Electron. Commun. Probab. **20** (2015), no. 70, 1–5.
DOI: 10.1214/ECP.v20-4034
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

A Gaussian martingale which is the sum of two independent Gaussian non-semimartingales*

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Abstract

In this paper two examples of two independent centered Gaussian processes are given such that at least one of them is not a semimartingale but their sum is a martingale.

Keywords: Martingales; semimartingales; Gaussian processes; Brownian bridges.
AMS MSC 2010: 60G15; 60G44.
Submitted to ECP on December 31, 2014, final version accepted on September 28, 2015.

1 Certain mixed Fractional Brownian motions are semimartingales

In his thesis, P. Cheridito [1, 2] obtained the following remarkable result: if $(B_t, t \geq 0)$ and $(B_t^{(H)}, t \geq 0)$ denote two independent Gaussian processes, the first one being a Brownian motion, and the second one a fractional Brownian motion with Hurst parameter $H \in]3/4, 1]$, i.e.,

$$E\left[B_t^{(H)}\right] = 0 \quad \text{and} \quad E\left[(B_t^{(H)} - B_s^{(H)})^2\right] = |t-s|^{2H}, \quad s, t \geq 0,$$

then, for every $\alpha \in \mathbb{R}$, the sum:

$$\Sigma_t^{(H)} = B_t + \alpha B_t^{(H)}, \quad t \geq 0,$$

is a semimartingale with respect to its own natural filtration.

Notice that, for $H = 1$, one has: $B_t^{(1)} = t\xi$, where ξ is a standard Gaussian variable, and consequently, $(\sum_t^{(1)}, t \geq 0)$ is a semimartingale in the filtration $\mathcal{B}_t^{(\xi)} := \sigma\{B_s, s \leq t; \xi\}$, made right continuous, hence, a fortiori, with respect to its own filtration. However, for $H \in]3/4, 1[$, $(B_t^{(H)}, t \geq 0)$ has zero quadratic variation, but infinite variation on any time interval, hence it is not a semimartingale with respect to its own filtration, which makes Cheridito's result remarkable.

Note: Throughout the rest of this paper, when we say that a process $(\Pi_t, t \geq 0)$ is a semimartingale with no further qualification, we mean: semimartingale with respect to its own filtration made right continuous and $\mathbb{P}$-complete.

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2 Some related questions

In the light of Cheridito's result, one may ask the following question:

$(*)$ *to give a "simpler" example of a pair of independent centered Gaussian processes,* $(X_t,\, t \geq 0)$ *and* $(Y_t,\, t \geq 0)$, *one of which at least is not a semimartingale, but such that the sum is a semimartingale.*

In Section 3, we shall give an example where $(X_t,\, t \geq 0)$ is constructed from a Brownian bridge, and is not a semimartingale whereas $(Y_t,\, t \geq 0)$ has bounded variation. In Section 4, pushing the construction of Section 3 one step further, we shall give another example of $(*)$, where neither (X_t) nor (Y_t) is a semimartingale. For the moment, we simply note that, in order to obtain some positive answer to $(*)$, at least one of the Gaussian processes (X_t) or (Y_t) must have some non-zero quadratic variation, i.e., $\sum_{\tau_n} (\Delta X_{t_i})^2$ does not converge to 0, where $\tau_n = \{0 = t_0 < t_1 < \cdots < t_{p_n} = 1\}$, $\Delta X_{t_i} = X_{t_i} - X_{t_{i-1}}$, and $\sup_{\tau_n}(t_i - t_{i-1}) \overset{(n\to\infty)}{\longrightarrow} 0$. This assertion follows from the

Lemma 2.1.

(i) *Assume that X and Y are two independent centered Gaussian processes, and τ is a subdivision of $[0,1]$. Then*

$$\begin{aligned}
&\max\left(E\left[\sum_\tau |\Delta X_{t_i}|\right]; E\left[\sum_\tau |\Delta Y_{t_i}|\right]\right)\\
\leq\ & E\left[\sum_\tau |\Delta (X+Y)_{t_i}|\right] \leq E\left[\sum_\tau |\Delta X_{t_i}| + \sum_\tau |\Delta Y_{t_i}|\right].
\end{aligned}$$

(ii) *If both, X and Y, have zero quadratic variation and at least one of them has infinite variation on a set of positive probability, then $X+Y$ also enjoys these two properties.*

Proof. (i) Only the LHS inequality needs to be proven; but this follows from

$$E\left[|\Delta (X+Y)_{t_i}|\right] = \sqrt{\frac{2}{\pi}}\, \|\Delta X_{t_i} + \Delta Y_{t_i}\|_2 \geq \sqrt{\frac{2}{\pi}}\, \|\Delta X_{t_i}\|_2 = E\left[|\Delta X_{t_i}|\right].$$

(ii) It is clear that $X+Y$ has zero quadratic variation. On the other hand, it follows from (i) and our hypothesis in (ii) that

$$E\left[\int_0^1 |d(X_s + Y_s)|\right] = \infty.$$

Now it follows from Fernique's integrability result for the norms of Gaussian vectors that $\int_0^1 |d(X_s+Y_s)|$ cannot be finite a.s. □

3 Brownian bridges and a first solution to $(*)$

Let $u > 0$, and denote by $(\eta_u(t), t \leq u)$ a Brownian bridge of length u, i.e., $(B_t, t \leq u)$ conditioned to be equal to 0 at time u. Recall that it can be realized as $\eta_u(t) = B_t - \frac{t}{u} B_u$, η_u is independent of B_u, and its canonical decomposition is

$$\eta_u(t) = \beta_t - \int_0^t ds \frac{\eta_u(s)}{u-s}, \quad t \leq u, \tag{3.1}$$

where $(\beta_t, t \leq u)$ is a Brownian motion in the filtration $(\mathcal{P}_t^{(u)}, t \leq u)$ of η_u. Furthermore, there is the following

Proposition 3.1. *Let* $f \in L^2([0,u])$. *Then*

(i) *The process*

$$\int_0^t f(s) d\eta_u(s) = \int_0^t f(s) d\beta_s - \int_0^t ds f(s) \frac{\eta_u(s)}{u-s}$$

is well defined for any $t \leq u$ *with*

$$\int_0^u f(s) d\eta_u(s) = (L^2 \text{ and a.s.}) \lim_{t \uparrow u} \int_0^t f(s) d\eta_u(s).$$

(ii) $\left(\int_0^t f(s) d\eta_u(s), t \leq u\right)$ *is a semimartingale with respect to* $(\mathcal{P}_t^{(u)}, t \leq u)$ *if and only if*

$$\int_0^u ds\, |f(s)| \frac{1}{\sqrt{u-s}} < \infty.$$

Proof. (i) The L^2 and a.s. convergence results are easily obtained from the representations of η_u as $\eta_u(t) = B_t - \frac{t}{u} B_u$.

(ii) The semimartingale property of $(\int_0^t f(s) d\eta_u(s), t \leq u)$ is clearly equivalent to

$$\int_0^u ds\, |f(s)| \frac{|\eta_u(s)|}{u-s} < \infty.$$

The arguments developed in the proof of Theorem 3 in Jeulin and Yor [3] show that this is equivalent to

$$\int_0^u ds\, |f(s)| \frac{1}{\sqrt{u-s}} < \infty.$$

□

In order to give explicit examples for $(*)$ in the sequel of this paper, let us point out that for $u \in]0,1]$ and $\alpha \in]1/2, 1]$, the function

$$\psi(s) = \frac{1}{\sqrt{u-s}} |\log(u-s)|^{-\alpha} 1_{(u/2 < s < u)}$$

satisfies

$$\int_0^u ds \psi^2(s) < \infty \quad \text{but} \quad \int_0^u ds \psi(s) \frac{1}{\sqrt{u-s}} = \infty.$$

To obtain a solution to $(*)$, we decompose a Brownian motion $(B_t, t \leq u)$ as

$$B_t = \eta_u(t) + \frac{t}{u} B_u, \quad t \leq u,$$

and we consider $f_* \in L^2([0,u])$ such that

$$\int_0^u ds |f_*(s)| \frac{1}{\sqrt{u-s}} = \infty \quad \text{and} \quad f_*(s) \neq 0 \text{ for every } s.$$

Then, taking

$$X_t = \int_0^t f_*(s) d\eta_u(s) \quad \text{and} \quad Y_t = \frac{B_u}{u} \int_0^t f_*(s) ds,$$

we obtain a solution to $(*)$ since X and Y are independent and $X_t + Y_t = \int_0^t f_*(s) dB_s$ is a martingale.

4 A "full" solution to $(*)$

Let $u \in]0,1[$. We shall use the same idea as in Section 3, but twice instead of once, by decomposing first $(B_t, t \le u)$ into $\eta_u(t) + \frac{t}{u} B_u$, and then

$$(\hat{B}_t \equiv B_{t+u} - B_u,\ t \le 1-u) \quad \text{into} \quad \hat{\eta}_{1-u}(t) + \frac{t}{1-u} \hat{B}_{1-u}. \tag{4.1}$$

Next, for $f \in L^2([0,1])$, we write

$$\begin{aligned} \int_0^t f(s) dB_s &= \int_0^t f(s) 1_{(s \le u)} dB_s + 1_{(u<t)} \int_u^t f(s) dB_s \\ = \quad & \int_0^t f(s) 1_{(s \le u)} d\eta_u(s) + \frac{B_u}{u} \int_0^t f(s) 1_{(s \le u)} ds \\ & + 1_{(u<t)} \int_u^t f(s) d\hat{\eta}_{1-u}(s-u) + 1_{(u<t)} \frac{B_1 - B_u}{1-u} \int_u^t f(s) ds. \end{aligned}$$

We then choose $f_* \in L^2([0,1])$ such that

$$\int_0^u |f_*(s)| \frac{ds}{\sqrt{u-s}} = \infty, \quad \int_u^1 |f_*(s)| \frac{ds}{\sqrt{1-s}} = \infty \quad \text{and} \quad f_*(s) \neq 0 \text{ for all } s < 1.$$

Then

$$X_t = \int_0^t f_*(s) 1_{(s \le u)} d\eta_u(s) + 1_{(u<t)} \frac{B_1 - B_u}{1-u} \int_u^t f_*(s) ds$$

and

$$Y_t = 1_{(u<t)} \int_u^t f_*(s) d\hat{\eta}_{1-u}(s-u) + \frac{B_u}{u} \int_0^t f_*(s) 1_{(s \le u)} ds$$

are two independent Gaussian processes such that $X_t + Y_t = \int_0^t f_*(s) dB_s$ is a martingale. Using the semimartingale characterization in part (ii) of Proposition 3.1, it is easily shown that neither X nor Y is a semimartingale. However, we give a few details:

Concerning (X_t), we see that $X_t = \tilde{X}_t$ for $t \le u$, where $\tilde{X}_t = \int_0^t f_*(s) 1_{(s \le u)} d\eta_u(s)$. Hence the non-semimartingale property of X follows from that of $\tilde{X}$ as discussed in Section 3.

Concerning (Y_t), we have

$$Y_u = \frac{B_u}{u} \int_0^u f_*(s) ds \quad \text{and} \quad Y_t - Y_u = \int_u^t f_*(s) d\hat{\eta}_{1-u}(s-u), \quad t \in [u,1].$$

Now Y, being a Gaussian process, could only be a semimartingale if it were a quasi-martingale; see, e.g., Stricker [4]. If

$$\mathcal{Y}_{u+t} = \sigma\{B_u,\, \hat{\eta}_{1-u}(s),\, s \le t\}$$

and $(\hat{\mathcal{P}}_t^{1-u})$ is the filtration of $\hat{\eta}_{1-u}$, it follows from the independence of B_u and $\hat{\eta}_{1-u}$ that for $s < t$:

$$E[Y_{u+t} - Y_{u+s} \mid \mathcal{Y}_{u+s}] = E[Y_{u+t} - Y_{u+s} \mid \hat{\mathcal{P}}_s^{1-u}].$$

From Section 3 we know that $(Y_t - Y_u)$ is not a $\hat{\mathcal{P}}^{1-u}$-semimartingale. So it is not a $\hat{\mathcal{P}}^{1-u}$-quasimartingale. It follows that (Y_t) is not a $\mathcal{Y}$-quasimartingale and therefore, also not a $\mathcal{Y}$-semimartingale.

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