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A spectral decomposition for the block counting process of the Bolthausen–Sznitman coalescent

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Abstract

A spectral decomposition for the generator and the transition probabilities of the block counting process of the Bolthausen–Sznitman coalescent is derived. This decomposition is closely related to the Stirling numbers of the first and second kind. The proof is based on generating functions and exploits a certain factorization property of the Bolthausen–Sznitman coalescent. As an application we derive a formula for the hitting probability $h(i,j)$ that the block counting process of the Bolthausen–Sznitman coalescent ever visits state j when started from state $i \geq j$. Moreover, explicit formulas are derived for the moments and the distribution function of the absorption time τ_n of the Bolthausen–Sznitman coalescent started in a partition with n blocks. We provide an elementary proof for the well known convergence of $\tau_n - \log\log n$ in distribution to the standard Gumbel distribution. It is shown that the speed of this convergence is of order $1/\log n$.

Keywords: absorption time; Bolthausen–Sznitman coalescent; Green matrix; hitting probabilities; spectral decomposition; Stirling numbers.
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1 Introduction and results

The Bolthausen–Sznitman coalescent [2] is the particular Λ-coalescent $\Pi = (\Pi_t)_{t\geq 0}$ with Λ being the uniform distribution on the unit interval $[0,1]$. Λ-coalescents are continuous time Markovian processes with state space $\mathcal{P}$, the set of partitions of $\mathbb{N} := \{1,2,\ldots\}$. During each transition blocks merge to form a single block. For more information on Λ-coalescents we refer the reader to [14] and [15]. For $t \in [0,\infty)$ let N_t denote the number of blocks of Π_t. It is well known that $(N_t)_{t\geq 0}$ is a Markovian process, called the block counting process of Π. Let $Q = (q_{ij})_{i,j\in\mathbb{N}}$ denote the generator of $(N_t)_{t\geq 0}$. It is well known that Q is a lower left triangular matrix with entries

$$q_{ij} \;=\; \frac{i}{(i-j)(i-j+1)}, \qquad \text{for } i > j, \tag{1.1}$$

$q_{ii} = -\sum_{j=1}^{i-1} q_{ij} = 1-i$ and $q_{ij} = 0$ for $i < j$. The quantities $q_i := -q_{ii} = i-1$, $i \in \mathbb{N}$, are (called) the total rates of the Bolthausen–Sznitman coalescent.

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In the following $s(i,j)$ and $S(i,j)$, $i,j\in\mathbb{N}_0:=\{0,1,2,\ldots\}$, denote the Stirling numbers of the first and second kind respectively. Note that $|s(i,j)|=(-1)^{i-j}s(i,j)$ is the number of permutations of $\{1,\ldots,i\}$ with j cycles, whereas $S(i,j)$ is the number of partitions of $\{1,\ldots,i\}$ into j nonempty subsets.

The main result (Theorem 1.1) provides a spectral decomposition for Q, which is closely related to the Stirling numbers. The proof of Theorem 1.1 is given in Section 2.

Theorem 1.1. (Spectral decomposition of the generator) *The infinitesimal generator $Q=(q_{ij})_{i,j\in\mathbb{N}}$ of the block counting process of the Bolthausen–Sznitman coalescent has spectral decomposition $Q=RDL$, where $D=(d_{ij})_{i,j\in\mathbb{N}}$ is the diagonal matrix with entries $d_{ij}=1-i$ for $i=j$ and $d_{ij}=0$ for $i\neq j$, and $R=(r_{ij})_{i,j\in\mathbb{N}}$ and $L=(l_{ij})_{i,j\in\mathbb{N}}$ are lower left triangular matrices with entries*

$$r_{ij} \;=\; \frac{(j-1)!}{(i-1)!}|s(i,j)| \qquad \text{and} \qquad l_{ij} \;=\; (-1)^{i+j}\frac{(j-1)!}{(i-1)!}S(i,j), \qquad i,j\in\mathbb{N}. \tag{1.2}$$

In particular, $r_{i1}=r_{ii}=l_{ii}=1$ and $l_{i1}=(-1)^{i-1}/(i-1)!$, $i\in\mathbb{N}$.

Remark 1.2. *For alternative formulas for r_{ij} and l_{ij} we refer the reader to (2.12), (2.13) and (2.15). The first entries of R and L are provided in (2.16).*

The next corollary provides the spectral decomposition of the transition matrix of the block counting process $(N_t)_{t\geq 0}$ of the Bolthausen–Sznitman coalescent.

Corollary 1.3. (Spectral decomposition of the transition matrix) *For all $t\in[0,\infty)$ the transition matrix $P(t):=(\mathbb{P}(N_t=j\,|\,N_0=i))_{i,j\in\mathbb{N}}$ of the block counting process $(N_t)_{t\geq 0}$ of the Bolthausen–Sznitman coalescent has spectral decomposition $P(t)=Re^{tD}L$, i.e.*

$$p_{ij}(t) \;:=\; \mathbb{P}(N_t=j\,|\,N_0=i) \;=\; \sum_{k=j}^{i} e^{-(k-1)t}r_{ik}l_{kj} \;=\; \frac{(j-1)!}{(i-1)!}\sum_{k=j}^{i} e^{-(k-1)t}|s(i,k)|S(k,j), \tag{1.3}$$

$i,j\in\mathbb{N}$, where D is the diagonal matrix defined in Theorem 1.1 and $R=(r_{ij})_{i,j\in\mathbb{N}}$ and $L=(l_{ij})_{i,j\in\mathbb{N}}$ are defined via (1.2).

Remark 1.4. *As a consequence of Corollary 1.3, the block counting process has resolvent (see, for example, Norris [13, p. 146]) $R_\lambda:=\int_0^\infty e^{-\lambda t}P(t)\,\mathrm{d}t=\int_0^\infty e^{-\lambda t}Re^{tD}L\,\mathrm{d}t=R(\int_0^\infty e^{-\lambda t}e^{tD}\,\mathrm{d}t)L=RD(\lambda)L$, $\lambda\in(0,\infty)$, where $D(\lambda)$ is the diagonal matrix with entries $d_{ii}(\lambda)=1/(\lambda+i-1)$, $i\in\mathbb{N}$.*

As an application we provide in the following a formula for the probability that the block counting process ever visits state $j\in\mathbb{N}$ when started from state $i\in\mathbb{N}$ with $i\geq j$.

Corollary 1.5. (Hitting probabilities) *The probability $h(i,j):=\mathbb{P}(N_t=j\text{ for some }t\geq 0\,|\,N_0=i)$ that the block counting process hits state $j\in\mathbb{N}$ started from state $i\in\mathbb{N}$ with $i\geq j$ is given by $h(i,1)=1$ and*

$$h(i,j) \;=\; (j-1)(-1)^{i+j}\frac{(j-1)!}{(i-1)!}\sum_{k=j}^{i}\frac{s(i,k)S(k,j)}{k-1}, \qquad 2\leq j\leq i.$$

Remark 1.6. *Further formulas for $h(i,j)$ are provided in [10, Theorem 2.1 with $\alpha=1$] and [11, Eq. (11)]. For the asymptotics of $h(i,j)$ as $i\to\infty$ we refer the reader to [7, Corollary 3.4] and [11, Theorem 1.1 and Corollary 1.3].*

As a further application we provide in the following formulas for the moments, the distribution function and the Laplace transform of the absorption time τ_n until the Bolthausen–Sznitman coalescent, started in a partition with $n \in \mathbb{N}$ blocks, reaches its absorbing state. In the biological context τ_n is called the time back to the most recent common ancestor of a sample of size n.

Corollary 1.7. (Moments of the absorption time) *The absorption time τ_n of the Bolthausen–Sznitman coalescent has moments*

$$\mathbb{E}(\tau_n^j) \;=\; \frac{j!}{(n-1)!}\sum_{k=2}^{n}(-1)^k\frac{|s(n,k)|}{(k-1)^j}, \qquad n,j\in\mathbb{N}. \tag{1.4}$$

Remark 1.8. *Note that (1.4) is simpler than the formula in [4, Proposition 1.1]. Some concrete values of the first and second moment of τ_n are listed in [4, p. 403, Table 1]. For the asymptotics of $\mathbb{E}(\tau_n^j)$ as $n\to\infty$ we refer the reader to [4, Theorem 1.1].*

Corollary 1.9. (Distribution and asymptotics of the absorption time) *The absorption time τ_n of the Bolthausen–Sznitman coalescent started in a partition with n blocks has distribution function*

$$\mathbb{P}(\tau_n\le t) \;=\; \prod_{j=1}^{n-1}\frac{j-e^{-t}}{j} \;=\; \binom{n-1-e^{-t}}{n-1} \;=\; \frac{\Gamma(n-e^{-t})}{\Gamma(n)\Gamma(1-e^{-t})}, \quad n\in\mathbb{N}, t\in(0,\infty) \tag{1.5}$$

and Laplace transform

$$\mathbb{E}(e^{-\lambda\tau_n}) \;=\; 1+\frac{(-1)^{n-1}}{(n-1)!}\sum_{k=2}^{n}s(n,k)\frac{\lambda}{k-1+\lambda}, \qquad n\in\mathbb{N}, \lambda\in[0,\infty). \tag{1.6}$$

In particular, $\tau_n-\log\log n\to\tau$ in distribution as $n\to\infty$, where τ is standard Gumbel distributed with distribution function $F(t):=e^{-e^{-t}}$, $t\in\mathbb{R}$.

Remark 1.10. *1. The first equation in (1.5) is implicitly contained in [14, Theorem 14, Eq. (17) with $k=1$]. The convergence in distribution of $\tau_n-\log\log n$ to the standard Gumbel distribution is well known from the literature (see, for example, [6, Proposition 3.4] or [4, Corollary 1.2]). Our alternative proof of this convergence is based on the last expression in (1.5) and, hence, rather elementary and follows by a straightforward application of Stirling's formula for $\Gamma(x)$ as $x\to\infty$.*

2. Let $\Psi(x) := (\mathrm{d}/\mathrm{d}x)(\log\Gamma(x)) = \Gamma'(x)/\Gamma(x)$ denote the digamma function (logarithmic derivative of the Gamma function). Taking the derivative with respect to t in (1.5) it is readily seen that for $n\ge 2$ the absorption time τ_n has density

$$\begin{aligned} f_{\tau_n}(t) &= \frac{e^{-t}\Gamma(n-e^{-t})}{\Gamma(n)\Gamma(1-e^{-t})}\big(\Psi(n-e^{-t})-\Psi(1-e^{-t})\big)\\ &= e^{-t}\mathbb{P}(\tau_n\le t)\sum_{k=1}^{n-1}\frac{1}{k-e^{-t}}, \qquad n\ge 2, t\in(0,\infty). \end{aligned}$$

Note that, for $n\ge 2$, $\mathbb{P}(\tau_n\le t)\sim t/(n-1)$ as $t\searrow 0$ and, hence, $\lim_{t\searrow 0} f_{\tau_n}(t)=1/(n-1)$.

Moreover, straightforward calculations show that $f_{\tau_n}(t+\log\log n)\to f(t)$ as $n\to\infty$, where f is the density of the standard Gumbel distribution, i.e. $f(t):=F'(t)=e^{-t}F(t)$ for all $t\in\mathbb{R}$. Note that this local convergence holds uniformly in $t\in\mathbb{R}$ due to the inversion formula for densities, so we have $\lim_{n\to\infty}\sup_{t\in\mathbb{R}}|f_{\tau_n}(t+\log\log n)-f(t)|=0$.

3. The fact that the distribution function (1.5) of the absorption time τ_n is known explicitly can be further exploited. For example, it is readily checked that for all $x\in\mathbb{R}$, $\mathbb{P}(\tau_n-\log\log n\le x)-F(x)\sim -\gamma e^{-x}F(x)/\log n$ as $n\to\infty$, where $\gamma\approx 0.577216$ denotes Euler's constant. Thus, the speed of the convergence of $\tau_n-\log\log n$ to the Gumbel distribution is of order $1/\log n$.

Final remark. For the Kingman coalescent the spectral decomposition $Q = RDL$ is well known (see Appendix). For the star-shaped coalescent the spectral decomposition $Q = RDL$ can be readily calculated and is omitted here. Finding the spectral decomposition of the generator Q of the block counting process for other exchangeable coalescents, for example for the Λ-coalescent with $\Lambda = \beta(2-\alpha,\alpha)$ being the beta-distribution with parameters $2-\alpha$ and α, $\alpha \in (0,2)\setminus\{1\}$, is an open problem. Note that the spectral decomposition $Q = RDL$ exists for any exchangeable coalescent, since the generator Q is lower left triangular. More precisely, R and L are recursively defined via the first equation in (2.3) and (2.4) respectively, where q_{ij} and q_i are the rates and total rates of the exchangeable coalescent.

2 Proofs

Before Theorem 1.1 will be proven, we mention two well known recursions for the Stirling numbers. These recursions are provided here, since they will be used in the proof of Theorem 1.1.

Lemma 2.1. *The absolute Stirling numbers of the first kind satisfy the recursion*

$$|s(i,j)| \;=\; (i-1)!\sum_{k=j-1}^{i-1}\frac{|s(k,j-1)|}{k!}, \qquad i,j\in\mathbb{N}, \tag{2.1}$$

whereas the Stirling numbers of the second kind satisfy the recursion

$$S(i,j) \;=\; \sum_{k=j-1}^{i-1}\binom{i-1}{k}S(k,j-1), \qquad i,j\in\mathbb{N}. \tag{2.2}$$

Proof. Let us verify (2.1) by induction on i. Obviously, (2.1) holds for $i=1$, since $|s(1,j)| = \delta_{j1} = |s(0,j-1)|$ for all $j\in\mathbb{N}$. The induction from i to $i+1$ works as follows. We have, by induction,

$$\begin{aligned} i!\sum_{k=j-1}^{i}\frac{|s(k,j-1)|}{k!} &= i!\sum_{k=j-1}^{i-1}\frac{|s(k,j-1)|}{k!} + |s(i,j-1)| \\ &= i!\frac{|s(i,j)|}{(i-1)!} + |s(i,j-1)| \;=\; i|s(i,j)| + |s(i,j-1)| \;=\; |s(i+1,j)|, \end{aligned}$$

by the well known recursion $|s(i+1,j)| = i|s(i,j)|+|s(i,j-1)|$. The induction is complete.

We verify (2.2) as follows. There are $\binom{i-1}{k}$ possibilities to choose a subset A of size k from $\{1,\ldots,i-1\}$ and there are $S(k,j-1)$ possibilities to partition this subset A into $j-1$ nonempty subsets. Together with the nonempty set $(\{1,\ldots,i-1\}\setminus A)\cup\{i\}$ one obtains, after summing over all possible values of k, the number $S(i,j)$ of partitions of $\{1,\ldots,i\}$ into j nonempty subsets. □

Let us now turn to the proof of the spectral decomposition $Q = RDL$ (see Theorem 1.1). The following proof is based on generating functions and has the advantage that the solution (1.2) for R and L does not need to be known in advance in order to perform the proof. The solution (1.2) pops up naturally during the proof as the coefficients of appropriately chosen generating functions. Crucial for the proof is the factorization formula (2.5), which essentially goes back to the particular form of the rates q_{ij} in (1.1).

Proof. (of Theorem 1.1) Let $D = (d_{ij})_{i,j\in\mathbb{N}}$ be the diagonal matrix with entries $d_{ii} := q_{ii} = -q_i = 1-i$, $i\in\mathbb{N}$, and $d_{ij} := 0$ for $i\neq j$. Furthermore, let $R = (r_{ij})_{i,j\in\mathbb{N}}$ be the lower left triangular matrix with entries recursively defined for each $j \in \mathbb{N}$ via $r_{jj} := 1$ and

$$r_{ij} \;:=\; \frac{1}{q_i - q_j}\sum_{k=j}^{i-1} q_{ik} r_{kj} \;=\; \frac{1}{i-j}\sum_{k=j}^{i-1} q_{ik} r_{kj}, \qquad i \in \{j+1, j+2, \ldots\}. \tag{2.3}$$

Since $q_{ii} = -q_i$, $i \in \mathbb{N}$, we conclude that $r_{ij} q_{jj} = \sum_{k=j}^{i} q_{ik} r_{kj}$. Thus, the entries r_{ij} of R are defined such that $RD = QR$. Define $L := R^{-1}$. Then, the spectral decomposition $Q = RDL$ holds. Moreover, $DL = LQ$ and, hence, $q_{ii} l_{ij} = \sum_{k=j}^{i} l_{ik} q_{kj}$, $i, j \in \mathbb{N}$. Since $q_{ii} = -q_i$, $i \in \mathbb{N}$, we obtain for each $i \in \mathbb{N}$ the backward recursion $l_{ii} = 1$ and

$$l_{ij} \;=\; \frac{1}{q_j - q_i}\sum_{k=j+1}^{i} l_{ik} q_{kj} \;=\; \frac{1}{j-i}\sum_{k=j+1}^{i} l_{ik} q_{kj}, \qquad j = i-1, i-2, \ldots, 2, 1. \tag{2.4}$$

Let us verify by induction on i ($\geq j$) that

$$r_{ij} \;\leq\; \prod_{k=j+1}^{i} \frac{q_k}{q_k - q_j} \;=\; \prod_{k=j+1}^{i} \frac{k-1}{k-j} \;=\; \binom{i-1}{j-1}, \qquad i, j \in \mathbb{N}, i \geq j,$$

with the convention that empty products are equal to 1. For $i = j$ this inequality obviously holds, since $r_{jj} = 1$. The induction step from $\{j, \ldots, i-1\}$ to i ($> j$) works as follows. By (2.3) and by induction,

$$\begin{aligned} r_{ij} \;&=\; \frac{1}{q_i - q_j}\sum_{k=j}^{i-1} q_{ik} r_{kj} \;\leq\; \frac{1}{q_i - q_j}\sum_{k=j}^{i-1} q_{ik} \prod_{l=j+1}^{k} \frac{q_l}{q_l - q_j} \\ &\leq\; \frac{1}{q_i - q_j}\sum_{k=j}^{i-1} q_{ik} \prod_{l=j+1}^{i-1} \frac{q_l}{q_l - q_j} \;\leq\; \frac{1}{q_i - q_j} q_i \prod_{l=j+1}^{i-1} \frac{q_l}{q_l - q_j} \;=\; \prod_{l=j+1}^{i} \frac{q_l}{q_l - q_j}, \end{aligned}$$

which completes the induction.

In order to solve the recursion (2.3) we exploit generating function techniques similar to those used in [4], [10] and [11]. Let $U := \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disc. For $j \in \mathbb{N}$ define the generating function $r_j : U \to \mathbb{C}$ via $r_j(z) := \sum_{i=j}^{\infty} r_{ij} z^i$, $z \in U$. Note that, for $j \in \mathbb{N}$ and $z \in U$, $|r_j(z)| \leq \sum_{i=j}^{\infty} r_{ij} |z|^i \leq \sum_{i=j}^{\infty} \binom{i-1}{j-1} |z|^j = |z|^j / (1 - |z|)^j < \infty$. Thus, for each $j \in \mathbb{N}$, the function $r_j : U \to \mathbb{C}$ is well defined. Consider the modified function $f_j : U \to \mathbb{C}$ defined via $f_j(z) := \sum_{i=j}^{\infty} (i-j) i^{-1} r_{ij} z^i$, $z \in U$. Clearly, $|f_j(z)| \leq \sum_{i=j}^{\infty} r_{ij} |z|^i = r_j(|z|) < \infty$, so $f_j : U \to \mathbb{C}$ is well defined. We have

$$f_j(z) \;=\; \sum_{i=j}^{\infty} r_{ij} z^i - j \sum_{i=j}^{\infty} \frac{r_{ij}}{i} z^i \;=\; r_j(z) - j \int_0^z \sum_{i=j}^{\infty} r_{ij} s^{i-1}\, \mathrm{d}s \;=\; r_j(z) - j \int_0^z \frac{r_j(s)}{s}\, \mathrm{d}s.$$

On the other hand, by the recursion (2.3), we obtain the factorization

$$\begin{aligned} f_j(z) \;&=\; \sum_{i=j+1}^{\infty} (i-j) r_{ij} \frac{z^i}{i} \;=\; \sum_{i=j+1}^{\infty} \sum_{k=j}^{i-1} q_{ik} r_{kj} \frac{z^i}{i} \;=\; \sum_{k=j}^{\infty} r_{kj} \sum_{i=k+1}^{\infty} \frac{q_{ik}}{i} z^i \\ &=\; \sum_{k=j}^{\infty} r_{kj} z^k \sum_{i=k+1}^{\infty} \frac{z^{i-k}}{(i-k)(i-k+1)} \\ &=\; \sum_{k=j}^{\infty} r_{kj} z^k \sum_{n=1}^{\infty} \frac{z^n}{n(n+1)} \;=\; r_j(z) a(z), \end{aligned} \tag{2.5}$$

where the function $a : U \to \mathbb{C}$ is defined via $a(z) := \sum_{n=1}^{\infty} z^n / (n(n+1)) = 1 - (1 - z) L(z)/z$, $z \in U$, with $L(z) := -\log(1 - z)$, $z \in U$. Thus, r_j satisfies the integral equation

$j\int_0^z r_j(s)/s\,\mathrm{d}s = (1-a(z))\,r_j(z)$. Differentiating with respect to z leads to $jr_j(z)/z = -a'(z)r_j(z)+(1-a(z))r_j'(z)$ or, equivalently, $(1-a(z))r_j'(z) = (a'(z)+j/z)r_j(z)$. Plugging in $1-a(z) = (1-z)L(z)/z$ and $a'(z) = L(z)/z^2 - 1/z$ leads to

$$r_j'(z) \;=\; \bigg(\frac{1}{z(1-z)} + \frac{j-1}{(1-z)L(z)}\bigg)r_j(z). \tag{2.6}$$

The solution of this homogeneous differential equation with initial conditions $r_j(0) = r_j'(0) = \cdots = r_j^{(j-1)}(0) = 0$ and $r_j^{(j)}(0) = j!$ is

$$r_j(z) \;=\; \frac{z}{1-z}(L(z))^{j-1}, \qquad j\in\mathbb{N}, |z|<1. \tag{2.7}$$

In the following, for a power series $f(z) = \sum_{n=0}^\infty f_n z^n$, $[z^n]f(z) := f_n$ denotes the coefficient of z^n in the series expansion of f. By [1, p. 824], $(L(z))^j/j! = \sum_{k=j}^\infty z^k|s(k,j)|/k!$. Thus, $[z^k](L(z))^j = j!|s(k,j)|/k!$. For the coefficient $r_{ij} = [z^i]r_j(z)$ we therefore obtain

$$\begin{aligned} r_{ij} &= [z^i]r_j(z) \;=\; [z^i]\bigg(\frac{z}{1-z}(L(z))^{j-1}\bigg) \;=\; \sum_{k=0}^{i-1}[z^{i-k}]\frac{z}{1-z}\,[z^k](L(z))^{j-1}\\ &= \sum_{k=0}^{i-1}[z^k](L(z))^{j-1} \;=\; (j-1)!\sum_{k=j-1}^{i-1}\frac{|s(k,j-1)|}{k!} \;=\; \frac{(j-1)!}{(i-1)!}|s(i,j)| \end{aligned} \tag{2.8}$$

by (2.1), which is the first formula in (1.2).

Let us now turn to $L := R^{-1}$. We have $(z,z^2,\ldots)R = (r_1(z),r_2(z),\ldots)$. Multiplying with L and noting that $RL = (\delta_{ij})_{i,j\in\mathbb{N}}$ it follows that $(z,z^2,\ldots) = (r_1(z),r_2(z),\ldots)L$. Thus, $z^j = \sum_{i=j}^\infty r_i(z)l_{ij} = z(1-z)^{-1}\sum_{i=j}^\infty(-\log(1-z))^{i-1}l_{ij}$, or, equivalently, $(1-z)z^{j-1} = \sum_{i=j}^\infty(-\log(1-z))^{i-1}l_{ij}$. Replacing z by $1-e^{-z}$ leads to $e^{-z}(1-e^{-z})^{j-1} = \sum_{i=j}^\infty l_{ij}z^{i-1}$. Thus,

$$l_j(z) \;:=\; \sum_{i=j}^\infty l_{ij}z^i \;=\; ze^{-z}(1-e^{-z})^{j-1}, \qquad j\in\mathbb{N}. \tag{2.9}$$

Let us extract the coefficient $l_{ij} = [z^i]l_j(z)$. We have

$$\begin{aligned} (1-e^{-z})^j &= \sum_{i=0}^j\binom{j}{i}(-e^{-z})^i \;=\; \sum_{i=0}^j\binom{j}{i}(-1)^i\sum_{k=0}^\infty\frac{(-zi)^k}{k!}\\ &= \sum_{k=0}^\infty\frac{z^k}{k!}(-1)^k\sum_{i=0}^j(-1)^i\binom{j}{i}i^k\\ &= \sum_{k=0}^\infty\frac{z^k}{k!}(-1)^{j+k}\sum_{i=0}^j(-1)^{j-i}\binom{j}{i}i^k \;=\; \sum_{k=0}^\infty\frac{z^k}{k!}(-1)^{j+k}j!S(k,j), \end{aligned}$$

where the last equality follows from the explicit formula

$$S(k,j) \;=\; \frac{1}{j!}\sum_{i=0}^j(-1)^{j-i}\binom{j}{i}i^k, \qquad k,j\in\mathbb{N}_0, \tag{2.10}$$

for the Stirling numbers of the second kind. Thus, $[z^k](1-e^{-z})^{j-1} = (-1)^{j+k-1}(j-$

$1)!S(k,j-1)/k!$ and, therefore,

$$\begin{aligned}
l_{ij} &= [z^i]l_j(z) \;=\; [z^i](ze^{-z}(1-e^{-z})^{j-1}) \;=\; \sum_{k=0}^{i-1}[z^{i-k}](ze^{-z})\,[z^k](1-e^{-z})^{j-1}\\
&= \sum_{k=j-1}^{i-1}\frac{(-1)^{i-1-k}}{(i-1-k)!}\frac{(-1)^{j+k-1}(j-1)!S(k,j-1)}{k!}\\
&= (-1)^{i+j}(j-1)!\sum_{k=j-1}^{i-1}\frac{S(k,j-1)}{(i-1-k)!k!}\\
&= (-1)^{i+j}\frac{(j-1)!}{(i-1)!}\sum_{k=j-1}^{i-1}\binom{i-1}{k}S(k,j-1) \;=\; (-1)^{i+j}\frac{(j-1)!}{(i-1)!}S(i,j)
\end{aligned}$$

by (2.2), which is the second formula in (1.2). The proof is complete. □

Remark 2.2. *1. Let us provide some additional information on R and L. Plugging in the formula*

$$s(i,j) \;=\; (-1)^{i+j}\frac{i!}{j!}\sum_{\substack{k_1,\ldots,k_j\in\mathbb{N}\\k_1+\cdots+k_j=i}}\frac{1}{k_1\cdots k_j},\qquad i,j\in\mathbb{N} \tag{2.11}$$

for the Stirling numbers of the first kind into the first formula in (1.2) leads to

$$r_{ij} \;=\; \frac{i}{j}\sum_{\substack{k_1,\ldots,k_j\in\mathbb{N}\\k_1+\cdots+k_j=i}}\frac{1}{k_1\cdots k_j},\qquad i,j\in\mathbb{N}. \tag{2.12}$$

In particular, $r_{i2}=(i/2)\sum_{k=1}^{i-1}1/(k(i-k))=(1/2)\sum_{k=1}^{i-1}(1/k+1/(i-k))=h_{i-1}$, $i\in\mathbb{N}$, where $h_0:=0$ and $h_n:=\sum_{k=1}^{n}1/k$ denotes the n-th harmonic number, $n\in\mathbb{N}$. Thus, $r_{i2}\sim\log i$ as $i\to\infty$. More generally, for arbitrary but fixed $j\in\mathbb{N}$ the absolute Stirling numbers of the first kind $|s(i,j)|$ satisfy the asymptotics $|s(i,j)|\sim(i-1)!(\log i)^{j-1}/(j-1)!$ as $i\to\infty$, see for example [1, p. 824]. Thus, for each $j\in\mathbb{N}$ we conclude from (1.2) that $r_{ij}\sim(\log i)^{j-1}$ as $i\to\infty$. For each $j\in\mathbb{N}$ it follows from the middle expression in (2.8) that the sequence $(r_{ij})_{i\in\mathbb{N}}$ is monotone increasing in i. Plugging in (2.10) into the second formula in (1.2) leads to

$$l_{ij} \;=\; (-1)^{i+j}\frac{1}{j(i-1)!}\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}k^i,\qquad i,j\in\mathbb{N}. \tag{2.13}$$

Alternatively one may also plug in the formula

$$S(i,j) \;=\; \frac{i!}{j!}\sum_{\substack{k_1,\ldots,k_j\in\mathbb{N}\\k_1+\cdots+k_j=i}}\frac{1}{k_1!\cdots k_j!},\qquad i,j\in\mathbb{N} \tag{2.14}$$

for the Stirling numbers of the second kind into the second formula in (1.2) leading to

$$l_{ij} \;=\; (-1)^{i+j}\frac{i}{j}\sum_{\substack{k_1,\ldots,k_j\in\mathbb{N}\\k_1+\cdots+k_j=i}}\frac{1}{k_1!\cdots k_j!},\qquad i,j\in\mathbb{N}. \tag{2.15}$$

Note that formula (2.15) for l_{ij} has structural similarities with formula (2.12) for r_{ij}. These similarities point to the spectral decomposition of the partition valued Bolthausen–Sznitman n-coalescent, which will be provided in [9].

2. Eqs. (2.12), (2.13) and (2.15) are useful to compute the entries of R and L numerically. One obtains

$$R = \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & \frac{3}{2} & 1 & & & \\ 1 & \frac{11}{6} & 2 & 1 & & \\ 1 & \frac{25}{12} & \frac{35}{12} & \frac{5}{2} & 1 & \\ \vdots & & & & & \ddots \end{pmatrix} \quad \textit{and} \quad L = \begin{pmatrix} 1 & & & & & \\ -1 & 1 & & & & \\ \frac{1}{2} & -\frac{3}{2} & 1 & & & \\ -\frac{1}{6} & \frac{7}{6} & -2 & 1 & & \\ \frac{1}{24} & -\frac{5}{8} & \frac{25}{12} & -\frac{5}{2} & 1 & \\ \vdots & & & & & \ddots \end{pmatrix}. \tag{2.16}$$

Proof. (of Corollary 1.3) Fix $t \in [0, \infty)$. By Theorem 1.1, $Q = RDL$, where R and $L := R^{-1}$ have entries (1.2). Therefore, the spectral decomposition of the transition matrix is $P(t) = e^{tQ} = e^{tRDL} = Re^{tD}L$. Thus, $p_{ij}(t) := \mathbb{P}(N_t = j \,|\, N_0 = i) = \sum_{k=j}^{i} e^{-q_k t} r_{ik} l_{kj}$ for all $i, j \in \mathbb{N}$. The last equality in (1.3) follows from (1.2). □

Proof. (of Corollary 1.5) The Green matrix G of $(N_t)_{t\geq 0}$ is given by $G := \int_0^\infty P(t)\,\mathrm{d}t$, cf. [13, p. 145]. Writing $G = (g(i,j))_{i,j\in\mathbb{N}}$ and using the spectral decomposition of $P(t)$ provided in Corollary 1.3 we obtain for $2 \leq j \leq i$

$$g(i,j) \ = \ \int_0^\infty p_{ij}(t)\,\mathrm{d}t \ = \ \sum_{k=j}^{i} r_{ik} l_{kj} \int_0^\infty e^{-(k-1)t}\,\mathrm{d}t \ = \ (-1)^{i+j} \frac{(j-1)!}{(i-1)!} \sum_{k=j}^{i} \frac{s(i,k)S(k,j)}{k-1},$$

where the last equality follows from (1.2). We have $g(i,j) = h(i,j)/(q_j(1 - f_j))$, where $h(i,j)$ is the probability of hitting j starting from i and f_j is the return probability for j. For $(N_t)_{t\geq 0}$ we evidently have $f_j = 0$ for all $j \geq 2$ and $q_j = j - 1$, $j \in \mathbb{N}$. Hence,

$$h(i,j) \ = \ q_j g(i,j) \ = \ (j-1)(-1)^{i+j} \frac{(j-1)!}{(i-1)!} \sum_{k=j}^{i} \frac{s(i,k)S(k,j)}{k-1}, \qquad 2 \leq j \leq i.$$

Clearly, $h(i,1) = 1$, since the block counting process, started at $i \in \mathbb{N}$, reaches its absorbing state 1 almost surely. □

Proof. (of Corollary 1.7) By Corollary 1.3, for all $t \in (0, \infty)$ and all $n \in \mathbb{N}$, $\mathbb{P}(\tau_n > t) = 1 - p_{n1}(t) = -\sum_{k=2}^{n} r_{nk} l_{k1} e^{-q_k t}$. Thus, for all $j \in \mathbb{N}$,

$$\mathbb{E}(\tau_n^j) \ = \ \int_0^\infty j t^{j-1} \mathbb{P}(\tau_n > t)\,\mathrm{d}t \ = \ -\sum_{k=2}^{n} r_{nk} l_{k1} \int_0^\infty j t^{j-1} e^{-q_k t}\,\mathrm{d}t \ = \ -\sum_{k=2}^{n} r_{nk} l_{k1} \frac{j!}{q_k^j}.$$

Plugging in the formulas $r_{nk} = (k-1)!|s(n,k)|/(n-1)!$ and $l_{k1} = (-1)^{k-1}/(k-1)!$ from Theorem 1.1 and noting that $q_k = k - 1$ the formula (1.4) for $\mathbb{E}(\tau_n^j)$, $n, j \in \mathbb{N}$, follows immediately. □

Proof. (of Corollary 1.9) From Corollary 1.3 it follows that the absorption time τ_n of the Bolthausen–Sznitman coalescent, started in a partition with $n \in \mathbb{N}$ blocks, has distribution function $\mathbb{P}(\tau_n \leq t) = p_{n1}(t) = \sum_{k=1}^{n} r_{nk} l_{k1} e^{-q_k t}$. By (1.2),

$$r_{nk} l_{k1} \ = \ \frac{(k-1)!}{(n-1)!} |s(n,k)| \frac{(-1)^{k-1}}{(k-1)!} \ = \ \frac{(-1)^{n-1}}{(n-1)!} s(n,k).$$

Thus, the absorption time τ_n has distribution function

$$\begin{aligned}\mathbb{P}(\tau_n \le t) &= \frac{(-1)^{n-1}}{(n-1)!}\sum_{k=1}^n s(n,k)\,e^{-(k-1)t} \;=\; \frac{(-1)^{n-1}}{(n-1)!}e^t\sum_{k=1}^n s(n,k)(e^{-t})^k \qquad (2.17)\\ &= \frac{(-1)^{n-1}}{(n-1)!}e^t(e^{-t})_n \;=\; \prod_{j=1}^{n-1}\frac{j-e^{-t}}{j} \;=\; \binom{n-1-e^{-t}}{n-1} \;=\; \frac{\Gamma(n-e^{-t})}{\Gamma(n)\Gamma(1-e^{-t})},\end{aligned}$$

$n \in \mathbb{N}$, $t \in (0,\infty)$, where we have used the formula $\sum_{k=1}^n s(n,k)x^k = (x)_n := x(x-1)\cdots(x-n+1)$, $n \in \mathbb{N}$, $x \in \mathbb{R}$. Note that (2.17) is in agreement with [14, Proposition 29, Eq. (44)], when taking [14, Eq. (49)] into account.

In particular, for all $x \in \mathbb{R}$ and n sufficiently large,

$$\begin{aligned}\mathbb{P}(\tau_n - \log\log n \le x) &= \mathbb{P}(\tau_n \le x + \log\log n) \;=\; \frac{\Gamma(n-e^{-x-\log\log n})}{\Gamma(n)\Gamma(1-e^{-x-\log\log n})}\\ &= \frac{\Gamma(n-e^{-x}/\log n)}{\Gamma(n)\Gamma(1-e^{-x}/\log n)} \;\sim\; \frac{\Gamma(n+c/\log n)}{\Gamma(n)}\end{aligned}$$

as $n \to \infty$, where the abbreviation $c := -e^{-x}$ is used for convenience. In order to see that the last expression converges to $e^c = e^{-e^{-x}} = F(x)$ as $n \to \infty$, one may apply Stirling's formula $\Gamma(n+1) \sim (n/e)^n\sqrt{2\pi n}$ as $n \to \infty$ to obtain

$$\begin{aligned}\frac{\Gamma(n+c/\log n)}{\Gamma(n)} &\sim \frac{\Gamma(n+c/\log n+1)}{\Gamma(n+1)} \;\sim\; \frac{((n+c/\log n)/e)^{n+c/\log n}\sqrt{2\pi(n+c/\log n)}}{(n/e)^n\sqrt{2\pi n}}\\ &\sim \frac{((n+c/\log n)/e)^{n+c/\log n}}{(n/e)^n} \;=\; e^{-c/\log n}(n+c/\log n)^{c/\log n}(1+c/(n\log n))^n\\ &\sim (n+c/\log n)^{c/\log n} \;=\; (1+c/(n\log n))^{c/\log n}n^{c/\log n} \;\sim\; n^{c/\log n} \;=\; e^c.\end{aligned}$$

Thus, for all $x \in \mathbb{R}$ it is shown that $\mathbb{P}(\tau_n - \log\log n \le x) \to F(x)$ as $n \to \infty$, so $\tau_n - \log\log n$ converges in distribution to the standard Gumbel distribution.

It remains to determine the Laplace transform of τ_n. Applying the formula

$$\mathbb{E}(f(\tau_n)) \;=\; f(0) + \int_0^\infty f'(t)\mathbb{P}(\tau_n > t)\,\mathrm{d}t$$

to the function $f(t) := e^{-\lambda t}$ it follows that τ_n has Laplace transform

$$\mathbb{E}(e^{-\lambda\tau_n}) \;=\; 1-\lambda\int_0^\infty e^{-\lambda t}(1-\mathbb{P}(\tau_n \le t))\,\mathrm{d}t, \qquad n \in \mathbb{N}, \lambda \in [0,\infty). \qquad (2.18)$$

Plugging in the formula (2.17) for the distribution function of τ_n it follows that

$$\mathbb{E}(e^{-\lambda\tau_n}) \;=\; 1-\lambda\int_0^\infty e^{-\lambda t}\bigg(1-\frac{(-1)^{n-1}}{(n-1)!}\sum_{k=1}^n s(n,k)e^{-(k-1)t}\bigg)\,\mathrm{d}t, \qquad n \in \mathbb{N}, \lambda \in [0,\infty).$$

Since $s(n,1) = (-1)^{n-1}(n-1)!$ we can rewrite this as

$$\begin{aligned}\mathbb{E}(e^{-\lambda\tau_n}) &= 1+\lambda\int_0^\infty e^{-\lambda t}\frac{(-1)^{n-1}}{(n-1)!}\sum_{k=2}^n s(n,k)e^{-(k-1)t}\,\mathrm{d}t\\ &= 1+\lambda\frac{(-1)^{n-1}}{(n-1)!}\sum_{k=2}^n s(n,k)\int_0^\infty e^{-(k-1+\lambda)t}\,\mathrm{d}t \;=\; 1+\frac{(-1)^{n-1}}{(n-1)!}\sum_{k=2}^n s(n,k)\frac{\lambda}{k-1+\lambda},\end{aligned}$$

$n \in \mathbb{N}$, $\lambda \in [0,\infty)$. The proof is complete. □

Remark 2.3. *Alternatively, the product formula (1.5) for the distribution function of τ_n follows from the construction of the Bolthausen–Sznitman coalescent via the random recursive tree due to Goldschmidt and Martin [6] as follows. Recall a random recursive tree with n vertices is a uniform tree in the set of labelled trees that have increasing labels along the branches from the root, labelled 1. A recursive construction is the following. Start with the tree reduced to a single vertex labelled 1. When the tree with $n-1$ vertices is built, the nth vertex is attached by an edge to a uniformly chosen vertex with label in $\{1,\ldots,n-1\}$. The Bolthausen–Sznitman coalescent is then obtained (see [6]) by cutting down this tree, and, in this construction, the event $\{\tau_n \le t\}$ coincides with*

$$\bigcap_{i=2}^{n}(\{i \text{ is a child of the root, } e_i \le t\} \cup \{i \text{ is not a child of the root}\}),$$

where $e_2,\ldots e_n$ are independent standard exponential random variables. These events, indexed by i, are furthermore independent. Formula (1.5) now follows by the straightforward calculation

$$\begin{aligned}\mathbb{P}(\tau_n \le t) &= \mathbb{P}(\bigcap_{i=2}^{n}(\{i \text{ is a child of the root, } e_i \le t\} \cup \{i \text{ is not a child of the root}\}))\\ &= \prod_{i=2}^{n}\bigg(\frac{1}{i-1}(1-e^{-t}) + \Big(1-\frac{1}{i-1}\Big)\bigg) = \prod_{j=1}^{n-1}\frac{j-e^{-t}}{j} = \frac{\Gamma(n-e^{-t})}{\Gamma(n)\Gamma(1-e^{-t})},\end{aligned}$$

$n \in \mathbb{N}$, $t \in (0,\infty)$.

The following third approach derives formula (1.5) via the Chinese restaurant process. It is known (see [14, Theorem 14]) that Π_t is $\mathrm{PD}(e^{-t},0)$-distributed. On the other hand the distribution $\mathrm{PD}(\alpha,\theta)$ is obtained by considering the ranked frequencies of the partition of an (α,θ)-Chinese restaurant process. Thus, the event $\{\tau_n \le t\}$ coincides with

$$\{\text{the first } n \text{ customers in a } (e^{-t},0)\text{-Chinese restaurant sit at table } 1\}.$$

Since any new customer joins the $m_k > 0$ customers at table k with probability $(m_k - e^{-t})/m$, where $m := \sum_k m_k$ denotes the number of already present customers, we obtain

$$\mathbb{P}(\tau_n \le t) = \prod_{i=2}^{n}\frac{i-1-e^{-t}}{i-1} = \frac{\Gamma(n-e^{-t})}{\Gamma(n)\Gamma(1-e^{-t})}$$

as required.

3 Appendix

For completeness we provide the spectral decomposition of the block counting process of the Kingman coalescent, which is the particular Λ-coalescent with $\Lambda = \delta_0$ being the Dirac measure at 0. The block counting process $(N_t)_{t\ge0}$ of the Kingman coalescent is a pure death process with total rates $q_i = i(i-1)/2$, $i \in \mathbb{N}$. The generator Q of $(N_t)_{t\ge0}$ has spectral decomposition (see, for example, [12, Appendix, Lemma 5.1] $Q = RDL$, where $D = (d_{ij})_{i,j\in\mathbb{N}}$ is the diagonal matrix with entries $d_{ij} = -i(i-1)/2$ for $i = j$ and $d_{ij} = 0$ for $i \ne j$, and $R = (r_{ij})_{i,j\in\mathbb{N}}$ and $L = (l_{ij})_{i,j\in\mathbb{N}}$ are lower left triangular matrices with entries

$$\begin{aligned} r_{ij} &= \prod_{k=j+1}^{i}\frac{q_k}{q_k-q_j} = \prod_{k=j+1}^{i}\frac{k(k-1)}{k(k-1)-j(j-1)}\\ &= \prod_{k=j+1}^{i}\frac{k(k-1)}{(k-j)(k+j-1)} = \frac{i!(i-1)!(2j-1)!}{j!(j-1)!(i-j)!(i+j-1)!}, \qquad i \ge j,\end{aligned}$$

and

$$l_{ij} = \prod_{k=j}^{i-1} \frac{q_{k+1}}{q_k - q_i} = \prod_{k=j}^{i-1} \frac{k(k+1)}{(k-i)(k+i-1)} = (-1)^{i-j} \frac{(i-1)!i!(i+j-2)!}{(j-1)!j!(i-j)!(2i-2)!}, \quad i \geq j.$$

The same coefficients r_{ij} and l_{ij}, $1 \leq i,j \leq N$, occur in the spectral decomposition of the transition matrix $P = (p_{ij})_{1 \leq i,j \leq N}$ of the Moran model with population size $N \in \mathbb{N}$ having eigenvalues $\lambda_i := 1 - i(i-1)/N^2$, $1 \leq i \leq N$, see [5, p. 635]. For the spectral expansion of the generator of the Kingman coalescent with mutation we refer the reader to [16, Appendix I]. For each $j \in \mathbb{N}$ the sequence $(r_{ij})_{i \in \mathbb{N}}$ is monotone increasing in i and converges to $r_j := (2j-1)!/(j!(j-1)!) = \binom{2j-1}{j}$ as $i \to \infty$. The absorption time τ_n of the Kingman coalescent restricted to a sample of size n has distribution function $\mathbb{P}(\tau_n \leq t) = \mathbb{P}(N_t = 1 \,|\, N_0 = n) = \sum_{k=1}^n r_{nk} l_{k1} e^{-q_k t}$, $t \in (0, \infty)$. Taking the limit $n \to \infty$ shows that the almost sure limit $\tau := \lim_{n \to \infty} \tau_n$ has distribution function $t \mapsto \sum_{k=1}^\infty r_k l_{k1} e^{-q_k t} = \sum_{k=1}^\infty (-1)^k (2k-1) e^{-k(k-1)t/2}$, $t \in (0, \infty)$, and, hence, density $t \mapsto \sum_{k=1}^\infty (-1)^k q_k (2k-1) e^{-q_k t}$, in agreement with Kingman [8, p. 37, Eq. (5.9)].

References

[1] Abramowitz, M. and Stegun, I. A.: *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. 9th printing. Dover, New York, 1972. MR-0757537

[2] Bolthausen, E. and Sznitman, A.-S.: On Ruelle's probability cascades and an abstract cavity method. *Commun. Math. Phys.* **197**, (1998), 247–276. MR-1652734

[3] Drmota, M., Iksanov, A., Möhle, M. and Rösler, U.: Asymptotic results concerning the total branch length of the Bolthausen–Sznitman coalescent. *Stoch. Process. Appl.* **117**, (2007), 1404–1421. MR-2353033

[4] Freund, F. and Möhle, M.: On the time back to the most recent common ancestor and the external branch length of the Bolthausen–Sznitman coalescent. *Markov Process. Related Fields* **15**, (2009), 387–416. MR-2554368

[5] Gladstien, K.: The characteristic values and vectors for a class of stochastic matrices arising in genetics. *SIAM J. Appl. Math.* **34**, (1978), 630–642. MR-0475977

[6] Goldschmidt, C. and Martin, J. B.: Random recursive trees and the Bolthausen–Sznitman coalescent. *Electron. J. Probab.* **10**, (2005), 718–745. MR-2164028

[7] Henard, O.: The fixation line. Preprint, (2013). arXiv:1307.0784

[8] Kingman, J.F.C.: On the genealogy of large populations. *J. Appl. Probab.* **19**, (1982), 27–43. MR-0633178

[9] Kukla, J., Miller, L. and Pitters, H.: A spectral decomposition for the Kingman and the Bolthausen–Sznitman coalescent. In preparation, (2014+).

[10] Möhle, M.: On hitting probabilities of beta coalescents and absorption times of coalescents that come down from infinity. *ALEA Lat. Am. J. Probab. Math. Stat.* **11**, (2014), 141–159.

[11] Möhle, M.: Asymptotic hitting probabilities for the Bolthausen–Sznitman coalescent. *J. Appl. Probab.* **51A**, (2014), to appear.

[12] Möhle, M. and Pitters, H.: Absorption time and tree length of the Kingman coalescent and the Gumbel distribution. Preprint, (2014).

[13] Norris, J. R.: *Markov Chains*. Cambridge University Press, Cambridge, 1997. MR-1600720

[14] Pitman, J.: Coalescents with multiple collisions. *Ann. Probab.* **27**, (1999), 1870–1902. MR-1742892

[15] Sagitov, S.: The general coalescent with asynchronous mergers of ancestral lines. *J. Appl. Probab.* **36**, (1999), 1116–1125. MR-1742154

[16] Tavare, S.: Line-of-descent and genealogical processes, and their applications in population genetics models. *Theor. Popul. Biol.* **26**, (1984), 119–164. MR-0770050

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