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Weighted moments for Mandelbrot's martingales*

Xingang Liang† Quansheng Liu‡§

Abstract

Let $(Y_n)_{n\geq 0}$ be a Mandelbrot's martingale defined as sums of products of random weights indexed by nodes of a Galton-Watson tree, and let Y be its limit. We show a necessary and sufficient condition for the existence of weighted moments of Y of the forms $\mathbb{E}Y^\alpha \ell(Y)$, where $\alpha > 1$ and ℓ is a positive function slowly varying at ∞. We also show a sufficient condition in the case of $\alpha = 1$. Our results complete those of Alsmeyer and Kuhlbusch (2010) for weighted branching processes by removing their extra conditions on ℓ.

Keywords: weighted moments; Mandelbrot's martingale; slowly varying function.
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1 Introduction and results

We consider a generalized Mandelbrot's martingale (Y_n) defined as sums of products of random weights indexed by nodes of a Galton-Watson tree. The study of this model is interesting due to a large number of applications and to its close connections with many problems arising in a variety of applied probability setting, such as branching random walks, infinite particle systems, Quicksort algorithms, and random fractals. For recent studies on the model, see for example Barral and Jin (2010, [3]), Barral and Peyrière (2014, [4]), and the references therein. For closely related problems arising in branching random walks, see for example the recent works by Hu and Shi (2009, [13]), Chen (2013, [10]), Barral, Hu and Madaule (2014, [2]) and Hu (2014, [14]).

As usual, we write $\mathbb{N} = \{0, 1, \ldots\}$, $\mathbb{N}^* = \{1, 2, \ldots\}$, $\mathbb{R}_+ = [0, \infty)$ and $U = \bigcup_{n=0}^{\infty} \mathbb{N}^{*n}$ be the set of finite sequences composed by $\mathbb{N}^*$, where $\mathbb{N}^{*0} = \{\emptyset\}$ contains the null sequence $\emptyset$. For $u, v \in U$, write $|u| = n$ for the length of u, and uv for the sequence obtained by juxtaposition. Let T_n be the set of sequences $u \in U$ with length $|u| = n$.

Suppose that $\{(N_u, A_{u1}, A_{u2}, \ldots) : u \in U\}$ is a sequence of independent and identically distributed random variables with values in $\mathbb{N} \times \mathbb{R}_+^{\mathbb{N}^*}$, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$; assume that for all $u \in U$ and $i > N_u$, we have $A_{ui} = 0$. For simplicity, we write $(N, A_1, A_2, \ldots)$ for $(N_\emptyset, A_{\emptyset 1}, A_{\emptyset 2}, \ldots)$. Assume that the initial distribution is normalized such that

$$\mathbb{E} \sum_{i=1}^{N} A_i = 1.$$

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†Beijing Technology and Business University, China. E-mail: liangxingang@th.btbu.edu.cn
‡Univ. Bretagne-Sud, UMR 6205, LMBA, Vannes, France. E-mail: quansheng.liu@univ-ubs.fr
§Changsha University of Science and Technology, College of Mathematics and Computing Science, China.

Set

$$X_\emptyset = 1, \quad X_u = A_{u_1} A_{u_1 u_2} \cdots A_{u_1 \dots u_n} \quad \text{for } u = u_1 u_2 \dots u_n \in U \text{ with } n \geq 1,$$

$$Y_0 = 1 \quad \text{and} \quad Y_n = \sum_{u \in T_n} X_u \quad \text{for } n \geq 1. \tag{1.1}$$

Then the sequence $(Y_n)_{n\geq 0}$ forms a nonnegative martingale, which is called *Mandelbrot's martingale*, with respect to the natural filtration

$$\mathcal{E}_0 = \{\emptyset, \Omega\} \quad \text{and} \quad \mathcal{E}_n = \sigma\{(N_u, A_{u1}, A_{u2}, \dots) : |u| < n\} \quad \text{for } n \geq 1.$$

It is also called *Mandelbrot's cascade*, and *weighted branching process*. Let

$$Y = \lim_{n\to\infty} Y_n \quad \text{and} \quad Y^* = \sup_{n\geq 0} Y_n, \tag{1.2}$$

where the limit exists a.s. by the martingale convergence theorem, and $\mathbb{E}Y \leq 1$ by Fatou's lemma.

The existence of moments of Y has been studied by many authors, see for example Bingham and Doney (1975), Kahane and Peyrière (1976), Durrett and Liggett (1983), Liu (2000) and Alsmeyer and Kuhlbusch (2010). Of particular interest are comparison theorems about weighted moments of Y_1 and Y. Let

$$R_0 = \left\{ \ell : \mathbb{R}_+ \to \mathbb{R}_+, \ell \text{ is measurable and } \lim_{x\to\infty} \frac{\ell(\lambda x)}{\ell(x)} = 1 \ \forall \lambda > 0 \right\}$$

be the set of functions slowly varying at ∞, and let $\ell \in R_0$. By the representation theorem (see [9, Theorem 1.3.1]), ℓ has the canonical representation of the form

$$\ell(x) = c(x) \exp\left(\int_{a_0}^{x} \frac{\epsilon(t)}{t} \mathrm{d}t \right) \qquad \text{for } x > a_0, \tag{1.3}$$

where $a_0 > 0$ is a constant, $c(\cdot)$ and $\epsilon(\cdot)$ are measurable with $c(x) \to c$ for some constant $c \in (0, \infty)$ and $\epsilon(x) \to 0$ as $x \to \infty$. For the Crump-Mode-Jirina process, Bingham and Doney (1975) showed via Tauberian theorems that when $\alpha > 1$ is not an integer,

$$\mathbb{E}Y^\alpha \ell(Y) < \infty \quad \text{if and only if} \quad \mathbb{E}Y_1^\alpha \ell(Y_1) < \infty. \tag{1.4}$$

For weighted branching processes, using convex inequalities on martingales, Alsmeyer and Kuhlbusch (2010) showed the equivalence (1.4) under the extra condition that the function $\epsilon(\cdot)$ in the canonical representation (1.3) of ℓ is positive and decreasing (in the wide sense) when $\alpha > 1$ is not a dyadic power. In this paper, we will show that *the equivalence is always true whenever $\alpha > 1$ without any additional assumption on ℓ.*

The case where $\alpha = 1$ will also be condidered: we will show a sufficient condition for the existence of $\mathbb{E}Y\ell(Y)$, which was found by Bingham and Doney (1975) and Alsmeyer and Kuhlbusch (2010) under some extra conditions on ℓ.

For any $x > 0$, write

$$\rho(x) = \mathbb{E} \sum_{i=1}^{N} A_i^x \qquad \text{and} \qquad \rho'(x) = \mathbb{E} \sum_{i=1}^{N} A_i^x \ln A_i \tag{1.5}$$

if the expectations exist in $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. Notice that $\rho'(x)$ is the usual derivative of ρ at x if ρ is derivable at x. It is well -known (see [20]) that Y is non-degenerate if and only if

$$\rho'(1) < 0 \quad \text{and} \quad \mathbb{E}Y_1 \log^+ Y_1 < \infty, \tag{1.6}$$

where $\log^+ x = \max(\log x, 0)$ is the positive part of $\log x$, and that $\mathbb{E}Y = 1$ while (1.6) holds.

Our main result is the following comparison theorem about the weighted moments of Y and Y_1. As usual, for a set A, we write Int$A$ for its interior.

Theorem 1.1. *Let* $\alpha \in \mathrm{Int}\{a > 1 : \rho(a) < 1\}$ *and* $\ell \in R_0$. *Then the following assertions are equivalent:*

(a) $\mathbb{E}Y_1^{\alpha}\ell(Y_1) < \infty$*;*
(b) $\mathbb{E}{Y^*}^{\alpha}\ell(Y^*) < \infty$*;*
(c) $\mathbb{E}Y = 1$ *and* $\mathbb{E}Y^{\alpha}\ell(Y) < \infty$*.*

Notice that, by the monotone convergence theorem and the dominated convergence theorem, the assumption that $\alpha \in \mathrm{Int}\{a > 1 : \rho(a) < 1\}$ is equivalent to the condition that

$$\rho(\alpha) < 1 \quad \text{and} \quad \rho(\alpha + \epsilon) < \infty \ \text{ for some } \ \epsilon > 0.$$

For the Crump-Mode-Jirina process (where $A_i \le 1$ for all i), the equivalence between (a) and (c) was shown by Bingham and Doney (1975) when $\alpha > 1$ is not an integer; when $\alpha > 1$ is an integer, they also showed that the equivalence remains true under the extra condition that the function $\epsilon(\cdot)$ in the canonical representation form (1.3) of ℓ is positive and slowly varying at ∞.

For the weighted branching process, Alsmeyer and Kuhlbusch (2010) showed the equivalence between (a) and (c) under the condition that $\epsilon(\cdot)$ is positive and decreasing (and vanishes at ∞) when $\alpha > 1$ is not a dyadic power; when $\alpha \in \{2^n : n \ge 1\}$ is a dyadic power, they also showed the equivalence between (a) and (c) under the additional assumption that $\epsilon(\cdot)$ is (positive and) slowly varying at ∞. Our result extends the corresponding ones of Alsmeyer and Kuhlbusch (2010) to the whole class R_0 of slowly varying functions (without any additional assumptions on ℓ) for $\alpha \in \mathrm{Int}\{a > 1 : \rho(a) < 1\}$.

The situation for the case where $\alpha = 1$ is different. For $\ell \in R_0$, define

$$\hat{\ell}(x) = \begin{cases} \int_1^x \frac{\ell(t)}{t}\mathrm{d}t, & \text{if} \quad x > 1; \\ 0, & \text{if} \quad x \le 1. \end{cases} \tag{1.7}$$

Then $\hat{\ell} \in R_0$ and $\lim\limits_{x\to\infty} \frac{\hat{\ell}(x)}{\ell(x)} = \infty$. Notice that if $\ell(x) = \log^{\beta}(x)$, then $\hat{\ell}(x) = \frac{1}{\beta+1}\log^{\beta+1}(x)$, where $\beta \ge 0$ and $x > 1$.

Theorem 1.2. *Let* $\ell \in R_0$ *be given in* (1.3) *with* $\epsilon(\cdot)$ *positive and slowly varying at* ∞*, and let* $\hat{\ell}$ *be defined by* (1.7)*. Assume* (1.6) *and that there exists some* $\delta > 0$ *such that* $\rho(1+\delta) < \infty$*. Then* $\mathbb{E}Y_1\hat{\ell}(Y_1) < \infty$ *implies*

$$\mathbb{E}Y^*\ell(Y^*) < \infty \qquad \textit{and} \qquad \mathbb{E}Y\ell(Y) < \infty.$$

Remark. In Theorem 1.2, the condition that $\epsilon(\cdot)$ is positive and slowly varying at ∞ can be relaxed to the condition that $\ell(x)$ is concave. This will be seen in the proof.

In the context of weighted branching processes, Theorem 1.2 was first proved by Alsmeyer and Kuhlbusch (2010) under the extra conditions that $\epsilon(\cdot)$ is positive and decreasing and ℓ is not bounded; moreover, under these extra conditions, they also showed the converse, that is, $\mathbb{E}Y\ell(Y) < \infty$ implies $\mathbb{E}Y_1\hat{\ell}(Y_1) < \infty$.

In the special case where $A_i \le 1$ for all $i \ge 1$, Theorem 1.2 was first proved by Bingham and Doney (1975) in the context of Crump-Mode-Jirina processes, under the extra conditions that $\epsilon(\cdot)$ is positive and slowly varying at ∞ and $\limsup_{n\to\infty} \ell(b^n)/\ell(a^n) < \infty$ for all $1 < a < b < \infty$; the last condition on the superior limit was removed by Iksanov and Rösler (2006). In fact, Iksanov and Rösler (2006) considered the slightly more general case where ℓ is increasing (in the wide sense) and concave on $(0,\infty)$, in the context of branching random walks with $A_i \le 1$ for all $i \ge 1$. We will consider this case in a more general setting, without assuming $A_i \le 1$ for all $i \ge 1$.

By the argument used in the proof of Theorem 1.2, we can give a new proof of the non-degeneration of Y, which we will show in Section 5.

This work is an extension of [17] where the Galton-Watson process (for which all the A_i are the same constant less than 1) was considered. Although the basic idea of the approach is the same, the technical treatment is much more delicate due to the appearance of random weights, as can be seen by the long paper (with 48 pages) of Alsmeyer and Kuhlbusch (2010, [1]) on the same topic. Our approach simplifies significantly the arguments in [1]; it leads to an uniform treatment for all $\alpha > 1$, and enables us to remove the extra conditions on ℓ used in [1].

2 Auxiliary lemma

The proofs of Theorems 1.1 and 1.2 are mainly based on the double martingale structure and convex inequalities for martingales, by a refinement of the martingale argument of Alsmeyer and Kuhlbusch (2010).

For $n \geq 1$, define

$$D_n = Y_n - Y_{n-1} = \sum_{u \in T_{n-1}} X_u B_u \quad \text{ where } \quad B_u = \sum_{i=1}^{N_u} A_{ui} - 1 \text{ for } u \in T_{n-1}. \tag{2.1}$$

Then $(D_n, \mathcal{E}_n)_{n \geq 1}$ forms a sequence of martingale differences, and

$$Y^* - 1 = \sup_{n \geq 1} (D_1 + \cdots + D_n). \tag{2.2}$$

For convenience, we write $\mathbb{P}_n$ for the conditional probability of $\mathbb{P}$ given $\mathcal{E}_n$, and $\mathbb{E}_n$ for the corresponding expectation. Let $\{u_k : k = 1, \ldots, \text{card } T_{n-1}\}$ be an enumeration of T_{n-1}, where card T_{n-1} denotes the cardinality of T_{n-1}. Since B_{u_k} are independent of each other under $\mathbb{P}_{n-1}$, with $\mathbb{E}_{n-1} B_{u_k} = 0$, we see that $\{B_{u_k} : k = 1, \ldots, \text{card } T_{n-1}\}$ forms a sequence of martingale differences under $\mathbb{P}_{n-1}$, with respect to the filtration

$$\mathcal{F}_{n-1,k} = \sigma\{(N_u, A_{u1}, A_{u2}, \ldots) : |u| < n-1; B_{u_1}, \ldots, B_{u_k}\} \quad \text{for } k \geq 0$$

with the convention that (for $k = 0$)

$$\mathcal{F}_{n-1,0} = \sigma\{(N_u, A_{u1}, A_{u2}, \ldots) : |u| < n-1\} = \mathcal{E}_{n-1}.$$

As X_{u_k} are $\mathcal{E}_{n-1}$ -measurable and $\mathcal{E}_{n-1} \subset \mathcal{F}_{n-1,k}$ ($|u_k| = n-1$), the random sequence $(X_{u_k} B_{u_k})_{u_k \in T_{n-1}}$ is also a martingale difference sequence with respect to $(\mathcal{F}_{n-1,k})_{k \geq 0}$. Hence under $\mathbb{P}_{n-1}$, the random variable D_n can be considered as the sum of martingale differences. Therefore Y_n and D_n constitute a double martingale structure, which has been used in [1].

For $\beta \in (1, 2]$, write

$$Y_n^{(\beta)} = \sum_{u \in T_n} X_u^\beta, \quad M_n = \sup_{u \in T_n} X_u^{\beta - 1} \quad \text{and} \quad M = \sum_{n=1}^\infty M_n. \tag{2.3}$$

Using the Burkholder-Davis-Gundy (BDG) inequalities (see [11], Chap. 11, Theorems 1 and 2) to (Y_n) and (D_n), we obtain the following lemma:

Lemma 2.1. *Let ϕ be a convex and increasing function on $\mathbb{R}_+$ with $\phi(0) = 0$, and $\phi(2x) \leq c\phi(x)$ for some constant $c \in (0, \infty)$ and all $x > 0$. Let $\beta \in (1, 2]$ and define $\phi_{1/\beta}(x) = \phi(x^{1/\beta})$.*

(a) *If the function* $\phi_{1/\beta}$ *is convex,* $\mathbb{E}Y_1^\beta < \infty$ *and* $\rho(\alpha) < 1$ *for some* $\alpha > 1$*, then* $\mathbb{E}(M^{\frac{\alpha}{\beta-1}}) < \infty$ *and*

$$\mathbb{E}\phi(|Y^* - 1|) \le C \sum_{n=1}^{\infty} \mathbb{E}\frac{M_{n-1}}{M}\phi_{1/\beta}(MY_{n-1}) + C\sum_{n=1}^{\infty}\mathbb{E}\sum_{u\in T_{n-1}}\frac{X_u^\beta}{Y_{n-1}^{(\beta)}}\phi_{1/\beta}\left(Y_{n-1}^{(\beta)}|B_u|^\beta\right), \tag{2.4}$$

where $C > 0$ *is a constant depending on* ϕ *and* β*.*

(b) *If the function* $\phi_{1/\beta}$ *is concave, then*

$$\mathbb{E}\phi(|Y^* - 1|) \le C \sum_{n=1}^{\infty}\mathbb{E}\sum_{u\in T_{n-1}}\phi(X_u|B_u|), \tag{2.5}$$

where $C > 0$ *is a constant depending on* ϕ *and* β*.*

Proof. (a) It follows from (2.2) that $Y^* - 1$ can be considered as the supremum of sum of the martingale differences (D_n). Hence by the BDG inequality (cf. [11], Chap. 11, p.427, Theorem 2), we have

$$\mathbb{E}\phi(|Y^* - 1|) \le B\mathbb{E}\phi_{1/\beta}\Big(\sum_{n=1}^{\infty}\mathbb{E}_{n-1}|D_n|^\beta\Big) + B\sum_{n=1}^{\infty}\mathbb{E}\phi(|D_n|), \tag{2.6}$$

where $B > 0$ is a constant depending on ϕ and β.

Recall that under $\mathbb{P}_{n-1}$, the random variable D_n is the sum of the martingale difference sequence $(X_uB_u)_{u\in T_{n-1}}$. Since $\mathbb{E}|B_u|^\beta = \mathbb{E}|B_\emptyset|^\beta \le 2^\beta\mathbb{E}Y_1^\beta < \infty$ for all $u \in U$, by another BDG inequality (cf. [11], Chap. 11, p.425, Theorem 1) or the Marcinkiewicz-Zygmund inequality to $(X_uB_u)_{u\in T_{n-1}}$, together with the subadditivity of the function $x \mapsto x^{\beta/2}$, we have

$$\begin{aligned}\mathbb{E}_{n-1}|D_n|^\beta &\le B\mathbb{E}_{n-1}\Big[\sum_{u\in T_{n-1}}|X_uB_u|^2\Big]^{\frac{\beta}{2}} \\ &\le B\mathbb{E}_{n-1}\sum_{u\in T_{n-1}}X_u^\beta|B_u|^\beta \le C_1 M_{n-1}\cdot Y_{n-1},\end{aligned} \tag{2.7}$$

where $C_1 = B\mathbb{E}|Y_1 - 1|^\beta$. To obtain the last inequality we have also used the fact that $X_u^\beta \le M_nX_u$ and that B_u is independent of X_u under $\mathbb{E}_{n-1}$, for each sequence u of length n. Notice that $\alpha > \beta - 1$, for the $L^{\frac{\alpha}{\beta-1}}$ norm $\|\cdot\|_{\frac{\alpha}{\beta-1}}$ (as usual for $p \in [1, \infty)$ and a random variable X we denote by $\|X\|_p = [\mathbb{E}|X|^p]^{1/p}$ the L^p norm of X), we have

$$\|M_{n-1}\|_{\frac{\alpha}{\beta-1}} = \Big[\mathbb{E}\Big(\sup_{u\in T_{n-1}}X_u^{\beta-1}\Big)^{\frac{\alpha}{\beta-1}}\Big]^{\frac{\beta-1}{\alpha}} \le \Big[\mathbb{E}\sum_{u\in T_{n-1}}X_u^\alpha\Big]^{\frac{\beta-1}{\alpha}} = [\rho(\alpha)]^{\frac{(\beta-1)(n-1)}{\alpha}}.$$

As $\rho(\alpha) < 1$, by the triangular inequality for the norm $\|\cdot\|_{\frac{\alpha}{\beta-1}}$, we see that

$$\|M\|_{\frac{\alpha}{\beta-1}} \le \sum_{n=1}^{\infty}\|M_{n-1}\|_{\frac{\alpha}{\beta-1}} \le \sum_{n=1}^{\infty}[\rho(\alpha)]^{\frac{(\beta-1)(n-1)}{\alpha}} < \infty.$$

In particular, this implies that $M < \infty$ almost surely. Since $\sum_{n=1}^{\infty}\frac{M_{n-1}}{M} = 1$, by the convexity of the function $\phi_{1/\beta}$, we obtain that

$$\begin{aligned}\phi_{1/\beta}\Big(\sum_{n=1}^{\infty}\mathbb{E}_{n-1}|D_n|^{\beta}\Big) &\leq \phi_{1/\beta}\Big(\sum_{n=1}^{\infty}\frac{M_{n-1}}{M}\cdot C_1 M Y_{n-1}\Big)\\ &\leq \sum_{n=1}^{\infty}\frac{M_{n-1}}{M}\phi_{1/\beta}(C_1 M Y_{n-1})\\ &\leq C_2\sum_{n=1}^{\infty}\frac{M_{n-1}}{M}\phi_{1/\beta}(M Y_{n-1}),\end{aligned} \tag{2.8}$$

where $C_2>0$ is a constant depending on C_1 and c.

For the second part of (2.6), as both ϕ and $\phi_{1/\beta}$ are convex, using BDG inequality to $\{X_u B_u\}_{u\in T_{n-1}}$, we have

$$\begin{aligned}\mathbb{E}_{n-1}\phi(|D_n|) &\leq B\mathbb{E}_{n-1}\phi_{1/\beta}\Big(\sum_{u\in T_{n-1}} X_u^{\beta}|B_u|^{\beta}\Big)\\ &\leq B\mathbb{E}_{n-1}\sum_{u\in T_{n-1}}\frac{X_u^{\beta}}{Y_{n-1}^{(\beta)}}\phi_{1/\beta}\left(Y_{n-1}^{(\beta)}|B_u|^{\beta}\right).\end{aligned} \tag{2.9}$$

Taking expectation on both sides, together with the inequalities (2.6) and (2.8), we get (2.4).

(b) Using the BDG inequality (cf. [11], Chap. 11, p.425, Theorem 1) to the martingale difference sequence $\{D_n\}$ and by the subadditivity of the function $\phi_{1/\beta}$ (which is implied by the concavity of $\phi_{1/\beta}$ and the condition that $\phi_{1/\beta}(0)=0$), we have

$$\mathbb{E}\phi(|Y^*-1|)\leq B\mathbb{E}\phi_{1/\beta}\Big(\sum_{n=1}^{\infty}|D_n|^{\beta}\Big)\leq B\sum_{n=1}^{\infty}\mathbb{E}\phi(|D_n|), \tag{2.10}$$

where $B>0$ is a constant depending on ϕ and β. Similarly, by the BDG inequality and the subadditivity of the function $\phi_{1/\beta}$, we see that

$$\mathbb{E}_{n-1}\phi(|D_n|)\leq B\mathbb{E}_{n-1}\phi_{1/\beta}\Big(\sum_{u\in T_{n-1}}X_u^{\beta}|B_u|^{\beta}\Big)\leq B\mathbb{E}_{n-1}\sum_{u\in T_{n-1}}\phi(X_u|B_u|). \tag{2.11}$$

Taking expectation on both sides, and from the inequality (2.10), we get (2.5). □

3 Proof of Theorem 1.1

To give the proof of Theorem 1.1, we shall need the following result on the existence of α-th moments of Y:

Lemma 3.1. *Let $\alpha>1$. Then the following assertions are equivalence: (a) $\rho(\alpha)<1$ and $\mathbb{E}Y_1^{\alpha}<\infty$; (b) $\mathbb{E}Y^{\alpha}<\infty$; (c) $\mathbb{E}Y^{*\alpha}<\infty$.*

The equivalence between (a) and (b) was shown by Liu (see [19], Theorem 2.1), and the equivalence between (b) and (c) can be obtained from the inequality

$$\mathbb{P}(Y^*\geq x)\leq C\mathbb{P}(Y\geq ax)\quad\forall x\geq 0 \tag{3.1}$$

for some constants $a, C>0$, valid when Y is not degenerate (see [6], Lemma 2).

Proof of Theorem 1.1. Let $\beta\in(1,2]$ with $\beta<\alpha$. Write $\phi(x)=x^{\alpha}\ell(x)$. We can assume that both the functions ϕ and $\phi_{1/\beta}$ are convex on $\mathbb{R}_+$, and $\ell(x)>0$ for all $x\geq 0$ (see [18]).

(i) We first show the implication from (a) to (b). By Lemma 2.1, to show the finiteness of $\mathbb{E}Y^{*\alpha}\ell(Y^*)$, it is enough to show the convergence of the two series in (2.4).

It follows from Potter's theorem (see [9], Theorem 1.5.6) that for any $\epsilon > 0$, there exists some constant $A = A(\ell, \epsilon) > 0$ such that $\ell(x) \le A\max(x^\epsilon, x^{-\epsilon})$ for all $x > 0$. Hence we obtain

$$\begin{aligned}\mathbb{E}\frac{M_{n-1}}{M}\phi_{1/\beta}(MY_{n-1}) \quad &\le \quad C\Big[\mathbb{E}M_{n-1}M^{\frac{\alpha+\epsilon}{\beta}-1}Y_{n-1}^{\frac{\alpha+\epsilon}{\beta}} + \mathbb{E}M_{n-1}M^{\frac{\alpha-\epsilon}{\beta}-1}Y_{n-1}^{\frac{\alpha-\epsilon}{\beta}}\Big] \\ &:= \quad C[I^+(n) + I^-(n)]. \end{aligned} \tag{3.2}$$

Using Hölder's inequality twice to $I^+(n)$, we see that

$$I^+(n) \le \left[\mathbb{E}M_{n-1}^{pq}\right]^{1/pq} \cdot \left[\mathbb{E}M^{(\frac{\alpha+\epsilon}{\beta}-1)pq^*}\right]^{1/pq^*} \cdot \left[\mathbb{E}Y_{n-1}^{\frac{\alpha+\epsilon}{\beta}p^*}\right]^{1/p^*}, \tag{3.3}$$

where $p^* = \frac{\beta(\alpha-\epsilon)}{\alpha+\epsilon}$, $q = \frac{\alpha+\epsilon}{\beta}$ and

$$\frac{1}{p} + \frac{1}{p^*} = \frac{1}{q} + \frac{1}{q^*} = 1.$$

Since $\alpha \in \mathrm{Int}\{a : \rho(a) < 1\}$, there exists some $\gamma > 0$ such that

$$\rho(\alpha + \gamma) < 1.$$

As $\rho(x)$ is convex with $\rho(1) = 1$, we have $\rho(x) < 1$ for all $x \in (1, \alpha + \gamma]$; in particular, $\rho(\alpha-\epsilon) < 1$ for $0 < \epsilon < \alpha-1$. Hence by Lemma 3.1 (noting that $\mathbb{E}Y_1^{\alpha-\epsilon} \le C(\mathbb{E}\phi(Y_1)+1) < \infty$),

$$\sup_{n\ge1}\mathbb{E}Y_{n-1}^{\frac{\alpha+\epsilon}{\beta}p^*} = \sup_{n\ge1}\mathbb{E}Y_{n-1}^{\alpha-\epsilon} \le \mathbb{E}(Y^*)^{\alpha-\epsilon} < \infty\,. \tag{3.4}$$

Let $\epsilon > 0$ be small enough such that

$$(\beta-1)pq \in (1, \alpha+\gamma], \quad \text{ so that } \rho((\beta-1)pq) < 1.$$

Notice that $M_{n-1} = \sup_{u\in T_{n-1}} X_u^{\beta-1} \le \sum_{u\in T_{n-1}} X_u^{\beta-1}$, we see that

$$\|M_{n-1}\|_{pq} = \left[\mathbb{E}\Big(\sup_{u\in T_{n-1}} X_u^{\beta-1}\Big)^{pq}\right]^{\frac{1}{pq}} \le \left[\mathbb{E}\sum_{u\in T_{n-1}} X_u^{(\beta-1)pq}\right]^{\frac{1}{pq}} = [\rho((\beta-1)pq)]^{\frac{n-1}{pq}}. \tag{3.5}$$

Moreover, by the triangular inequality for the norm $\|\cdot\|_{pq}$, as $\rho((\beta-1)pq) < 1$, we have

$$\|M\|_{pq} \le \sum_{n=1}^{\infty}\|M_{n-1}\|_{pq} \le \sum_{n=1}^{\infty}[\rho((\beta-1)pq)]^{\frac{n-1}{pq}} < \infty\,. \tag{3.6}$$

It follows from (3.3), (3.4), (3.5) and (3.6) that

$$\sum_{n=1}^{\infty} I^+(n) \le \sum_{n=1}^{\infty}[\rho((\beta-1)pq)]^{(n-1)/pq} \cdot \|M\|_{pq}^{q/q^*} \cdot \left[\mathbb{E}(Y^*)^{\alpha-\epsilon}\right]^{1/p^*} < \infty\,.$$

Similarly we can prove that $I^-(n)$ is also summable on n. Hence we show from (3.2) that the first series in (2.4) converges.

We now consider the second series in (2.4). Again by Potter's theorem, for all $\epsilon > 0$, there exists $C > 0$ such that $\ell(xy) \le C\ell(x)\cdot\max(y^\epsilon, y^{-\epsilon})$ for all $x, y > 0$. Since B_u is independent of X_u and has the same distribution as $B_\emptyset = Y_1 - 1$, we have

$$\begin{aligned}\mathbb{E}\sum_{u\in T_{n-1}}\frac{X_u^\beta}{Y_{n-1}^{(\beta)}}\phi_{1/\beta}(Y_{n-1}^{(\beta)}|B_u|^\beta) \quad &\le \quad C\mathbb{E}\sum_{u\in T_{n-1}}\frac{X_u^\beta}{Y_{n-1}^{(\beta)}}\cdot|B_u|^\alpha\ell(|B_u|)\left[(Y_{n-1}^{(\beta)})^{\frac{\alpha+\epsilon}{\beta}} + (Y_{n-1}^{(\beta)})^{\frac{\alpha-\epsilon}{\beta}}\right] \\ &= \quad C\mathbb{E}\phi(|B_\emptyset|)\left[\mathbb{E}(Y_{n-1}^{(\beta)})^{(\alpha+\epsilon)/\beta} + \mathbb{E}(Y_{n-1}^{(\beta)})^{(\alpha-\epsilon)/\beta}\right]. \end{aligned} \tag{3.7}$$

By Hölder's inequality,

$$\mathbb{E}(Y_{n-1}^{(\beta)})^{(\alpha+\epsilon)/\beta} \le \mathbb{E}(M_{n-1}Y_{n-1})^{(\alpha+\epsilon)/\beta} \le \left[\mathbb{E}M_{n-1}^{p(\alpha+\epsilon)/\beta}\right]^{1/p} \cdot \left[\mathbb{E}Y_{n-1}^{p^*(\alpha+\epsilon)/\beta}\right]^{1/p^*}, \quad (3.8)$$

where p^* and p are now defined by $p^* = \frac{\beta(\alpha-\epsilon)}{(\alpha+\epsilon)} > 1$ and $\frac{1}{p} + \frac{1}{p^*} = 1$. Let $\epsilon > 0$ be small enough such that

$$a := (\beta-1)p(\alpha+\epsilon)/\beta \in (1, \alpha+\gamma), \quad \text{so that } \rho(a) < 1.$$

As $M_{n-1}^{p(\alpha+\epsilon)/\beta} \le \sum_{u\in T_{n-1}} X_u^{(\beta-1)p(\alpha+\epsilon)/\beta} = \sum_{u\in T_{n-1}} X_u^a$, it follows that

$$\mathbb{E}M_{n-1}^{p(\alpha+\epsilon)/\beta} \le [\rho(a)]^{n-1}. \quad (3.9)$$

Recalling that $\rho(\alpha-\epsilon) < 1$, by the definition of p^* and Lemma 3.1, we have

$$\mathbb{E}Y_{n-1}^{p^*(\alpha+\epsilon)/\beta} = \mathbb{E}Y_{n-1}^{\alpha-\epsilon} \le \mathbb{E}(Y^*)^{\alpha-\epsilon} < \infty.$$

Therefore it follows from (3.8) and (3.9) that

$$\sum_{n=1}^{\infty} \mathbb{E}(Y_{n-1}^{(\beta)})^{(\alpha+\epsilon)/\beta} \le \sum_{n=1}^{\infty} [\rho(a)]^{\frac{n-1}{p}} \cdot \left[\mathbb{E}(Y^*)^{\alpha-\epsilon}\right]^{\frac{1}{p^*}} < \infty\,.$$

Similarly, we can prove that $\mathbb{E}(Y_{n-1}^{(\beta)})^{(\alpha-\epsilon)/\beta}$ is summable on n. Together with (3.7) and the fact that $\mathbb{E}\phi(|B_\emptyset|) \le C[\mathbb{E}\phi(Y_1) + \phi(1)] < \infty$, we see that the second series in (2.4) converges.

Therefore, we have $\mathbb{E}\phi(|Y^*-1|) < \infty$, which is equivalent to $\mathbb{E}\phi(Y^*) < \infty$.

ii) We now show the implication from (b) to (c). It is evident that $\mathbb{E}\phi(Y) \le \mathbb{E}\phi(Y^*) < \infty$; by the dominated convergence theorem, Y_n convergence to Y in L^1, so that $\mathbb{E}Y = 1$. Thus (c) holds true.

iii) We finally show the implication from (c) to (a). Notice that Y satisfies the distributional equation

$$Y = \sum_{i=1}^{N} A_i Y^{(i)}, \quad (3.10)$$

where $(Y^{(i)})$ are independent of each other and are independent of $(N, A_1, A_2, \ldots)$, each has the same law as Y. By Jensen's inequality, we have

$$\mathbb{E}\phi(Y) = \mathbb{E}\phi\Big[\sum_{i=1}^{N} A_i Y^{(i)}\Big] \ge \mathbb{E}\phi\Big[\sum_{i=1}^{N} \mathbb{E}_1 A_i Y^{(i)}\Big] = \mathbb{E}\phi(\mathbb{E}Y \cdot Y_1) = \mathbb{E}\phi(Y_1), \quad (3.11)$$

hence $\mathbb{E}\phi(Y_1) < \infty$. □

4 Proof of Theorem 1.2

In the remark after Theorem 1.2, we mentioned that the condition in Theorem 1.2 that $\epsilon(x)$ is positive and slowly varying at ∞ can be relaxed to the condition that $\ell(x)$ is concave, as shown by the following lemma. As usual, we write

$$f(x) \asymp g(x) \quad \text{if} \quad 0 < \liminf_{x\to\infty} \frac{f(x)}{g(x)} \le \limsup_{x\to\infty} \frac{f(x)}{g(x)} < \infty\,,$$

and $f(x) \sim g(x)$ if $\lim_{x\to\infty} f(x)/g(x) = 1$.

Lemma 4.1. *Let $\epsilon(\cdot)$ be positive and slowly varying at ∞. Then there exists a concave function ℓ_1 such that $\ell_1(x) \asymp \ell(x)$.*

Proof. Notice that any slowly varying function ℓ posses a smoothed version ℓ_2 in the sense that $\ell(x) \sim \ell_2(x)$ as $x \to \infty$, with ℓ_2 of the form

$$\ell_2(x) = c\ \exp\bigg(\int_{a_0}^{x} \frac{\epsilon_2(t)}{t} dt\bigg), \quad x > a_0, \tag{4.1}$$

$\epsilon_2(\cdot)$ infinitely differentiable on (a_0, ∞), and $\lim_{x\to\infty} \epsilon_2(x) = 0$ (see [9, Theorem 1.3.3]). The value of a_0 and those of $\ell(x)$ on $[0, a_0]$ will not be important for our purpose. Therefore, without loss of generality, we suppose from now on that ℓ is of the form (1.3) with $a_0 = 1$, $c(x) = 1$, and $\epsilon(\cdot)$ is infinitely differentiable.

Since $\epsilon(\cdot)$ is continuous, we have $\ell'(x) = x^{-1}\ell(x)\cdot\epsilon(x)$. As both $\ell(\cdot)$ and $\epsilon(\cdot)$ are positive and slowly varying at ∞, the function $\ell'(\cdot)$ is regularly varying with order -1 (which means that $x\ell'(x)$ is a slowly varying function at ∞). Let $\psi(x) = \inf\{\ell'(t) : 1 \le t \le x\}$ for $x \ge 1$, then we have $\ell'(x) \sim \psi(x)$ when $x \to \infty$ (see [9, Theorem 1.5.3]). Since $\psi(\cdot)$ is positive and decreasing, we see that $\ell(x) \asymp \ell_1(x) := \int_1^x \psi(t)dt$ ($x \ge 1$) and $\ell_1(\cdot)$ is a positive concave function on $[1, \infty)$. $\square$

Proof of Theorem 1.2. By Lemma 4.1, it suffices to consider the case where ℓ is concave. So we suppose that ℓ is concave, and write $\phi(x) = x\ell(x)$. By Lemma 3.2 of [18], we can and we assume that the function ϕ is convex, the functions $x \mapsto \phi_{1/2}(x) = \phi(x^{1/2})$ and ℓ are concave with $\ell(0) = 0$.

Since ρ is convex on $(1, 1+\delta)$ with $\rho'(1) < 0$ and $\rho(1+\delta) < \infty$, there exists some $\epsilon \in (0, \delta)$ such that

$$\rho(1+\epsilon) := a < 1.$$

Pick $b \in (a^{1/\epsilon}, 1)$. As the function $\phi_{1/2}$ is concave, by Lemma 2.1, we have

$$\mathbb{E}\phi(|Y^* - 1|) \le C \sum_{n=1}^{\infty} \mathbb{E} \sum_{u\in T_{n-1}} X_u|B_u|\ell(X_u|B_u|) := C\sum_{n=1}^{\infty} \left[I_1(n) + I_1'(n)\right], \tag{4.2}$$

where

$$I_1(n) \quad = \quad \mathbb{E} \sum_{u\in T_{n-1}} X_u|B_u|\ell(X_u|B_u|)\mathbf{1}_{\{X_u \le b^{n-1}\}}$$

and

$$I_1'(n) \quad = \quad \mathbb{E} \sum_{u\in T_{n-1}} X_u|B_u|\ell(X_u|B_u|)\mathbf{1}_{\{X_u > b^{n-1}\}}.$$

In the following, we will show that both $I_1(n)$ and $I_1'(n)$ are summable on n. In fact, as ℓ is increasing and B_u are independent of X_u for all $u \in T_{n-1}$, we have

$$\begin{aligned} I_1(n) \quad &\le \quad \mathbb{E} \sum_{u\in T_{n-1}} X_u|B_u|\ell(b^{n-1}|B_u|) \\ &= \quad \mathbb{E}|B_\emptyset|\ell(b^{n-1}|B_\emptyset|) \le C\mathbb{E}|B_\emptyset| \int_{b^{n-1}|B_\emptyset|}^{b^{n-2}|B_\emptyset|} \frac{\ell(t)}{t}\mathrm{d}t. \end{aligned}$$

Taking sum on n, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} I_1(n) \quad &\le \quad C\mathbb{E}|B_\emptyset| \int_0^{|B_\emptyset|} \frac{\ell(t)}{t}\mathrm{d}t \\ &\le \quad C[\mathbb{E}|B_\emptyset|\hat{\ell}(|B_\emptyset|) + 1] \le C[\mathbb{E}Y_1\hat{\ell}(Y_1) + 1] < \infty\,, \end{aligned} \tag{4.3}$$

which shows the summability of $I_1(n)$. For $I_1'(n)$, as ℓ is increasing and slowly varying at ∞, by Potter's theorem and the independence between B_u and X_u ($u \in T_{n-1}$), we have

$$\begin{aligned} I_1'(n) &\leq C\mathbb{E} \sum_{u \in T_{n-1}} X_u |B_u| \ell(b^{n-1}|B_u|) \cdot (b^{1-n} X_u)^\epsilon \\ &\leq C\mathbb{E}|B_\emptyset| \ell(|B_\emptyset|) \cdot b^{(1-n)\epsilon} \cdot \mathbb{E} \sum_{u \in T_{n-1}} X_u^{1+\epsilon} = C\mathbb{E}|B_\emptyset|\ell(|B_\emptyset|) \cdot (ab^{-\epsilon})^{n-1}. \end{aligned}$$

By the definition of b, we have $ab^{-\epsilon} < 1$, hence

$$\sum_{n=1}^{\infty} I_1'(n) \leq C\mathbb{E}|B_\emptyset|\ell(|B_\emptyset|) \cdot \sum_{n=1}^{\infty} (ab^{-\epsilon})^{n-1} < \infty\,, \tag{4.4}$$

which shows the summability of $I_1'(n)$.

It follows from (4.2), (4.3) and (4.4) that $\mathbb{E}\phi(|Y^* - 1|) < \infty$, which is equivalent to $\mathbb{E}\phi(Y^*) < \infty$. Evidently $\mathbb{E}\phi(Y) \leq \mathbb{E}\phi(Y^*) < \infty$ as ℓ is increasing. This ends the proof of Theorem 1.2. □

5 New proof of non-degeneration of Y

The argument in the proof of Theorem 1.2 can be used to study the non-degeneration of Y, leading to a new proof of the following result of Biggins (1977).

Proposition 5.1. *Assume* (1.6) *and* $\mathbb{E}\sum_{i=1}^N A_i (\ln^+ A_i)^2 < \infty$. *Then*

$$\mathbb{E}Y^* < \infty \qquad \text{and} \qquad \mathbb{E}Y = 1.$$

In fact, the condition $\mathbb{E}\sum_{i=1}^N A_i(\ln^+ A_i)^2 < \infty$ in Proposition 5.1 can be removed (see [20]) as mentioned earlier. However, we need this condition in the following proof as in the approach of Biggins (1977). Notice that the conclusions $\mathbb{E}Y^* < \infty$ and $\mathbb{E}Y = 1$ are equivalent by (3.1).

Proof of Proposition 5.1. Let

$$\ell(x) = \begin{cases} 1 - \frac{1}{2x}, & \text{if} \quad x > 1; \\ \frac{x}{2}, & \text{if} \quad x \leq 1. \end{cases}$$

It is easy to see that the function $\phi(x) = x\ell(x)$ is convex and the function $\phi_{1/2}(x) = \phi(x^{1/2})$ is concave. Hence by Lemma 2.1, we have

$$\mathbb{E}\phi(|Y^* - 1|) \leq C \sum_{n=1}^{\infty} \mathbb{E} \sum_{u \in T_{n-1}} X_u |B_u| \ell(X_u |B_u|). \tag{5.1}$$

Pick $b \in (e^{\rho'(1)}, 1)$. We divide the domain of integration above into two parts according to $\{X_u \leq b^{n-1} : u \in T_{n-1}\}$ and $\{X_u > b^{n-1} : u \in T_{n-1}\}$. For the first part, with the same argument as in (4.3), we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbb{E} \sum_{u \in T_{n-1}} X_u |B_u| \ell(X_u|B_u|) \mathbf{1}_{\{X_u \leq b^{n-1}\}} &\leq C\mathbb{E}|B_\emptyset| \int_0^{|B_\emptyset|} \frac{\ell(t)}{t} \mathrm{d}t \\ &\leq C\mathbb{E}|B_\emptyset| \hat{\ell}(|B_u|) < \infty\,. \end{aligned} \tag{5.2}$$

Here, $\mathbf{1}_A$ is the indicative function of the Borel set A. For the second part, as ℓ is bounded by 1 and B_u is independent of X_u ($u \in T_{n-1}$), we have

$$\sum_{n=1}^{\infty} \mathbb{E} \sum_{u \in T_{n-1}} X_u |B_u| \ell(X_u |B_u|) \mathbf{1}_{\{X_u > b^{n-1}\}} \leq \mathbb{E}|B_\emptyset| \cdot \sum_{n=1}^{\infty} \mathbb{E} \sum_{u \in T_{n-1}} X_u \mathbf{1}_{\{X_u > b^{n-1}\}} \tag{5.3}$$

Define the probability measure $\mathbb{Q}$ on $\mathcal{B}(\mathbb{R})$ by

$$\mathbb{Q}(B) = \mathbb{E}\sum_{i=1}^{N} A_i \mathbf{1}_B(A_i), \qquad B \in \mathcal{B}(\mathbb{R}),$$

where $\mathcal{B}(\mathbb{R})$ is the Borel field on $\mathbb{R}$. Let $\tilde{A}_1, \tilde{A}_2, \ldots$ be i.i.d. random variables with common distribution $\mathbb{Q}$, and define $\Pi_0 = 1$ and $\Pi_n = \prod_{i=1}^{n} \tilde{A}_i$ for $n \geq 1$. By Lemma 4.1 (iii) of [7] (see also [1], Lemma 4.1), for any nonnegative measurable function f, we have

$$\mathbb{E}f(\Pi_{n-1}) = \mathbb{E}\sum_{u \in T_{n-1}} X_u f(X_u);$$

in particular,

$$\mathbb{E}\sum_{u \in T_{n-1}} X_u \mathbf{1}_{\{X_u > b^{n-1}\}} = \mathbb{Q}(\Pi_{n-1} > b^{n-1}). \tag{5.4}$$

As $\mathbb{E}\ln\tilde{A}_1 = \rho'(1) < \ln b$ and $\mathbb{E}(\ln^+\tilde{A}_1)^2 = \mathbb{E}\sum_{i=1}^{N} A_i(\ln^+ A_i)^2 < \infty$, by Lemma 6.1 of [18], we see that

$$\sum_{n=1}^{\infty} \mathbb{Q}(\Pi_{n-1} > b^{n-1}) < \infty.$$

Together with (5.4), this shows the convergence of the left series in (5.3).

It follows from (5.1), (5.2) and (5.3) that $\mathbb{E}\phi(|Y^* - 1|) < \infty$, which is equivalent to $\mathbb{E}Y^* < \infty$. This implies that $\mathbb{E}Y = 1$ by the dominated convergence theorem, as $Y_n \leq Y^*$. □

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