Electron. Commun. Probab. **20** (2015), no. 75, 1–14.
DOI: 10.1214/ECP.v20-4130
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Large deviations for processes on half-line*

F.C. Klebaner† A.V. Logachov‡ A.A. Mogulskii§

Abstract

We consider a sequence of processes $X_n(t)$ defined on the half-line $0 \le t < \infty$, $n = 1, 2, \ldots$. We give sufficient conditions for Large Deviation Principle (LDP) to hold in the space of continuous functions with metric

$$\rho_\kappa(f, g) = \sup_{t \ge 0} \frac{|f(t) - g(t)|}{1 + t^{1+\kappa}}, \quad \kappa \ge 0.$$

LDP is established for Random Walks and Diffusions defined on the half-line. LDP in this space is “more precise" than that with the usual metric of uniform convergence on compacts.

Keywords: Large Deviations, Random Walk, Diffusion processes.
AMS MSC 2010: 60F10; 60G50; 60H10; 60J60.
Submitted to ECP on February 22, 2015, final version accepted on October 19, 2015.
Supersedes arXiv:1502.06342v2.

1 Introduction

In this work we give sufficient conditions for a sequence of stochastic processes $X_n(t)$; $0 \le t < \infty$, $n = 1, 2, \ldots$, to satisfy the Large Deviation Principle (LDP) in the space of continuous functions on $[0, \infty)$, which we denote by $\mathbb{C}$. In the recent literature [18], [9], [12] the space $\mathbb{C}$ is considered with the metric

$$\rho^{(P)}(f, g) := \sum_{k=1}^{\infty} 2^{-k} \min\{ \sup_{0 \le t \le k} |f(t) - g(t)|, 1\}. \tag{1.1}$$

Theorem 2.6 of [19] gives sufficient conditions for X_n to satisfy LDP in the space $(\mathbb{C}, \rho^{(P)})$. As noted in [19], convergence $f_n \to f$ in metric $\rho^{(P)}$ is equivalent to convergence in $\mathbb{C}[0, T]$ with uniform metric for any $T \ge 0$. A considerable drawback of metric $\rho^{(P)}$ is that it is “ not sensitive" to behaviour of functions as $t \to \infty$.

We consider the space $\mathbb{C}$ with metric

$$\rho(f, g) = \rho_\kappa(f, g) := \sup_{t \ge 0} \frac{|f(t) - g(t)|}{1 + t^{1+\kappa}},$$

*Support: RFFI 13–01–12415 ofi-m, 14–01–0020–a; Australian Research Council Grant DP120102728.
†Monash University, Australia. E-mail: fima.klebaner@monash.edu
‡Novosibirsk State University, Russia. E-mail: omboldovskaya@mail.ru
§Sobolev Institute of Mathematics, Russia. E-mail: mogul@math.nsc.ru

for a fixed $\kappa \geq 0$. It is obvious that $(\mathbb{C}, \rho)$ is a complete separable metric (Polish) space. As we shall see in §2, the LDP in the space $(\mathbb{C}, \rho)$ is "more precise" than the LDP in $(\mathbb{C}, \rho^{(P)})$.

Here we treat continuous processes on infinite interval, a treatment of discontinuous processes on infinite interval will need a metric essentially different to ρ, (see [11], for the LDP for Compound Poisson processes on infinite interval). Note that in [10], Theorem 1.3.27, LDP for Wiener process in space $(\mathbb{C}, \rho_\kappa)$ when $\kappa = 0$ is given, while in [7] the Law of Iterated Logarithm is proved for Wiener process in this space.

The paper is organised as follows. Sufficient conditions for LDP in the space $(\mathbb{C}, \rho)$ are given in §2, Theorem 2.1. We also compare Theorem 2.1 and Theorem 2.6 of [19], and show that Theorem 2.1 is more precise. We apply Theorem 2.1 to Random Walks and Diffusions on the half line. Only Random Walks case is given here, and the reader is referred to the Arxiv for other examples.

2 Main Result

To formulate the main result we need the following definitions and notations. For any $T \in (0, \infty)$ denote by $\mathbb{C}[0, T]$ the metric space of real continuous functions $f = f(t);\ 0 \leq t \leq T$, with metric

$$\rho_T(f, g) := \sup_{0 \leq t \leq T} \frac{|f(t) - g(t)|}{1 + t^{1+\kappa}},$$

where $\kappa \geq 0$ is fixed. We say that

$$I_0^T = I_0^T(f) :\ \mathbb{C}[0, T] \to [0, \infty],$$

is a (good) rate function in space $\mathbb{C}[0, T]$ if:
(i) it is lower semi-continuous: *for any* $f \in \mathbb{C}[0, T]$

$$\varliminf_{f_n \to f} I_0^T(f_n) \geq I_0^T(f); \tag{2.1}$$

(ii) *for any* $r \geq 0$ *the set*

$$B_{T,r} := \{f \in \mathbb{C}[0, T] :\ I_0^T(f) \leq r\}$$

is a compact in $\mathbb{C}[0, T]$.

For a non-empty measurable set $B \subset \mathbb{C}[0, T]$ let

$$I_0^T(B) := \inf_{f \in B} I_0^T(f), \ \text{ with } \ I_0^T(\emptyset) := \infty.$$

$(f)_{T,\varepsilon}$ and $(B)_{T,\varepsilon}$ denote ε-neighbourhood in metric ρ_T in space $\mathbb{C}[0, T]$ of $f \in \mathbb{C}[0, T]$ and a measurable set $B \subset \mathbb{C}[0, T]$ respectively. The interior and the closure of B is denoted by $(B)_T$ and $[B]_T$ respectively.

Note that lower semi-continuity (2.1) can be written as: *for any* $f \in \mathbb{C}[0, T]$

$$\lim_{\varepsilon \to 0} I_0^T((f)_{T,\varepsilon}) = I_0^T(f). \tag{2.2}$$

For a function $f \in \mathbb{C}$, $f^{(T)}$ denotes its projection on $\mathbb{C}[0, T]$,

$$f^{(T)} = f^{(T)}(t) := f(t);\ 0 \leq t \leq T.$$

Denote by $\mathbb{C}_0 \subset \mathbb{C}$ – the class of functions $f \in \mathbb{C}$, such that $f(0) = 0$, $\lim_{t \to \infty} \frac{f(t)}{1+t^{1+\kappa}} = 0$.

Let now $X_n(t);\ t \in [0, \infty)$, $n = 1, 2, \ldots$, be a sequence of processes in space $\mathbb{C}_0$. We assume the following conditions.

I. *For any $T \in (0,\infty)$ processes $X_n^{(T)}$ satisfy LDP in space $\mathbb{C}[0,T]$ with good rate function I_0^T, i.e. for any measurable set $B \subset \mathbb{C}[0,T]$*

$$\varlimsup_{n\to\infty} \frac{1}{n} \ln \mathbf{P}(X_n^{(T)} \in B) \le -I_0^T([B]_T),$$

$$\varliminf_{n\to\infty} \frac{1}{n} \ln \mathbf{P}(X_n^{(T)} \in B) \ge -I_0^T((B)_T).$$

Moreover, for any $f \in \mathbb{C}[0,T]$ there is $g = g_f \in \mathbb{C}_0$, such that $g^{(T)} = f$, and for any $U \ge T$ it holds

$$I_0^U(g^{(U)}) = I_0^T(f). \tag{2.3}$$

Condition (2.3) means that one can extend any $f \in \mathbb{C}[0,T]$ for $t > T$ such that the rate function will stay the same. It is natural to call the function $g = g_f$ the most likely extension of f beyond $[0,T]$.

II. *For any $r \ge 0$*

$$\lim_{T\to\infty} \sup_{f\in B_r^+} \sup_{t\ge T} \frac{|f(t)|}{1+t^{1+\kappa}} = 0,$$

where

$$B_r^+ := \{f \in \mathbb{C} : \varlimsup_{T\to\infty} I_0^T(f^{(T)}) \le r\}.$$

III. *For any $N < \infty$ and $\varepsilon > 0$ there is $T = T_{N,\varepsilon} < \infty$ such that*

$$\varlimsup_{n\to\infty} \frac{1}{n} \ln \mathbf{P}(\sup_{t\ge T} \frac{|X_n(t)|}{1+t^{1+\kappa}} > \varepsilon) \le -N.$$

Theorem 2.1. *Assume Conditions* **I, II and III.** *Then for any $f \in \mathbb{C}$ there exists*

$$\lim_{T\to\infty} I_0^T(f^{(T)}) =: I(f), \tag{2.4}$$

and it is a good rate function in the space $(\mathbb{C},\ \rho)$. The sequence X_n satisfies LDP in this space with rate function $I(f)$, i.e. for any measurable $B \subset \mathbb{C}$

$$\varlimsup_{n\to\infty} \frac{1}{n} \ln \mathbf{P}(X_n \in B) \le -I([B]), \tag{2.5}$$

$$\varliminf_{n\to\infty} \frac{1}{n} \ln \mathbf{P}(X_n \in B) \ge -I((B)), \tag{2.6}$$

where

$$I(B) := \inf_{f\in B} I(f), \quad \textit{with} \quad I(\emptyset) = \infty.$$

Before we give the proof, we compare our Theorem to Theorem 2.6 in [19], which gives sufficient conditions for LDP in the space $(\mathbb{C}, \rho^{(P)})$. Note that if a set B is such that $I([B]) = I((B)) (= I(B))$, then inequalities (2.5), (2.6) can be replaced by the equality

$$\lim_{n\to\infty} \frac{1}{n} \ln \mathbf{P}(X_n \in B) = -I(B).$$

Hence the difference

$$D(B) := I((B)) - I([B]) \ge 0$$

describes the precision of LDP: the smaller the difference the more precise is the theorem.

The rate functions in both theorems are the same. This is because projections $X_n^{(T)}$ on $[0,T]$ satisfy LDP in the space $\mathbb{C}[0,T]$ with uniform metric and common rate function $I_0^T(f)$. Therefore we can compare these theorems by comparing differences

$$D(B) := I((B)) - I([B]) \quad \text{and} \quad D^{(P)}(B) := I((B)^{(P)}) - I([B]^{(P)}),$$

where $[B]^{(P)}$, $(B)^{(P)}$ denote the closure and the interior of B in metric $\rho^{(P)}$.

As noted earlier, convergence in $\rho^{(P)}$ is equivalent to convergence in ρ_T for any $T > 0$.

Therefore $\rho(f_n, f) \to 0$ implies $\rho^{(P)}(f_n, f) \to 0$. It is easy to see that the opposite is not true.

Thus

$$[B] \subset [B]^{(P)}, \quad (B)^{(P)} \subset (B),$$

and therefore

$$I([B]^{(P)}) \le I([B]), \quad I((B)) \le I((B)^{(P)}),$$

so that we always have $D(B) \le D^{(P)}(B)$. Below we give an example of B satisfying simultaneously

$$I([B]) = I((B)) \in (0, \infty), \quad I([B]^{(P)}) = 0.$$

Hence Theorem 2.1 allows to give "precise" logarithmic asymptotic for $\mathbf{P}(X_n \in B)$, while Theorem 2.6 in [19] does not. Thus we conclude that *LDP in the space* $(\mathbb{C}, \rho)$ *is more precise than in the space* $(\mathbb{C}, \rho^{(P)})$.

Example 2.2. Consider Wiener process $w = w(t)$ on $[0, \infty)$. Denote

$$w_n = w_n(t) := \frac{1}{\sqrt{n}} w(t), \quad t \ge 0.$$

Since conditions **I—III** are easily checked, then LDP follows from Theorem 2.1 with rate function

$$I(f) = \begin{cases} \frac{1}{2}\int_0^\infty (f'(t))^2 dt, & \text{if} \quad f(0) = 0, \quad f \text{ is absolutely continuous,} \\ \infty & \text{otherwise.} \end{cases}$$

$$B = \overline{(f_0)_1} := \left\{ g \in \mathbb{C} : \sup_{t \ge 0} \frac{|g(t)|}{1+t} \ge 1 \right\}, \quad f_0 = f_0(t) \equiv 0.$$

Since it is a complement to an open set $(f_0)_1$, it is closed in $(\mathbb{C}, \rho)$, and therefore

$$I([B]) = I(B) = \inf_{g \in B} I(g).$$

By Cauchy-Schwarz-Bunyakovski inequality

$$1 \le \sup_{t \ge 0} \frac{|g(t)|}{1+t} = \sup_{t \ge 0} \frac{|\int_0^t g'(s)ds|}{1+t} \le \sup_{t \ge 0} \left| \int_0^t (g'(s))^2 ds \right|^{1/2} \sup_{t \ge 0} \frac{t^{1/2}}{1+t} = \frac{\sqrt{2I(g)}}{2}.$$

Hence $I(g) \ge 2$ for all $g \in B$.

Take now $f(t) = 2tI(0 \le t \le 1) + 2I(t \ge 1)$. It is easy to see that $f \in B$ and $I(f) = 2$. Therefore $I([B]) = I(B) = 2$.

Taking $f_n(t) = (2 + 1/n)tI(0 \le t \le 1) + 2 + 1/nI(t \ge 1)$, we can see that $f_n \in (B)$ and $I([B]) = I(B) = I((B)) = 2$. Hence,

$$\lim_{n \to \infty} \frac{1}{n} \ln \mathbf{P}(w_n \in B) = -2.$$

Consider now $[B]^{(P)}$, the closure of B in metric $\rho^{(P)}$. By taking $g_n(t) = \frac{t^2}{n}$ it is easy to see that $g_n \in B$ for all n and $\lim_{n \to \infty} \rho^{(P)}(g_n, f_0) = 0$. Therefore, $f_0 \in [B]^{(P)}$, and $I([B]^{(P)}) = 0$. Hence the upper bound for this set is trivial, and does not allow to find logarithmic asymptotic of the required probability.

3 Proof of Theorem 2.1

Denote by $(f)_\varepsilon$ and $(B)_\varepsilon$ the ε-neighborhoods ($\varepsilon > 0$) of $f \in \mathbb{C}$, and set $B \subset \mathbb{C}$. The proof of the Theorem 2.1 consists of three steps. The first step in Lemma 3.1 proves that $I(f)$ is a good rate function. The second step proves the local LDP for X_n in $(\mathbb{C}_0, \rho)$ in Lemma 3.2 and Corollary 3.3. The third step proves a weaker form of exponential tightness for X_n in Lemma 3.4 . The proof is then concluded as follows.

The upper bound is obtained by Lemmas 3.2 and 3.4 from general results on LDP in metric spaces: for $B \subset \mathbb{C}_0$ and $\varepsilon > 0$ it holds (e.g. [4], Theorem 3.1)

$$\varlimsup_{n\to\infty} \frac{1}{n} \ln \mathbf{P}(X_n \in B) \le -I((B)_\varepsilon).$$

It is also known (e.g. [4], Lemma 2.1), that a good rate function $I(f)$ satisfies

$$\lim_{\varepsilon\to 0} I((B)_\varepsilon) = I([B]),$$

and the upper bound (2.5) follows. Lower bound (2.6) follows from Lemma 3.2.

Lemma 3.1. *The rate function $I(f)$ (defined in (2.4)) is a good rate function, i.e. it is lower semi-continuous and for any $r \ge 0$ the set $B_r := \{f \in \mathbb{C} : \ I(f) \le r\}$ is a compact in $\mathbb{C}$.*

Proof. First we show that the limit exists. It is known (see e.g. [4], Theorem 3.1 or Lemma 1.3), that LDP implies local LDP: for any $f \in \mathbb{C}[0, T]$

$$-I_0^T(f) \ge \lim_{\varepsilon\to 0} \varlimsup_{n\to\infty} \frac{1}{n} \ln \mathbf{P}(X_n^{(T)} \in (f)_{T,\varepsilon}) \ge \lim_{\varepsilon\to 0} \varliminf_{n\to\infty} \frac{1}{n} \ln \mathbf{P}(X_n^{(T)} \in (f)_{T,\varepsilon}) \ge -I_0^T(f).$$

For $U \ge T$, with obvious notations, we have for $f \in \mathbb{C}$

$$\{X_n^{(U)} \in (f^{(U)})_{U,\varepsilon}\} \subset \{X_n^{(T)} \in (f^{(T)})_{T,\varepsilon}\},$$

therefore

$$\begin{aligned} -I_0^T(f^{(T)}) \quad &\ge \quad \lim_{\varepsilon\to 0} \varlimsup_{n\to\infty} \frac{1}{n} \ln \mathbf{P}(X_n^{(T)} \in (f^{(T)})_{T,\varepsilon}) \\ &\ge \quad \lim_{\varepsilon\to 0} \varliminf_{n\to\infty} \frac{1}{n} \ln \mathbf{P}(X_n^{(U)} \in (f^{(U)})_{U,\varepsilon}) \ge -I_0^U(f^{(U)}). \end{aligned}$$

Thus we established that $I_0^T(f^{(T)})$ is non-decreasing in T, and (2.4) follows.

Next, we show lower semi-continuity, that is if $f_n \to f$, then

$$\varliminf_{n\to\infty} I(f_n) \ge I(f). \tag{3.1}$$

For any $N < \infty$, $\varepsilon > 0$ there is $T = T_{N,\varepsilon} < \infty$ such that

$$I_0^T(f) \ge \min\{I(f), N\} - \varepsilon.$$

Since $f_n \to f$, $\rho_T(f_n, f) \to 0$. Thanks to Condition **I** the rate function $I_0^T(f)$ is lower semi-continuous in $(\mathbb{C}[0, T], \rho_T)$, therefore

$$\varliminf_{n\to\infty} I(f_n) \ge \varliminf_{n\to\infty} I_0^T(f_n) \ge I_0^T(f) \ge \min\{I(f), N\} - \varepsilon.$$

This implies (3.1), since N and ε are arbitrary.

We show next that the set B_r is completely bounded. Due to Condition **II** for any $\varepsilon > 0$ there is $T = T_r < \infty$ such that for any $f \in B_r$

$$\sup_{t\ge T} \frac{|f(t)|}{1 + t^{1+\kappa}} < \varepsilon. \tag{3.2}$$

Denote

$$B_r^{(T)} := \{f^{(T)} :\ f \in B_r\},$$

so that

$$B_r^{(T)} \subset B_{T,r},$$

where we recall that $B_{T,r} := \{f \in \mathbb{C}[0,T] :\ I_0^T(f) \le r\}$.

Since by Condition **I** the set $B_{T,r}$ is a compact in $\mathbb{C}[0,T]$, it is possible to find finite ε-net:

$$B_r^{(T)} \subset B_{T,r} \subset \cup_{i=1}^M (f_i)_{T,\varepsilon}.$$

Now for $f \in \mathbb{C}[0,T]$ define $f^{(T+)} \in \mathbb{C}_0$ as

$$f^{(T+)}(t) := \begin{cases} f(t), & \text{if } 0 \le t \le T, \\ f(T), & \text{if } t \ge T \end{cases}$$

For any $f \in B_r$ there is $i \in \{1, \cdots, M\}$ such that

$$\sup_{0 \le t \le T} \frac{|f(t) - f_i^{(T+)}(t)|}{1 + t^{1+\kappa}} < \varepsilon < 3\varepsilon.$$

We have for this i due to (3.2)

$$\sup_{t \ge T} \frac{|f(t) - f_i^{(T+)}(t)|}{1 + t^{1+\kappa}} \le \sup_{t \ge T} \frac{|f(t)|}{1 + t^{1+\kappa}} + \sup_{t \ge T} \frac{|f(T)|}{1 + t^{1+\kappa}} + \sup_{t \ge T} \frac{|f(T) - f_i^{(T+)}(T)|}{1 + t^{1+\kappa}} \le 3\varepsilon,$$

therefore the collection $\{f_1^{(T+)}, \cdots, f_M^{(T+)}\}$ represents a 3ε-net in the set B_r. Thus we have shown that the set B_r is completely bounded in $\mathbb{C}_0$.

From lower semi-continuity of $I(f)$, established earlier, it follows that B_r is closed in $\mathbb{C}_0$. Since *a closed completely bounded subset of a Polish space is a compact* (e.g. [15], Theorem 3, p. 109), we have shown that B_r is a compact in $\mathbb{C}_0$, thus completing the proof of Lemma 3.1. □

Lemma 3.2. *For any* $f \in \mathbb{C}_0$, $\varepsilon > 0$

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \ln \mathbf{P}(X_n \in (f)_\varepsilon) \le -I((f)_{2\varepsilon}), \tag{3.3}$$

$$\underline{\lim_{n\to\infty}} \frac{1}{n} \ln \mathbf{P}(X_n \in (f)_\varepsilon) \ge -I(f). \tag{3.4}$$

Proof. First we prove the lower bound (3.4) as it is also used in the proof of the upper bound. If $I(f) = \infty$, then (3.4) is trivially satisfied. Let now $I(f) < \infty$. For any $T \in (0, \infty)$ we have

$$\{X_n \in (f)_\varepsilon\} \supset A_n(T) \cap B_n(T) \cap C_n(T) \cap D(T) = A_n(T) \cap B_n(T) \cap D(T),$$

where

$$A_n(T) := \left\{ \sup_{0 \le t \le T} \frac{|X_n(t) - f(t)|}{1 + t^{1+\kappa}} < \varepsilon \right\}, \quad B_n(T) := \left\{ \sup_{t \ge T} \frac{|X_n(t) - X_n(T)|}{1 + t^{1+\kappa}} < \varepsilon/4 \right\},$$

$$C_n(T) := \left\{ \frac{|X_n(T)|}{1 + T^{1+\kappa}} < \varepsilon/4 \right\}, \quad D(T) := \left\{ \sup_{t \ge T} \frac{|f(t)|}{1 + t^{1+\kappa}} < \varepsilon/2 \right\}.$$

For a large T the event $D(T)$ is a certainty (due to $I(f) < \infty$). Therefore there exists $T_0 < \infty$, such that for all $T \ge T_0$

$$\mathbf{P}(X_n \in (f)_\varepsilon) \ge \mathbf{P}(A_n(T) \cap B_n(T)) \ge \mathbf{P}(A_n(T)) - \mathbf{P}(\overline{B_n}(T)), \tag{3.5}$$

where $\overline{B_n}(T)$ is a complement of $B_n(T)$. Due to Condition **III** there is $T \geq T_0$ such that

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \ln \mathbf{P}(\overline{B_n}(T)) \leq -2I(f), \tag{3.6}$$

and for this T due to Condition **I** we have

$$\underline{\lim}_{n\to\infty} \frac{1}{n} \ln \mathbf{P}(A_n(T)) \geq -I_0^T(f^{(T)}) \geq -I(f). \tag{3.7}$$

(3.4) now follows from (3.5) by using (3.6), (3.7).
Next we prove the upper bound (3.3). It is obvious that for any $T \in (0, \infty)$

$$\mathbf{P}(X_n \in (f)_\varepsilon) \leq \mathbf{P}(X_n^{(T)} \in (f^{(T)})_{T,\varepsilon}),$$

where we recall that $(f)_{T,\varepsilon}$ denote ε-neighbourhood in metric ρ_T in space $\mathbb{C}[0,T]$ of $f \in \mathbb{C}[0,T]$.

Due to Condition **I** for any $\delta > 0$

$$\begin{aligned} L(\varepsilon) := \overline{\lim_{n\to\infty}} \frac{1}{n} \ln \mathbf{P}(X_n \in (f)_\varepsilon) \quad &\leq \quad \overline{\lim_{n\to\infty}} \frac{1}{n} \ln \mathbf{P}(X_n^{(T)} \in (f^{(T)})_{T,\varepsilon}) \\ &\leq \quad -I_0^T([(f^{(T)})_{T,\varepsilon}]_T) \leq -I_0^T((f^{(T)})_{T,\varepsilon+\delta}). \end{aligned}$$

For any $T \in (0,\infty)$ and chosen ε and δ, in this way we have the inequality

$$L(\varepsilon) \leq -I_0^T((f^{(T)})_{T,\varepsilon+\delta}). \tag{3.8}$$

Choose now $T < \infty$ so large, that simultaneously the following holds:

$$\sup_{t\geq T} \frac{|f(t)|}{1+t^{1+\kappa}} < \delta; \tag{3.9}$$

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \ln \mathbf{P}(X_n \in R(T,\varepsilon)) \leq -N, \tag{3.10}$$

where $N < \infty$ is arbitrary, and

$$R(T,\varepsilon) := \left\{ g \in \mathbb{C}_0 : \ \sup_{t\geq T} \frac{|g(t)|}{1+t^{1+\kappa}} > \varepsilon \right\}.$$

Denote

$$((f^{(T)})_{T,\varepsilon+\delta})^{(T+)} := \{g \in \mathbb{C}_0 : \ g^{(T)} \in (f^{(T)})_{T,\varepsilon+\delta}\}.$$

Next we show that

$$I_0^T(B) = I(B^{(T+)}), \tag{3.11}$$

where for $B \subset \mathbb{C}[0,T]$

$$B^{(T+)} := \{g \in \mathbb{C}_0 : \ g^{(T)} \in B\}.$$

Indeed, for any $\varepsilon > 0$ let $f \in B$ be such that

$$I_0^T(f) \leq I_0^T(B) + \varepsilon.$$

Then due to (2.3) in condition **I** there is $g \in \mathbb{C}_0$ such that $g^{(T)} = f$ (consequently $g \in B^{(T+)}$) with $I(g) = I_0^T(f)$. Therefore

$$I_0^T(B) + \varepsilon \geq I_0^T(f) = I(g) \geq I(B^{(T+)}).$$

Since $\varepsilon > 0$ is arbitrary,

$$I_0^T(B) \geq I(B^{(T+)}). \tag{3.12}$$

Let now $g \in B^{(T+)}$ such that

$$I(g) \le I(B^{(T+)}) + \varepsilon.$$

Then $g^{(T)} \in B$ with $I_0^T(g^{(T)}) \le I(g)$. Therefore

$$I(B^{(T+)}) + \varepsilon \ge I(g) \ge I_0^T(g^{(T)}) \ge I_0^T(B),$$

and

$$I_0^T(B) \le I(B^{(T+)}). \tag{3.13}$$

Inequalities (3.12), (3.13) now prove equality (3.11).

Due to (3.11) we have

$$I_0^T((f^{(T)})_{T,\varepsilon+\delta}) = I(((f^{(T)})_{T,\varepsilon+\delta})^{(T+)}),$$

therefore due to (3.8)

$$L(\varepsilon) \le -I(((f^{(T)})_{T,\varepsilon+\delta})^{(T+)}). \tag{3.14}$$

Take an arbitrary $g \in ((f^{(T)})_{T,\varepsilon+\delta})^{(T+)}$. Then either

$$\sup_{t\ge T} \frac{|g(t) - f(t)|}{1 + t^{1+\kappa}} < \varepsilon + 2\delta,$$

and then

$$g \in (f)_{\varepsilon+2\delta}; \tag{3.15}$$

or

$$\sup_{t\ge T} \frac{|g(t) - f(t)|}{1 + t^{1+\kappa}} \ge \varepsilon + 2\delta, \tag{3.16}$$

and then

$$\sup_{t\ge T} \frac{|g(t)|}{1 + t^{1+\kappa}} \ge \varepsilon + \delta, \tag{3.17}$$

and

$$g \in R(T, \varepsilon). \tag{3.18}$$

To see how (3.17) follows from (3.16), note that if the inequality (3.17) is not true, then the opposite holds

$$\sup_{t\ge T} \frac{|g(t)|}{1 + t^{1+\kappa}} < \varepsilon + \delta,$$

and due to (3.9)

$$\sup_{t\ge T} \frac{|g(t) - f(t)|}{1 + t^{1+\kappa}} \le \sup_{t\ge T} \frac{|g(t)|}{1 + t^{1+\kappa}} + \sup_{t\ge T} \frac{|f(t)|}{1 + t^{1+\kappa}} < \varepsilon + \delta + \delta = \varepsilon + 2\delta,$$

which contradicts (3.16). We have proved (see (3.15) and (3.18)), that

$$((f^{(T)})_{T,\varepsilon+\delta})^{(T+)} \subset (f)_{\varepsilon+2\delta} \cup R(T, \varepsilon).$$

From the latter we obtain

$$I(((f^{(T)})_{T,\varepsilon+\delta})^{(T+)}) \ge \min\{I((f)_{\varepsilon+2\delta}), I(R(T, \varepsilon))\}. \tag{3.19}$$

Further, due to (3.10)

$$-N \ge \overline{\lim_{n\to\infty}} \frac{1}{n} \ln \mathbf{P}(X_n \in R(T, \varepsilon)) \ge \underline{\lim_{n\to\infty}} \frac{1}{n} \ln \mathbf{P}(X_n \in R(T, \varepsilon)) \ge -I(R(T, \varepsilon)),$$

where the last inequality for an open set $R(T,\varepsilon)$ follows from the established lower bound (3.4). Therefore $I(R(T,\varepsilon)) \geq N$, and, in view of (3.19),

$$I(((f^{(T)})_{T,\varepsilon+\delta})^{(T+)}) \geq \min\{I((f)_{\varepsilon+2\delta}), N\}.$$

Going back to (3.14), we obtain the inequality

$$L(\varepsilon) \leq -\min\{I((f)_{\varepsilon+2\delta}), N\},$$

in which $\delta > 0$ and $N < \infty$ are arbitrary. Taking $2\delta = \varepsilon$ and sending N to ∞, we obtain the required upper bound

$$L(\varepsilon) \leq -I((f)_{2\varepsilon}).$$

□

Local LDP in $\mathbb{C}_0$ follows from Lemma 3.2, and is stated as a corollary.

Corollary 3.3. *For any* $f \in \mathbb{C}_0$

$$\lim_{\varepsilon\to 0} \overline{\lim_{n\to\infty}} \frac{1}{n} \ln \mathbf{P}(X_n \in (f)_\varepsilon) \leq -I(f), \quad \lim_{\varepsilon\to 0} \underline{\lim_{n\to\infty}} \frac{1}{n} \ln \mathbf{P}(X_n \in (f)_\varepsilon) \geq -I(f).$$

Next result proves a weaker form of exponential tightness: for any N there is a completely bounded set K_N in $(\mathbb{C}_0, \rho)$ such that

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \ln \mathbf{P}(X_n \notin K) \leq -N.$$

Lemma 3.4. *For any* $N < \infty$ *and* $\varepsilon > 0$ *there is a finite collection of* $g_1, \cdots, g_M \in \mathbb{C}_0$ *such that*

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \ln \mathbf{P}(X_n \notin \cup_{i=1}^M (g_i)_\varepsilon) \leq -N. \tag{3.20}$$

Proof. Denote by

$$R_T(\varepsilon) := \left\{ f \in \mathbb{C}_0 : \sup_{t\geq T} \frac{|f(t)|}{1+t^{1+\kappa}} \leq \varepsilon \right\}.$$

Then due to condition **III** there is $T < \infty$ such that

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \ln \mathbf{P}(X_n \notin R_T(\varepsilon)) \leq -N. \tag{3.21}$$

For this T, thanks to condition **I**, processes $X_n^{(T)}$ satisfy LDP in the space $\mathbb{C}[0,T]$. Therefore for a chosen N by the Puhalskii theorem on exponential tightness ([19], Theorem 2.1) there is a compact $K \subset \mathbb{C}[0,T]$ such that

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \ln \mathbf{P}(X_n^{(T)} \notin K) \leq -N.$$

For a given $\varepsilon > 0$ take a finite ε-net $f_1, \cdots, f_M \in \mathbb{C}[0,T]$ in K:

$$K \subset \cup_{i=1}^M (f_i)_{T,\varepsilon}.$$

Then

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \ln \mathbf{P}(X_n^{(T)} \notin \cup_{i=1}^M (f_i)_{T,\varepsilon}) \leq -N. \tag{3.22}$$

Denote for all $i = 1, \cdots, M$

$$g_i(t) := \begin{cases} f_i(t), & \text{if } 0 \leq t \leq T, \\ f_i(T), & \text{if } t \geq T. \end{cases}$$

Define the set $\mathcal{M}_\varepsilon := \{i \in \{1,\cdots,M\} : \frac{|f_i(T)|}{1+T^{1+\kappa}} \le 2\varepsilon\}$. Then

$$P := \mathbf{P}(X_n \notin \cup_{i=1}^M (g_i)_{3\varepsilon}) \le \mathbf{P}(X_n \notin R_T(\varepsilon)) + \mathbf{P}(X_n \in R_T(\varepsilon),\ X_n^{(T)} \notin \cup_{i=1}^M (f_i)_{T,\varepsilon}) +$$
$$\mathbf{P}(X_n \in R_T(\varepsilon),\ X_n^{(T)} \in \cup_{i=1}^M (f_i)_{T,\varepsilon},\ \ X_n \notin \cup_{i=1}^M (g_i)_{3\varepsilon}) =: P_1 + P_2 + P_3.$$

We bound P_3 as follows:

$$P_3 \le \sum_{i\in\mathcal{M}_\varepsilon} \mathbf{P}\left(\sup_{t\ge T} \frac{|X_n(t) - f_i(T)|}{1+t^{1+\kappa}} > 3\varepsilon\right) \le M\mathbf{P}\left(\sup_{t\ge T} \frac{|X_n(t)|}{1+t^{1+\kappa}} > \varepsilon\right) \le M\mathbf{P}(X_n \notin R_T(\varepsilon)).$$

Since

$$P_1 \le \mathbf{P}(X_n \notin R_T(\varepsilon)),$$

we obtain

$$P \le \mathbf{P}(X_n^{(T)} \notin \cup_{i=1}^M (f_i)_{T,\varepsilon}) + (M+1)\mathbf{P}(X_n \notin R_T(\varepsilon)). \tag{3.23}$$

The required inequality in (3.20) now follows from (3.21), (3.22) and (3.23). □

4 Large Deviations for Random Walks

4.1 Large Deviation Principle for Random Walks on half-line.

Let ξ be a non-degenerate random variable satisfying the following condition

$[\mathbf{C}_\infty]$. *For any* $\lambda \in \mathbb{R}$ $\ \mathbf{E}e^{\lambda\xi} < \infty$.

Denote by $A(\lambda) := \ln \mathbf{E}e^{\lambda\xi}$ the log moment generating function, and by $\Lambda(\alpha)$ the Deviation function of ξ (Legendre-Fenchel transform of A)

$$\Lambda(\alpha) := \sup_\lambda\{\lambda\alpha - A(\lambda)\}.$$

Denote $S_0 := 0, \quad S_k := \xi_1 + \cdots + \xi_k \quad$ for $\quad k \ge 1$, where $\{\xi_n\}$ is a sequence of i.i.d. copies of ξ. Consider a random piece-wise linear function $s_n = s_n(t) \in \mathbb{C}$, going through the nodes

$$\left(\frac{k}{n},\ \frac{S_k}{x}\right), \quad k = 0, 1, \cdots,$$

where $x = x(n)$ is a sequence of positive constants such that $x \sim n$, $n \to \infty$.

Theorem 4.1. *Assume* $[\mathbf{C}_\infty]$. *Then the sequence of processes* $s_n(t)$ *satisfies LDP in space* $(\mathbb{C},\ \rho_\kappa)$ *with* $\kappa = 0$ *and rate function* I*,*

$$I(f) := \begin{cases} \int_0^\infty \Lambda(f'(t))dt, & f(0) = 0, \quad f \ \text{ is absolutely continuous}, \\ +\infty, \text{otherwise.} \end{cases}$$

Proof. Without loss of generality we can take $\mathbf{E}\xi = 0$. This is because the Deviation function for $\xi^{(0)} := \xi - a$ is given by $\Lambda^{(0)}(\alpha) = \Lambda(\alpha + a)$. (superscript ${}^{(0)}$ denotes quantities for the centered random variable). Therefore the rate function for $s_n^{(0)}$, is given by $I^{(0)}(f) = I(f + e_a)$, where $e_a = e_a(t) := at;\ t \ge 0$. Clearly, $s_n = s_n^{(0)} + e_a$, $\mathbf{P}(s_n \in B) = \mathbf{P}(s_n^{(0)} \in B - e_a)$, where $B - e_a := \{f - e_a :\ \ f \in B\}$. It is obvious that $[B - e_a] = [B] - e_a, \quad (B - e_a) = (B) - e_a$, implying $I^{(0)}([B - e_a]) = I([B]), \quad I^{(0)}((B - e_a)) = I((B))$. Hence LDP for $s_n^{(0)}$ with rate function $I^{(0)}$ implies LDP for s_n with rate function I.

The rest of the proof consists in checking Conditions $\mathbf{I} - \mathbf{III}$. Condition $\mathbf{I}$ follows from the LDP for s_n in $\mathbb{C}[0, 1]$ (Theorem 9 of [5] or [4], Section 6.2).

Proof of $\mathbf{II}$. By $[\mathbf{C}_\infty]$, with $\mathbf{E}\xi = 0$ it follows that there exists a non-decreasing continuous function $h(t);\ t \ge 0$, such that for some $\delta > 0$, $h(t) = \delta t$, if $0 \le t \le 1$, $\lim_{t\to\infty} h(t) = \infty$, and that for all $\alpha \in \mathbb{R}$ the following inequality holds

$$\Lambda(\alpha) \ge h(|\alpha|)|\alpha|. \tag{4.1}$$

Indeed, for $\alpha \to 0$ (see e.g. [5], p.21) $\Lambda(\alpha) \sim \frac{\alpha^2}{2\sigma^2}$; and for any $\lambda > 0$, $\alpha > 0$

$$\Lambda(\alpha) \geq \lambda\alpha - A(\lambda), \quad \Lambda(-\alpha) \geq \lambda\alpha - A(-\lambda).$$

Therefore $\varliminf\limits_{|\alpha|\to\infty} \frac{\Lambda(\alpha)}{|\alpha|} \geq \lambda$, so that $\varliminf\limits_{|\alpha|\to\infty} \frac{\Lambda(\alpha)}{|\alpha|} = \infty$, and (4.1) follows. Denote by

$$f_T(t) := t\frac{f(T)}{T}, \quad t \in [0,T].$$

The function f_T "straightens" function f on $[0,T]$:

$$I_0^T(f) \geq I_0^T(f_T) = \int_0^T \Lambda\left(\frac{f(T)}{T}\right) dt = T\Lambda\left(\frac{f(T)}{T}\right).$$

Therefore by (4.1) for $f \in B_r$

$$r \geq T\Lambda\left(\frac{f(T)}{T}\right) \geq T\frac{|f(T)|}{T} h\left(\frac{|f(T)|}{T}\right),$$

so that

$$\frac{|f(T)|}{T} \leq \frac{r}{Th\left(\frac{|f(T)|}{T}\right)}. \tag{4.2}$$

Let $c := \sqrt{\frac{r}{\delta}}$, and T be such that $\frac{c}{\sqrt{T}} \leq 1$. Assume that $\frac{|f(T)|}{T} > \frac{c}{\sqrt{T}}$. Then it follows from (4.2)

$$\frac{|f(T)|}{T} \leq \frac{r}{Th(\frac{c}{\sqrt{T}})} = \frac{r}{T\delta\frac{c}{\sqrt{T}}} = \frac{c}{\sqrt{T}},$$

which is a contradiction. Thus for $T \geq c^2 = \frac{r}{\delta}$ it holds

$$|f(T)| \leq \sqrt{\frac{r}{\delta}}\sqrt{T}.$$

Clearly, $B_r = B_r^+$. Therefore we have proved

$$\sup_{f\in B_r^+}\sup_{t\geq T}\frac{|f(t)|}{1+t} \leq \sqrt{\frac{r}{\delta}}\frac{\sqrt{T}}{1+T} \leq \sqrt{\frac{r}{\delta}}\frac{1}{\sqrt{T}}.$$

Condition **II** now follows.

Check now condition **III**. Let $T_n := \max\{\frac{k}{n} \leq T: \ k = 1, 2, \cdots\}$, to have

$$\begin{aligned}
\mathbf{P}\left(\sup_{t\geq T}\frac{|s_n(t)|}{1+t} > \varepsilon\right) &\leq \mathbf{P}\left(\sup_{t\geq T_n}\frac{|s_n(t)|}{1+t} > \varepsilon\right)\\
&\leq \mathbf{P}\left(\sup_{t\geq T_n}\frac{|s_n(t)-s_n(T_n)|}{1+t} > \varepsilon/2\right) + \mathbf{P}\left(\sup_{t\geq T_n}\frac{|s_n(T_n)|}{1+t} > \varepsilon/2\right)\\
&= \mathbf{P}\left(\sup_{u\geq 0}\frac{|s_n(u)|}{1+T_n+u} > \varepsilon/2\right) + \mathbf{P}\left(\sup_{t\geq T_n}\frac{|s_n(T_n)|}{1+T_n} > \varepsilon/2\right)\\
&\leq \mathbf{P}\left(\sup_{k\geq 1}\frac{|s_n(\frac{k}{n})|}{T+\frac{k}{n}} > \varepsilon/2\right) + \mathbf{P}\left(\frac{|s_n(T_n)|}{T_n} > \varepsilon/2\right) =: P_1(n) + P_2(n).
\end{aligned}$$

To bound $P_1(n)$ use the exponential Chebyshev's (Chernoff's) inequality

$$\begin{aligned}
P_1(n) &\leq \sum_{k\geq 1}\mathbf{P}\left(\frac{|S_k|}{x(T+\frac{k}{n})} > \varepsilon/2\right)\\
&\leq \sum_{k\geq 1}\mathbf{P}\left(\frac{S_k}{x(T+\frac{k}{n})} > \varepsilon/2\right) + \sum_{k\geq 1}\mathbf{P}\left(\frac{S_k}{x(T+\frac{k}{n})} < -\varepsilon/2\right)\\
&\leq \sum_{k\geq 1}e^{-k\Lambda(R)} + \sum_{k\geq 1}e^{-k\Lambda(-R)},
\end{aligned}$$

where $R := \frac{x}{k}(T + \frac{k}{n})$. Since for all n and T large enough

$$R \geq \varepsilon/4, \quad kR \geq T\varepsilon/4 + k\varepsilon/4,$$

we have due to (4.1) for $\varepsilon/4 \in (0,1)$

$$k\Lambda(\pm R) \geq kRh(R) \geq (T\varepsilon/4 + k\varepsilon/4)\delta\varepsilon/4 = T\frac{\delta\varepsilon^2}{16} + k\frac{\delta\varepsilon^2}{16}.$$

Therefore

$$P_1(n) \leq 2e^{-T\delta_1} \sum_{k\geq 1} e^{-k\delta_1} = C_1 e^{-T\delta_1},$$

where $\delta_1 := \frac{\delta\varepsilon^2}{16}$, $\quad C_1 := 2\frac{e^{-\delta_1}}{1-e^{-s_1}}$. Similarly we obtain the bound $P_2(n) \leq C_2 e^{-T\delta_2}$ for some $\delta_2 > 0$, $C_2 < \infty$. Condition **III** follows. $\square$

4.2 Moderate Deviation Principle for Random Walks on half-line.

Let $s_n = s_n(\cdot) \in \mathbb{C}$ be defined as in previous section, and let $\mathbf{E}\xi = 0$. Assume Cramer's condition

$[\mathbf{C_0}]$. *For some* $\delta > 0 \quad \mathbf{E}e^{\delta|\xi|} < \infty$.

Let a sequence $x = x(n)$, used in the construction of s_n, satisfy as $n \to \infty$

$$\frac{x}{\sqrt{n}} \to \infty, \quad \frac{x}{n} \to 0.$$

Theorem 4.2. *Let $\mathbf{E}\xi = 0$, $\mathbf{E}\xi^2 =: \sigma^2$, and condition $[\mathrm{C}_0]$ holds. Then the sequence of processes $s_n(t)$ satisfies LDP with speed $\frac{x^2}{n}$ in space $(\mathbb{C},\ \rho_\kappa)$ with $\kappa = 0$, and rate function I_0*

$$I_0(f) := \begin{cases} \frac{1}{2\sigma^2}\int_0^\infty (f'(t))^2 dt, & \text{if} \quad f(0) = 0, \quad f \quad \text{is absolutely continuous} \\ \infty & \text{otherwise}, \end{cases}$$

i.e. for any measurable set $B \subset \mathbb{C}$

$$\overline{\lim_{n\to\infty}} \frac{n}{x^2} \ln \mathbf{P}(s_n \in B) \leq -I_0([B]), \quad \underline{\lim_{n\to\infty}} \frac{n}{x^2} \ln \mathbf{P}(s_n \in B) \geq -I_0((B)).$$

The proof is similar to that of Theorem 4.1 with replacing n by $\frac{x^2}{n}$.

Condition **I** is verified with help of [17] (Theorem 1) or [6] (Theorem 2.2). Condition **II** is obvious. Only Condition **III** requires clarification, which is done by using the following form of Kolmogorov's inequality ([1], p. 295, lemma 11.2.1): for any $x \geq 0$, $y \geq 0$, $n \geq 1$

$$\mathbf{P}(\max_{1\leq m\leq n} |S_m| \geq x + y) \leq \frac{\mathbf{P}(|S_n| \geq x)}{\min\limits_{1\leq m\leq n} \mathbf{P}(|S_m| \leq y)}. \tag{4.3}$$

It is clear that

$$\begin{aligned} \mathbf{P}\left(\sup_{t\geq T} \frac{|s_n(t)|}{1+t} \geq \varepsilon\right) &\leq \sum_{K\geq T} \mathbf{P}\left(\sup_{K\leq t\leq K+1} |s_n(t)| \geq K\varepsilon\right) \\ &\leq \sum_{K\geq T} \mathbf{P}(\sup_{K\leq t\leq K+1} |s_n(t) - s_n(K)| \geq K\varepsilon/2) + \sum_{K\geq T} \mathbf{P}(|s_n(K)| \geq K\varepsilon/2) \\ &\leq \sum_{K\geq T} P_1(K,n) + \sum_{K\geq T} P_2(K,n), \end{aligned} \tag{4.4}$$

where

$$P_1(K,n) := \mathbf{P}(\sup_{0\leq t\leq 1} |s_n(t)| \geq K\varepsilon/2), \text{ and } P_2(K,n) := \mathbf{P}(|s_n(K)| \geq K\varepsilon/2).$$

We bound $P_2(K,n)$ using exponential Chebyshev's (Chernoff) inequality:

$$P_2(K,n) = \mathbf{P}\left(\frac{|S_{nK}|}{nK} \geq \frac{x\varepsilon}{2n}\right) \leq e^{-nK\Lambda(\frac{x\varepsilon}{2n})} + e^{-nK\Lambda(-\frac{x\varepsilon}{2n})}.$$

It follows by (4.1) that

$$\sum_{K\geq T} P_2(K,n) \leq 2\frac{e^{-\frac{x^2}{n}T\delta_1}}{1-e^{-\frac{x^2}{n}\delta_1}}, \tag{4.5}$$

as for n large enough $\frac{x\varepsilon}{2n} \leq 1$, and $nK\Lambda\left(\pm\frac{x\varepsilon}{2n}\right) \geq \frac{x^2}{n}K\delta_1$, with $\delta_1 := \frac{\delta\varepsilon^2}{4}$.

By (4.3)

$$P_1(K,n) = \mathbf{P}\left(\max_{1\leq m\leq n}\frac{|S_m|}{xK} \geq \varepsilon/4 + \varepsilon/4\right) \leq \frac{\mathbf{P}\left(\frac{|S_n|}{xK} \geq \varepsilon/4\right)}{\min_{1\leq m\leq n}\mathbf{P}\left(\frac{|S_m|}{xK} < \varepsilon/4\right)}.$$

Since

$$\min_{1\leq m\leq n}\mathbf{P}\left(\frac{|S_m|}{xK} < \varepsilon/4\right) \geq \min_{1\leq m\leq n}\mathbf{P}\left(\frac{|S_m|}{\sqrt{m}} < \frac{xT}{\sqrt{n}}\varepsilon/4\right) \to 1$$

as $n\to\infty$, for n large enough

$$P_1(K,n) \leq 2\mathbf{P}\left(\frac{|S_n|}{xK} \geq \varepsilon/4\right) \leq 2e^{-n\Lambda(\frac{xK}{n}\varepsilon/4)} + 2e^{-n\Lambda(-\frac{xK}{n}\varepsilon/4)}.$$

It now follows by (4.1)

$$\sum_{K\geq T} P_1(K,n) \leq 4\frac{e^{-\frac{x^2}{n}T\delta_1}}{1-e^{-\frac{x^2}{n}\delta_1}}. \tag{4.6}$$

The desired inequality in **III** follows from (4.5), (4.6) and (4.4): for $T \geq \frac{N}{\delta_1}$

$$\varlimsup_{n\to\infty}\frac{n}{x^2}\ln\mathbf{P}\left(\sup_{t\geq T}\frac{|s_n(t)|}{1+t} \geq \varepsilon\right) \leq -N.$$

References

[1] Borovkov A.A. *Probability Theory.* Moscow, Editorial URSS, 2009, 470 p. (in Russian)

[2] Borovkov A.A. *Asymptotic Analysis of Random Walks. Quikly Decreasing Jumps.* Fizmathlit, Moscow, 2013. (in Russian)

[3] Borovkov A.A. (1967) Boundary problems for random walks and large deviations in function spaces. *Theor. Probability Appl.* **12**, 575-595. MR-0243585

[4] Borovkov A.A. and Mogulskii A.A. (2010) On large deviation principles in metric spaces. *Sibirsk. Mat. Zh.* **51**, 1251-1269. MR-2797595

[5] Borovkov A.A. Mogulskii A.A. Large deviations and testing statistical hypothesis. *Nauka*, Novosibirsk, 1992, 222 pp. (in Russian) MR-1261864

[6] Borovkov A.A. and Mogulskii A.A. (2013) Moderately large deviation principles for trajectories of random walks and processes with independent increments. *Theory Probab. Appl.* **58**, 648-671. (in Russian)

[7] Budkov D.S. and Makhno S.Ya. (2007) Functional iterated logarithm law for a Wiener process. *Theory of Stochastic Processes.* **13**, 22-28. MR-2396061

[8] Delbaen F. and Shirakawa H.. (2002) A note on option pricing for the constant elasticity of variance model. *Asia-Pacific Financial Markets*, **13**, 85-99.

[9] Dembo A. Zeitouni O. Large Deviations Techniques and Applications. *Springer*, 2nd edition, 1998. MR-1619036

[10] Deuschel J.D. and Stroock D.W. Large Deviations. *Academic Press*, Boston 1989. MR-0997938

[11] Dobrushin R.L. and Pecherskij E.A. (1998) Large deviations for random processes with independent increments on infinite intervals. *Probl. Inf. Transm.* **34**, 354-382. MR-1793496

[12] Feng J. and Kurtz T. Large deviations for stochastic processes. *American Mathematical Society*, 2006. MR-2260560

[13] Freidlin M. and Wentzell A. Random Perturbations of Dynamical Systems. *Springer-Verlag*, New York, 1998. MR-1652127

[14] Gihman I.I. and Skorokhod A.V. Stochastic Differential Equations. *Naukova dumka*, Kiev, 1968, 355 pp. (in Russian) MR-0263172

[15] Kolmogorov A.N. and Fomin S.V. Elements of the theory of functions and functional analysis. *Nauka*, Moscow, 1976, 543 pp. (in Russian) MR-0435771

[16] Kulik A.M. and Soboleva D.D. (2012) Large deviations for one-dimensional SDE with discontinuous diffusion coefficient. *Theory of Stochastic Processes* **18**, 101-110. MR-3124766

[17] Mogulskii A.A. (1976) Large Deviations for Trajectories of Multi-Dimensional Random Walks. *Theory Probab. Appl.* **21**, 300-315. MR-0420798

[18] Puhalskii A.A. Large deviations and idempotent probability. *Chapman and Hall* CRC Monographs and Surveys in Pure and Applied Mathematics, 119. Boca Raton, FL, 2001. MR-1851048

[19] Puhalskii A.A. Large deviations for stochastic processes. LMS/EPSRC *Short Course: Stochastic Stability, Large Deviations and Coupling Methods.* Heriot-Watt University, Edinburgh, 4-6 September 2006.

Acknowledgments. The authors are grateful to the Referee for comments and suggestions.