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Analyticity of Scattering for the φ^4 Theory

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Abstract. We consider scattering for the equation $(\Box + m^2)\varphi + \lambda\varphi^3 = 0$ on four-dimensional Minkowski space. For $m > 0$, one-to-one and onto wave operators $W_\lambda{}^\pm : \mathbf{H} \to \mathbf{H}$ are known to exist for all $\lambda \geqq 0$, where $\mathbf{H}$ denotes the Hilbert space of finite-energy Cauchy data. We prove that the maps $(\lambda, u) \mapsto W_\lambda{}^\pm(u)$ and $(\lambda, u) \mapsto (W_\lambda{}^\pm)^{-1}(u)$ are continuous from $[0, \infty) \times \mathbf{H}$ to $\mathbf{H}$, and extend to real-analytic functions from an open neighborhood of $\{0\} \times \mathbf{H} \cup \mathbb{R} \times \{0\} \subset \mathbb{R} \times \mathbf{H}$ to the Hilbert space $\mathbf{H}_{-1}$ of Cauchy data with Poincaré-invariant norm. For $m = 0$, wave operators $W_\lambda{}^\pm$ are known to exist as diffeomorphisms of $\mathbf{H}$ for all $\lambda \geqq 0$, where here $\mathbf{H}$ denotes the Hilbert space of finite Einstein energy Cauchy data. In this case we prove that the maps $(\lambda, u) \mapsto W_\lambda{}^\pm(u)$ and $(\lambda, u) \mapsto (W_\lambda{}^\pm)^{-1}(u)$ extend to real-analytic functions from a neighborhood of $[0, \infty) \times \mathbf{H} \subset \mathbb{R} \times \mathbf{H}$ to $\mathbf{H}$.

1. Introduction

The classical φ^4 theory is the Poincaré-invariant nonlinear wave equation:

$$(\Box + m^2)\varphi + \lambda\varphi^3 = 0, \quad m, \lambda \geqq 0,$$

where $\Box$ denotes the D'Alembertian on Minkowski space, $\mathbf{M}_0 \cong \mathbb{R}^4$, and φ is a real-valued function on $\mathbf{M}_0$. Its main interest is as a simple classical analogue of the equations describing interacting relativistic quantum fields, which in four dimensions have so far resisted attempts at a rigorous formulation. The possibility of the existence of wave and scattering operators for this equation as transformations of the Hilbert space $\mathbf{H}$ of finite-energy solutions of the free equation ($\lambda = 0$) was suggested by Segal, who first published results in this direction in 1966 [10]. The problem inspired a large amount of research, most focusing on the massive case ($m > 0$). In 1978 Strauss [12] proved for this case the existence of wave operators $W^\pm : \mathbf{H} \to \mathbf{H}$ such that

$$\lim_{t \to \pm\infty} \| U(t)u - V(t)W^\pm u \| = 0$$

for each $u \in \mathbf{H}$, where $U(t)$ is the unitary group on $\mathbf{H}$ corresponding to time evolution by the free equation and $V(t)$ is the nonlinear group action corresponding to time evolution by the interacting ($\lambda > 0$) equation. In 1983 Brenner [5] proved the invertibility of the $W^{\pm}$, hence the existence of the scattering operator $S = (W^+)^{-1} W^-$.

In the massless case, use of the Hilbert space of finite-energy data is plagued by "infrared divergences," i.e., by the fact that the spectrum of the free Hamiltonian is not bounded away from zero. However, the massless case is conformally invariant, and this additional symmetry makes natural the use of the "Einstein energy norm," the weakest norm dominating the Minkowski energy norm relative to which the conformal group acts continuously on solutions of the free Eq. [8]. Wave and scattering operators as transformations of the Hilbert space of finite Einstein energy data were recently proved to exist by Segal and the present authors [3].

The initial problem having been solved, it is now reasonable to study the regularity properties of these wave and scattering operators, and determine for example whether they are continuous, smooth, or even real-analytic, as functions of the Cauchy datum or of the coupling constant λ. Some work along these lines has already been done. In the massive case, Morawetz Strauss have proved that the scattering operator is a diffeomorphism of a Banach space with norm stronger than the energy norm, and have used certain analyticity properties of this map [6, 7]. These analyticity properties were expanded upon by Raczka and Strauss [15]. Strauss has also proved that the wave and scattering operators are homeomorphisms in a neighborhood of zero [13]. More recently, one of the present authors proved for the massless case that the wave and scattering operators are diffeomorphisms of the space of finite Einstein energy data [2].

Here we show that in the massive case the maps $(\lambda, u) \mapsto W_\lambda{}^{\pm} u$, $(\lambda, u) \mapsto (W_\lambda{}^{\pm})^{-1} u$, and $(\lambda, u) \mapsto S_\lambda u$ are continuous from $[0, \infty) \times \mathbf{H}$ to $\mathbf{H}$, where $\mathbf{H}$ denotes the space of finite-energy data. Moreover, they extend to real-analytic functions from an open neighborhood of $\{0\} \times \mathbf{H} \cup \mathbb{R} \times \{0\}$ in $\mathbb{R} \times \mathbf{H}$ to the space $H^{1/2}(\mathbb{R}^3) \oplus H^{-1/2}(\mathbb{R}^3)$. For the massless case, taking $\mathbf{H}$ to denote the space of finite Einstein energy data, the maps $(\lambda, u) \mapsto W_\lambda{}^{\pm} u$, $(\lambda, u) \mapsto (W_\lambda{}^{\pm})^{-1} u$, and $(\lambda, u) \mapsto S_\lambda u$ extend to real-analytic functions from an open neighborhood of $[0, \infty) \times \mathbf{H} \subset \mathbb{R} \times \mathbf{H}$ to $\mathbf{H}$. Perhaps the most surprising aspect of these results is analyticity at $\lambda = 0$.

2. Analytic Nonlinear Semigroups

For the study of scattering, nonlinear wave equations are most conveniently formulated in terms of groups of nonlinear time evolution operators on a Banach space of initial data. We begin with a useful condition for a semigroup of nonlinear operators to be analytic in the initial datum, or analytic in parameters such as coupling constants. (Here and in what follows, "Banach space" means "real Banach space" and "analytic" means "real-analytic" unless otherwise specified; also, all function spaces are of real-valued functions.)

Let $\mathbf{X}$ and $\mathbf{Y}$ be Banach spaces, and suppose that $Q \subseteq \mathbf{X}$ is open and $f: Q \to \mathbf{Y}$. We define f to be "smooth" if for all $x \in Q$ and all $n \geq 0$ the Frechét derivative $D^n f(x)$ of f at x exists; $D^n f(x)$ is an element of $L^n(\mathbf{X}, \mathbf{Y})$, the Banach space of continuous multilinear maps from $\mathbf{X}$ to $\mathbf{Y}$, with norm as in [4]. Given $h \in \mathbf{X}$, let

$D^n f(x; h)$ denote the result of evaluating $D^n f(x)$ on the n-tuple $(h, \ldots, h)$. We define f to be "analytic" if f is smooth and for each $x \in Q$ there exists $\varepsilon > 0$ such that the sum $\sum_{n \geqq 0} (n!)^{-1} D^n f(x; h)$ converges to $f(x+h)$ for all $h \in \mathbf{X}$ with $\|h\| < \varepsilon$. The following is an analytic version of the implicit function theorem:

Lemma 1. *Suppose that* $\mathbf{X}, \mathbf{Y}$, *and* $\mathbf{Z}$ *are Banach spaces and* Q *is an open neighbourhood of the point* $(x, y) \in \mathbf{X} \times \mathbf{Y}$. *Suppose that* $f: Q \rightarrow \mathbf{Z}$ *is analytic,* $f(x, y) = 0$, *and* $D_2 f(x, y): \mathbf{X} \rightarrow \mathbf{Z}$ *has a left inverse, where the subscript 2 indicates the derivative in the second argument. Then for some open set* P *containing* x *there is a unique analytic function* $g: P \rightarrow \mathbf{Y}$ *such that* $g(x) = y$ *and* $f(x', g(x')) = 0$ *for all* $x' \in P$.

Proof. We may extend f to a complex-analytic function $\bar{f}$ in a neighborhood $\bar{Q}$ of $(x, y) \in \mathbb{C}\mathbf{X} \times \mathbb{C}\mathbf{Y}$ [1]. $D_2 \bar{f}(x, y)$ is the complex-linear extension of $D_2 f(x, y)$, hence it has a left inverse. By the complex-analytic implicit function theorem [4], for some neighborhood $\bar{P}^U \mathbb{C}\mathbf{X}$ of x there exists a complex-analytic function $\bar{g}: \bar{P} \rightarrow \mathbb{C}\mathbf{Y}$ such that $\bar{g}(x) = y$ and $\bar{f}(x', \bar{g}(x')) = 0$ for all $x' \in \bar{P}$. The restriction of $\bar{g}$ to $P = \bar{P} \cap \mathbf{X}$ has the desired properties; uniqueness follows from the usual implicit function theorem. □

Let $\mathbf{X}$ and $\mathbf{Y}$ be Banach spaces. Given $f: \mathbf{X} \rightarrow \mathbf{Y}$, we define f to be "boundedly Lipschitzian" if for each bounded $B \subset \mathbf{X}$ there exists $M > 0$ such that

$$\|f(x) - f(x')\| \leqq M \|x - x'\|. \tag{1}$$

holds for all $x, x' \in B$. Given a set S and a function $f: S \times \mathbf{X} \rightarrow \mathbf{Y}$, we write f_λ for the function $f(\lambda, \cdot): \mathbf{X} \rightarrow \mathbf{Y}$, and say that the functions $\{f_\lambda\}_{\lambda \in S}$ are "uniformly boundedly Lipschitzian" if for each bounded $B \subset \mathbf{X}$ there exists $M > 0$ such that (1) holds for all $x, x' \in B$ for all the functions f_λ.

Proposition 2. *Suppose that* $\mathbf{X}$ *is a Banach space and* $U: [0, \infty) \times \mathbf{X} \rightarrow \mathbf{X}$ *is a strongly continuous semigroup of bounded linear operators on* $\mathbf{X}$. *Suppose that* S *is an open subset of a Banach space* $\mathbf{Z}$, *the function* $N: S \times \mathbf{X} \rightarrow \mathbf{X}$ *is analytic, and the functions* $\{N_\lambda\}_{\lambda \in S}$ *are uniformly boundedly Lipschitzian. Then for each bounded open* $B \subset \mathbf{X}$ *there exists* $\varepsilon > 0$ *such that given* $\lambda \in S, x \in B$ *there is a unique* $f \in C([0, \varepsilon], \mathbf{X})$ *satisfying:*

$$f(t) = U(t)x + \int_0^t U(t-s) N_\lambda(f(s)) ds. \tag{2}$$

The map $(\lambda, x) \mapsto f$ *is analytic from* $S \times B$ *to* $C([0, \varepsilon], \mathbf{X})$.

Proof. For small enough $\varepsilon > 0$, the existence and uniqueness of $f \in C([0, \varepsilon], \mathbf{X})$ satisfying (2) given any $\lambda \in S$, $x \in B$ is a basic result of the theory of nonlinear semigroups [9]. Moreover, f is the unique fixed point of the contraction $T_{\lambda,x}$ mapping the open set

$$Q = \{g \in C([0, \varepsilon], \mathbf{X}): \|g(t) - x\| < 1\}$$

into itself, given by

$$(T_{\lambda,x} g)(t) = U(t)x + \int_0^t U(t-s) N_\lambda(g(s)) ds.$$

The hypotheses imply that the map $T: S \times B \times C([0, \varepsilon], \mathbf{X}) \rightarrow C([0, \varepsilon], \mathbf{X})$ given by

$T(\lambda, x, g) = T_{\lambda,x}g$ is analytic. Since each map $T_{\lambda,x}$ is a contraction, the Frechét derivative $D_3 T(\lambda, x, g)$ of T with respect to g has norm < 1 for all (λ, x, g). Defining $R: S \times B \times C([0, \varepsilon], \mathbf{X}) \to C([0, \varepsilon], \mathbf{X})$ by

$$R(\lambda, x, g) = T(\lambda, x, g) - g,$$

it follows that R is analytic and $D_3 R(\lambda, x, g)$ has a left inverse for all (λ, x, g). By Lemma 1 this implies that $f \in C([0, \varepsilon], \mathbf{X})$, which satisfies

$$R(\lambda, x, f) = 0,$$

depends analytically on $(\lambda, x) \in S \times B$. $\square$

Proposition 3. *Assume the hypotheses of Proposition 2, and suppose* $f \in C([0, T], \mathbf{X})$ *satisfies 2) for some* $(\lambda, x) \in S \times \mathbf{X}$. *Then for every* (λ', x') *in some neighborhood* $Q \subset S \times \mathbf{X}$ *of* (λ, x) *there is a unique* $g \in C([0, T], \mathbf{X})$ *satisfying*

$$g(t) = U(t)x' + \int_0^t U(t-s) N_{\lambda'}(g(s))\, ds \tag{3}$$

and the map $(\lambda', x') \mapsto g$ *is analytic* Q *to* $C([0, T], \mathbf{X})$.

Proof. Choose $\delta > 0$ such that the closed ball of radius δ about λ is contained in S. Let $r = \sup_{t \in [0,T]} \| f(t) \| + 1$ and let $C = \sup_{t \in [0,T]} \| U(t) \|$. Choose $M > 0$ such that

$$\| N_{\lambda'}(y) - N_{\lambda'}(y') \| \leqq M \| y - y' \|$$

for all $y, y' \in \mathbf{X}$ with $\| y \|, \| y' \| \leqq r$ and all λ' with $\| \lambda' - \lambda \| < \delta$. Choose $\varepsilon > 0$ such that $\varepsilon C e^{CMT} < 1$. Let

$$Q = \{ (\lambda', x') \in S \times \mathbf{X} : \| \lambda' - \lambda \| < \delta, \| x' - x \| < \varepsilon \}.$$

Suppose that $(\lambda', x') \in Q$; we claim that there exists a unique $g \in C([0, T], \mathbf{X})$ satisfying (3). The uniqueness of g follows directly from Proposition 2. To prove existence it suffices by Proposition 2 to show that given $T_0 < T$ and $g \in C([0, T_0], \mathbf{X})$ satisfying (3), we have:

$$\sup_{t \in [0, T_0]} \| g(t) \| \leqq r.$$

The proof is by contradiction. Suppose that for some $t \in [0, T_0]$ we have $\| g(t) \| > r$, and let $\tau = \inf \{ t \in [0, T_0] : \| g(t) \| > r \}$. By the continuity of g, $\| g(\tau) \| = r$. On the other hand, 2) and 3) imply that for all $t \in [0, \tau]$,

$$\| g(t) - f(t) \| \leqq C\varepsilon + CM \int_0^t \| g(s) - f(s) \|\, ds,$$

so by Gronwall's inequality and our choice of ε it follows that

$$\| g(\tau) - f(\tau) \| \leqq \varepsilon C e^{CM\tau} < 1,$$

hence

$$\| g(\tau) \| < \| f(\tau) \| + 1 \leqq r.$$

Finally, to prove that $g \in C([0, T], \mathbf{X})$ is analytic as a function of $(\lambda', x') \in Q$, it suffices to make repeated application of Proposition 2, using the semigroup property of the nonlinear time evolution. $\square$

3. The Massless Case

Because of its conformal invariance, scattering for the equation

$$\Box\varphi + \lambda\varphi^3 = 0 \quad (\lambda \geqq 0) \tag{4}$$

may be treated using the universal cover of conformally compactified Minkowski space. We begin by recalling the essential aspects of this method, as developed in [2, 3].

In addition to the usual "Minkowski energy norm" on the Cauchy data space $C_0^\infty(\mathbb{R}^3) \oplus C_0^\infty(\mathbb{R}^3)$:

$$\|(u_1, u_2)\|_M^2 = \tfrac{1}{2} \int_{\mathbb{R}^3} (\nabla u_1)^2 + u_2{}^2,$$

we define the "Einstein energy norm":

$$\|(u_1, u_2)\|_E^2 = \tfrac{1}{2} \int_{\mathbb{R}^3} (r^2/4 + 1)[(\nabla u_1)^2 + u_2{}^2] - \tfrac{1}{2} u_1{}^2.$$

Though not immediately obvious from this definition, $\|\cdot\|_E$ is indeed a norm, and in fact $\|u\|_E \geqq \|u\|_M$ for all $u \in C_0^\infty(\mathbb{R}^3) \oplus C_0^\infty(\mathbb{R}^3)$ [8]. Let **H** denote the completion of $C_0^\infty(\mathbb{R}^3) \oplus C_0^\infty(\mathbb{R}^3)$ with respect to $\|\cdot\|_E$; **H** is a real Hilbert space. For equation (4) with $\lambda = 0$, the time evolution of Cauchy data $(\varphi, \partial_t\varphi) \in \mathbf{H}$ is given by the action of a strongly continuous group $U(t)$ of bounded linear operators on **H**.

Given $u \in \mathbf{H}$ there is a unique global distributional solution φ of 4) on $\mathbf{M}_0$ with $u = (\varphi, \partial_t\varphi)|_{t=0}$, and for all $s \in \mathbb{R}$ the restriction $(\varphi, \partial_t\varphi)|_{t=s}$ is well-defined as an element of **H**. Letting $V(s)u = (\varphi, \partial_t\varphi)|_{t=s}$, V is a strongly continuous group of diffeomorphisms of **H**, and for any $u \in \mathbf{H}$ there exist $u_+, u_- \in \mathbf{H}$ such that

$$\lim_{t \to \pm\infty} \| U(t)u - V(t)u_\pm \|_M = 0.$$

We define wave operators $W^\pm : \mathbf{H} \to \mathbf{H}$ such that $W^\pm(u) = u_\pm$. These wave operators are diffeomorphisms, so there exists a smooth scattering operator $S = (W^+)^{-1} W^- : \mathbf{H} \to \mathbf{H}$. When we wish to make explicit the dependence of $W^\pm$ and S on the coupling constant λ we write them as $W_\lambda{}^\pm$ and S_λ. We caution the reader that the wave operators as defined here are the inverses of those in [2, 3].

The existence of these operators is proved via the following correspondence. Let $\tilde{\mathbf{M}}$ denote the universal cover of the conformal compactification of $\mathbf{M}_0$, which may be identified with the "Einstein universe," that is, $\mathbb{R} \times S^3$ given the metric $dt^2 - ds^2$, where t denotes the $\mathbb{R}$-valued coordinate on $\tilde{\mathbf{M}}$ (the "Einstein time") and ds^2 denotes the Riemannian metric on the unit sphere $S^3 \subset \mathbb{R}^4$. Let $\mathbf{H}(S) = H^1(S^3) \oplus L^2(S^3)$, with norm given by

$$\|(u_1, u_2)\|^2 = \tfrac{1}{2} \int_{S^3} (\nabla u_1)^2 + u_1{}^2 + u_2{}^2.$$

Let A be the skew-adjoint operator on $\mathbf{H}(S)$ given by

$$A(u_1, u_2) = (u_2, -(\Delta + 1)u_1)$$

on the domain $H^2(S^3) \oplus H^1(S^3)$, and let $N_\lambda : \mathbf{H}(S) \to \mathbf{H}(S)$ be the map given by

$$N_\lambda(u_1, u_2) = (0, -\lambda u_1{}^3).$$

Given $u \in \mathbf{H}(S)$ and $\lambda \geqq 0$ there exists a unique continuous function $f:\mathbb{R} \to \mathbf{H}(S)$ such that

$$f(t) = e^{At}u + \int_0^t e^{A(t-s)} N_\lambda(f(s))ds. \tag{5}$$

Given $u \in \mathbf{H}(S)$, let $f(t) = (f_1(t), f_2(t))$, and let $\tilde{\varphi}$ be the function defined a.e. on $\tilde{\mathbf{M}}$ by

$$\tilde{\varphi}(t, x) = f_1(t)(x). \tag{6}$$

Let $C_\pm = (\rho \pm t = \pi) \subset \tilde{\mathbf{M}}$, where ρ denotes the arclength from the point $(1, 0, 0, 0) \in S^3$. Let $\mathbf{H}(C_\pm)$ denote the Sobolev spaces $H^1(C_\pm)$, defined using the identifications of $C_\pm$ with S^3 arising from the projection $\tilde{\mathbf{M}} \to S^3$, with norm

$$\|u\|^2 = \tfrac{1}{2} \int_{S^3} (\nabla u)^2 + u^2.$$

Then the restrictions $\tilde{\varphi}|C_\pm$ are well-defined as elements of $\mathbf{H}(C_\pm)$; moreover, the maps $(\tilde{W}^\pm)^{-1}: \mathbf{H} \to \mathbf{H}(C_\pm)$ given by

$$(\tilde{W}^\pm)^{-1}(u) = \tilde{\varphi}|C_\pm$$

are diffeomorphisms. When we wish to make explicit the dependence of these operators on the coupling constant λ we shall write them as $(\tilde{W}_\lambda{}^\pm)^{-1}$. Finally, for certain orthogonal isomorphisms $U: \mathbf{H} \to \mathbf{H}(S)$, $U_\pm: \mathbf{H} \to \mathbf{H}(C_\pm)$ (independent of λ), there is a commutative diagram:

$$\begin{array}{ccc} \mathbf{H} & \xrightarrow{W_\lambda{}^\pm} & \mathbf{H} \\ {\scriptstyle U_\pm}\downarrow & & \downarrow{\scriptstyle U} \\ \mathbf{H}(C_\pm) & \xrightarrow[\tilde{W}_\lambda{}^\pm]{} & \mathbf{H}(S) \end{array} \tag{6}$$

The analyticity properties of the wave and scattering operators are given by:

Theorem 4. *The functions* $(\lambda, u) \mapsto W_\lambda{}^\pm u$, $(\lambda, u) \mapsto (W_\lambda{}^\pm)^{-1} u$, *and* $(\lambda, u) \mapsto S_\lambda u$ *extend to analytic functions from an open neighborhood of* $[0, \infty) \times \mathbf{H} \subset \mathbb{R} \times \mathbf{H}$ to $\mathbf{H}$.

Proof. We begin by extending the function $(\lambda, u) \mapsto (\tilde{W}_\lambda{}^+)^{-1} u$ to an analytic function on an open neighborhood of $[0, \infty) \times \mathbf{H}(S)$. If $\lambda < 0$, there may not be a global solution of Eq. (5). Note, however, that $N: \mathbb{R} \times \mathbf{H}(S) \to \mathbf{H}(S)$ is analytic, and the functions $\{N_\lambda\}$ are uniformly boundedly Lipschitzian as λ ranges over any bounded set in $\mathbb{R}$ (these follow immediately from the Sobolev inequalities and the fact that all derivatives of N above the fourth vanish). Thus by Proposition 3, there exists an open set $P \subset \mathbb{R} \times \mathbf{H}(S)$ containing $[0, \infty) \times \mathbf{H}(S)$ such that for each $(\lambda, u) \in P$ there is a unique $f \in C([0, \pi], \mathbf{H}(S))$ satisfying (5), and the map $(\lambda, u) \mapsto f$ is analytic from P to $C([0, \pi], \mathbf{H}(S))$.

Let R denote the region in $\tilde{\mathbf{M}}$ defined by $\{0 < \tau < \pi - \rho)$. As in Theorem 5 of [2], there is a bounded linear operator $W: \mathbf{H}(S) \oplus L^2(R) \to \mathbf{H}(C_+)$ such that

$$(\tilde{W}_\lambda{}^+)^{-1} u = W(u, -\lambda \tilde{\varphi}^3 | R),$$

where $\tilde{\varphi}$ is defined by (6). Since $(\lambda, u) \mapsto f$ is analytic from P to $C([0, \pi], \mathbf{H}(S))$, the

Sobolev inequalities imply that $(\lambda, u) \mapsto -\lambda\tilde{\varphi}^3|R$ is analytic from P to $L^2(R)$. It follows that the map $F: P \to \mathbf{H}(C_+)$ defined by

$$F(\lambda, u) = W(u, -\lambda\tilde{\varphi}^3|R)$$

is analytic, and extends the map $(\lambda, u) \mapsto (\tilde{W}_\lambda{}^+)^{-1}u$.

Next we extend the function $(\lambda, u) \mapsto \tilde{W}_\lambda{}^+ u$ to an analytic function on an open neighborhood of $[0, \infty) \times \mathbf{H}(C_+)$. $\tilde{W}_\lambda{}^+$ is an orthogonal linear operator for $\lambda = 0$ [3]. This implies that there is an open set $Q \subseteq P$ containing $[0, \infty) \times \mathbf{H}(S)$ such that for each $(\lambda, u) \in Q$ with $\lambda < 0$ the Frechét derivative $DF_\lambda(u)$ satisfies

$$\| DF_\lambda(u) - (\tilde{W}_0{}^+)^{-1} \| < \tfrac{1}{2}. \tag{7}$$

One can choose Q such that each "slice" $Q \cap \{\lambda = c\}$ for $c < 0$ is convex. We now restrict F to Q, still denoting this restriction as F.

We claim that for each λ, F_λ is a diffeomorphism onto its open range. For $\lambda > 0$ this follows from $F_\lambda = (\tilde{W}_\lambda{}^+)^{-1}$. For $\lambda < 0$, the inverse function theorem, the inequality (7), and the fact that $(\tilde{W}_0{}^+)^{-1}$ is orthogonal imply that the range of F_λ is open. It thus suffices to show that F_λ is one-to-one for $\lambda < 0$. Suppose that $\lambda < 0$ and $u, v \in \mathbf{H}(S)$ are in the domain of F_λ, a convex subset of $\mathbf{H}(S)$. Then by (7),

$$\begin{aligned} &\| F_\lambda(u) - F_\lambda(v) \| \\ &\geqq \| (\tilde{W}_0{}^+)^{-1}(u-v) \| - \int_0^1 \| DF_\lambda(v + t(u-v); u-v) - (\tilde{W}_0{}^+)^{-1}(u-v) \| \, dt \geqq \tfrac{1}{2} \| u - v \|, \end{aligned}$$

so F_λ is one-to-one.

Defining $G: Q \to \mathbb{R} \times \mathbf{H}(C_+)$ by

$$G(\lambda, u) = (\lambda, F_\lambda(u)),$$

it follows that G is analytic and a diffeomorphism onto its open range. By an analytic version of the inverse function theorem (a corollary of Lemma 1) the function G^{-1} is analytic from $\operatorname{Ran} G$ to Q. Since $G^{-1}(\lambda, u) = (\lambda, F_\lambda{}^{-1}(u))$, the function $(\lambda, u) \mapsto F_\lambda{}^{-1}(u)$ is analytic from $\operatorname{Ran} G$ to $\mathbf{H}(S)$. It is easy to check that the map $(\lambda, u) \mapsto F_\lambda{}^{-1}(u)$ extends the map $(\lambda, u) \mapsto \tilde{W}_\lambda{}^+ u$ to an open set containing $[0, \infty) \times \mathbf{H}(C_+)$.

By symmetry, analogous arguments prove the existence of analytic maps with open domains extending the maps $(\lambda, u) \mapsto \tilde{W}_\lambda{}^- u$ and $(\lambda, u) \mapsto (\tilde{W}_\lambda{}^-)^{-1}u$. The statement of the theorem follows from diagram (6) and the definition of S_λ.

4. The Massive Case

Scattering for the massive φ^4 theory,

$$(\Box + m^2)\varphi + \lambda\varphi^3 = 0 \quad m > 0, \lambda \geqq 0, \tag{8}$$

presents problems of greater technical subtlety than the massless case. We begin by briefly reviewing the relevant results of [5, 9, 12]. The Hilbert space of finite-energy Cauchy data, $\mathbf{H}$, is defined to be $H^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ with norm given by:

$$\| (u_1, u_2) \|^2 = \tfrac{1}{2} \int_{\mathbb{R}^3} (\nabla u_1)^2 + m^2 u_1{}^2 + u_2{}^2.$$

Time evolution of finite-energy Cauchy data for the free equation is given by a strongly continuous group of orthogonal operators $U(t)$ acting on $\mathbf{H}$. Given $u \in \mathbf{H}$ there is a unique distributional solution φ of (8) on $\mathbf{M}_0$ with $u = (\varphi, \partial_t \varphi)|_{t=0}$, and for all $s \in \mathbb{R}$ the restriction $(\varphi, \partial_t \varphi)|_{t=s}$ is well-defined as an element of $\mathbf{H}$. Letting $V_\lambda(s)u = (\varphi, \partial_t \varphi)|_{t=s}$, V_λ is a strongly continuous group of diffeomorphisms of $\mathbf{H}$, and for any $u \in \mathbf{H}$ there exist $u_+, u_- \in \mathbf{H}$ such that

$$\lim_{t \to \pm\infty} \| U(t)u - V_\lambda(t)u_\pm \| = 0.$$

Wave operators $W_\lambda^\pm : \mathbf{H} \to \mathbf{H}$ may thus be defined such that $W_\lambda^\pm(u) = u_\pm$. These wave operators are one-to-one and onto, so there exists a scattering operator $S_\lambda = (W_\lambda^+)^{-1} W_\lambda^- : H \to \mathbf{H}$.

We now prove a continuity result for the wave and scattering operators:

Theorem 5. *The functions* $(\lambda, u) \mapsto W_\lambda^\pm u$, $(\lambda, u) \mapsto (W_\lambda^\pm)^{-1} u$, *and* $(\lambda, u) \mapsto S_\lambda u$ *are continuous from* $[0, \infty) \times \mathbf{H}$ to $\mathbf{H}$.

Proof. The proof will be divided into a sequence of lemmas; Lemma 6 proves the continuity of $(\lambda, u) \mapsto (W_\lambda^-)^{-1} u$, and Lemma 10 proves it for $(\lambda, u) \mapsto W_\lambda^- u$. Analogous arguments imply corresponding continuity results for W_λ^+ and $(W_\lambda^+)^{-1}$ and the continuity of $(\lambda, u) \mapsto S_\lambda u$ follows.

We begin with some definitions and facts. Define N_λ on pairs of measurable functions on $\mathbb{R}^3$ by $N_\lambda(u_1, u_2) = (0, -\lambda u_1^3)$. The function $(\lambda, u_1, u_2) \mapsto N_\lambda(u_1, u_2)$ is analytic from $\mathbb{R} \times \mathbf{H}$ to $\mathbf{H}$, since the Sobolev and Hölder inequalities imply it is smooth, and all derivatives above the fourth vanish. Also, the functions $\{N_\lambda\}$ are uniformly boundedly Lipschitzian from $\mathbb{R} \times \mathbf{H}$ to $\mathbf{H}$ as λ ranges over bounded subsets of $\mathbb{R}$. The nonlinear time evolution group $V_\lambda(t)$ satisfies

$$V_\lambda(t)u = U(t)u + \int_0^t U(t-s) N_\lambda(V_\lambda(s)u)\, ds. \tag{9}$$

Let the "potential energy," $G_\lambda : \mathbf{H} \to \mathbb{R}$, be the continuous function given by

$$G_\lambda(u_1, u_2) = \tfrac{1}{4}\lambda \int_{\mathbb{R}^3} u_1^4. \tag{10}$$

The following conservation of energy results hold [12]: given $u \in \mathbf{H}$ and $\lambda > 0$, for all t,

$$\| V_\lambda(t)u \|_{\mathbf{H}}^2 + G_\lambda(V_\lambda(t)u) = \| u \|_{\mathbf{H}}^2 + G_\lambda(u) = \| W_\lambda^\pm)^{-1} u \|_{\mathbf{H}}^2. \tag{11}$$

Lemma 6. *The function* $(\lambda, u) \mapsto (W_\lambda^-)^{-1} u$ *is continuous from* $[0, \infty) \times \mathbf{H}$ *to* $\mathbf{H}$.

Proof. For convenience let $F(\lambda, u) = (W_\lambda^-)^{-1} u$. Suppose that $\lambda' \to \lambda$ in $[0, \lambda)$ and $u' \to u$ in $\mathbf{H}$. Let $f(t) = V_\lambda(t)u$ and let $f'(t) = V_{\lambda'}(t)u'$. Given $g \in C_0^\infty(\mathbb{R}^3) \oplus C_0^\infty(\mathbb{R}^3) \subset \mathbf{H}$, then by (9) and the definition of the wave operator:

$$\begin{aligned} |\langle g, F(\lambda, u) - F(\lambda', u') \rangle| &= \lim_{t \to -\infty} |\langle U(t)g, (f(t) - f'(t)) \rangle| \\ &\leqq \| g \| \, \| u - u' \| + \int_{-\infty}^0 |\langle U(s)g, N_\lambda(f(s)) - N_{\lambda'}(f'(s)) \rangle| \, ds. \end{aligned}$$

The first term goes to zero as $u' \to u$, and the Sobolev and Hölder inequalities imply that for some constant k,

$$\int_{-\infty}^{0} |\langle U(s)g, N_\lambda(f(s)) - N_{\lambda'}(f'(s))\rangle| ds$$

$$\leqq k(\lambda + \lambda') \int_{-\infty}^{0} \|(U(s)g)_2\|_\infty (\|f(s)\|_{\mathbf{H}} + \|f'(s)\|_{\mathbf{H}})^2 \|f(s) - f'(s)\|_{\mathbf{H}} ds.$$

By an L^∞ decay estimate for solutions of the free Klein–Gordon equation with $C_0{}^\infty$ Cauchy data [10], for some constant $c > 0$,

$$\|(U(s)g)_2\|_\infty \leqq c(|s| + 1)^{-3/2}.$$

This function of s is integrable. By Proposition 3, $\|f(s) - f'(s)\|_{\mathbf{H}} \to 0$ for all $s \in \mathbb{R}$. Moreover, (11) implies the bound

$$(\|f(s)\|_{\mathbf{H}} + \|f'(s)\|_{\mathbf{H}})^2 \|f(s) - f'(s)\|_{\mathbf{H}} < K$$

for all $s \in \mathbb{R}$ and all u, u', λ, λ' in given bounded sets. By the dominated convergence theorem, the integral above approaches zero. It follows that $\langle g, F(\lambda, u)\rangle$ is a continuous function of (λ, u) for all $g \in C_0{}^\infty(\mathbb{R}^3) \oplus C_0{}^\infty(\mathbb{R}^3)$. Since $C_0{}^\infty(\mathbb{R}^3) \oplus C_0{}^\infty(\mathbb{R}^3)$ is dense in $\mathbf{H}$ and $F(\lambda, u)$ is locally bounded by (11), this implies that $\langle g, F(\lambda, u)\rangle$ is a continuous function of (λ, u) for all $g \in \mathbf{H}$.

To prove that F is continuous from $[0, \infty) \times \mathbf{H}$ to $\mathbf{H}$ it thus suffices to show that $\|F(\lambda', u')\|_{\mathbf{H}} \to \|F(\lambda, u)\|_{\mathbf{H}}$. This follows from (11), as

$$\|F(\lambda, u)\|_{\mathbf{H}}{}^2 = \|u\|_{\mathbf{H}}{}^2 + G_\lambda(u), \quad \|F(\lambda', u')\|_{\mathbf{H}}{}^2 = \|u'\|_{\mathbf{H}}{}^2 + G_{\lambda'}(u'),$$

and $G_\lambda(u)$ is continuous as a function of $(\lambda, u) \in [0, \infty) \times \mathbf{H}$. □

The techniques of the rest of the proof are based on ideas of Strauss [12, 13]. We introduce the additional spaces $\mathbf{X} = L^4(\mathbb{R}^3) \oplus L^{4,-1}(\mathbb{R}^3)$ and $\mathbf{Y} = \{0\} \oplus L^{4/3}(\mathbb{R}^3)$. The function $(\lambda, u_1, u_2) \mapsto N_\lambda(u_1, u_2)$ is analytic from $\mathbb{R} \times \mathbf{X}$ to $\mathbf{Y}$, since the Hölder inequalities imply it is smooth, and all derivatives above the fourth vanish. The space $\mathbf{H}$ is dense and compactly embedded in $\mathbf{X}$, and the potential energy as defined in (10) is continuous as a function of $(\lambda, u) \in [0, \infty) \times \mathbf{X}$. Note that $\mathbf{H} \cap \mathbf{Y}$ is dense in $\mathbf{Y}$; as shown in [12], each $U(t)$ with $t > 0$ extends uniquely to a continuous map from $\mathbf{Y}$ to $\mathbf{X}$, also denoted $U(t)$, which satisfies

$$\|U(t)x\|_{\mathbf{X}} \leqq ct^{-1/2} \|x\|_{\mathbf{Y}}. \tag{12}$$

Lemma 7. *Let $I \subseteqq \mathbb{R}$ be an interval. Given $\lambda \in \mathbb{R}$ and $f \in L^4(I, \mathbf{X})$, the function*

$$M(\lambda, f)(t) = \int_{I \cap \{s \leqq t\}} U(t-s) N_\lambda(f(s)) ds$$

also belongs to $L^4(I, \mathbf{X})$. Moreover, $M: \mathbb{R} \times L^4(I, \mathbf{X}) \to L^4(I, \mathbf{X})$ is analytic.

Proof. Our proof follows that of Lemma 1 to [12]. Suppose that $f \in L^4(I, \mathbf{X})$. Since $(\lambda, u) \mapsto N_\lambda(u)$ is analytic from $\mathbb{R} \times \mathbf{X}$ to $\mathbf{Y}$ and satisfies $\|N_\lambda(x)\|_{\mathbf{Y}} = \lambda \|x\|_{\mathbf{X}}{}^3$, the map $(\lambda, f) \mapsto N_\lambda \circ f$ is analytic from $\mathbb{R} \times L^4(I, \mathbf{X})$ to $L^{4/3}(I, \mathbf{Y})$. It thus suffices to show that the map from g to the function

$$h(t) = \int_{I \cap \{s \leqq t\}} U(t-s)g(s)ds$$

is bounded from $L^{4/3}(I, \mathbf{Y})$ to $L^4(I, \mathbf{X})$. This follows from (12) and the fact that convolution with $|t|^{-1/2}$ is bounded from $L^{4/3}(\mathbb{R})$ to $L^4(\mathbb{R})$ [11]. □

Lemma 8. *For each open ball B about the origin of $\mathbb{R} \times \mathbf{H}$, there exists $T \in \mathbb{R}$ such that given $(\lambda, u) \in B$, there is a unique $f \in L^4(I, \mathbf{X})$ with*

$$f(t) = U(t)u + \int_{-\infty}^{t} U(t-s)N_\lambda(f(s))ds$$

where $I = (-\infty, T]$. The map $(\lambda, u) \mapsto f$ is analytic from B to $L^4(I, \mathbf{X})$. If $\lambda \geqq 0$, f satisfies

$$f(t) = V_\lambda(t)W_\lambda^{-}(u).$$

Proof. The first statement follows from Theorem 1 of [12], as does the expression for f for $\lambda \geqq 0$. Define $R: \mathbb{R} \times \mathbf{H} \times L^4(I, \mathbf{X}) \to L^4(I, \mathbf{X})$ by

$$R(\lambda, u, f)(t) = U(t)u + \int_{-\infty}^{t} U(t-s)N_\lambda(f(s))ds.$$

There is a bounded linear operator from $u \in \mathbf{H}$ to $U(\cdot)u \in L^4(I, \mathbf{X})$ [14]. By this fact and Lemma 7 it follows that R is analytic.

Let B be an open ball about the origin of $\mathbb{R} \times \mathbf{H}$. As shown in the proof of Lemma 2 of [12], there exists T such that for all $(\lambda, u) \in B$ the map $R(\lambda, u, \cdot)$ is a contraction on

$$B' = \{g \in L^4(I, \mathbf{X}) : \|g\|_{L^4(I, \mathbf{X})} < 2\|U(\cdot)u\|_{L^4(I, \mathbf{X})}\}.$$

Let $f \in B'$ be the unique element such that $R(\lambda, u, f) = f$; it is clear that this definition of f agrees with that in the statement of the lemma. By Lemma 1, the map $(\lambda, u) \mapsto f$ is analytic from B to $L^4(I, X)$. □

Lemma 9. *With the same notation as in Lemma 8, given $(\lambda, u) \in B$ with $\lambda \geqq 0$, then $f \in C(I, \mathbf{H})$. Moreover, the map $(\lambda, u) \mapsto f$ is continuous from $B \cap \{\lambda \geqq 0\}$ to $C(I, \mathbf{H})$ with its weak$^-$* topology.*

Proof. Lemma 8 implies that $f \in C(I, \mathbf{H})$ if $(\lambda, u) \in B$ and $\lambda \geqq 0$. Suppose that $(\lambda_i, u_i) \to (\lambda, u)$ in $B \cap \{\lambda \geqq 0\}$ and $f_i \in C(I, \mathbf{H})$ are defined as in Lemma 8. Then by (11), $\{f_i\}$ is a bounded sequence in $C(I, \mathbf{H})$. Moreover any weak$^-$* accumulation point of $\{f_i\} \in C(I, \mathbf{H})$ must equal f, since $f_i \to f$ in $L^4(I, \mathbf{X})$ by Lemma 8, and $\mathbf{H} \subset \mathbf{X}$. It follows that $f_i \to f$ in the weak $-$ * topology of $C(I, \mathbf{H})$. □

Lemma 10. *The function $(\lambda, u) \mapsto W_\lambda^{-}u$ is continuous from $[0, \infty) \times \mathbf{H}$ to $\mathbf{H}$.*

Proof. We shall show that for each open ball B about the origin in $\mathbb{R} \times \mathbf{H}$ the map $(\lambda, u) \mapsto f(T)$, where f is as in Lemmas 8 and 9, is continuous from $B \cap \{\lambda \geqq 0\}$ to $\mathbf{H}$. The lemma then follows from continuity result for time evolution over a finite interval given by Proposition 3.

Suppose that $(\lambda_i, u_i) \to (\lambda, u)$ in $B \cap \{\lambda \geqq 0\}$ and $f_i \in C(I, \mathbf{H})$ are defined as in Lemma 8. By Lemma 9, $f_i(T) \to f(T)$ weakly in $\mathbf{H}$, so it suffices to prove that

$\|f_i(T)\|_{\mathbf{H}} \to \|f(T)\|_{\mathbf{H}}$. By (11) we have

$$\|f(T)\|_{\mathbf{H}}^2 + G_\lambda(f(T)) = \|u\|_{\mathbf{H}}^2$$

and

$$\|f_i(T)\|_{\mathbf{H}}^2 + G_{\lambda_i}(f_i(T)) = \|u_i\|_{\mathbf{H}}^2,$$

so it suffices to prove that $G_{\lambda_i}(f_i(T)) \to G_\lambda(f(T))$. This follows from the facts that $f_i(T) \to f(T)$ weakly in $\mathbf{H}$, $\mathbf{H}$ is compactly embedded in $\mathbf{X}$, and $G_\lambda(u)$ is continuous as a function of $(\lambda, u) \in [0, \infty) \times \mathbf{X}$. □

Corollary 11. *Let $B(\mathbb{R}, \mathbf{H})$ denote the space of bounded continuous functions from $\mathbb{R}$ to $\mathbf{H}$. Given $(\lambda, u) \in [0, \infty) \times \mathbf{H}$ and defining f by $f(t) = V_\lambda(t) W_\lambda^{-}(u)$, then the map $(\lambda, u) \mapsto$ is continuous from $[0, \infty) \times \mathbf{H}$ to $B(\mathbb{R}, \mathbf{H})$ with the topology of uniform convergence on compact sets.*

Proof. This is a consequence of Proposition 3 and Theorem 5. □

Lemma 8 gives a sense in which the solution as a function on space-time (in the far past) depends analytically on λ and the datum at $t = -\infty$. We conclude with a related result on the analyticity of the wave and scattering operators for small values of λ or the datum. We define the Hilbert space of Cauchy data $\mathbf{H}_{-1} = H^{1/2}(\mathbb{R}^3) \oplus H^{-1/2}(\mathbb{R}^3)$; this space is of importance because regarded as a space of solutions of the free equation it has a Poincaré-invariant norm.

Theorem 12. *The functions $(\lambda, u) \mapsto W_\lambda^{\pm} u$, $(\lambda, u) \mapsto (W_\lambda^{\pm})^{-1} u$, and $(\lambda, u) \mapsto S_\lambda u$ extend to analytic functions from an open neighborhood of $\{0\} \times \mathbf{H} \cup \mathbb{R} \times \{0\} \subset \mathbb{R} \times \mathbf{H}$ to H_{-1}.*

Proof. We first prove this for S_λ. Taking the interval I to be $\mathbb{R}$, the map $M: \mathbb{R} \times L^4(I, \mathbf{X}) \to L^4(I, \mathbf{X})$ given in Lemma 7 has $\|D_2 M\| < \frac{1}{2}$ in a neighborhood of the set of $\{0\} \times L^4(I, \mathbf{X}) \cup \mathbb{R} \times \{0\}$, since on this set $D_2 M = 0$. We may choose this neighborhood, say P, such that each slice $P \cap \{\lambda = c\}$ is an open ball of radius $r(c)$ about the origin of $L^4(I, \mathbf{X})$, with $r(0) = \infty$.

Let k be the norm of the bounded operator mapping $u \in \mathbf{H}$ to $U(\cdot) u \in L^4(I, \mathbf{X})$ [14]. Define $Q \subseteq \mathbb{R} \times \mathbf{H}$ by requiring that the slice $Q \cap \{\lambda = c\}$ is an open ball of radius $\frac{1}{2} k^{-1} r(c)$ about the origin of $\mathbf{H}$. Q is an open neighborhood of $\{0\} \times \mathbf{H} \cup \mathbb{R} \times \{0\}$, and we shall show that $(\lambda, u) \mapsto S_\lambda u$ extends to an analytic function F from Q to $\mathbf{H}_{-1}$.

Define $R: \mathbb{R} \times \mathbf{H} \times L^4(I, \mathbf{X}) \to L^4(I, \mathbf{X})$ by

$$R(\lambda, u, f)(t) = U(t) u + M(\lambda, f)(t).$$

We claim that for $(\lambda, u) \in Q$, the map $R(\lambda, u, \cdot)$ is an analytic contraction of the open ball of radius $r(\lambda)$ about the origin of $L^4(I, \mathbf{X})$. The map R is analytic by the same argument as in Lemma 8, and if $f \in L^4(I, \mathbf{X})$ has $\|f\| < r(\lambda)$, then:

$$\|R(\lambda, u, f)\| < \tfrac{1}{2} r(\lambda) + \int_0^1 \|D_2 M(\lambda, tf; f)\| \, dt < r(\lambda).$$

Similarly, if $f, g \in L^4(I, \mathbf{X})$ have $\|f\|, \|g\| < r(\lambda)$, then:

$$\| R(\lambda, u, f) - R(\lambda, u, g) \| \leqq \int_0^1 \| D_2 M(\lambda, g + t(f-g); f-g) \| dt \leqq \tfrac{1}{2} \| f - g \| .$$

It follows that for $(\lambda, u) \in Q$, $R(\lambda, u, \cdot)$ has a unique fixed point f in the ball of radius $r(\lambda)$ about the origin of $L^4(I, \mathbf{X})$, and as in Lemma 8 the map $(\lambda, u) \mapsto f$ is analytic from Q to $L^4(I, \mathbf{X})$. By Lemma 8, if $\lambda \geqq 0$ then f satisfies $f(t) = V_\lambda(t) W_\lambda{}^-(u)$, hence

$$S_\lambda(u) = u + \lim_{t \to +\infty} U(-t) \int_{-8}^{t} U(t-s) N_\lambda(f(s)) ds, \tag{13}$$

with the limit being taken in $\mathbf{H}$, but the integral taken in $\mathbf{X}$.

Note that we may identify the dual of $\mathbf{H}_{-1}$ with the space $\mathbf{H}_1 = H^{3/2}(\mathbb{R}^3) \oplus H^{1/2}(\mathbb{R}^3)$ via the formula for the inner product $\langle \cdot, \cdot \rangle_{\mathbf{H}}$. We claim that for all $(\lambda, u) \in Q$ and $g \in \mathbf{H}_1$ the integral $\int_{-\infty}^{\infty} \langle U(s) g, N_\lambda(f(s)) \rangle_{\mathbf{H}} ds$ is absolutely convergent, that

$$g \mapsto \langle g, u \rangle_{\mathbf{H}} + \int_{-\infty}^{\infty} \langle U(s) g, N_\lambda(f(s)) \rangle_{\mathbf{H}} ds \tag{14}$$

defines an element $F(\lambda, u)$ of $\mathbf{H}_1{}^* \cong \mathbf{H}_{-1}$, and that $F: Q \to \mathbf{H}_{-1}$ is analytic. Given these, it is easily checked using (13) that $F(\lambda, u) = S_\lambda u$ for all $(\lambda, u) \in Q$ with $\lambda \geqq 0$ and $u \in C_0{}^\infty(\mathbb{R}^3) \oplus C_0{}^\infty(\mathbb{R}^3)$, hence by continuity for all $(\lambda, u) \in Q$ with $\lambda \geqq 0$. F is thus the desired extension of $(\lambda, u) \mapsto S_\lambda u$.

To show absolute convergence of the integral it suffices to note that $N_\lambda \circ f \in L^{4/3}(I, \mathbf{Y})$ and $(U(\cdot) g)_2 \in L^4(I, L^4(\mathbb{R}^3))$, where the subscript 2 denotes the second component. The latter follows from the result of Stricharz [14] that the linear operator $g \mapsto (U(\cdot) g)_2$ is bounded from $\mathbf{H}_1$ to $L^4(I, L^4(\mathbb{R}^3))$. The observations also imply that (14) defines an element $F(\lambda, u)$ of $\mathbf{H}_{-1}$. Since $N_\lambda \circ f \in L^{4/3}(I, \mathbf{Y})$ depends analytically on $(\lambda, u) \in Q$, the function $F: Q \to \mathbf{H}_{-1}$ is analytic.

Taking I to be $(-\infty, 0]$, a completely analogous argument proves the existence of the desired extension of $(\lambda, u) \mapsto W_\lambda{}^- u$, and taking I to be $[0, \infty)$, we can similarly treat $(W_\lambda{}^+)^{-1}$. The other cases follow by symmetry. □

5. Conclusions

A number of interesting questions about the massive case remain open. One might boldly ask if the functions $(\lambda, u) \mapsto W_\lambda{}^\pm u$ or $(\lambda, u) \mapsto (W_\lambda{}^\pm)^{-1} u$ extend to analytic functions from an open neighborhood of $[0, \infty) \times \mathbf{H}$ to $\mathbf{H}$. More modestly, it is still not known whether these functions are locally Lipschitzian, either in λ or in u. In particular, for S_λ to be locally Lipschitzian would be an interesting stability property.

It is also natural to investigate the map from the data $u \in \mathbf{H}$ at $t = -\infty$ to the solution $f \in B(\mathbb{R}, \mathbf{H})$, as given in Corollary 11. It is not known if this map is continuous when $B(\mathbb{R}, \mathbf{H})$ is given the topology of uniform convergence.

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