepsilon}{\sqrt{\Delta_m C}}\right).$$

By iterating the above computation m times we see that

$$P_C\left(\bigcap_{i=1}^{m} A_i\right)\leq \prod_{i=1}^{m} P_C\left(\sup_{0\leq s\leq 1}|B(s)|\leq \frac{b_i\varepsilon}{\sqrt{\Delta_i C}}\right)$$

$$\leq \left(\frac{4}{\pi}\right)^m \exp\left(-\frac{\pi^2}{8\varepsilon^2}\sum_{i=1}^{m}\frac{\Delta_i C}{b_i^2}\right)$$

where the second inequality follows from the upper bound in (2.1). Taking the expectation of both sides yields (2.2). $\square$

We now move towards obtaining the lower bounds with the following lemma.

Lemma 2.4. *Fix $\gamma > 0$, and let $0 < \delta < \gamma$ be such that $a_i(1+\delta) < b_i(1-\delta)$. Also let $f_i = f_i(\varepsilon, \delta)$ and $g_i = g_i(\varepsilon, \delta)$ be given by*

$$f_i := P_C\left(\sup_{0\leq s\leq 1}|B(s)|\leq \frac{b_i(1-\delta)\varepsilon}{\sqrt{\Delta_i C}}\right) \text{ and } g_i := P_C\left(\sup_{0\leq s\leq 1}|B(s)|\leq \frac{a_i(1+\delta)\varepsilon}{\sqrt{\Delta_i C}}\right)$$

and set

$$\Phi := \{\phi = \{\phi_i\}_{i=1}^{m} : \phi_i \in \{f_i, g_i\} \text{ and at least one } \phi_i = g_i\}.$$

Then

$$\mathbb{P}\left(\bigcap_{i=1}^{m}\{a_i\varepsilon\leq M(t_i)\leq b_i\varepsilon\}, |Z(t_m)|\leq b_m\gamma\varepsilon\right)$$

$$\geq \mathbb{E}\left[\prod_{i=1}^{m}f_i P_C\left(|B(1)|\leq \frac{\Delta_i b\delta\varepsilon}{\sqrt{\Delta_i C}}\right)\right] - \sum_{\phi\in\Phi}\mathbb{E}\left[\prod_{i=1}^{m}\phi_i\right],$$

where $\Delta_i b = b_i - b_{i-1}$ with $b_0 = 0$.

Proof. Define

$$\Upsilon_i := \left\{a_i\varepsilon\leq \sup_{t_{i-1}\leq s\leq t_i}|Z(s)|\leq b_i\varepsilon, |Z(t_i)|\leq b_i\delta\varepsilon\right\}.$$

Since $\sup\limits_{t_{i-1} \leq s \leq t_i} |Z(s)| \leq M(t_i)$ and $b_i \gamma \varepsilon \geq b_i \delta \varepsilon$ for all i, we have that

$$\bigcap_{i=1}^{m} \{a_i \varepsilon \leq M(t_i) \leq b_i \varepsilon\} \cap \{|Z(t_m)| \leq b_m \gamma \varepsilon\} \supset \bigcap_{i=1}^{m} \Upsilon_i.$$

Define

$$A_i := \left\{ a_i(1+\delta)\varepsilon \leq \sup_{t_{i-1} \leq s \leq t_i} |Z(s) - Z(t_{i-1})| \leq b_i(1-\delta)\varepsilon, \right.$$

$$\left. |Z(t_i) - Z(t_{i-1})| \leq (\Delta_i b)\delta\varepsilon \right\}.$$

By P_C independent increments,

$$P_C\left(\bigcap_{i=1}^{m} \Upsilon_i\right) \geq P_C\left(\bigcap_{i=1}^{m-1} \Upsilon_i \cap A_m\right) = P_C\left(\bigcap_{i=1}^{m-1} \Upsilon_i\right) P_C(A_m),$$

and repeating this computation m times gives that

$$P_C\left(\bigcap_{i=1}^{m} \Upsilon_i\right) \geq \prod_{i=1}^{m} P_C(A_i).$$

Again we use the stationary and scaling properties of Brownian motion, as well as Šidak's Lemma (see for example, Corollary 4.6.2 of [4]), to show that

$$P_C(A_i) = P_C\left(\frac{a_i(1+\delta)\varepsilon}{\sqrt{\Delta_i C}} \leq \sup_{0 \leq s \leq 1} |B(s)| \leq \frac{b_i(1-\delta)\varepsilon}{\sqrt{\Delta_i C}}, |B(1)| \leq \frac{\Delta_i b \delta \varepsilon}{\sqrt{\Delta_i C}}\right)$$

$$\geq P_C\left(\frac{a_i(1+\delta)\varepsilon}{\sqrt{\Delta_i C}} \leq \sup_{0 \leq s \leq 1} |B(s)| \leq \frac{b_i(1-\delta)\varepsilon}{\sqrt{\Delta_i C}}\right) P_C\left(|B(1)| \leq \frac{\Delta_i b \delta \varepsilon}{\sqrt{\Delta_i C}}\right)$$

$$= (f_i - g_i) P_C\left(|B(1)| \leq \frac{\Delta_i b \delta \varepsilon}{\sqrt{\Delta_i C}}\right).$$

Thus, taking expectations we have that

$$\mathbb{P}\left(\bigcap_{i=1}^{m} \Upsilon_i\right) \geq \mathbb{E}\left[\prod_{i=1}^{m} f_i P_C\left(|B(1)| \leq \frac{\Delta_i b \delta \varepsilon}{\sqrt{\Delta_i C}}\right) - \sum_{\Phi} \prod_{i=1}^{m} \phi_i P_C\left(|B(1)| \leq \frac{\Delta_i b \delta \varepsilon}{\sqrt{\Delta_i C}}\right)\right]$$

$$\geq \mathbb{E}\left[\prod_{i=1}^{m} f_i P_C\left(|B(1)| \leq \frac{\Delta_i b \delta \varepsilon}{\sqrt{\Delta_i C}}\right)\right] - \sum_{\Phi} \mathbb{E}\left[\prod_{i=1}^{m} \phi_i\right]$$

as desired. $\qquad \square$

Now the next three lemmas give the necessary estimates on the terms appearing in Lemma 2.4.

Lemma 2.5. *Let f_i, g_i, and Φ be as in Lemma 2.4. Then for any $\phi \in \Phi$*

$$\limsup_{\varepsilon \downarrow 0} \varepsilon^{2\alpha/(1+\alpha)} |\log \varepsilon|^{\beta/(1+\alpha)} \log \mathbb{E}\left[\prod_{i=1}^{m} \phi_i(\varepsilon)\right]$$

$$\leq -2^{-\beta/(1+\alpha)}(1+\alpha)^{1+\beta/(1+\alpha)} \left(\frac{\pi^2}{8\alpha}\right)^{\alpha/(1+\alpha)} \sum_{i=1}^{m} \left(\frac{K(t_{i-1}, t_i)}{d_i^{\phi}(\delta)^{2\alpha}}\right)^{1/(1+\alpha)}$$

where $d_i^{\phi}(\delta) := \begin{cases} b_i(1-\delta) & \text{if } \phi_i = f_i \\ a_i(1+\delta) & \text{if } \phi_i = g_i \end{cases}$.

Proof. By the upper bound in (2.1),

$$\prod_{i=1}^{m} \phi_i = \prod_{i=1}^{m} P_C \left(\sup_{0 \le s \le 1} |B(s)| \le \frac{d_i^\phi(\delta)\varepsilon}{\sqrt{\Delta_i C}} \right)$$

$$\le \left(\frac{4}{\pi} \right)^m \exp \left(-\frac{\pi^2}{8\varepsilon^2} \sum_{i=1}^{m} \frac{\Delta_i C}{d_i^\phi(\delta)^2} \right).$$

Then applying (1.3) completes the proof. $\qquad\square$

Lemma 2.6. *Suppose that $\{\eta_i\}_{i=1}^{m}$ are nonnegative random variables such that, for any $\beta_1, \ldots, \beta_m > 0$, there exists $\alpha > 0$, $\beta \in \mathbb{R}$, and $K > 0$ such that*

$$\lim_{\varepsilon \downarrow 0} \varepsilon^\alpha |\log \varepsilon|^\beta \log \mathbb{P} \left(\sum_{i=1}^{m} \beta_i \eta_i \le \varepsilon \right) \ge -K.$$

Let $P_\eta = \mathbb{P}(\cdot \mid \eta_1, \ldots, \eta_m)$. Then, if G is a standard normal random variable and $\gamma_1, \ldots, \gamma_m > 0$, we have that

$$\liminf_{\lambda \to \infty} \lambda^{-\alpha/(1+\alpha)} (\log \lambda)^{\beta/(1+\alpha)} \mathbb{E} \left[\exp \left(-\lambda \sum_{i=1}^{m} \beta_i \eta_i \right) \prod_{i=1}^{m} P_\eta \left(|G| \le \frac{\gamma_i}{\sqrt{\lambda \eta_i}} \right) \right]$$

$$\ge -(1+\alpha)^{1+\beta/(1+\alpha)} \alpha^{-\alpha/(1+\alpha)} K^{1/(1+\alpha)}.$$

Proof. For any $L > 0$, when $\sum_{i=1}^{m} \beta_i \eta_i \le L$, the positivity of all parameters implies that $\eta_i \le L/\beta_i$ for each i and thus

$$\min_{1 \le i \le m} \frac{\gamma_i}{\sqrt{\eta_i}} \ge \min_{1 \le i \le m} \gamma_i \sqrt{\frac{\beta_i}{L}} > 0.$$

Also, note that for all sufficiently small $x > 0$, one may choose $K' > 0$ such that $\mathbb{P}(|G| \le x) \ge K'x$. Thus, for sufficiently large λ, there exists $K'' > 0$ such that

$$\mathbb{E} \left[\exp \left(-\lambda \sum_{i=1}^{m} \beta_i \eta_i \right) \prod_{i=1}^{m} P_\eta \left(|G| \le \frac{\gamma_i}{\sqrt{\lambda \eta_i}} \right) \right]$$

$$\ge \mathbb{E} \left[\exp \left(-\lambda \sum_{i=1}^{m} \beta_i \eta_i \right) \left(\min_{1 \le i \le m} P_\eta \left(|G| \le \frac{\gamma_i}{\sqrt{\lambda \eta_i}} \right) \right)^m ; \sum_{i=1}^{m} \beta_i \eta_i \le L \right]$$

$$\ge \left(\frac{K''}{\sqrt{\lambda}} \right)^m \mathbb{E} \left[\exp \left(-\lambda \sum_{i=1}^{m} \beta_i \eta_i \right) ; \sum_{i=1}^{m} \beta_i \eta_i \le L \right]. \qquad (2.3)$$

Thus, for any $\xi > 0$, we may take $\theta(\lambda) = \xi \lambda^{-1/(1+\alpha)} (\log \lambda)^{-\beta/(1+\alpha)}$, and we have

$$\liminf_{\lambda \to \infty} \lambda^{-\alpha/(1+\alpha)} (\log \lambda)^{\beta/(1+\alpha)} \log \mathbb{E} \left[\exp \left(-\lambda \sum_{i=1}^{m} \beta_i \eta_i \right) ; \sum_{i=1}^{m} \beta_i \eta_i \le L \right]$$

$$\ge \liminf_{\lambda \to \infty} \lambda^{-\alpha/(1+\alpha)} (\log \lambda)^{\beta/(1+\alpha)} \log \mathbb{E} \left[\exp \left(-\lambda \sum_{i=1}^{m} \beta_i \eta_i \right) ; \sum_{i=1}^{m} \beta_i \eta_i \le \theta(\lambda) \right]$$

$$\ge \liminf_{\lambda \to \infty} \lambda^{-\alpha/(1+\alpha)} (\log \lambda)^{\beta/(1+\alpha)} \left(-\theta(\lambda)\lambda + \log \mathbb{P} \left(\sum_{i=1}^{m} \beta_i \eta_i \le \theta(\lambda) \right) \right)$$

$$= -\xi + \xi^{-\alpha} (1+\alpha)^\beta \liminf_{\lambda \to \infty} \theta(\lambda)^\alpha |\log \theta(\lambda)|^\beta \log \mathbb{P} \left(\sum_{i=1}^{m} \beta_i \eta_i \le \theta(\lambda) \right)$$

$$\ge -\xi - \xi^{-\alpha} (1+\alpha)^\beta K$$

In particular, combining this inequality with (2.3) and taking $\xi = (K\alpha(1+\alpha)^\beta)^{1/(1+\alpha)}$ completes the proof. $\qquad\square$

Lemma 2.7. *Let f_i be as in Lemma 2.4. Then*

$$\liminf_{\varepsilon\downarrow 0} \varepsilon^{2\alpha/(1+\alpha)} |\log\varepsilon|^{\beta/(1+\alpha)} \log \mathbb{E}\left[\prod_{i=1}^m f_i(\varepsilon) P_C\left(|B(1)| \leq \frac{\Delta_i b\delta\varepsilon}{\sqrt{\Delta_i C}}\right)\right]$$
$$\geq -2^{-\beta/(1+\alpha)}(1+\alpha)^{1+\beta/(1+\alpha)} \left(\frac{\pi^2}{8\alpha}\right)^{\alpha/(1+\alpha)} \sum_{i=1}^m \left(\frac{K(t_{i-1}, t_i)}{b_i^{2\alpha}(1-\delta)^{2\alpha}}\right)^{1/(1+\alpha)}.$$

Proof. The lower bound in (2.1) implies that

$$\prod_{i=1}^m f_i = \prod_{i=1}^m P_C\left(\sup_{0\leq s\leq 1} |B(s)| \leq \frac{b_i(1-\delta)\varepsilon}{\sqrt{\Delta_i C}}\right)$$
$$\geq \left(\frac{2}{\pi}\right)^m \exp\left(-\frac{\pi^2}{8\varepsilon^2} \sum_{i=1}^m \frac{\Delta_i C}{b_i^2(1-\delta)^2}\right).$$

Using this estimate to bound the desired expectation and applying Lemma 2.6 and equation (1.3) completes the proof. $\qquad\square$

Proposition 2.8. *Under the hypotheses of Theorem 1.3, we have that*

$$\liminf_{\varepsilon\downarrow 0} \varepsilon^{2\alpha/(1+\alpha)} |\log\varepsilon|^{\beta/(1+\alpha)} \log \mathbb{P}\left(\bigcap_{i=1}^m \{a_i\varepsilon \leq M(t_i) \leq b_i\varepsilon\}\right)$$
$$\geq -2^{-\beta/(1+\alpha)}(1+\alpha)^{1+\beta/(1+\alpha)} \left(\frac{\pi^2}{8\alpha}\right)^{\alpha/(1+\alpha)} \sum_{i=1}^m \left(\frac{K(t_{i-1}, t_i)}{b_i^{2\alpha}}\right)^{1/(1+\alpha)}.$$

Proof. Clearly, for any $\gamma > 0$,

$$\mathbb{P}\left(\bigcap_{i=1}^m \{a_i\varepsilon \leq M(t_i) \leq b_i\varepsilon\}\right) \geq \mathbb{P}\left(\bigcap_{i=1}^m \{a_i\varepsilon \leq M(t_i) \leq b_i\varepsilon\}, |Z(t_m)| \leq b_m\gamma\varepsilon\right).$$

Thus, by Lemma 2.4, for any $0 < \delta < \gamma$ with δ sufficiently small that $a_i(1+\delta) < b_i(1-\delta)$ for each i, we have that

$$\mathbb{P}\left(\bigcap_{i=1}^m \{a_i\varepsilon \leq M(t_i) \leq b_i\varepsilon\}\right) \geq \mathbb{E}\left[\prod_{i=1}^m f_i P_C\left(|B(1)| \leq \frac{\Delta_i b\delta\varepsilon}{\sqrt{\Delta_i C}}\right)\right] - \sum_{\phi\in\Phi} \mathbb{E}\left[\prod_{i=1}^m \phi_i\right].$$

Now, given any $\phi \in \Phi$, the associated sequence $\{d_i^\phi(\delta)\}_{i=1}^m$ (as defined in Lemma 2.5) must satisfy $d_i^\phi(\delta) = a_i(1+\delta)$ for at least one i. Thus, for any $\phi \in \Phi$ we have that

$$\sum_{i=1}^m \frac{K(t_{i-1}, t_i)}{b_i(1-\delta)} < \sum_{i=1}^m \frac{K(t_{i-1}, t_i)}{d_i^\phi(\delta)}.$$

Given this strict inequality, Lemmas 2.5 and 2.7 imply that, for each $\phi \in \Phi$

$$\frac{\mathbb{E}\left[\prod_{i=1}^m \phi_i(\varepsilon)\right]}{\mathbb{E}\left[\prod_{i=1}^m f_i(\varepsilon) P_C\left(|B(1)| \leq \frac{\Delta_i b\delta\varepsilon}{\sqrt{\Delta_i C}}\right)\right]} \to 0$$

as $\varepsilon \downarrow 0$. This fact, combined with the identity $\log(A - B) = \log A + \log(1 - B/A)$ and again applying Lemma 2.7 gives the desired result with b_i replaced by $b_i(1-\delta)$. Since $\delta > 0$ was arbitrary, allowing $\delta \downarrow 0$ completes the proof. $\qquad\square$

As alluded to in the discussion from Section 1, a brief review of the proof shows that conditioning easily determines the first-order ($m = 1$) asymptotics of $Z = X \circ C$ for other self-similar processes X satisfying their own small ball estimates. The following statement could also be inferred from the proofs of [2].

Proposition 2.9. *Suppose that $\{\hat{C}(t)\}_{t \geq 0}$ is continuous non-negative non-decreasing and $\{X(t)\}_{t \geq 0}$ is an H-self-similar process (that is, $\{X(ct)\}_{t \geq 0} \stackrel{d}{=} \{c^H X(t)\}_{t \geq 0}$ for any $c > 0$) which is independent of $\hat{C}$. If there exist $\alpha, \theta, \kappa > 0$ and $K : (0, \infty) \to (0, \infty)$ such that*

$$\lim_{\varepsilon \downarrow 0} \varepsilon^\alpha \log \mathbb{P}\left(\hat{C}(t) \leq \varepsilon\right) = -K(t)$$

for all $t > 0$ and

$$\lim_{\varepsilon \downarrow 0} \varepsilon^\theta \log \mathbb{P}\left(\sup_{s \in [0,1]} |X(s)| \leq \varepsilon\right) = -\kappa,$$

then

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{\alpha\theta/(\rho+\alpha)} \log \mathbb{P}\left(\sup_{s \in [0,t]} |X(\hat{C}(s))| \leq \varepsilon\right) = -(\rho + \alpha)(\kappa^\alpha \rho^{-\rho} \alpha^{-\alpha} K(t)^\rho)^{1/(\rho+\alpha)}$$

where $\rho = \theta H$.

Proof. Under the assumptions on X, for any $\delta > 0$, there exists $\varepsilon_0 = \varepsilon_0(\delta)$ such that for any $\varepsilon \in (0, \varepsilon_0)$

$$\exp\left(-(1+\delta)\kappa\varepsilon^{-\theta}\right) \leq \mathbb{P}\left(\sup_{s \in [0,1]} |X(s)| \leq \varepsilon\right) \leq \exp\left(-(1-\delta)\kappa\varepsilon^{-\theta}\right).$$

Thus, there exist $c_1, c_2 \in (0, \infty)$ depending only on ε_0 so that, for all $\varepsilon > 0$,

$$c_1 \exp\left(-(1+\delta)\kappa\varepsilon^{-\theta}\right) \leq \mathbb{P}\left(\sup_{s \in [0,1]} |X(s)| \leq \varepsilon\right) \leq c_2 \exp\left(-(1-\delta)\kappa\varepsilon^{-\theta}\right).$$

Then continuity of $\hat{C}$ and self-similarity of X implies that

$$P_{\hat{C}}\left(\sup_{s \in [0,t]} |X(\hat{C}(s))| \leq \varepsilon\right) = P_{\hat{C}}\left(\sup_{s \in [0,\hat{C}(t)]} |X(s)| \leq \varepsilon\right)$$

$$= P_{\hat{C}}\left(\sup_{s \in [0,1]} \hat{C}(t)^H |X(s)| \leq \varepsilon\right) \leq c_2 \exp\left(-(1-\delta)\kappa\hat{C}(t)^\rho \varepsilon^{-\theta}\right).$$

Taking expectations and applying the asymptotics of $\hat{C}$ gives

$$\limsup_{\varepsilon \downarrow 0} \varepsilon^{\alpha\theta/(\rho+\alpha)} \log \mathbb{P}\left(\sup_{s \in [0,t]} |X(\hat{C}(s))| \leq \varepsilon\right)$$

$$= -(\rho + \alpha)(((1-\delta)\kappa)^\alpha \rho^{-\rho} \alpha^{-\alpha} K(t)^\rho)^{1/(\rho+\alpha)}.$$

Letting $\delta \downarrow 0$ proves the upper bound. The lower bound follows in a similar manner. □

Remark 2.10. *Note that this result can be more general than that for the two-sided diffusions in [2] where they require that $\theta H = 1$. This equality is often satisfied with the supremum norm, but there are basic processes in this setting for which this does not hold. For example, the process C defined in (1.4) is 2-self-similar, but by Theorem 1.2 satisfies a small ball estimate with $\alpha = 1$. (And more generally, for $\tilde{\rho} \equiv 1$ and general $p \in [1, \infty)$, $\alpha = 2/p$ and $H = (p+2)/2$.)*

Remark 2.11. *Note also that we could have again allowed a slowly varying factor in the asymptotics of $\hat{C}$, but we have omitted it for ease.*

Remark 2.12. *It is in the iterative arguments for Theorem 1.3 that one uses, for example, the Gaussian properties of Brownian motion. It is clear that some of these estimates may be extended to other more general processes. For example, there is a known analogue of the Anderson inequality that holds for symmetric α-stable processes (see for example Lemma 2.1 of [8]) that one could use to extend the proof of Proposition 2.3.*

3   Applications to second order chaos

Here we apply the results of the previous section to prove small deviations estimates for stochastic integrals of the form

$$Z_t = \int_0^t \omega(W_s, dW_s),$$

where W is an infinite-dimensional Brownian motion and ω is an anti-symmetric continuous bilinear form. Small deviations have been studied for analogous integrals of finite-dimensional Brownian motions in [25] and [13].

First we define the integral processes we study. We will then prove that these processes are equal in distribution to a Brownian motion under an independent time-change, and we establish a small ball estimate for the relevant random clock. Then by applying the results of Section 2, we are able to prove small deviations results for Z. We fix the following notation for the sequel.

Notation 3.1. *Let $(\mathcal{W}, \mathcal{H}, \mu)$ be a real abstract Wiener space (see for example [14] and [4]). We will let*

$$\mathcal{H}_* := \{h \in \mathcal{H} : \langle h, \cdot \rangle \text{ extends to a continuous linear functional on } \mathcal{W}\}.$$

Let $\{W_t\}_{t \geq 0}$ be a Brownian motion on $\mathcal{W}$ with variance determined by

$$\mathbb{E}[\langle W_s, h \rangle \langle W_s, k \rangle] = \langle h, k \rangle_H \min(s, t)$$

for all $s, t \geq 0$ and $h, k \in \mathcal{H}_*$. Let $\omega : \mathcal{W} \times \mathcal{W} \to \mathbb{R}$ be a anti-symmetric continuous bilinear map.

Remark 3.2. *It is standard that continuity for a bilinear map ω on $\mathcal{W} \times \mathcal{W}$ implies that the restriction of ω to $\mathcal{H} \times \mathcal{H}$ is Hilbert-Schmidt, that is,*

$$\|\omega\|_{HS}^2 := \|\omega|_{\mathcal{H} \times \mathcal{H}}\|_{\mathcal{H}^{\otimes 2}}^2 := \sum_{i,j=1}^\infty |\omega(h_i, h_j)|^2 < \infty$$

where $\{h_i\}_{i=1}^\infty$ is any orthonormal basis of $\mathcal{H}$; see for example Proposition 3.14 of [10].

Associated to any abstract Wiener space is a class of canonical projections. Suppose that $P : \mathcal{H} \to \mathcal{H}$ is a finite-rank orthogonal projection such that $P\mathcal{H} \subset \mathcal{H}_*$. Let $\{e_j\}_{j=1}^n$ be an orthonormal basis for $P\mathcal{H}$. Then we may extend P to a (unique) continuous operator from $\mathcal{W} \to \mathcal{H}$ (still denoted by P) by letting

$$Pw := \sum_{j=1}^n \langle w, e_j \rangle_{\mathcal{H}} e_j \tag{3.1}$$

for all $w \in \mathcal{W}$.

Notation 3.3. *Let* $\mathrm{Proj}(\mathcal{W})$ *denote the collection of finite-rank projections on* $\mathcal{H}$ *such that* $P\mathcal{H} \subset \mathcal{H}_*$ *and* $P|_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$ *is an orthogonal projection (that is, P has the form given in equation (3.1)).*

For P as in (3.1) and $\{W_t\}_{t \geq 0}$ a Brownian motion on $\mathcal{W}$ as in Notation 3.1, $\{PW_t\}_{t \geq 0}$ is a Brownian motion on the finite-dimensional space $\mathrm{Range}(P)$ and thus may be expressed as $PW_t = \sum_{j=1}^n W_t^j e_j$ where the W^j's are independent real-valued Brownian motions. We will let $\{Z_t^P\}_{t \geq 0}$ denote the process defined by

$$Z_t^P := \int_0^t \omega(PW_s, dPW_s).$$

Note that, by the bilinearity and anti-symmetry of ω, we may write

$$Z_t^P = \int_0^t \omega \left(\sum_{j=1}^n W_s^j e_j, \sum_{k=1}^n dW_s^k e_k \right)$$

$$= \sum_{j,k=1}^n \omega(e_j, e_k) \int_0^t W_s^j dW_s^k = \sum_{j<k} \omega(e_j, e_k) \int_0^t W_s^j dW_s^k - W_s^k dW_s^j;$$

thus, $\{Z_t^P\}_{t \geq 0}$ is a continuous L^2-martingale.

It is well-known that $\mathcal{H}_*$ contains an orthonormal basis of $\mathcal{H}$. Thus, we may always take a sequence $P_n \in \mathrm{Proj}(\mathcal{W})$ so that $P_n|_{\mathcal{H}} \uparrow I_{\mathcal{H}}$.

Proposition 3.4. *If* $\{P_n\}_{n=1}^\infty \subset \mathrm{Proj}(\mathcal{W})$ *is an sequence of projections such that* $P_n|_{\mathcal{H}} \uparrow I_{\mathcal{H}}$ *and* $Z_t^n := Z_t^{P_n}$, *then there exists an* L^2-*martingale* $\{Z_t\}_{t \geq 0}$ *such that, for all* $p \in [1, \infty)$ *and* $T > 0$,

$$\lim_{n \to \infty} \mathbb{E} \left[\sup_{0 \leq t \leq T} |Z_t^n - Z_t|^p \right] = 0, \tag{3.2}$$

and $\{Z_t\}_{t \geq 0}$ *is independent of the sequence of projections. Thus, we will denote the limiting process by*

$$Z_t = \int_0^t \omega(W_s, dW_s).$$

The quadratic variation of Z is given by

$$\langle Z \rangle_t = \int_0^t \|\omega(W_s, \cdot)\|_{\mathcal{H}}^2 \, ds := \int_0^t \sum_{j=1}^\infty |\omega(W_s, e_j)|^2 \, ds, \tag{3.3}$$

where $\{e_j\}_{j=1}^\infty$ *is an orthonormal basis of* $\mathcal{H}$, *and, for all* $p \in [1, \infty)$ *and* $T > 0$, $\{Z_t\}_{t \geq 0}$ *satisfies*

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |Z_t|^p \right] < \infty.$$

Proof. First note that, for P as in (3.1),

$$\mathbb{E}|Z_t^P|^2 = \mathbb{E} \langle Z^P \rangle_t = \sum_{j=1}^n \int_0^t \mathbb{E} \left| \omega(PW_s, e_j) \right|^2 \, ds$$

$$= \sum_{j,k=1}^n \int_0^t \int_0^{s_1} |\omega(e_k, e_j)|^2 \, ds_2 \, ds_1 \leq \frac{1}{2} t^2 \|\omega\|_{HS}^2.$$

Let $P, P' \in \mathrm{Proj}(\mathcal{W})$, and let $\{h_j\}_{j=1}^N$ be an orthonormal basis for $P\mathcal{H} + P'\mathcal{H}$. We then have that

$$
\begin{aligned}
\mathbb{E}\left|Z_t^P - Z_t^{P'}\right|^2 &= \mathbb{E}\left|\int_0^t \left(\omega(PW_s, dPW_s) - \omega(P'W_s, dP'W_s)\right)\right|^2 \\
&= \mathbb{E}\left|\int_0^t \left(\omega((P-P')W_s, dPW_s) + \omega(P'W_s, d(P-P')W_s)\right)\right|^2 \\
&\leq 2\mathbb{E}\left[\left|\int_0^t \omega((P-P')W_s, dPW_s)\right|^2 + \left|\int_0^t \omega(P'W_s, d(P-P')W_s)\right|^2\right] \\
&= t^2 \sum_{j,k=1}^N \left(|\omega((P-P')h_k, Ph_j)|^2 + |\omega(P'h_k, (P-P')h_j)|^2\right) \\
&= t^2 \sum_{j,k=1}^\infty \left(|\omega((P-P')e_k, Pe_j)|^2 + |\omega(P'e_k, (P-P')e_j)|^2\right).
\end{aligned}
\tag{3.4}
$$

Taking $P = P_n$ and $P' = P_m$ for $m \leq n$ gives

$$
\mathbb{E}\left[|Z_t^n - Z_t^m|^2\right] \leq t^2 \left(\sum_{j=1}^n \sum_{k=m+1}^n |\omega(e_k, e_j)|^2 + \sum_{j=m+1}^n \sum_{k=1}^m |\omega(e_k, e_j)|^2\right) \to 0
$$

as $m, n \to \infty$ since $\sum_{j,k=1}^\infty |\omega(e_k, e_j)|^2 = \|\omega\|_{HS}^2 < \infty$. Since the space of continuous L^2-martingales on $[0, T]$ is complete in the norm $N \mapsto \mathbb{E}|N_T|^2$, and, by Doob's maximal inequality, there exists $c < \infty$ such that

$$
\mathbb{E}\left[\sup_{0 \leq t \leq T} |N_t|^p\right] \leq c\mathbb{E}|N_T|^p,
$$

it follows that there exists an L^2-martingale $\{Z_t\}_{t \geq 0}$ such that (3.2) holds with $p = 2$. For $p > 2$, since Z is a chaos expansion of order 2, a theorem of Nelson (see Lemma 2 of [24] and pp. 216-217 of [23]) implies that, for each $j \in \mathbb{N}$, there exists $c_j < \infty$ such that

$$
\mathbb{E}|Z_t^n - Z_t|^{2j} \leq c_j \left(\mathbb{E}|Z_t^n - Z_t|^2\right)^j,
$$

and again this combined with Doob's maximal inequality is sufficient to prove (3.2).

One may similarly use (3.4) to show that, for $\{e_j'\}_{j=1}^\infty \subset \mathcal{H}_*$ another orthonormal basis of $\mathcal{H}$ and P_n' a corresponding sequence of orthogonal projections, that

$$
\lim_{n \to \infty} \mathbb{E}\left[\sup_{0 \leq t \leq T} \left|Z_t^{P_n} - Z_t^{P_n'}\right|^p\right] = 0
$$

and thus Z is independent of choice of basis.

Since the quadratic variation of Z^n is given by

$$
\langle Z^n \rangle_t = \int_0^t |\omega(P_n B_s, dP_n B_s)|^2 = \int_0^t \sum_{j=1}^n |\omega(P_n B_s, e_j)|^2 \, ds
$$

and

$$
\begin{aligned}
\mathbb{E}|\langle Z \rangle_t - \langle Z^n \rangle_t| &\leq \sqrt{\mathbb{E}|\langle Z - Z^n \rangle_t| \cdot \mathbb{E}|\langle Z + Z^n \rangle_t|} \\
&= \sqrt{\mathbb{E}|Z_t - Z_t^n|^2 \cdot \mathbb{E}|Z_t + Z_t^n|} \to 0
\end{aligned}
$$

as $n \to \infty$ and (3.3) follows.

$\square$

More general integrals of the form above are considered in [10], in the context of Brownian motions on certain infinite-dimensional Lie groups, and the above proposition is a special case of Proposition 4.1 of that reference. In particular, processes like the Z_t defined in Proposition 3.4 appear as the central component of hypoelliptic Brownian motions on infinite-dimensional Heisenberg groups with one-dimensional center, and more generally as a term in Brownian motions on infinite-dimensional nilpotent Lie groups.

We give the following basic example of the type of process Z we study here.

Example 3.5. *Let $q = \{q_j\}_{j=1}^{\infty} \in \ell^1(\mathbb{R}^+)$ and set*

$$\mathcal{W} := \ell_q^2(\mathbb{C}) := \left\{ v \in \mathbb{C}^{\mathbb{N}} : \sum_{j=1}^{\infty} q_j |v_j|^2 < \infty \right\}$$

and $\mathcal{H} = \ell^2(\mathbb{C})$ where both $\mathcal{W}$ and $\mathcal{H}$ are considered as vector spaces over $\mathbb{R}$. Then $(\mathcal{W}, \mathcal{H})$ determines an abstract Wiener space (for example, Example 3.9.7 of [4]). Define $\omega : \mathcal{W} \times \mathcal{W} \to \mathbb{R}$ by

$$\omega(w, w') = \sum_{j=1}^{\infty} q_j \mathrm{Im}(\bar{w}_j w_j') = \sum_{j=1}^{\infty} q_j (x_j y_j' - y_j x_j')$$

where $w_j = x_j + i y_j$ for each j. Then for a Brownian motion $W = \{X^j + iY^j\}_{j=1}^{\infty}$, where $\{X^j, Y^j\}_{j=1}^{\infty}$ are independent standard real-valued Brownian motions, we have that

$$Z(t) = \int_0^t \omega\left(W_s, dW_s\right) = \sum_{j=1}^{\infty} q_j \int_0^t X_s^j dY_s^j - Y_s^j dX_s^j$$

is an infinite weighted sum of independent Lévy areas. (Note that, since the weights $\{q_j\}$ are ℓ^1, this expression for Z makes sense. Indeed, in order for the Brownian motion W to make sense on $\mathcal{W}$, these weights must be ℓ^1. See [4] for more details.)

Remark 3.6. *Since Z is a martingale with*

$$
\begin{aligned}
\langle Z \rangle_t &= \int_0^t \|\omega(W_s, \cdot)\|_{\mathcal{H}}^2 \, ds = \int_0^t \sum_{j=1}^{\infty} |\omega(W_s, e_j)|^2 \, ds \\
&= \int_0^t \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |W_s^k \omega(e_k, e_j)|^2 \, ds = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |\omega(e_k, e_j)|^2 \int_0^t (W_s^k)^2 \, ds \\
&= \sum_{k=1}^{\infty} \|\omega(e_k, \cdot)\|_{\mathcal{H}}^2 \int_0^t (W_s^k)^2 \, ds,
\end{aligned}
$$

we know there exists a (not necessarily independent) real-valued Brownian motion B such that $Z(t) = B(\langle Z \rangle_t)$ by the Dubins-Schwarz representation (see for example Theorem 34.1 on page 64 of [26]). We will show in the next section that this representation in fact holds with B an independent Brownian motion.

3.1 A representation theorem

In this section, we show that $Z \overset{d}{=} B(\langle Z \rangle)$ for an independent Brownian motion B. This representation is well-known for Z the standard stochastic Lévy area for two-dimensional Brownian motion (see for example Example 6.1 on page 470 of [11]), and was also proved for more general stochastic integrals of finite-dimensional Brownian motions in [13]. We summarize the latter result now; see Section 3 of [13] for a proof.

Lemma 3.7. *Let W be a standard Brownian motion in $\mathbb{R}^n$ and A be a real non-zero skew-symmetric $n \times n$ matrix with non-zero eigenvalues $\{\pm i a_j\}_{j=1}^r$ (where $2r \leq n$ and 0 is an eigenvalue of multiplicity $n - 2r$). For $t > 0$, let*

$$L(t) := \int_0^t \langle AW_s, dW_s \rangle$$

and

$$\tilde{L}(t) := B\left(\sum_{j=1}^r a_j^2 \int_0^t (X_s^j)^2 + (Y_s^j)^2 \, ds \right),$$

where B and $\{X_j, Y_j\}_{j=1}^r$ are independent standard real-valued Brownian motions. Then the law of $\{L(t)\}_{t \geq 0}$ is equal to the law of $\{\tilde{L}(t)\}_{t \geq 0}$.

Remark 3.8. *In particular, this lemma implies that each of the finite-dimensional approximations Z^n to Z has such a representation, in the following way. By Remark 3.2, the continuity assumption for ω implies that its restriction to the Cameron-Martin space is Hilbert-Schmidt, and thus the Riesz representation theorem implies the existence of an anti-symmetric Hilbert-Schmidt operator $Q = Q_\omega : \mathcal{H} \to \mathcal{H}$ such that*

$$\omega(h, k) = \langle Qh, k \rangle_{\mathcal{H}}, \quad \text{for all } h, k \in \mathcal{H}.$$

Thus,

$$Z_t^P = \int_0^t \omega(PB_s, dPB_s) = \int_0^t \langle QPB_s, dPB_s \rangle_{\mathcal{H}} = \int_0^t \langle (PQP)PB_s, dPB_s \rangle_{\mathcal{H}},$$

and we may apply Lemma 3.7 to Z^P, as PB is a Brownian motion on the finite-dimensional space $P\mathcal{H} \subset \mathcal{H}$ and $A = PQP$ is a skew-symmetric linear operator on $P\mathcal{H}$.

We will use this representation for the finite-dimensional approximations to show that an analogous statement is true for Z. First we record the following simple lemma.

Lemma 3.9. *Let $Q : \mathcal{H} \to \mathcal{H}$ be a Hilbert-Schmidt operator, and let P_n be an increasing sequence of orthogonal projections on $\mathcal{H}$ such that $P_n|_{\mathcal{H}} \uparrow I_{\mathcal{H}}$. Then, as $n \to \infty$, $P_n Q P_n \to Q$ in Hilbert-Schmidt norm.*

Proof. Let $\{e_i\}_{i=1}^\infty$ be an orthonormal basis of $\mathcal{H}$ so that $\{e_i\}_{i=1}^{r_n}$ is an orthonormal basis of $P_n \mathcal{H}$. We have

$$
\begin{aligned}
\|P_n Q P_n - Q\|_{HS}^2 &= \sum_{i=1}^\infty \|(P_n Q P_n - Q)e_i\|_{\mathcal{H}}^2 \\
&= \sum_{i=1}^{r_n} \|(P_n - I)Qe_i\|_H^2 + \sum_{i=r_n+1}^\infty \|Qe_i\|_{\mathcal{H}}^2 \\
&\leq \sum_{i=1}^\infty \|(P_n - I)Qe_i\|_{\mathcal{H}}^2 + \sum_{i=r_n+1}^\infty \|Qe_i\|_{\mathcal{H}}^2.
\end{aligned}
$$

The second term goes to zero since it is the tail of the convergent sum $\sum_{i=1}^\infty \|Qe_i\|_{\mathcal{H}}^2 = \|Q\|_{HS}^2 < \infty$. For the first term, we may use the dominated convergence theorem: since $P_n \to I$ strongly we have $\|(P_n - I)Qe_i\|_{\mathcal{H}}^2 \to 0$ for each i, and $\|(P_n - I)Qe_i\|_{\mathcal{H}}^2 \leq 4\|Qe_i\|_{\mathcal{H}}^2$ which is summable. $\square$

Now we may prove the desired representation for Z.

Theorem 3.10. *Let* $Z(t) = \int_0^t \omega(W_s, dW_s)$ *be as defined in Proposition 3.4, and let* $Q = Q_\omega$ *be the linear operator on* $\mathcal{H}$ *such that* $\omega(h, k) = \langle Qh, k \rangle_{\mathcal{H}}$ *for all* $h, k \in \mathcal{H}$ *as in Remark 3.8. Let* $\{X^j, Y^j\}_{j=1}^\infty$ *be independent standard real-valued Brownian motions,* $\{\pm i q_j\}_{j=1}^\infty$ *be the eigenvalues of* Q *so that* $\{q_j\}_{j=1}^\infty$ *is ordered from largest to smallest, and define for* $t \geq 0$

$$C(t) := \sum_{j=1}^\infty q_j^2 \int_0^t (X_s^j)^2 + (Y_s^j)^2 \, ds.$$

(Note that $C(t)$ *is well-defined and finite almost surely for each* t*.) Then the law of* $\{Z(t)\}_{t\geq 0}$ *is equal to the law of* $\{\tilde{Z}(t)\}_{t\geq 0}$ *where* $\tilde{Z}(t) = B(C(t))$ *for* B *a standard Brownian motion independent of* $\{X^j, Y^j\}_{j=1}^\infty$*.*

Proof. Let $\{P_n\}_{n=1}^\infty \subset \mathrm{Proj}(W)$ be such that $P_n|_{\mathcal{H}} \uparrow I_{\mathcal{H}}$ and

$$Z^n(t) = \int_0^t \omega(P_n W_s, dP_n W_s) = \int_0^t \langle (P_n Q P_n) P_n W_s, dP_n W_s \rangle_{\mathcal{H}},$$

as in Proposition 3.4, Then Lemma 3.7 implies that, for each n, the law of $\{Z^n(t)\}_{t\geq 0}$ is equal to the law of $\{\tilde{Z}^n(t)\}_{t\geq 0}$ where

$$\tilde{Z}^n(t) := B(C_t^n) := B\left(\sum_{j=1}^{r_n} q_{nj}^2 \int_0^t (X_s^j)^2 + (Y_s^j)^2 \, ds \right)$$

where $\{\pm i q_{nj}\}_{j=1}^{r_n}$ are the non-zero eigenvalues of $P_n Q P_n$. For each n, we will assume the q_{nj} are ordered in j from largest to smallest. Clearly, Proposition 3.4 and in particular (3.2) imply that $Z^n \Rightarrow Z$ and the collection $\{Z^n\}_{n=0}^\infty$ is tight. Equality in distribution then implies that $\tilde{Z}^n \Rightarrow Z$ and $\{\tilde{Z}^n\}_{n\geq 0}$ is tight.

Now we also have that, for each fixed $t > 0$,

$$\mathbb{E}|\tilde{Z}(t) - \tilde{Z}^n(t)|^2 = \mathbb{E}[\mathbb{E}[|\tilde{Z}(t) - \tilde{Z}^n(t)|^2 | C, C^n]] = \mathbb{E}|C(t) - C^n(t)|$$

$$= \mathbb{E}\left| \sum_{j=1}^\infty q_j^2 \int_0^t (X_s^j)^2 + (Y_s^j)^2 \, ds - \sum_{j=1}^{r_n} q_{nj}^2 \int_0^t (X_s^j)^2 + (Y_s^j)^2 \, ds \right|$$

$$= \mathbb{E}\left| \sum_{j=1}^{r_n} (q_j^2 - q_{nj}^2) \int_0^t (X_s^j)^2 + (Y_s^j)^2 \, ds - \sum_{j=r_n+1}^\infty q_j^2 \int_0^t (X_s^j)^2 + (Y_s^j)^2 \, ds \right|$$

$$\leq \mathbb{E}\left| \sum_{j=1}^{r_n} (q_j^2 - q_{nj}^2) \int_0^t (X_s^j)^2 + (Y_s^j)^2 \, ds \right| + \mathbb{E}\left| \sum_{j=r_n+1}^\infty q_j^2 \int_0^t (X_s^j)^2 + (Y_s^j)^2 \, ds \right|.$$

Note that

$$\|Q_n\|_{HS}^2 = 2 \sum_{j=1}^{r_n} q_{nj}^2 \quad \text{and} \quad \|Q\|_{HS}^2 = 2 \sum_{j=1}^\infty q_j^2 < \infty.$$

Thus, for the second term,

$$\mathbb{E}\left| \sum_{j=r_n+1}^\infty q_j^2 \int_0^t (X_s^j)^2 + (Y_s^j)^2 \, ds \right| = t^2 \sum_{j=r_n+1}^\infty q_j^2 \to 0,$$

clearly, since this is the tail of a convergent sequence. For the first term,

$$\mathbb{E}\left|\sum_{j=1}^{r_n}(q_j^2 - q_{nj}^2)\int_0^t (X_s^j)^2 + (Y_s^j)^2\, ds\right| \leq t^2 \sum_{j=1}^{r_n}|q_j^2 - q_{nj}^2| = t^2 \sum_{j=1}^{r_n}(q_j^2 - q_{nj}^2)$$

$$\leq t^2\left(\sum_{j=1}^{\infty}q_j^2 - \sum_{j=1}^{r_n}q_{nj}^2\right) \to 0,$$

where the equality follows from the min-max theorem which implies that

$$q_{nj} = \sup_{\substack{S \subset H \\ \dim(S) = j}} \min_{h \in S} \frac{\|Q_n h\|}{\|h\|}$$

$$\leq \sup_{\substack{S \subset H \\ \dim(S) = j}} \min_{h \in P_n S} \frac{\|Qh\|}{\|h\|} = \sup_{\substack{S \subset P_n H \\ \dim(S) = j}} \min_{h \in S} \frac{\|Qh\|}{\|h\|} \leq q_j$$

and the limit follows from Lemma 3.9 which implies that the Hilbert-Schmidt norms of Q_n converge to the Hilbert-Schmidt norm of Q. Thus, for any $(t_1, \ldots, t_m) \in (\mathbb{R}^+)^m$,

$$\mathbb{E}\left|(\tilde{Z}_{t_1}^n, \ldots, \tilde{Z}_{t_m}^n) - (\tilde{Z}_{t_1}, \ldots, \tilde{Z}_{t_m})\right|^2 = \sum_{i=1}^{m}\mathbb{E}|\tilde{Z}^n(t_i) - \tilde{Z}(t_i)|^2 \to 0$$

as $n \to \infty$, and the finite-dimensional distributions of $\tilde{Z}^n$ converge to those of $\tilde{Z}$. Combining this with the tightness of $\{\tilde{Z}^n\}$ implies that $\tilde{Z}^n \Rightarrow \tilde{Z}$. However, since $\tilde{Z}^n \Rightarrow Z$ also, it must be that $\{\tilde{Z}(t)\}_{t\geq 0}$ and $\{Z(t)\}_{t\geq 0}$ are equal in distribution. $\square$

Noting that, for $\{e_k\}_{k=1}^{\infty}$ an orthonormal basis of $\mathcal{H}$,

$$\sum_{k=1}^{\infty}\|\omega(e_k, \cdot)\|_{\mathcal{H}}^2 = \sum_{k=1}^{\infty}\|\langle Qe_k, \cdot\rangle_{\mathcal{H}}\|_{\mathcal{H}}^2 = \sum_{k=1}^{\infty}\|Qe_k\|_{\mathcal{H}}^2 = 2\sum_{k=1}^{\infty}q_k^2,$$

we see that indeed $C(t) = \langle Z\rangle_t$ up to a reordering of terms. Given this last theorem, in order to prove small deviations for Z, it suffices to prove them for $\tilde{Z}$. The results of Section 2 lead us to find a small ball estimate for the process $\langle Z\rangle$.

3.2 Small deviations for $\langle Z\rangle_t$ and applications

Note again that, as in Remark 3.6 we may write $\langle Z\rangle_t = \sum_{k=1}^{\infty}\|\omega(e_k, \cdot)\|_{\mathcal{H}}^2 \xi_k(t)$ where $\{\xi_k\}_{k=1}^{\infty}$ are i.i.d. copies of

$$\xi(t) := \int_0^t B_s^2\, ds. \tag{3.5}$$

Recall that, if $\{\zeta_j\}_{j=1}^m$ are independent positive random variables satisfying small ball estimates with the same exponents α and β for coefficients $\{K_j\}_{j=1}^m$, then

$$\lim_{\lambda \to \infty}\lambda^{-\alpha/(1+\alpha)}(\log\lambda)^{\beta/(1+\alpha)}\log\mathbb{E}\left[e^{-\lambda\sum_{j=1}^m \zeta_j}\right]$$

$$= -(1+\alpha)^{1+\beta/(1+\alpha)}\alpha^{-\alpha/(1+\alpha)}\sum_{j=1}^m K_j^{1/(1+\alpha)}$$

and equivalently

$$\lim_{\varepsilon \downarrow 0}\varepsilon^{\alpha}|\log\varepsilon|^{\beta}\log\mathbb{P}\left(\sum_{j=1}^m\zeta_j \leq \varepsilon\right) = -\left(\sum_{j=1}^m K_j^{1/(1+\alpha)}\right)^{(1+\alpha)}.$$

In particular, if $\{\eta_j\}_{j=1}^m$ are positive i.i.d. random variables satisfying small ball estimates with $K_j = K$ for each j and $\zeta_j = a_j \eta_j$ for some $a_j > 0$, then we have that

$$\lim_{\varepsilon \downarrow 0} \varepsilon^\alpha |\log \varepsilon|^\beta \log \mathbb{P}\left(\sum_{j=1}^m a_j \eta_j \le \varepsilon\right) = -K\left(\sum_{j=1}^m a_j^{\alpha/(1+\alpha)}\right)^{(1+\alpha)}.$$

Equivalently,

$$\limsup_{\lambda \to \infty} \lambda^{-\alpha/(1+\alpha)} (\log \lambda)^{\beta/(1+\alpha)} \log \mathbb{E}\left[e^{-\lambda \sum_{j=1}^\infty a_j \zeta_j}\right]$$

$$= -(1+\alpha)^{1+\beta/(1+\alpha)} \alpha^{-\alpha/(1+\alpha)} K^{1/(1+\alpha)} \sum_{j=1}^m a_j^{\alpha/(1+\alpha)}. \quad (3.6)$$

In the event this sum is actually infinite with a summable sequence of coefficients $\{a_j\}_{j=1}^\infty$, analogous results hold under additional requirements on the coefficients. Small deviations of random variables of the form

$$S = \sum_{j=1}^\infty a_j \zeta_j$$

where $\{a_j\} \in \ell^1(\mathbb{R}^+)$ and $\{\zeta_j\}$ are non-negative i.i.d. random variables, have been studied in [1, 7, 22, 27, 28, 18]. In particular, we present the following theorem (Theorem 3.1 of [7]) without proof.

Theorem 3.11. *Suppose that ζ is a non-negative random variable such that there exist $\alpha > 0$ and a slowly varying function L so that*

$$\log \mathbb{P}(\zeta < \varepsilon) \sim -\varepsilon^{-\alpha} L(\varepsilon)$$

as $\varepsilon \downarrow 0$, and there exist $\kappa, \delta > 0$ so that

$$\mathbb{E}\left[\zeta^{(1/(\gamma+\kappa))+\delta}\right] < \infty,$$

where $\gamma = \frac{1+\alpha}{\alpha}$. Then, given a sequence $\{a_j\}_{j=1}^\infty \subset \mathbb{R}^+$ such that $a_j = O(j^{-(\gamma+\kappa)})$ and $\{\zeta_j\}_{j=1}^\infty$ i.i.d. copies of ζ,

$$\log \mathbb{P}\left(\sum_{j=1}^\infty a_j \zeta_j < \varepsilon\right) \sim -\left(\sum_{j=1}^\infty a_j^{\alpha/(1+\alpha)}\right)^{(1+\alpha)} \varepsilon^{-\alpha} L(\varepsilon)$$

as $\varepsilon \downarrow 0$.

Remark 3.12. *For any $\{a_j\} \in \ell^1(\mathbb{R}^+)$, we easily obtain the upper bound for (3.6) in the following way. First, note that*

$$\mathbb{E}\left[e^{-\lambda \sum_{j=1}^\infty a_j \zeta_j}\right] = \prod_{j=1}^\infty \mathbb{E}\left[e^{-\lambda a_j \zeta_j}\right]$$

by independence and bounded convergence. Thus,

$$\limsup_{\lambda \to \infty} \lambda^{-\alpha/(1+\alpha)} (\log \lambda)^{\beta/(1+\alpha)} \log \mathbb{E}\left[e^{-\lambda \sum_{j=1}^\infty a_j \zeta_j}\right]$$

$$\le \sum_{j=1}^\infty \limsup_{\lambda \to \infty} \lambda^{-\alpha/(1+\alpha)} (\log \lambda)^{\beta/(1+\alpha)} \log \mathbb{E}\left[e^{-\lambda a_j \zeta_j}\right]$$

$$= -(1+\alpha)^{1+\beta/(1+\alpha)} \alpha^{-\alpha/(1+\alpha)} K^{1/(1+\alpha)} \sum_{j=1}^\infty a_i^{\alpha/(1+\alpha)}$$

by reverse Fatou's lemma for non-positive functions.

Remark 3.13. *When $\{a_j\} \in \ell^1(\mathbb{R}^+)$ are such that $a_j \leq \tilde{a}_j$ for $\{\tilde{a}_j\}$ a sequence as in Theorem 3.11, then we may easily obtain a lower bound in terms of $\{\tilde{a}_j\}$. Since $a_j \leq \tilde{a}_j$,*

$$S := \sum_{j=1}^\infty a_j \zeta_j \leq \sum_{j=1}^\infty \tilde{a}_j \zeta_j =: \tilde{S}$$

and thus $\mathbb{P}(\tilde{S} \leq \varepsilon) \leq \mathbb{P}(S \leq \varepsilon)$. It follows that

$$\liminf_{\varepsilon \downarrow 0} \varepsilon^\alpha \log \mathbb{P}(S \leq \varepsilon) \geq \liminf_{\varepsilon \downarrow 0} \varepsilon^\alpha \log \mathbb{P}(\tilde{S} \leq \varepsilon) = -K \left(\sum_{j=1}^\infty \tilde{a}_j^{\alpha/(1+\alpha)} \right)^{(1+\alpha)}.$$

Similarly,

$$\liminf_{\lambda \to \infty} \lambda^{-\alpha/(1+\alpha)} \log \mathbb{E}\left[e^{-\lambda S^q} \right] \geq \liminf_{\lambda \to \infty} \lambda^{-\alpha/(1+\alpha)} \log \mathbb{E}\left[e^{-\lambda S^a} \right]$$

$$= -(1+\alpha) \alpha^{-\alpha/(1+\alpha)} K \sum_{j=1}^\infty a_j^{\alpha/(1+\alpha)}.$$

Proposition 3.14. *Let $Z(t) = \int_0^t \omega(W_s, dW_s)$ be as in Proposition 3.4, and suppose that $\|\omega(e_j, \cdot)\|_{\mathcal{H}} = O(j^{-r})$ for $r > 1$. Then $C = \langle Z \rangle$ satisfies Assumption 1.1 with $\alpha = 1$, $\beta = 0$, and*

$$K(t_{i-1}, t_i) = \frac{1}{8} \|\omega\|_1^2 (\Delta_i t)^2$$

where $\Delta_i t := t_i - t_{i-1}$ and

$$\|\omega\|_1 := \sum_{j=1}^\infty \|\omega(e_j, \cdot)\|_{\mathcal{H}} < \infty.$$

That is, for any $m \in \mathbb{N}$, $0 = t_0 < t_1 < \cdots < t_m$, and $\{d_i\}_{i=1}^m$ a decreasing sequence,

$$\lim_{\varepsilon \downarrow 0} \varepsilon \log \mathbb{P}\left(\sum_{i=1}^m d_i \Delta_i \langle Z \rangle \leq \varepsilon \right) = -\frac{1}{8} \|\omega\|_1^2 \left(\sum_{i=1}^m d_i^{1/2} \Delta_i t \right)^2.$$

Proof. By Equation (3.5), we have that $\sum_{i=1}^m d_i \Delta_i C = \sum_{j=1}^\infty \|\omega(e_j, \cdot)\|_{\mathcal{H}}^2 \zeta_j$ where $\{\zeta_j\}_{j=1}^\infty$ are i.i.d. copies of

$$\zeta := \sum_{i=1}^m d_i \int_{t_{i-1}}^{t_i} (B_s)^2 \, ds.$$

Theorem 1.2 implies that

$$\lim_{\varepsilon \downarrow 0} \varepsilon \log \mathbb{P}(\zeta \leq \varepsilon) = -\frac{1}{8} \left(\sum_{i=1}^m d_i^{1/2} \Delta_i t \right)^2.$$

Thus, under the assumptions on ω, the desired result follows from Theorem 3.11 with $\alpha = 1$ and $a_j = \|\omega(e_j, \cdot)\|_{\mathcal{H}}^2$. $\qquad\square$

Now combining this result with Theorem 3.10 and Theorem 1.3 with $\alpha = 1$, $\beta = 0$, and $K(t_{i-1}, t_i) = \frac{1}{2} \|\omega\|_1^2 (\Delta_i t)^2$ immediately yields the following.

Theorem 3.15. *Let $Z(t) = \int_0^t \omega(W_s, dW_s)$ with $\|\omega(e_j, \cdot)\|_{\mathcal{H}} = O(j^{-r})$ for $r > 1$. Then, for any $m \in \mathbb{N}$, $0 = t_0 < t_1 < \cdots < t_m$, and $0 \leq a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_m < b_m$,*

$$\lim_{\varepsilon \downarrow 0} \varepsilon \log P \left(\bigcap_{i=1}^m \{ a_i \varepsilon \leq \sup_{0 \leq s \leq t_i} |Z(s)| \leq b_i \varepsilon \} \right) = -\frac{\pi}{4} \|\omega\|_1 \sum_{i=1}^m \frac{\Delta_i t}{b_i}.$$

Remark 3.16. *Note that the logarithmic asymptotics determined by Remarks 3.12 and 3.13 are sufficient to show that Theorem 1.3 has the correct order of asymptotics for general ω satisfying $\|\omega\|_1 < \infty$.*

As was done in [13], Theorem 3.15 may be used to prove a functional law of the iterated logarithm for Z. This immediately implies a Chung-like law of the iterated logarithm, or one may prove this directly from the first-order small deviations estimates proved in Theorem 3.15 as was done in [25]. The proofs follow exactly analogously to the finite-dimensional cases in [25] and [13], so we omit the proofs here.

Theorem 3.17. *Let $Z(t) = \int_0^t \omega(W_s, dW_s)$ with $\|\omega(e_j, \cdot)\|_{\mathcal{H}} = O(j^{-r})$ for $r > 1$. Then*

$$\mathbb{P}\left(\liminf_{t \to \infty} \frac{\log \log t}{t} \sup_{0 \le s \le t} |Z(s)| = \frac{\pi}{4}\|\omega\|_1\right) = 1.$$

Theorem 3.18. *Let $Z(t) = \int_0^t \omega(W_s, dW_s)$ with $\|\omega(e_j, \cdot)\|_{\mathcal{H}} = O(j^{-r})$ for $r > 1$. Let*

$$\eta_n(t) := \frac{\log \log n}{\frac{\pi}{4}\|\omega\|_1 n} \sup_{0 \le s \le nt} |Z(s)|,$$

and let $\mathcal{M}$ denote the set of non-negative, non-decreasing continuous functions such that $f(0) = 0$ and $\lim_{t \to \infty} f(t) = \infty$. Then, with probability 1, $\{\eta_n\}$ is relatively compact in $\mathcal{M}$ and the set of cluster points of $\{\eta_n\}$ is

$$\left\{f \in \mathcal{M} : \int_0^\infty f^{-1}(s)\, ds \le 1\right\}.$$

From here it is possible to obtain various occupation measure results for the maximal process of Z, as was done in [8], [12], and [13].

Remark 3.19. *Note that Theorems 3.15, 3.17, and 3.18 also include the finite-dimensional stochastic integrals already studied in [25] and [13]. The difference in factors of 2 arises from the fact that the non-zero singular values of Q necessarily have multiplicity which is a factor of 2 and the sum in $\|\omega\|_1$ counts all of these.*

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