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Complete Ricci-Flat Kähler Manifolds of Infinite Topological Type

Michael T. Anderson [1,*], Peter B. Kronheimer [2,**], and Claude LeBrun [1,*]

[1] Department of Mathematics, SUNY at Stony Brook, Stony Brook, NY 11790, USA
[2] School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540, USA and Merton College, Oxford, OX1 4JD, United Kingdom

Abstract. We display an infinite dimensional family of complete Ricci-flat Kähler manifolds of complex dimension 2, for which the second homology is infinitely generated. These are obtained from the Gibbons-Hawking Ansatz [2] by using infinitely many, sparsely distributed centers.

Introduction

In [2], Gibbons and Hawking construct families of complete Ricci-flat Kähler metrics on a class of non-compact 4-manifolds N_k. The metrics are asymptotically locally Euclidean in the sense that $\partial N_k \approx S^3/\mathbb{Z}_k$, and the metrics approach, at infinity, the locally Euclidean metric on the cone $C(S^3/\mathbb{Z}_k)$. Another description of these metrics was given by Hitchin [3]. Further examples, with boundary a spherical space form S^3/Γ, $\Gamma \subset SU(2)$, and a characterization of these metrics (Torelli theorem) among asymptotically locally Euclidean metrics were obtained by Kronheimer [5, 6].

In this paper, we show that one may also obtain complete Ricci-flat Kähler metrics corresponding to the case "$k=\infty$" of the Gibbons-Hawking metrics. These metrics are no longer asymptotically locally Euclidean, or of finite action, and are carried by a 4-manifold whose 2nd homology is infinitely generated. It is only recently (7) that examples of complete metrics of non-negative Ricci curvature have been exhibited on manifolds of infinite topological type.

The example shows that a complex 2-manifold supporting a complete Ricci-flat Kähler metric need not be the complement of a divisor in a compact complex surface since the homology of such a complement is certainly finitely generated. This indicates that a conjecture of Yau [8, 9] concerning the existence of such compactifications is not true without some strengthening of the hypothesis.

These metrics also provide the first example for which the moduli space of complete Ricci-flat metrics on a given manifold is infinite dimensional.

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1. Construction of the Manifold

We begin by considering any divergent sequence of distinct points $p_j \in \mathbb{R}^3, j \in \mathbb{N}$. We will construct a 4-manifold M and a smooth map $\pi: M \to \mathbb{R}^3$ such that $\pi^{-1}(p_j)$ is a point for all j, but $\pi^{-1}(p) \approx S^1$ for $p \in \mathbb{R}^3 - \{p_j\}$. To begin, we let π_0 $M_0 \to \mathbb{R}^3 - \{p_j\}$ be the principal S^1 bundle whose Chern class is -1 when restricted to a small sphere around any p_j; here "small" means of radius less than $r_j = \min_{k \neq j} \|p_k - p_j\|$. Since

$$H_2(\mathbb{R}^3 - \{p_j\}, \mathbb{Z}) \simeq \bigoplus_{j=1}^{\infty} \mathbb{Z}$$

is the free abelian group generated by the homology classes of these small spheres, this uniquely determines the Chern class in

$$H^2(\mathbb{R}^3 - \{p_j\}, \mathbb{Z}) \simeq \prod_{j=1}^{\infty} \mathbb{Z},$$

and thus determines a unique principal S^1 bundle. Thus $\pi_0^{-1}(B_{r_j}(p_j))$ is diffeomorphic to a punctured 4-ball $\hat{B}_j - \{0\} \subset \mathbb{R}^4$ in a manner such that the S^1 action becomes the action of $S^1 \subset \mathbb{C}$ on $\mathbb{C}^2 = \mathbb{R}^4$ by scalar multiplication. We then define

$$M = M_0 \cup \bigcup_{j=1}^{\infty} \hat{B}_j := M_0 \sqcup \bigsqcup_{j=1}^{\infty} \hat{B}_j / \sim,$$

where the equivalence relation $\sim$ identifies $\hat{B}_j - \{0\}$ with $\pi_0^{-1}(B_{r_j}(p_j))$. The map $\pi_0: M_0 \to \mathbb{R}^3$ clearly extends to a smooth map $\pi: M \to \mathbb{R}^3$. Note that there is an S^1 action on M and π is just the projection to the orbit space, with $\{p_j\}$ corresponding to the fixed points of the action.

To understand better the topology of M, consider the case in which the points p_j in the description above are given by $p_j = (x_j, 0, 0)$, with $x_j < x_{j+1}$ and let $D_j = \pi^{-1}([p_j, p_{j+1}])$ be the inverse image of the line segment $[p_j, p_{j+1}] \subset \mathbb{R}^3$. Each D_j is a smoothly embedded 2-sphere with self-intersection -2, meeting D_{j+1} transversely at the point $\pi^{-1}(p_{j+1})$. Clearly, the manifold M is diffeomorphic to the open subset $N \subset M$ consisting of the tubular neighborhood of these spheres. It follows that M is simply connected and

$$H_q(M, \mathbb{Z}) = \begin{cases} \mathbb{Z} & q = 0 \\ \bigoplus_{j=1}^{\infty} \mathbb{Z} & q = 2 \\ 0 & \text{otherwise.} \end{cases}$$

This description can be summarized by saying that M is the result of plumbing an infinite family of 2-spheres according to the "Cartan matrix"

$$A_\infty = \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & 1 & \cdot & \cdot \\ & & & \cdot & \cdot \end{bmatrix}.$$

Note that the Gibbons-Hawking metrics with k centers correspond to plumbing a collection of $(k-1)$ 2-spheres according to the Cartan matrix of A_k, cf. [1, 3].

2. The Gibbons-Hawking Metric

We now restrict somewhat the above choice of the sequence $\{p_j\}_1^\infty$ in $\mathbb{R}^3$. Namely, we impose the extra condition that, for some point $p_0 \in \mathbb{R}^3$ we have

$$\sum_{j=1}^{\infty} \frac{1}{\|p_0 - p_j\|} < \infty ;$$

for example, we might take $p_j = (j^2, 0, 0)$ and let $p_0 = (0, 0, 0)$. It then follows that $V: \mathbb{R}^3 - \{p_j\} \to \mathbb{R}$ defined by

$$V(p) = \frac{1}{2} \sum_{j=1}^{\infty} \frac{1}{\|p - p_j\|}$$

is a smooth function on $\mathbb{R}^3 - \{p_j\}$. Clearly, V is a solution of the Laplace equation

$$\Delta V = d * dV = 0 ,$$

where $*$ is the Hodge $*$ operator on $\mathbb{R}^3$. Further, it is easily verified that the cohomology class of the closed 2-form $\frac{1}{2\pi} * dV$ represents the Chern class of the principal S^1 bundle π_0: $M_0 \to \mathbb{R}^3 - \{p_j\}$ in deRham cohomology. There is therefore a connection on π_0: $M_0 \to \mathbb{R}^3 - \{p_j\}$ with curvature $*dV$. Let $\omega \in \Omega^1(M_0)$ be the connection 1-form for such a connection, so that

$$\pi_0^* (*dV) = d\omega .$$

The form ω is then unique up to gauge transformations, since $\mathbb{R}^3 - \{p_j\}$ is simply connected. The *Gibbon-Hawking metric* on M_0 is given by

$$g = \frac{1}{V} \omega \odot \omega + V \pi_0^* ds^2 ,$$

where ds^2 is the Euclidean metric on $\mathbb{R}^3$. It has anti-self dual curvature tensor, as follows from $d\omega = \pi_0^*(*dV)$, see for example [4]. In particular g is Ricci-flat. Since M_0 is simply connected, it follows that M_0 is hyperkähler, (cf. [4]), i.e. there is an entire 2-sphere's worth of complex structures for which g is a Kähler metric.

To display these parallel complex structures explicitly, let e_1, e_2, e_3 be any oriented orthonormal basis for $\mathbb{R}^3$. Consider these as constant vector fields on $\mathbb{R}^3$ and let $\hat{e}_1, \hat{e}_2, \hat{e}_3$ be their horizontal lifts to M_0 via the connection ω. Further, let X denote the generator of the S^1 action on M_0. Then

$$V^{1/2} X,\ V^{-1/2} \hat{e}_1,\ V^{-1/2} \hat{e}_2,\ V^{-1/2} \hat{e}_3 ,$$

is an orthonormal frame for M_0. Relative to this frame, the matrix

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

defines an almost complex structure, depending only on the choice of e_1, which one may verify to be parallel, and hence integrable.

The Gibbons-Hawking metric now continues smoothly across the isolated points $\pi^{-1}(p_j)$. Indeed, near p_j, we have $V = \frac{1}{2r} + f \equiv V_0 + f$, where $r(p) = \|p - p_j\|$, and where f is smooth. If ω_0 is the connection form on $\pi_0^{-1}(B_\varepsilon(p_j) - \{p_j\})$ with $d\omega_0 = \pi_0^*(*dV_0)$, then it is easily seen that the metric

$$g_0 = \frac{1}{V_0} \omega_0 \odot \omega_0 + V_0 \pi_0^*(ds^2)$$

extends smoothly over $\pi_0^{-1}\{p_j\} \subset M$. In fact, g_0 is just the flat metric defined near $\pi_0^{-1}(p_j)$, as one sees by performing the coordinate change $r \to \sqrt{2r}$. Clearly, the metrics g and g_0 differ by a smooth bilinear form, depending on f only, so that g extends smoothly to M.

It follows that the curvature tensor in again anti-self dual, and, since M is simply connected, this makes M hyperkähler. As a consequence, any of the parallel complex structures on M_0 extends as a parallel complex structure to M. Choosing one makes (M, g) a Ricci-flat Kähler surface.

As a particular case, suppose again that $p_j = (x_j, 0, 0)$, with $x_j < x_{j+1}$ and $\Sigma 1/|x_j| < \infty$. If e_1 points along the x-axis, then the 2-sphere D_j described in Sect. 1 is a holomorphic curve with respect to the complex structure defined above. If, on the other hand, we consider the complex structure corresponding to any other direction in $\mathbb{R}^3$, then M contains no holomorphic curves: for example, if e_1 points along the z-axis, then M becomes biholomorphically equivalent to the hypersurface in $\mathbb{C}^3$ defined by the equation

$$\zeta_1 \cdot \zeta_2 = \prod_{j=1}^{\infty} \left(1 - \frac{\zeta_3}{x_j}\right).$$

Briefly, to see this, note that the projection, $\bar{\pi}$, of M onto the (e_2, e_3) plane, thought of as $\mathbb{C}$, is holomorphic. This defines the coordinate ζ_3 above. The fibre $\bar{\pi}^{-1}(\zeta_3)$ is generically one orbit of the $\mathbb{C}^*$ action, the complexification of the S^1 action on M defined in Sect. 1. Note that $\mathbb{C}^* \simeq \{\zeta_1 \cdot \zeta_2 = 1\} \subset \mathbb{C}^2$. The only exception is where ζ_3 is the image of one of the p_j, in which case $\bar{\pi}^{-1}(\zeta_3)$ is the curve $\zeta_1 \cdot \zeta_2 = 0$.

This description is precisely analogous to the description [3] of the complex manifolds arising from the Gibbons-Hawking ansatz in the case of finitely many centers. In view of the classification scheme of [6], one might expect a similar limit for the family of gravitational instantons corresponding to the Cartan matrices D_k as $k \to \infty$.

3. Completeness

Let $z_n \in M$ be a Cauchy sequence with respect to g; let $y_n = \pi(z_n)$ denote the projection of the sequence to $\mathbb{R}^3$. We claim that $\{z_n\}$ converges. If not, we have $y_n \neq p_1$ for all but finitely many n, so without loss of generality, $y_n \neq p_1$ for all n.

Let δ denote distance in $\mathbb{R}^3 - \{p_1\}$ with respect to the metric $(ds^2)/2r$, where $r(p) = \|p - p_1\|$; let $\hat{\delta}$ denote distance in M with respect to g. Then for any $a, b \in M$ we have

$$\hat{\delta}(a,b) > \delta(\pi(a), \pi(b)).$$

Indeed, it suffices to observe that for any curve γ in M_0, the length of $\pi\gamma$ with respect to $V ds^2$ is less than that of γ with respect to g, since g was constructed so as to make $(M_0, g) \to (\mathbb{R}^3 - \{p_j\}, V ds^2)$ a Riemannian submersion. But since $V > 1/2r$, $\pi\gamma$ is even shorter with respect to $(ds^2)/2r$. Since $\hat{\delta}(a,b)$ is just the infimum of the lengths of curves joining a and b in M_0, the inequality follows.

Thus y_n is a Cauchy sequence with respect to $ds^2/2r$. We claim that $\|p_1 - y_n\|$ is therefore bounded. Indeed, we know that for some C,

$$\delta(y, y_n) < C$$

for all n, since the sequence is Cauchy; but for any curve $\alpha: [a, b] \to \mathbb{R}^3 - \{p_1\}$ we have that the length of α with respect to $\frac{ds^2}{2r}$ satisfies

$$L(\alpha) = \int_a^b \frac{1}{\sqrt{2r}} \|\alpha'(t)\| \, dt \geqq \int_{r(a)}^{r(b)} \frac{|dr|}{\sqrt{2r}} \geqq \sqrt{2}\, |\sqrt{r(b)} - \sqrt{r(a)}|,$$

so that $\delta(y, y_n) \geqq \sqrt{2}\, |\sqrt{r(y_n)} - \sqrt{r(y_1)}|$.

Hence $r(y_n) < \left(\frac{C}{\sqrt{2}} + \sqrt{r(y_1)}\right)^2 = R$ for all n, and y_n is a bounded sequence.

It follows that for some R the sequence $\{z_n\}$ is contained in $\pi^{-1}(\overline{B_R(p_1)})$. Since this is compact, it follows that $\{z_n\}$ converges.

To summarize, we have proved

Theorrem. *(M, g) is a complete, hyperkähler 4-manifold with infinitely generated homology group H_2.*

We note finally that we have produced an infinite-dimensional family of such metrics on M. Indeed, as in the finite case [3], the configuration of points $\{p_j\}$ can be uniquely recovered from the metric g, to within an isometry of $\mathbb{R}^3$. One way to prove this is to observe first that the natural isometric circle action on M is uniquely determined as being the only circle action to preserve all the complex structures. (Hyperkähler 4-manifolds with more than one such circle action can be classified, and ours is not on the list.) The projection $\pi: M \to \mathbb{R}^3$ is then the momentum mapping for this action, in the sense of [4], and the configuration $\{p_j\}$ is the image of the fixed point set.

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