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A characterization of the Poisson process revisited

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Abstract

We show that the splitting-characterization of the Poisson point process is an immediate consequence of the Mecke-formula.

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1 Notation

Let X be a Polish space, $\mathcal{B}(X)$ resp. $\mathcal{B}_0(X)$ denote the Borel resp. bounded Borel sets. $\mathcal{M}(X)$ is the space of locally finite measures on X, i.e., Radon measures on X, which is Polish for the vague topology. $\mathcal{M}^{\cdot\cdot}(X)$ denotes the closed and thereby measurable subspace of Radon point measures and $\mathcal{M}^{\cdot}(X)$ denotes the measurable subspace of simple Radon point measures. A law $P \in \mathcal{P}(\mathcal{M}^{\cdot\cdot}(X))$ on $\mathcal{M}^{\cdot\cdot}(X)$ resp. $\mathcal{M}^{\cdot}(X)$ is called point process resp. simple point process. The first moment measure of a point process P will be denoted by

$$\nu_P(B) = \int_{\mathcal{M}^{\cdot\cdot}(X)} P(\mathrm{d}\,\mu)\,\mu(B), \quad B \in \mathcal{B}(X).$$

We say that a point process P is of first order if $\nu_P \in \mathcal{M}(X)$. By U we denote the set of non negative measurable test functions on X with a compact support. Remark that for $f \in U$, $\zeta_f : \mu \mapsto \mu(f)$ is a well defined measurable function on $\mathcal{M}^{\cdot\cdot}(X)$ and let $\mathcal{L}_P(f) = P(e^{-\zeta_f})$, $f \in U$, be the Laplace transform of a point process P. Furthermore we let $\Gamma_q(P)$ be the independent q-thinning of a point process P, where $q \in (0,1)$ is some fixed constant, denoting the survival probability. That is

$$\Gamma_q(P) = \int_{\mathcal{M}^{\cdot\cdot}(X)} P(\mathrm{d}\,\mu) \mathop{*}_{x\in\mu} ((1-q)\,\delta_{\mathbf{0}} + q\,\delta_{\delta_x}).$$

Here $*$ denotes ordinary convolution and $\mathbf{0}$ is the zero measure on X.

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2 Introduction

Having observed a realization $\nu \in \mathcal{M}^{\cdot\cdot}(X)$ of an independent q-thinning of a point process P one can ask the following question: What is the distribution of deleted point configurations given the realization $\nu \in \mathcal{M}^{\cdot\cdot}(X)$? This conditional probability will be called splitting kernel and will be denoted by $\Upsilon_q^\nu(P)$ in the sequel (see also section 6.3 in [7]). In case P is a finite point process, that is P is concentrated on the set of finite Radon point measures, Karr obtained in [4] a representation of the splitting kernel in terms of the reduced Palm distributions $P^!_{\delta_{x_1}+\ldots+\delta_{x_n}}$, $x_1, \ldots, x_n \in X$, $n \geq 1$, of P. That is

$$\Upsilon_q^\nu(P)(\varphi) = \frac{1}{\int (1-q)^{\mu(X)} P^!_\nu(\mathrm{d}\,\mu)} \int \varphi(\mu)\,(1-q)^{\mu(X)} P^!_\nu(\mathrm{d}\,\mu),$$

where $\varphi \in F_+$ is some non negative measurable test function on the space of finite point measures and ν denotes a finite point measure (see also proposition 6.3.5 in [7]). To obtain a representation for $\Upsilon_q^\nu(P)$ in case P is a general point process seems to be an open problem.

In this note we want to prove the following: Assume you have a point process P such that $\Upsilon_q^\nu(P)$ does not depend on the observed point configuration $\nu \in \mathcal{M}^{\cdot\cdot}(X)$. That is, there is some point process Q_q such that $\Upsilon_q^\nu(P) = Q_q$ for all $\nu \in \mathcal{M}^{\cdot\cdot}(X)$. Then P can only be a Poisson point process. This result is a corollary of Fichtner's main theorem (Satz 1) in [3]. Fichtner's arguments were quite involved so Assunção and Ferrari [1] gave a simpler proof of the result (in the present setting) using a characterization of the Poisson distribution and the fact that a simple point process is determined by its avoidance function. Note that in [1] the result is stated for general point processes (meaning elements of $\mathcal{P}(\mathcal{M}^{\cdot\cdot}(X))$) but i.e. Brown and Xia have shown in [2] that one can in general not conclude that the point process is Poisson if its counting variables ζ_B, $B \in \mathcal{B}_0(X)$, are Poisson distributed. So in the present setting we have to resort to different techniques. The most important one will be Mecke's characterization of the Poisson point process (Satz 3.1 in [5]).
Let us introduce the notion of a Papangelou kernel (also sometimes called conditional intensity) π of a point process P. π is a kernel from $\mathcal{M}^{\cdot\cdot}(X)$ to $\mathcal{M}(X)$, that is for any $\mu \in \mathcal{M}^{\cdot\cdot}(X)$ we have $\pi(\mu, \mathrm{d}\,x) \in \mathcal{M}(X)$, so that P satisfies the equation

$$C_P(h) := \int\limits_{\mathcal{M}^{\cdot\cdot}(X)} \int\limits_X h(x,\mu)\,\mu(\mathrm{d}\,x) P(\mathrm{d}\,\mu) = \int\limits_{\mathcal{M}^{\cdot\cdot}(X)} \int\limits_X h(x, \mu+\delta_x)\,\pi(\mu, \mathrm{d}\,x) P(\mathrm{d}\,\mu),$$

for all non negative measurable test functions h on the product space $X \times \mathcal{M}^{\cdot\cdot}(X)$. In the first equation the definition of the Campbell measure of P is provided.
One direction of Mecke's result can now be formulated as follows: Assume P is a point process whose Papangelou kernel $\pi(\mu, \mathrm{d}\,x)$ does not depend on $\mu \in \mathcal{M}^{\cdot\cdot}(X)$, that is, there is some $\varrho \in \mathcal{M}(X)$ such that $\pi(\mu, \cdot) = \varrho$ for all $\mu \in \mathcal{M}^{\cdot\cdot}(X)$ then P is the Poisson point process with first moment measure given by ϱ.

Let us introduce for a given $\mu \in \mathcal{M}^{\cdot\cdot}(X)$ and $q \in (0,1)$ the point process

$$T_q^\mu = \mathop{*}_{x \in \mu} ((1-q)\,\delta_{\mathbf{0}} + q\,\delta_{\delta_x}).$$

So T_q^μ describes the deletion operation of the point realization $\mu \in \mathcal{M}^{\cdot\cdot}(X)$. Furthermore we need the so called splitting law $S_q(P)$ of a point process P

$$S_q(P)(h) = \int\limits_{\mathcal{M}^{\cdot\cdot}(X)} \int\limits_{\mathcal{M}^{\cdot\cdot}(X)} P(\mathrm{d}\,\mu) T_q^\mu(\mathrm{d}\,\nu)\, h(\nu, \mu-\nu),$$

where h is some non negative measurable test function on $\mathcal{M}^{\cdot\cdot}(X) \times \mathcal{M}^{\cdot\cdot}(X)$. So $S_q(P)$ is a law on $\mathcal{M}^{\cdot\cdot}(X) \times \mathcal{M}^{\cdot\cdot}(X)$, which realizes tuples (ν, η) such that ν is the point configuration which survived the thinning and η is the collection of deleted points. The marginal laws of $S_q(P)$ are given by

$$S_q(P)(\varphi \otimes \mathbf{1}) = \Gamma_q(P)(\varphi) \text{ and } S_q(P)(\mathbf{1} \otimes \varphi) = \Gamma_{1-q}(P)(\varphi),$$

where φ is a non negative test function on $\mathcal{M}^{\cdot\cdot}(X)$ and $\mathbf{1}$ denotes the function, which is constantly one. Thus for any $N \in \mathcal{B}(\mathcal{M}^{\cdot\cdot}(X))$ we have that $S_q(P)(\cdot \times N)$ is absolutely continuous to $\Gamma_q(P)$. Thus by the theory of disintegration we obtain the existence of the splitting kernel $\Upsilon_q^\nu(P)$, that is

$$S_q(P)(\mathrm{d}\,\nu\,\mathrm{d}\,\eta) = \Gamma_q(P)(\mathrm{d}\,\nu)\,\Upsilon_q^\nu(P)(\mathrm{d}\,\eta).$$

The last tool we will need is the Pólya difference process $P_{z,\mu}^-$, where $z \in (0, +\infty)$ and $\mu \in \mathcal{M}^{\cdot\cdot}(X)$ as introduced in [8]. It is a point process with independent increments (meaning ζ_{B_1} and ζ_{B_2} are independent under $P_{z,\mu}^-$ for $B_1, B_2 \in \mathcal{B}_0(X)$, $B_1 \cap B_2 = \emptyset$) and the counting variables ζ_B for $B \in \mathcal{B}_0(X)$ are Binomial distributed:

$$P_{z,\mu}^-\{\zeta_B = k\} = \binom{\mu(B)}{k} \left(\frac{z}{1+z}\right)^k \left(\frac{1}{1+z}\right)^{\mu(B)-k}, \quad k \in \mathbb{N}_0.$$

By definition T_q^μ has also independent increments and the distribution of the counting variables ζ_B for $B \in \mathcal{B}_0(X)$ are also Binomial distributed:

$$T_q^\mu\{\zeta_B = k\} = \binom{\mu(B)}{k} q^k\,(1-q)^{\mu(B)-k}, \quad k \in \mathbb{N}_0.$$

So we have $T_q^\mu = P^-_{\frac{q}{1-q},\mu}$ for $q \in (0,1)$ and $\mu \in \mathcal{M}^{\cdot\cdot}(X)$. In [8] Zessin and Nehring established that $P_{z,\mu}^-$ for $z \in (0, +\infty)$ and $\mu \in \mathcal{M}^{\cdot\cdot}(X)$ has a Papangelou kernel π given by

$$\pi(\kappa, \mathrm{d}\,x) = z\,(\mu - \kappa)(\mathrm{d}\,x).$$

Remark 1. *For all $q \in (0,1)$ and $\mu \in \mathcal{M}^{\cdot\cdot}(X)$, T_q^μ has a Papangelou kernel given by*

$$\pi(\kappa, \mathrm{d}\,x) = \frac{q}{1-q}\,(\mu - \kappa)(\mathrm{d}\,x), \quad \kappa \in \mathcal{M}^{\cdot\cdot}(X).$$

Note that T_q^μ realizes only sub configurations of μ so $\mu - \kappa$ is T_q^μ - a.s. $[\kappa]$ in $\mathcal{M}^{\cdot\cdot}(X)$.

3 A Characterization

We are now ready to state the result.

Theorem 1. *Let P be a point process of first order. Then P is a Poisson point process if and only if the splitting law factorizes into its marginals, that is*

$$(\mathcal{F}) \qquad S_q(P) = \Gamma_q(P) \otimes \Gamma_{1-q}(P),$$

for some $q \in (0,1)$.

Proof. Let us denote by Π_ϱ the Poisson point process with first moment measure $\varrho \in \mathcal{M}(X)$. First assume $P = \Pi_\varrho$. To establish the identity $(\mathcal{F})$ it suffices to show the equality of both sides on the class of test functions $h = e^{-\zeta_f} \otimes e^{-\zeta_g}$, where $f, g \in U$. We have

$$S_q(\Pi_\varrho)(h) = \int \Pi_\varrho(\mathrm{d}\,\mu)\, e^{-\mu(g)}\, T_q^\mu(e^{\zeta_{g-f}}).$$

Note that by definition of T_q^μ

$$T_q^\mu(e^{\zeta_{g-f}}) = \prod_{x\in\mu}(1-q+qe^{(g-f)(x)}) = \exp(\mu(\log(1-q+q\,e^{g-f})))$$

and remark that only finitely many factors in the above product are different from one. So we obtain $S_q(\Pi_\varrho)(h) = \mathcal{L}_{\Pi_\varrho}(v)$, where $v = g - \log(1-q+q\,e^{g-f})$. One straightforwardly checks that $v \in U$. In fact $v = -\log((1-q)e^{-\zeta_g} + qe^{-\zeta_f})$. By using the representation of the Laplace transform of Π_ϱ one obtains

$$\mathcal{L}_{\Pi_\varrho}(v) = \mathcal{L}_{\Pi_{q\varrho}}(f)\, \mathcal{L}_{\Pi_{(1-q)\varrho}}(g),$$

which establishes the identity $(\mathcal{F})$, since it is well known that $\Gamma_s(\Pi_\varrho) = \Pi_{s\varrho}$ for any $s \in (0,1)$.
Assume now from the contrary that P solves $(\mathcal{F})$ for some $q \in (0,1)$. Let us compute the Campbell measure of $\Gamma_q(P)$. Take h to be a non negative measurable test function on $X \times \mathcal{M}^{\cdot\cdot}(X)$ then we have

$$\begin{aligned}
C_{\Gamma_q(P)}(h) &\overset{(i)}{=} \int P(\mathrm{d}\,\mu) C_{T_q^\mu}(h)\\
&\overset{(ii)}{=} \int P(\mathrm{d}\,\mu) T_q^\mu(\mathrm{d}\,\kappa)\, \frac{q}{1-q}(\mu-\kappa)(\mathrm{d}\,x)\, h(x, \kappa+\delta_x)\\
&\overset{(iii)}{=} \int S_q(P)(\mathrm{d}\,\kappa\,\mathrm{d}\,\eta)\, \frac{q}{1-q}\eta(\mathrm{d}\,x)\, h(x, \kappa+\delta_x)\\
&\overset{(iv)}{=} \int \Gamma_q(P)(\mathrm{d}\,\kappa)\Gamma_{1-q}(P)(\mathrm{d}\,\eta)\, \frac{q}{1-q}\eta(\mathrm{d}\,x)\, h(x, \kappa+\delta_x)\\
&\overset{(v)}{=} \int \Gamma_q(P)(\mathrm{d}\,\kappa) \frac{q}{1-q}\nu_{\Gamma_{1-q}(P)}(\mathrm{d}\,x)\, h(x, \kappa+\delta_x)\\
&\overset{(vi)}{=} \int \Gamma_q(P)(\mathrm{d}\,\kappa)\, q\,\nu_P(\mathrm{d}\,x)\, h(x, \kappa+\delta_x).
\end{aligned}$$

(i) follows by definition of $\Gamma_q(P)$. (ii) is due to remark 1 in the introductory section. (iii) follows by definition of the splitting law $S_q(P)$. Since P is assumed to satisfy $(\mathcal{F})$ (iv) holds. In (v) the definition of the first moment measure has been used. Finally (vi) holds true because $\nu_{\Gamma_{1-q}(P)} = (1-q)\,\nu_P$. So by Mecke's characterization (Satz 3.1 in [5]) it follows that $\Gamma_q(P) = \Pi_{q\,\nu_P}$. Lemma 9 in [6] states that $\Gamma_q : \mathcal{P}(\mathcal{M}^{\cdot\cdot}(X)) \to \mathcal{P}(\mathcal{M}^{\cdot\cdot}(X))$ is an injective mapping. Therefore $P = \Pi_{\nu_P}$.
□

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