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Coupling for drifted Brownian motion on an interval with redistribution from the boundary*

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Abstract

We answer a question by Kolb and Wubker [7] on the threshold drift for Brownian Motion on an interval with redistribution from the boundary. We do that by constructing an efficient coupling.

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1 Introduction

Consider an elliptic diffusion on a bounded domain $D \subset \mathbb{R}^d$, with infinitesimal generator L, that upon hitting the boundary ∂D is redistributed in D according to some prescribed probability measure (that could depend on the point of exit), restarting afresh, and repeated indefinitely. The redistribution is independent of the past. Under some standard smoothness assumptions on the boundary (and the redistribution measure as a function of the boundary point), the process is ergodic and converges exponentially fast to its invariant distribution in total variation. We call the resulting process diffusion with redistribution, or DR in short. This process has been studied by several authors: [4] [5] (analysis for BM with fixed deterministic redistribution), [3] [2](ergodicity, characterization, comparison), [8] (analysis of spectrum), [9] (holding times at boundary), and most recently [6][7] (coupling approach). The DR is never reversible and the problems of characterizing and estimating the exponential rate of convergence are typically non-trivial. Unsurprisingly, the exponential rate is equal to the “spectral-gap" for $-L$ with the nonlocal boundary conditions imposed by the redistribution, that is, the minimal real part among all non-zero eigenvalues, yet this result is far from trivial to prove. As for estimation and comparison with other quantities, the main obstacle is that the corresponding eigenvalue may not be real. Yet, if it is real, then the spectral gap is bounded below by the principal eigenvalue for $-L$ with the Dirichlet boundary condition. This realness condition was shown to hold under certain conditions and in some concrete examples. This has led to the question whether the principal Dirichlet eigenvalue is a lower bound when the realness condition is relaxed, at least when the underlying diffusion process is reversible. The question was formulated by Pinsky and the author of this note in [2].

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Recently, Kolb and Wubker [7] answered the question negatively by finding a counterexample. Their counterexample is obtained from drifted Brownian Motion on an interval $(0,\ell)$, generated by $L = L_{\sigma,\mu}$ where

$$L_{\sigma,\mu} = \frac{\sigma^2}{2}\frac{d^2}{dx^2} + \mu\frac{d}{dx},$$

for some constant σ and μ, and deterministic redistribution to the center of the interval, $\frac{\ell}{2}$. The underlying diffusion is reversible, and a straightforward calculation shows that the principal eigenvalue for $-L_{\sigma,\mu}$ with Dirichlet boundary condition tends to infinity as $|\mu| \to \infty$. Kolb and Wubker showed that for all sufficiently large $|\mu|$ the spectral gap is constant, thus answering the question negatively. Their approach is essentially probabilistic and the core of the argument is construction of efficient coupling for the DR for large values of the drift coefficient. A problem they left open is the threshold value for μ above which the spectral gap is constant, and the authors conjectured it is equal to $\sqrt{3}\frac{2\pi\sigma^2}{\ell}$.

Let $X = (X_t : t \geq 0)$ be the DR process on the interval $(0,\ell)$, with underlying diffusion generated by $L_{\sigma,\mu}$ and redistribution from the boundary $\{0,\ell\}$ to $\frac{\ell}{2}$. We refer the reader to [3] for the construction of the process. We denote the corresponding probability and expectation with initial distribution ρ by P_ρ and E_ρ, and if $\rho = \delta_x$ for some $x \in (0,\ell)$, then we abbreviate and write P_x and E_x, respectively. Let $P_\rho(t)$ denote the distribution of X_t under P_ρ, and for distributions μ_1, μ_2, let $d_t(\mu_1,\mu_2) = \|P_{\mu_1}(t) - P_{\mu_2}(t)\|_{TV}$, where $\|\cdot\|_{TV}$ is the total variation norm,

$$\|\mu_1 - \mu_2\|_{TV} = \sup_{f\geq 0, \|f\|_\infty = 1}\left(\int f d\mu_1 - \int f d\mu_2\right).$$

We write x for δ_x when it appears as a parameter of $d_t(\cdot,\cdot)$, and define

$$d_t := \sup_{\mu_1,\mu_2} d_t(\mu_1,\mu_2) = \sup_{x,y} d_t(x,y).$$

Let

$$\mathcal{D}_\ell = \left\{u \in C^2((0,\ell)) \cap L^\infty((0,\ell)), \lim_{\zeta\to 0^+} u(\zeta) = \lim_{\zeta\to\ell^-} u(\zeta) = u(\frac{\ell}{2})\right\},$$

and

$$\Sigma_{\sigma,\mu,\ell} = \{\lambda \in \mathbb{C} : \exists u \in \mathcal{D}_\ell, L_{\sigma,\mu}u = -\lambda u\}.$$

We define the spectral gap

$$\gamma_1(\sigma,\mu,\ell) = \inf\Big\{\mathrm{Re}(\lambda) : \lambda \in \Sigma_{\sigma,\mu,\ell} \setminus \{0\}\Big\}.$$

Similarly, let

$$\mathcal{D}_\ell^D = \{u \in C^2((0,\ell)) \cap C([0,\ell]), u(0) = u(\ell) = 0\},$$

and

$$\Sigma^D_{\sigma,\mu,\ell} = \{\lambda \in \mathbb{C} : \exists u \in \mathcal{D}^D_\ell, L_{\sigma,\mu}u = -\lambda u\}.$$

It is easy to see that the three parameters σ,μ,ℓ can be reduced to one parameter by scaling. Indeed, it follows directly from the definition of $\Sigma_{\sigma,\mu,\ell}$ that for $\ell_1 > 0$,

$$\Sigma_{\sigma,\mu,\ell} = \frac{\ell_1^2\sigma^2}{\ell^2}\Sigma_{1,\frac{\ell\mu}{\ell_1\sigma^2},\ell_1}. \tag{1.1}$$

The analogous identity holds for $\Sigma^D_\cdot$ as well. In fact, we will prove our results for $\sigma = 1,\ \ell = 2\pi$, and derive the general statements from these scaling identities.

As it is well-known,

$$\Sigma^D_{\sigma,\mu,\ell} = \left\{\lambda^D_k(\sigma,\mu,\ell) = \frac{\sigma^2\pi^2(k+1)^2}{2\ell^2} + \frac{\mu^2}{2\sigma^2} : k \in \mathbb{Z}_+\right\}.$$

The eigenvalue

$$\lambda^D_0(\sigma,\mu,\ell) = \frac{\sigma^2\pi^2}{2\ell^2} + \frac{\mu^2}{2\sigma^2} \tag{1.2}$$

is called the principal eigenvalue.

Here is a summary of the relevant results.

Theorem 1.1 ([3],[7]).

a. X has a unique stationary distribution ν, and

$$\lim_{t\to\infty} \frac{1}{t} \ln d_t(x,\nu) = -\gamma_1(\sigma,\mu,\ell) < 0, \ x \in (0,\ell).$$

b. There exists a constant $\mu_0 = \mu_0(\sigma,\ell)$ such that if $\mu \ge \mu_0$, then

$$\gamma_1(\sigma,\mu,\ell) = \frac{8\sigma^2\pi^2}{\ell^2},$$

and there exists an efficient coupling.

We briefly recall the notion of coupling and efficient coupling. In this paper we will only consider Markovian couplings. A Markovian coupling for X with initial distributions μ_1 and μ_2 is a Markov process $((X^1_t, X^2_t) : t \ge 0)$ on $(0,\ell)\times(0,\ell)$, such that the marginals $(X^1_t : t \ge 0)$ and $(X^2_t : t \ge 0)$ are copies of X, the distribution of X^1_0 is μ_1, and the distribution of X^2_0 is μ_2. Given a coupling, the coupling time (or meeting time) τ_C is defined through

$$\tau_C = \inf\{t \ge 0 : X^1_t = X^2_t\}.$$

As it is well known and easy to verify, the tail of the coupling time dominates $d_t(\mu_1,\mu_2)$. That is,

$$d_t(\mu_1,\mu_2) \le P(\tau_C > t).$$

As a result, couplings provide lower bounds on $\gamma_1(\sigma,\mu,\ell)$. Coupling for X with initial distributions μ_1,μ_2 is efficient if the coupling time decays at an exponential rate equal to $\gamma_1(\sigma,\mu,\ell)$:

$$\lim_{t\to\infty} \frac{1}{t} \ln P(\tau_C > t) = -\gamma_1(\sigma,\mu,\ell).$$

The first part of the theorem (for a general domain, diffusion and redistribution) was proved in [3], and the second part was proved in [7]. We refer the reader to [7] for the explicit formula for the ν. The proof of part b. of the theorem has led Kolb and Wubker to conjecture that

$$\mu_0(\sigma,\ell) = \sqrt{3}\frac{2\pi\sigma^2}{\ell}. \tag{1.3}$$

We are ready to state our main result.

Theorem 1.2.

$$\gamma_1(\sigma,\mu,\ell) = \lambda^D_0(\sigma,\mu,\ell/2) \wedge \lambda^D_0(\sigma,0,\ell/4), \tag{1.4}$$

and there exists an efficient coupling.

As it turns out, the first statement follows from a straightforward eigenvalue calculation, see Proposition 1.4 below. But this does not provide any insight or explanation. The more substantial result is the existence of efficient coupling. This coupling both proves (1.4) and explains the origin of each of the terms. In addition, the proof gives upper and lower bounds on $d_t(x,y)$ for any $x,y\in(0,\ell)$ in terms of tails of exit times of BM or drifted BM from an interval.

The coupling used to prove the theorem is fairly simple. When the two copies of the DR are exactly $\ell/2$ units apart, they have the same increments. This guarantees that they meet when the copy in $(0,\ell/2)$ exits this interval. When the distance between the two copies is different than $\ell/2$, then the (non-drifted) Brownian components of the increments are of the same magnitude but with opposite signs. The main idea is then to exploit the symmetry of the model to show that with this coupling, the first time when either the copies meet or are $\ell/2$ apart (note that both events can occur after a large number of redistributions), coincides with the exit time for BM from an interval of length $\ell/4$.

Suppose that under P_x, the process $Y=(Y_t:t\ge 0)$ is BM with diffusivity σ and drift μ, with $Y_0=x$, and let $\tau_\ell=\inf\{t\ge 0:|Y_t|\ge\frac{\ell}{2}\}$. Recall [10], that for $|x|<\ell/2$

$$\lim_{t\to\infty}\frac{1}{t}\ln P_x(\tau_\ell>t)=\lim_{t\to\infty}\frac{1}{t}\sup_{|y|<\ell/2}\ln P_y(\tau_\ell>t)=-\lambda_0^D(\sigma,\mu,\ell). \tag{1.5}$$

Since for all θ, $E_xe^{\theta\tau_\ell}=1+\theta\int_0^\infty e^{\theta t}P_x(\tau_\ell>t)dt$, it follows from (1.5) that

$$\lambda_0^D(\sigma,\mu,\ell)=\inf\left\{\theta:E_xe^{\theta\tau_\ell}=\infty\text{ for some }x\in(-\frac{\ell}{2},\frac{\ell}{2})\right\}. \tag{1.6}$$

From (1.2) we obtain

$$\lambda_0^D(\sigma,\mu,\ell/2)=\frac{2\sigma^2\pi^2}{\ell^2}+\frac{\mu^2}{2\sigma^2}\text{ and }\lambda_0^D(\sigma,0,\ell/4)=\frac{8\sigma^2\pi^2}{\ell^2}.$$

Combining this with Theorem 1.2 gives:

Corollary 1.3.

$$\gamma_1(\sigma,\mu,\ell)=\begin{cases}\frac{2\sigma^2\pi^2}{\ell^2}+\frac{\mu^2}{2\sigma^2} & |\mu|\le\sqrt{3}\frac{2\pi\sigma^2}{\ell};\\ \frac{8\sigma^2\pi^2}{\ell^2} & \textit{otherwise.}\end{cases}$$

This proves (1.3).

The analytic proof to (1.3) is an immediate consequence to the following standard calculation.

Proposition 1.4. *Let* $b=\frac{\ell\mu}{2\pi\sigma^2}$. *Then*

$$\Sigma_{\sigma,\mu,\ell}=\frac{4\pi^2\sigma^2}{\ell^2}\left\{0,2k^2+ibk,\frac{b^2+k^2}{2}:k\in\mathbb{Z}-\{0\}\right\}.$$

2 Proof of Theorem 1.2

We will reduce the problem from three parameters, σ,μ,ℓ to one parameter by scaling, using the identity (1.1) and its analog for $\Sigma^D_{\sigma,\mu,\ell}$. It follows that

$$\gamma_1(\sigma,\mu,\ell)=\frac{\ell_1^2\sigma^2}{\ell^2}\gamma_1(1,\frac{\ell\mu}{\ell_1\sigma^2},\ell_1),\text{ and }\lambda_0^D(\sigma,\mu,\ell)=\frac{\ell_1^2\sigma^2}{\ell^2}\lambda_0^D(1,\frac{\ell\mu}{\ell_1\sigma^2},\ell_1).$$

Therefore if we prove the theorem for $\sigma = 1$ and $\ell = 2\pi$, then for the general case we obtain:

$$\begin{aligned}
\gamma_1(\sigma,\mu,\ell) &= \frac{(2\pi)^2\sigma^2}{\ell^2}\gamma_1(1,\frac{\ell\mu}{2\pi\sigma^2},2\pi)\\
&= \frac{(2\pi)^2\sigma^2}{\ell^2}\left(\lambda_0^D(1,\frac{\ell\mu}{2\pi\sigma^2},\pi)\wedge\lambda_0^D(1,0,\frac{\pi}{2})\right)\\
&= \frac{(2\pi)^2\sigma^2}{\ell^2}\min\left(\frac{(\ell/2)^2}{\sigma^2\pi^2}\lambda_0^D(\sigma,\mu,\ell/2),\frac{(\ell/4)^2}{\sigma^2(\pi/2)^2}\lambda_0^D(\sigma,0,\ell/4)\right)\\
&= \lambda_0^D(\sigma,\mu,\ell/2)\wedge\lambda_0^D(\sigma,0,\ell/4).
\end{aligned}$$

In light of the above, in the remainder of this section we will assume the diffusivity $\sigma = 1$, and $\ell = 2\pi$. We will use b for the drift coefficient, and without loss of generality, assume $b \geq 0$. We let P_x^b denote the probability measure under which the process $Y = (Y_t : t \geq 0)$ is BM on $\mathbb{R}$ starting from x with diffusivity 1 and drift b. Recall that $\tau_\ell = \inf\{t \geq 0 : |Y_t| \geq \frac{\ell}{2}\}$. We also denote by $B = (B_t : t \geq 0)$ standard BM on $\mathbb{R}$.

2.1 Lower bound on d_t

The lower bound is very simple, but it suggests the couplings to be used for the upper bound.

Lemma 2.1. $d_t \geq P_0^b(\tau_\pi > t) \vee P_0^0(\tau_{\pi/2} > t)$.

Thus from Theorem 1.1, Lemma 2.1 and (1.5), we obtain the following:

Corollary 2.2.
$$\gamma_1(1,b,2\pi) \leq \lambda_0^D(1,b,\pi)\wedge\lambda_0^D(1,0,\pi/2).$$

Proof. Define the coupled processes X^1 and X^2 on $(0,2\pi)$ by letting

$$X_t^1 = \frac{\pi}{2} + B_t + bt,\ X_t^2 = \pi + X_t^1,\ t \geq 0.$$

Let X be a DR starting at π. Let $\tau = \inf\{t : X_t^1 \in \{0,\pi\}\}$. Then τ has the same distribution as τ_π (defined above (1.5)), under P_0^b. Observe that at time τ, $X_\tau^2 = \pi$ or 2π according to whether $X_\tau^1 = 0$ or $X_\tau^2 = \pi$. We continue the coupling for $t \geq \tau$ by redefining X^1 and X^2 through:

$$X_t^2 = X_t^1 = X_{t-\tau},\ t \geq \tau.$$

Thus X^1 and X^2 are copies of the DR process, meeting at time τ. Let $f = \mathbf{1}_{[0,\pi)}$. This gives

$$d_t \geq E_{\frac{\pi}{2}} f(X_t) - E_{\frac{3\pi}{2}} f(X_t) = E\left(f(X_t^1) - f(X_t^2)\right) = P_0^b(\tau_\pi > t).$$

To prove the second bound, observe that if $f : \mathbb{R} \to \mathbb{R}$ is piecewise continuous and π-periodic, then under P_x, $f(X_t)$ has the same distribution as $f(x+B_t+bt)$. Furthermore, for a fixed t, let $g(u) = f(u+bt)$. Note that this transformation is one-to-one and onto from the set of piecewise continuous π-periodic functions to itself, and that under P_x, the distribution of $f(X_t)$ coincides with the distribution of $g(x+B_t)$. Thus, if in addition $0 \leq f \leq 1$, then $d_t(x,y) \geq E_x f(X_t) - E_y f(X_t) = Eg(x+B_t) - Eg(y-B_t)$. Choose now $x = \frac{\pi}{2}$, $y = \pi$, and let g be the π-periodic function equal to $\mathbf{1}_{[\frac{\pi}{4},\frac{3\pi}{4}]}$ on $[0,\pi)$. Let $\tau = \inf\{t : |B_t| = \frac{\pi}{4}\}$. Then τ has the same distribution as $\tau_{\pi/2}$ under P_0^0. Observe that for $t < \tau$, $g(x+B_t) = 1$, and $g(y-B_t) = 0$. We define the coupling

$$B_t^1 = x + B_t,\ B_0^2 = y,\ dB_t^2 = \begin{cases} -dB_t & t < \tau; \\ dB_t & t \geq \tau. \end{cases}$$

Observe that either $B^1_\tau = B^2_\tau = \frac{3\pi}{4}$, or $B^1_\tau = \frac{\pi}{4}$ and $B^2_\tau = \pi + \frac{\pi}{4}$. Thus, according to the construction of the coupling and the choice of g, it follows that $g(B^1_t) = g(B^2_t)$ for all $t \geq \tau$. In particular,

$$d_t \geq Eg(x + B_t) - Eg(y - B_t) = E\left(\mathbf{1}_{\{\tau > t\}}\left(g(B^1_t) - g(B^2_t)\right)\right) = P^0_0(\tau_{\pi/2} > t).$$

□

2.2 An upper bound on d_t

We will use the following well-known lemma, which is an immediate corollary to the eigenvalue expansion for the transition function of drifted BM (e.g. [1, p. 94, Theorem 5.9]). The lemma could be avoided, but helps us obtain tighter upper bounds on d_t.

Lemma 2.3. *Let Y be the drifted BM generated by $L_{1,b}$ on $(-\frac{\ell}{2}, \frac{\ell}{2})$. Then*

$$\sup_{|x|<\frac{\ell}{2}} \left| e^{\lambda^D_0(1,b,\ell)t} P^b_x(\tau_\ell > t) - \frac{2\ell}{\pi} C^2 e^{-bx} \cos(\frac{\pi x}{\ell}) \right| = e^{-(\lambda^D_1(1,b,\ell) - \lambda^D_0(1,b,\ell))t} O_t(1),$$

where $C = \sqrt{\frac{2b}{1-e^{-b\ell}} \frac{(b\ell)^2+4\pi^2}{4\pi^2}}$, $\lambda^D_0(1,b,\ell) = \frac{1}{2}\left(\frac{\pi^2}{\ell^2} + b^2\right)$ and $\lambda^D_1(1,b,\ell) - \lambda^D_0(1,b,\ell) = \frac{3\pi^2}{2\ell^2}$.

If (X, Y) is a coupling for the DR starting from x and y respectively, then we denote the joint distribution by $P_{x,y}$ and the corresponding expectation by $E_{x,y}$. Let τ_C denote the coupling time,

$$\tau_C = \inf\{t \geq 0 : X_t = Y_t\}.$$

Then for $f \geq 0$, $E_x f(X_t) - E_y f(X_t) = E_{x,y}\big(f(X_t) - f(Y_t)\big) \leq \|f\|_\infty P_{x,y}(\tau_C > t)$. In particular,

$$d_t(x, y) \leq P_{x,y}(\tau_C > t).$$

From the triangle inequality, $d_t(x, y) \leq d_t(x, \pi) + d_t(\pi, y) \leq 2\sup_x d_t(x, \pi)$. Thus,

$$d_t \leq 2 \sup_\theta P_{\theta,\pi}(\tau_C > t). \tag{2.1}$$

We will now obtain an upper bound on $P_{\theta,\pi}(\tau_C > t)$ through coupling of two copies of the DR. As will be shown below, we only need to consider $\theta \in (\pi, 2\pi)$. The coupling is constructed and analyzed in the proof of Lemma 2.4 below. This coupling consists of two stages.

Stage 1. The two copies have Brownian increments which are of the same magnitude and opposite signs. We begin this stage with one copy at π and another copy at $\theta \in (\pi, 2\pi)$. We stop this stage at time T_1, which is when either:

(a) The two copies meet, and then $\tau_C = T_1$; or
(b) The distance between the copies is π. In this case we move to the second stage.

The first stage continues as long as none of the above conditions is met. While in the first stage, the first redistribution event, if occurs, can only occur when the copy starting in $(\pi, 2\pi)$ hits $2\pi^-$ (this will be explained below). Since the initial distance is strictly less than π, and condition (b) was not met, at this time the second copy is in $(\pi, 2\pi)$. As a result, at the redistribution, we again have one copy (the redistributed one) at π and another copy in $(\pi, 2\pi)$. By induction, this holds for all redistributions during the first stage.

Stage 2. In the second stage, the two copies share the same Brownian increments, and in particular, their distance remains π until the first redistribution event, at which they meet. This is the coupling time τ_C.

Here are the details of the coupling. Fix $\theta \in (\pi, 2\pi)$. Let X^1 and X^2 be given by

$$X_t^1 = \theta - B_t + bt,\ X_t^2 = \pi + B_t + bt,\ t \geq 0.$$

Define the stopping times

$$\sigma_1 = \inf\{t : X_t^1 = X_t^2\},\ \sigma_2 = \inf\{t : X_t^1 - X_t^2 = \pi\},\ \text{and } \sigma = \sigma_1 \wedge \sigma_2.$$

Observe that

$$X_t^1 = X_t^2 \text{ if and only if } B_t = \frac{\theta - \pi}{2}.$$

Similarly,

$$X_t^1 - X_t^2 = \pi \text{ if and only if } B_t = \frac{\theta - 2\pi}{2}.$$

Therefore, σ is the exit time for B from the interval $(\frac{\theta-2\pi}{2}, \frac{\theta-\pi}{2})$.
Next, let

$$\tau_1 = \inf\{t : X_t^1 = 2\pi\},\ \tau_2 = \inf\{t : X_t^2 = 0\},\ \text{and } \tau = \tau_1 \wedge \tau_2.$$

We have that

$$X_t^1 = 2\pi \text{ if and only if } B_t = \theta - 2\pi + bt.$$

Similarly,

$$X_t^2 = 0 \text{ if and only if } B_t = -\pi - bt.$$

Since $-\pi < \frac{\theta-2\pi}{2}$ and $b \geq 0$, it follows that $\tau_2 \geq \sigma_2$. Therefore $\sigma \wedge \tau = \sigma \wedge \tau_1$.

We continue the construction inductively. For this we need a definition. For $j \in \mathbb{N}$, let $J_j = 1$ if j is odd and $J_j = 2$ if j is even. Also, let $\tau^1 = \tau$ and $\sigma^1 = \sigma$. Starting from $j = 1$, if $\sigma^j \leq \tau^j$, then the first stage ends, and we let $T_1 = \sigma^j$. Otherwise, that is on the event $\{\tau^j < \sigma^j\}$, we let $\theta^j = X_{\tau^j}^{J_{j+1}}$ and redefine

$$X_t^{J_j} = \pi + (-1)^j (B_t - B_{\tau^j}) + b(t - \tau^j),\ X_t^{J_{j+1}} = \theta^j + (-1)^{j+1}(B_t - B_{\tau^j}) + b(t - \tau^j),\ t \geq \tau^j$$

We also let

$$\sigma_1^{j+1} = \inf\{t > \tau^j : X_t^{J_{j+1}} - X_t^{J_j} = 0\},\ \sigma_2^{j+1} = \inf\{t > \tau^j : X_t^{J_{j+1}} - X_t^{J_j} = \pi\},\ \sigma^{j+1} = \sigma_1^{j+1} \wedge \sigma_2^{j+1};$$

and

$$\tau_1^{j+1} = \inf\{t > \tau^j : X_t^{J_{j+1}} = 2\pi\},\ \tau_2^{j+1} = \inf\{t > \tau^j : X_t^{J_j} = 0\},\ \tau^{j+1} = \tau_1^{j+1} \wedge \tau_2^{j+1}.$$

Similar to the argument given in the paragraph above, $\sigma^{j+1} \wedge \tau^{j+1}$ is attained by σ_1^{j+1}, σ_2^{j+1} or by τ_1^{j+1}. This completes the construction of the coupling.

Lemma 2.4. *If $X_0^1 = \theta \in (\pi, 2\pi)$ and $X_0^2 = \pi$, then $T_1 = \inf\left\{t \geq 0 : B_t \in \{\frac{\theta-2\pi}{2}, \frac{\theta-\pi}{2}\}\right\}$. That is, the distribution of T_1 coincides with the distribution of $\tau_{\pi/2}$ under $P^0_{\frac{3\pi-2\theta}{4}}$.*

Proof. Let $\theta^0 = \theta$, $\tau^0 = 0$ and $\sigma^0 = \infty$. In terms of B, on the event $\bigcap_{i \leq j}\{\tau^i < \sigma^i\},\ j \geq 0$, we have

$$\tau^{j+1} = \inf\left\{t \geq \tau^j : B_t - B_{\tau^j} = (-1)^{j+1}(2\pi - \theta^j - b(t - \tau^j))\right\},$$

and

$$\sigma^{j+1} = \inf\left\{t \geq \tau^j : B_t - B_{\tau^j} \in (-1)^j \{\frac{\theta^j - \pi}{2}, \frac{\theta^j - 2\pi}{2}\}\right\}.$$

We will show that

$$\sigma^{j+1} = \inf\left\{ t > \tau^j : B_t \in \{\frac{\theta^0 - 2\pi}{2}, \frac{\theta^0 - \pi}{2}\}\right\}.$$

From this it follows that T_1, the time the coupling is stopped, coincides with the exit time of B from $(\frac{\theta^0-2\pi}{2}, \frac{\theta^0-\pi}{2})$. To prove the claim, we first derive some identities. On $\{\tau^j < \sigma^j\}$, we have

$$\begin{aligned} 2\pi &= \theta^j + (-1)^{j+1}(B_{\tau^{j+1}} - B_{\tau^j}) + b(\tau^{j+1} - \tau^j) \\ \theta^{j+1} &= \pi + (-1)^j(B_{\tau^{j+1}} - B_{\tau^j}) + b(\tau^{j+1} - \tau^j). \end{aligned}$$

It follows that $2(-1)^j(B_{\tau^{j+1}} - B_{\tau^j}) = -3\pi + \theta^{j+1} + \theta^j$. Multiplying both sides by $(-1)^j$ and summing over $j = 0, \dots, k-1$, it follows that on the event $\bigcap_{j=0}^{k-1}\{\tau^j < \sigma^j\}$ we have

$$2B_{\tau^k} = -3\pi\delta_{k \text{ odd}} + \theta^0 - (-1)^k\theta^k.$$

On rewriting this, we have

$$(-1)^k\theta^k = -3\pi\delta_{k \text{ odd}} + \theta^0 - 2B_{\tau^k},$$

and consequently,

$$(-1)^k\left\{\frac{\theta^k - \pi}{2}, \frac{\theta^k - 2\pi}{2}\right\} = \left\{\frac{\theta^0 - 2\pi}{2}, \frac{\theta^0 - \pi}{2}\right\} - B_{\tau^k}.$$

That is, in terms of the BM, the stopping time σ^{k+1} is the first time $t \geq \tau^k$ such that

$$B_t - B_{\tau^k} \in \left\{\frac{\theta^0 - 2\pi}{2}, \frac{\theta^0 - \pi}{2}\right\} - B_{\tau^k},$$

completing the proof. □

Lemma 2.5. *Let* $\theta \in (0, 2\pi)$. *Then*

$$d_t = O_t(1)e^{-(\lambda_0^D(1,0,\frac{\pi}{2}) \wedge \lambda_0^D(1,b,\pi))t}\begin{cases} t & \lambda_0^D(1,b,\pi) = \lambda_0^D(1,0,\frac{\pi}{2}); \\ 1 & \textit{otherwise,} \end{cases}$$

and for all $t > 1$*, the function* $O_t(1)$ *is independent of* θ*.*

Proof. We first prove the lemma for $\theta \in (\pi, 2\pi)$. At time T_1 either the two copies coincide and coupling is achieved or that they are π units away. In the latter case, we continue the coupling by letting

$$X_t^1 = X_{T_1}^1 + (B_t - B_{T_1}) + b(t - T_1),\ X_t^2 = X_{T_1}^2 + (B_t - B_{T_1}) + b(t - T_1).$$

A coupling will occur when one of the copies hits 0 or 2π. Since their distance is π and one, say X^1, is in $(0, \pi)$, the coupling will occur exactly when X^1 exits the interval $(0, \pi)$. Summarizing, τ_C is bounded above by the sum $T_1 + T_2$ where $T_2 = \inf\{t \geq 0 : X_{T_1+t}^1 \in$

$\{0,\pi\}\}$. In particular, $P_{\theta,\pi}(\tau_C > t) \le P_{\theta,\pi}(T_1 + T_2 > t)$. Therefore

$$
\begin{aligned}
d_t(\theta,\pi) &\le P_{\theta,\pi}(T_1+T_2>t) \\
&\le \sum_{k=0}^{\lfloor t-1\rfloor} P_{\theta,\pi}(T_1 > t-k-1, T_2 \in (k,k+1]) \\
&\le \sum_{k=0}^{\lfloor t-1\rfloor} E_{\theta,\pi}\left(\mathbf{1}_{\{T_1>t-k-1\}} P^b_{X^1_{T_1}\wedge X^2_{T_1}-\pi/2}(\tau_\pi > k)\right) \\
&\le O_t(1) \sum_{k=0}^{\lfloor t-1\rfloor} e^{-\lambda_0^D(1,0,\frac{\pi}{2})(t-k-1)} e^{-\lambda_0^D(1,b,\pi)k} \\
&= O_t(1) e^{-(\lambda_0^D(1,0,\frac{\pi}{2})\wedge\lambda_0^D(1,b,\pi))t} \begin{cases} t & \lambda_0^D(1,b,\pi)=\lambda_0^D(1,0,\frac{\pi}{2}); \\ \left(1-e^{-|\lambda_0^D(1,b,\pi)-\lambda_0^D(1,0,\frac{\pi}{2})|}\right)^{-1} & \text{otherwise,} \end{cases}
\end{aligned} \tag{2.2}
$$

where the inequality on the fourth line follows from Lemma 2.3, with $O_t(1)$ independent of θ. This completes the proof for $\theta \in (\pi, 2\pi)$.

Suppose now that $\theta \in (0,\pi)$. The Consider the coupling

$$X^1_t = \theta + B_t + bt,\ X^2_t = \pi + X^1_t,\ t \ge 0.$$

Let $\tau = \inf\{t : X^1_t \in \{0,\pi\}\}$. At time τ, $X^2_\tau = \pi$ or 2π according to whether $X^1_\tau = 0$ or $X^2_\tau = \pi$. We continue the coupling for $t \ge \tau$ by redefining X^1 and X^2 through

$$X^2_t = X^1_t = X_{t-\tau},\ t \ge \tau.$$

Thus, X^1 and X^2 are coupled by time τ, and τ is the exit time of BM with drift b, starting from θ from the interval $(0,\pi)$. Lemma 2.3 gives

$$d_t(\theta,\theta+\pi) \le P^b_{\theta-\pi/2}(\tau_\pi > t) \le O_t(1)e^{-\lambda_0^D(1,b,\pi)t}.$$

From the triangle inequality,

$$d_t(\theta,\pi) \le d_t(\theta,\theta+\pi) + d_t(\theta+\pi,\pi),$$

and the result follows from (2.2). □

3 Proof of Proposition 1.4

Proof. Similarly to Section 2 we will consider the DR with diffusivity $\sigma = 1$, drift $b = \frac{\ell\mu}{2\pi\sigma^2}$ on the interval $\ell = (0, 2\pi)$. The general case follows from the scaling identity (1.1) applied to this result.

Suppose $L_{1,b}u = -\lambda u$, and $u \in \mathcal{D}_{2\pi}$, and let $\mathbf{v}(x) = \binom{u'(x)}{u(x)}$. Then since $(u')' = -2bu' - 2\lambda u$, we have

$$\mathbf{v}' = \mathcal{A}\mathbf{v} \text{ where } \mathcal{A} = \begin{pmatrix} -2b & -2\lambda \\ 1 & 0 \end{pmatrix}.$$

Note that a priori λ may not be real. This is crucial. The solution is of the form $\mathbf{v}(x) = e^{xA}\mathbf{v}(0)$. If $\mathcal{A}$ is diagonalizable, with eigenvalues $\lambda_1,\ \lambda_2$, then this implies $u(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$. Otherwise, $u(x) = e^{\lambda_1 x}(A + Bx)$. The trace of $\mathcal{A}$ gives $\lambda_1 + \lambda_2 = -2b$.

Since b is real, $\Im\lambda_1 + \Im\lambda_2 = 0$. The determinant gives $\lambda_1\lambda_2 = 2\lambda$.

We continue according to the cases.

1. Inequality, $\lambda_1 \neq \lambda_2$. Then $\mathcal{A}$ is diagonalizable, and the boundary condition is

$$A + B = Ae^{\pi\lambda_1} + Be^{\pi\lambda_2} = Ae^{2\pi\lambda_1} + Be^{2\pi\lambda_2}.$$

There are two cases to consider.

$\mathbf{AB} = \mathbf{0}$. Without loss of generality, $A = 0$ and $B = 1$. The boundary condition reduces to $1 = e^{\pi\lambda_2} = e^{2\pi\lambda_2}$, therefore $\pi\lambda_2 = 2\pi ik$ for some $k \in \mathbb{Z}$, and then the trace implies $\lambda_1 = -2b - 2ik$, with $b \neq 0$ or $k \neq 0$ (otherwise $\lambda_1 = \lambda_2$). The determinant then gives $2\lambda = (-2b - 2ik)2ik$. In particular, $\lambda = 2k^2 - 2bki$, and $k \neq 0$ or $b \neq 0$.
$\mathbf{AB} \neq \mathbf{0}$. Let $\alpha = e^{\pi\lambda_1}$ and let $\beta = e^{\pi\lambda_2}$. The boundary condition could be rewritten as

$$A(1-\alpha) = -B(1-\beta), \text{ and } A(1-\alpha^2) = -B(1-\beta^2).$$

If $\alpha = 1$, then $\beta = 1$, and this is equivalent to $\lambda_1 = 2ik_1$, and $\lambda_2 = 2ik_2$ for $k_1, k_2 \in \mathbb{Z}$. The trace holds if and only if $b = 0$ and $k_2 = -k_1 \neq 0$ (again, because $\lambda_1 \neq \lambda_2$), in which it follows from the determinant that $\lambda = 2k_1^2$. This eigenvalue already appeared for the previous case. If $\alpha \neq 1$, then $\beta \neq 1$ and we can divide the second equation by the first to obtain $\alpha = \beta$. That is $\pi\lambda_2 = \pi\lambda_1 + 2\pi ik$ for some $k \in \mathbb{Z} - \{0\}$. From the trace, we obtain $-2b = 2\lambda_1 + 2ki$, hence $\lambda_1 = -b - ik$. The determinant gives $2\lambda = (-b-ik)(-b+ik)$. That is, $\lambda = \frac{b^2+k^2}{2},\ k \in \mathbb{Z} - \{0\}$.

Summarizing, the eigenvalues obtained when $\lambda_1 \neq \lambda_2$ are $2k^2 - 2bki$ with $k \neq 0$ or $b \neq 0$, and $\frac{b^2+k^2}{2}$ for $k \neq 0$.

2. Equality, $\lambda_1 = \lambda_2$. This clearly holds if and only if $\lambda_1 = -b$, and an eigenvector $\phi = \binom{\phi_1}{\phi_2}$ for $\mathcal{A}$ must satisfy $\phi_1 = -b\phi_2$. Thus, the eigenspace is one-dimensional and spanned by the vector $(-b, 1)$. In particular, the boundary condition reads

$$A = e^{-\pi b}(A + \pi B), \text{and } A = e^{-2\pi b}(A + 2\pi B).$$

If $A = 0$ then $B = 0$, therefore there is no loss of generality assuming $A = 1$. The first equation becomes $e^{\pi b} = 1 + \pi B$, and the second equation becomes $e^{2\pi b} = 1 + 2\pi B = 1 + 2(e^{\pi b} - 1) = 2e^{\pi b} - 1$. Clearly, this equation holds if and only if $b = 0$.

Summarizing this case, the only eigenvalue obtained is 0, corresponding to $b = 0$. □

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