A METHOD FOR STUDYING THE INTEGRAL FUNCTIONALS OF STOCHASTIC PROCESSES WITH APPLICATIONS, III

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1. Introduction

This paper is a continuation of the results presented in two earlier papers [20], [21] and may be read as the sequel. A brief account of their results will, however, be given here in order to make this paper selfcontained. The subject under study is the distribution of the integrals of the form

$$Y(t) = \int_0^t f(X(\tau), \tau)\, d\tau, \tag{1.1}$$

where $X(t)$, $t \geqq 0$, is a continuous time parameter stochastic process defined on a probability space $(\Omega, \mathscr{A}, \mathscr{P})$, with $\mathscr{X}$ as its state space, and f is a nonnegative (measurable) function defined on $\mathscr{X} \times [0, \infty)$. Here it is assumed that the integral $Y(t)$ exists and is finite almost surely for every $t > 0$.

The integrals $Y(t)$ arise in several domains of application such as in the theory of inventories and storage (see Moran [13], Naddor [14]), and in the study of the cost of the flow stopping incident involved in the automobile traffic jams (see Gaver [9], Daley [4], and Daley and Jacbos [5]). Such integrals are also encountered in certain stochastic models suitable for the study of response time distributions arising in various live situations (see Puri [16], [18], [19]). In fact in [18], it was shown that such a distribution is equivalent to the study of an integral of the type (1.1).

In [20], the work done by several authors in the past on the integral functionals of stochastic processes was briefly surveyed. But more importantly a method was introduced for obtaining the distribution of $Y(t)$. This method is based on a "quantal response process" $Z(t)$ defined for a hypothetical animal. By definition $Z(t)$ equals one if the animal is alive at time t and is equal to zero otherwise. In particular, it is assumed that

$$P(Z(t + \Delta t = 0 | Z(t) = 1, X(t) = x) = \delta f(x, t)\Delta t + o(\Delta t), \tag{1.2}$$

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with $Z(0) = 1$ and δ a nonnegative constant. Here the state "zero" is an absorption state for the process $Z(t)$. It is easy to establish by using a standard argument that

$$P(Z(t) = 1) = E\left(\exp\left\{-\delta\int_0^t f(X(\tau), \tau)\,d\tau\right\}\right), \tag{1.3}$$

which in turn gives the Laplace transform (L.t.) of the integral $Y(t)$. Thus, the study of the distribution of the integral $Y(t)$ can be carried out equivalently by studying the process $Z(t)$. Note that the quantal response process $Z(t)$ does not influence the process $X(t)$ in any way, rather, as is clear from (1.2), it is influenced itself by the growth of the process $X(t)$. Again, as was pointed out in [20], f is assumed to be nonnegative without loss of generality. Finally, in [20] and [21], this method was applied to the case of Markov chains. The results obtained there are summarized in the next section for later use.

2. The case of Markov chains

Consider a time homogeneous Markov chain (M.c.) $X(t)$ with $\mathscr{X} = \{1, 2, \cdots\}$, constructively defined as follows. If $X(t_1) = i$ at some epoch t_1, the value of $X(t)$ will remain constant for an interval $t_1 \leqq t < t_1 + \tau$, whose random duration τ is exponentially distributed with density function $c_i \exp\{-c_i x\}$, $c_i \geqq 0$; the probability that $X(t_1 + \tau) = j$ is p_{ij}, where the matrix $\mathbf{p} = (p_{ij})$, is a stochastic transition matrix. By assumption, the quantities c_i and p_{ij} are independent of time. Also, we assume that $c_i < \infty$ for all i so that all the states of $\mathscr{X}$ are stable. The sample paths of the process are assumed to be right continuous. Since the process is defined constructively, it is separable. Also, it is evident from the construction that the process $(X(t), Z(t))$ is a Markov process with state space $\mathscr{X} = \{(i, r); i = 1, 2, 3, \cdots; r = 0, 1\}$. In [20] and [21], the above method was applied to Markov chains under the assumption that f depends only on $X(t)$ and not explicitly on t, in which case, in order to specify f, we are given a sequence of numbers $f(i) = f_i$, with $0 \leqq f_i < \infty$, $i = 1, 2, \cdots$. Let

$$\begin{aligned} P_{ij}(t) &= P(X(t) = j \mid X(0) = i), \\ \tilde{P}_{ij}(t) &= P(X(t) = j, Z(t) = 1 \mid X(0) = i, Z(0) = 1); \end{aligned} \tag{2.1}$$

$$\begin{aligned} \pi_{ij}(\alpha) &= \int_0^\infty \exp\{-\alpha t\} P_{ij}(t)\,dt, \\ \tilde{\pi}_{ij}(\alpha) &= \int_0^\infty \exp\{-\alpha t\} \tilde{P}_{ij}(t)\,dt, \\ \boldsymbol{\pi}(\alpha) &= (\pi_{ij}(\alpha)), \qquad \tilde{\boldsymbol{\pi}}(\alpha) = (\tilde{\pi}_{ij}(\alpha)); \end{aligned} \tag{2.2}$$

$$(2.3) \qquad \begin{aligned} \mathbf{C} &= (\delta_{ij} c_i), & \mathbf{I} &= (\delta_{ij}), \\ \mathbf{1} &= (1, 1, 1, \cdots), & \mathbf{f} &= (\delta_{ij} f_i); \end{aligned}$$

where $i, j = 1, 2, \cdots$; $\alpha > 0$, and δ_{ij} is the Kronecker delta.

It is known that the probabilities P_{ij} in terms of their L.t. $\pi_{ij}(\alpha)$ satisfy the backward Kolmogorov system of equations (see Feller [7])

$$(2.4) \qquad (\alpha \mathbf{I} + \mathbf{C})\boldsymbol{\pi}(\alpha) = \mathbf{I} + \mathbf{C}\mathbf{p}\boldsymbol{\pi}(\alpha).$$

If the solution of (2.4) satisfies $\alpha\boldsymbol{\pi}(\alpha)\mathbf{1} = \mathbf{1}$ for $\alpha > 0$, then $\boldsymbol{\pi}$ is the unique solution of (2.4) and is also the unique solution of the forward system of equations given by

$$(2.5) \qquad \boldsymbol{\pi}(\alpha)(\alpha \mathbf{I} + \mathbf{C}) = \mathbf{I} + \boldsymbol{\pi}(\alpha)\mathbf{C}\mathbf{p}.$$

Analogous to (2.4) $\tilde{\boldsymbol{\pi}}$ satisfies the backward system

$$(2.6) \qquad (\alpha \mathbf{I} + \mathbf{C} + \delta \mathbf{f})\tilde{\boldsymbol{\pi}}(\alpha) = \mathbf{I} + \mathbf{C}\mathbf{p}\tilde{\boldsymbol{\pi}}(\alpha).$$

In [20], it was shown that there always exists a solution of (2.6), which is minimal among all its solutions and which also is the minimal solution of the forward system, analogue of (2.5)

$$(2.7) \qquad \tilde{\boldsymbol{\pi}}(\alpha)(\alpha \mathbf{I} + \mathbf{C} + \delta \mathbf{f}) = \mathbf{I} + \tilde{\boldsymbol{\pi}}(\alpha)\mathbf{C}\mathbf{p}.$$

Let $I_j(t)$ denote the indicator function of the set $[X(t) = j]$. Since

$$(2.8) \qquad \tilde{P}_{i,j}(t) = E\left(\exp\left\{-\delta \int_0^t f(X(\tau))\, d\tau\right\} I_j(t) \,\middle|\, X(0) = i\right),$$

it is evident that knowledge of the $\tilde{P}_{i,j}$ is equivalent to that of the joint distribution of $X(t)$ and $Y(t)$. With this in mind, in [20] the problem of existence and uniqueness of the solution of (2.6) and (2.7) was studied in some detail. In particular, if the chain is finite with $\mathscr{X} = \{1, 2, \cdots, N\}$, $N < \infty$, then it can be easily seen that for all $\alpha > 0$, the matrix $(\alpha\mathbf{I} + \mathbf{C} + \delta\mathbf{f} - \mathbf{C}\mathbf{p})$ has an inverse, so that from (2.6) and (2.7) we have the explicit solution for $\tilde{\boldsymbol{\pi}}(\alpha)$ as

$$(2.9) \qquad \tilde{\boldsymbol{\pi}}(\alpha) = (\alpha \mathbf{I} + \mathbf{C} + \delta \mathbf{f} - \mathbf{C}\mathbf{p})^{-1},$$

valid for $\alpha > 0$ and $\delta \geqq 0$. Let $\boldsymbol{\psi}' = (\psi_1, \psi_2, \cdots, \psi_N)$, where

$$(2.10) \qquad \psi_i(\alpha) = \int_0^\infty \exp\{-\alpha t\}\, E\left(\exp\left\{-\delta \int_0^t f(X(\tau))\, d\tau\right\} \,\middle|\, X(0) = i\right) dt.$$

We then have

$$(2.11) \qquad \boldsymbol{\psi}(\alpha) = \tilde{\boldsymbol{\pi}}(\alpha)\mathbf{1} = (\alpha \mathbf{I} + \mathbf{C} + \delta \mathbf{f} - \mathbf{C}\mathbf{p})^{-1}\mathbf{1}.$$

The L.t. $\psi_i(\alpha)$ is in general a rational function of α and can therefore be easily inverted to yield $E(\exp\{-\delta \int_0^t f(X(\tau))\, d\tau\} \mid X(0) = i)$.

Again in [20] and [21], under certain assumptions, we proved the identity

$$\boldsymbol{\pi} - \tilde{\boldsymbol{\pi}} = \delta \tilde{\boldsymbol{\pi}} \mathbf{f} \boldsymbol{\pi}, \tag{2.12}$$

which connects $\tilde{\boldsymbol{\pi}}$ and $\boldsymbol{\pi}$, allowing us to obtain the desired $\tilde{\boldsymbol{\pi}}(\alpha)$ in terms of $\boldsymbol{\pi}(\alpha)$, which may be known. The identity (2.12) is found very useful in applications particularly because of the manner in which $\mathbf{f}$ appears. In particular in [21], we used this identity to obtain the joint distribution of times spent by the Markov chain in various states of a given finite set before the process hits a given taboo set.

REMARK. A formulation alternative to the consideration of the M.c. $\{X(t), Z(t)\}$ would be to consider a modified time homogeneous M.c. $\hat{X}(t)$ with state space $\{a, 1, 2, 3, \cdots\}$ with new exponential parameters, say $\hat{c}_i$, given by

$$\hat{c}_i = (c_i + \delta f_i)(1 - \delta_{ia}), \qquad i = a, 1, 2, 3, \cdots, \tag{2.13}$$

and the new transition matrix $\hat{p}_{ij}$ given by

$$\hat{p}_{ij} = \begin{cases} c_i p_{ij}(c_i + \delta f_i)^{-1} & \text{for } i, j = 1, 2, \cdots, \\ \delta f_i (c_i + \delta f_i)^{-1} & \text{for } j = a, i = 1, 2, \cdots, \\ \delta_{aj} & \text{for } i = a, \end{cases} \tag{2.14}$$

so that the state a is an absorption state. However, since for each formulation, the relevant information concerning the distribution of $X(t)$ and $Y(t)$ is contained in the equations (2.6) and (2.7), we find no essential gain in considering this alternative formulation.

In [20], it was pointed out that in the past most of the researchers in the area touched by this paper have exploited the backward system such as (2.6) (see for instance, Gaver [9], Daley [4], Daley and Jacobs [5], and McNeil [12]). Forward equations (2.7) were not used possibly because of lack of probabilistic interpretation. The present method via the quantal response process $Z(t)$ has the advantage over the past ones in that it provides the needed probabilistic interpretation. In the present paper, we shall exploit the forward system a great deal, by applying it to the case of certain well-known processes arising in several live situations. In [20], [21] and also in the applications of the method exhibited in the present paper, we have restricted outselves mostly to Markov chains with countable state space. However, it is evident that the method is applicable to almost all types of continuous time stochastic processes. The application to certain processes such as semi-Markov processes will be dealt with elsewhere.

Finally, it may be remarked that the above method has some resemblance with the work of Kemperman [11] and also with the method of collective marks due to van Dantzig [6]; in the present case, however, the approach was motivated by the author's work on the response time distribution arising in certain biological situations (see Puri [16], [18], and [19]).

3. Birth processes

This section will be devoted to the case where the M.c. $X(t)$ is a birth process. Section 3.1 deals with the time homogeneous case, while Section 3.2 deals with linear nonhomogeneous birth processes.

3.1. *Time homogeneous birth processes.* We shall use here the notation of Section 2. Let $X(t)$ be a time homogeneous birth process with $p_{jk} = \delta_{j+1,k}$ and $X(0) = i$. Also let

$$N = \min\{j; j \geqq i, c_j = 0\}; \tag{3.1}$$

if $c_j > 0$ for all $j \geqq i$, then $N = \infty$. If $N < \infty$, the M.c. is a finite one (with $c_j > 0, j = i, i + 1, \cdots, N - 1$, and $c_N = 0$), a case which was already considered in Section 2 with an explicit answer given by (2.9). However, if $N = \infty$, we assume that $\Sigma_{j=i}^{\infty} c_j^{-1} = \infty$, so that with probability one only a finite number of jumps of the chain are allowed in any finite time interval. For the present case, the systems of equations (2.6) and (2.7) are given by

$$(\alpha + c_i + \delta f_i)\tilde{\pi}_{ik}(\alpha) - c_i\tilde{\pi}_{i+1,k}(\alpha) = \delta_{ik} \tag{3.2}$$

and

$$(\alpha + c_k + \delta f_k)\tilde{\pi}_{ik}(\alpha) - c_{k-1}\tilde{\pi}_{i,k-1}(\alpha) = \delta_{ik}, \tag{3.3}$$

respectively, with $\tilde{\pi}_{ij}(\alpha) = 0$ for $j < i$ and for $j > N$. Each of these systems can be uniquely solved for $\tilde{\pi}_{ik}(\alpha)$ recursively, yielding

$$\tilde{\pi}_{ik}(\alpha) = \begin{cases} r_i & \text{for } k = i, \\ \rho_i\rho_{i+1}\cdots\rho_{k-1}r_k & \text{for } k > i, \\ 0 & \text{for } k < i \text{ and for } k > N, \end{cases} \tag{3.4}$$

where $r_j = (\alpha + c_j + \delta f_j)^{-1}$ and $\rho_j = c_j r_j$. Because of the condition $\Sigma_{j=i}^{N} c_j^{-1} = \infty$, it is now easily seen that for $\alpha > 0$,

$$\begin{aligned} \int_0^\infty \exp\{-\alpha t\} E\left(\exp\left\{-\delta\int_0^t f(X(\tau))\,d\tau\right\}\middle| X(0) = i\right) dt \\ = \sum_{k=i}^{N} \tilde{\pi}_{ik}(\alpha) = \lim_{n\to N} \sum_{k=i}^{n} \tilde{\pi}_{ik}(\alpha) \\ = \frac{1}{\alpha}\lim_{n\to N}\left[1 - \rho_i\rho_{i+1}\cdots\rho_n - \delta\sum_{k=i}^{n} f_k\rho_i\rho_{i+1}\cdots\rho_{k-1}r_k\right] \\ = \frac{1}{\alpha}\left[1 - \delta\sum_{k=i}^{N} f_k\rho_i\rho_{i+1}\cdots\rho_{k-1}r_k\right], \end{aligned} \tag{3.5}$$

where if $N < \infty$, the limit of a sum as $n \to N$ is taken to be the appropriate finite sum. Here in (3.5), we have used the fact that $\alpha\,\Sigma_{k=i}^{N}\tilde{\pi}_{ik}(\alpha) \leqq 1$, for all $\alpha > 0$, which is known from the general theory of Markov chains. The fact that $\lim_{n\to N}(\rho_i\rho_{i+1}\cdots\rho_n) = 0$ follows from the condition $\Sigma_{j=i}^{N} c_j^{-1} = \infty$.

Expressions (3.4) can easily be inverted to yield expressions for $\tilde{p}_{ik}(t)$. For instance, if $c_j + \delta f_j$ are all distinct for $j = i, i + 1, \cdots, N$, it can be easily shown that

$$(3.6) \quad \tilde{p}_{ik}(t) = \left(\prod_{j=1}^{k-1} c_j\right) \sum_{j=i}^{k} \left[\prod_{\ell=i,\ell\neq j}^{k} (c_\ell + \delta f_\ell - c_j - \delta f_j)^{-1}\right] \exp\{-(c_j + \delta f_j)t\},$$

where by convention $(\Pi_{j=i}^{k-1} c_j) = \Pi_{\ell\neq j}^{k} (c_\ell + \delta f_\ell - c_j - \delta f_j)^{-1} = 1$ for $k = j = i$. Summing (3.6) over k, we finally obtain

$$(3.7) \quad E\left(\exp\left\{-\delta \int_0^t f(X(\tau))\, d\tau\right\}\middle| X(0) = i\right)$$
$$= \sum_{k=i}^{N} \left(\prod_{j=i}^{k-1} c_j\right) \sum_{j=i}^{k} \left[\prod_{\ell=i,\ell\neq j}^{k} (c_\ell + \delta f_\ell - c_j - \delta f_j)^{-1}\right] \exp\{-(c_j + \delta f_j)t\}.$$

Note that for the case when $N = \infty$, an interchange of the two summation signs on the right side of (3.7) is not always valid. Also, one could instead invert the L.t. given in (3.5) and obtain a different yet equivalent expression for (3.7).

Again, since $Y(t)$ is a monotone nondecreasing function of t, it almost surely converges to a random variable, say Y, as $t \to \infty$. By using a Tauberian argument it follows from (3.5) that

$$(3.8) \quad E(\exp\{-\delta Y\}|X(0) = i) = \lim_{\alpha\to 0} \alpha \sum_{k=i}^{N} \tilde{\pi}_{ik}(\alpha)$$
$$= 1 - \lim_{n\to N} \delta \sum_{k=i}^{n} f_k \rho_i^* \rho_{i+1}^* \cdots \rho_{k-1}^* r_k^*,$$

where $r_i^* = (c_i + \delta f_i)^{-1}$ and $\rho_i^* = c_i r_i^*$. Now after some manipulation with the right side of (3.8), it can be shown that

$$(3.9) \quad E(\exp\{-\delta Y\}|X(0) = i) = \lim_{n\to N} (\rho_i^* \cdots \rho_n^*) = \prod_{j=i}^{N} \left[1 + \delta\left(\frac{f_j}{c_j}\right)\right]^{-1}.$$

If $N < \infty$, this is zero (keeping in mind that $c_N = 0$) unless $f_N = 0$. Thus, if $N < \infty$ and $f_N > 0$, $Y = \infty$ a.s. Again if $N = \infty$, (3.9) is equal to zero if and only if $\Sigma_{j=i}^{\infty} (f_j/c_j) = \infty$, in which case also $Y = \infty$ a.s. Let $\Sigma_{j=i}^{\infty} (f_j/c_j) < \infty$. This means that $f_N = 0$ whenever $N < \infty$. Then (3.9) is positive for all $\delta \geqq 0$, and in particular is equal to one when $\delta = 0$, and hence Y is finite a.s. Furthermore, it is clear from (3.9) that Y has a density. In fact, if we assume that $f_j/c_j, j = i, i + 1, \cdots, N$, are all distinct, then, at least when $N < \infty$, one easily obtains the density function of Y as

$$(3.10) \quad g(y) = \sum_{j=i}^{N} \frac{c_j}{f_i} \left[\prod_{\ell=i,\ell\neq j}^{N} \left(1 - \frac{c_j f_\ell}{f_j c_\ell}\right)^{-1}\right] \exp\left\{-\frac{c_j y}{f_j}\right\}, \qquad y > 0,$$

by inverting the L.t. (3.9). As expected, it follows from (3.9) that Y can be expressed as the sum $\Sigma_{j=i}^{N} f_j \tau_j$, where $\tau_i, \tau_{i+1}, \cdots$, are independently negative exponentially distributed with parameters $c_i, c_{i+1}, \cdots$. Here $\tau_j, j = i, i+1, \cdots$, are essentially the random lengths of times that the process spends in various states.

In the following subsections, we consider certain special cases of homogeneous birth processes that arise in practice.

3.1.1. *Case of a simple epidemic.* Jerwood [10] has recently considered the case of a simple epidemic which starts with $X(0) = i$ infectives and $S(0) = N - i$ susceptibles at time $t = 0$, where $N < \infty$. If $X(t)$ denotes the number of infectives at time t, then $X(t)$ is a birth process as considered by Bailey [1] with the finite state space $(i, i+1, \cdots, N)$, N being the absorption state and

$$c_j = \beta_j(N - j), \qquad j = i, i+1, \cdots, N. \tag{3.11}$$

Jerwood [10] considers the distribution of the cost of the epidemic exhibited by

$$C_i = aW_i + bT_i, \qquad a > 0, \quad b > 0, \tag{3.12}$$

where

$$W_i = \int_0^{T_i} X(\tau)\, d\tau, \qquad T_i = \inf\{t : X(t) = N \mid X(0) = i\}, \tag{3.13}$$

a is the cost per unit time per infective, and b is the cost per unit time both over the period T_i. Perhaps a more realistic situation is where the rate of the first cost varies with the number of infectives at time t, so that one would like to obtain the distribution of

$$\tilde{C}_i = aW_i' + bT_i, \qquad a > 0, \quad b > 0, \tag{3.14}$$

where

$$W_i' = \int_0^{T_i} h(X(\tau))\, d\tau, \tag{3.15}$$

with $0 \leqq h(j) < \infty, j = i, i+1, \cdots, N-1$. Since we are concerned with the epidemic only until the first passage time T_i to state N, without loss of generality, we may take $h(N) = 0$. In order to fit this into our situation above, all we need to take is

$$f_j = \begin{cases} ah(j) + b, & j = i, i+1, \cdots, N-1, \\ 0, & \text{otherwise.} \end{cases} \tag{3.16}$$

Now since the passage to the absorption state N occurs with probability one, it is easy to see that

$$\begin{aligned} E(\exp\{-\delta \tilde{C}_i\}) &= \lim_{t\to\infty} E\left(\exp\left\{-\delta \int_0^t f(X(\tau))\, d\tau\right\} \middle| X(0) = i\right) \\ &= E(\exp\{-\delta Y\} \mid X(0) = i) = \prod_{j=i}^{N-1}\left[1 + \delta\left(\frac{f_j}{c_j}\right)\right]^{-1}. \end{aligned} \tag{3.17}$$

The last equality follows from (3.9). If the f_j/c_j are all distinct, then the density function of $\tilde{C}_i$ is given by

$$(3.18)\quad g(y) = \left\{\prod_{j=1}^{N-1} \frac{c_j}{f_j}\right\} \sum_{j=1}^{N-1} \left[\prod_{\ell=i,\ell\neq j}^{N-1} \left(\frac{c_\ell}{f_\ell} - \frac{c_j}{f_j}\right)^{-1}\right] \exp\left\{-\frac{c_j y}{f_j}\right\}, \qquad y > 0,$$

the expected cost being $E(\tilde{C}_i) = \Sigma_{j=i}^{N-1}(f_j/c_j)$. Incidentally using (3.16) in (3.17), we have, with $\alpha_1 = a\delta$ and $\alpha_2 = b\delta$,

$$(3.19)\quad E(\exp\{-\alpha_1 W_i' + \alpha_2 T_i\}) = \prod_{j=i}^{N-1}\left[1 + \frac{\alpha_1 h(j) + \alpha_2}{c_j}\right]^{-1}, \alpha_1 > 0, \alpha_2 > 0,$$

which, by treating α_1 and α_2 as the dummy variables, is the joint L.t. of the random variables W_i' and T_i.

3.1.2. *Poisson process.* This process is a special case of the birth process of Section 3.1, with $c_j = \lambda > 0$ for all j. In this case, if $f_j, j = i, i + 1, \cdots$, are all distinct, then it follows from (3.6) that

$$(3.20)\quad E\left(\exp\left\{-\delta\int_0^t f(X(\tau))\,d\tau\right\} I_k(t)\,\middle|\, X(0) = i\right) = \tilde{P}_{ik}(t)$$

$$= \left(\frac{\lambda}{\delta}\right)^{k-i} \exp\{-\lambda t\} \sum_{j=i}^{k}\left[\prod_{\ell=i,\ell\neq j}^{k} (f_\ell - f_j)^{-1}\right] \exp\{-\delta f_j t\},$$
$$k = i, i + 1, \cdots.$$

A special case with $f_j = j$ has been considered elsewhere by the author (see [18]). It was shown there that for $|s| \leqq 1$ and $\delta \geqq 0$ and with $X(0) = 0$,

$$(3.21)\quad E\left(s^{X(t)} \exp\left\{-\delta\int_0^t X(\tau)\,d\tau\right\}\,\middle|\, X(0) = 0\right) = \exp\left\{-\lambda t + \frac{\lambda s}{\delta}(1 - e^{-\delta t})\right\}.$$

With $s = 1$, this can be easily inverted to yield the distribution function of $Y(t)$ given by

$$(3.22)\qquad H_t(y) = \exp\{-\lambda t\} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} F_t^{*(n)}(y),$$

where $F_t^{*(n)}$ stands for the n fold convolution of F_t, the distribution function of a random variable uniformly distributed over $(0, t)$.

3.2. *Nonhomogeneous birth processes.* For the case of nonhomogeneous Markov processes, in general, one finds it more convenient to use the forward Kolmogorov system of equations than the backward. Consider a birth process $X(t)$ with $X(0) = i$, such that for $j = i, i + 1, \cdots$,

$$(3.23)\qquad \begin{aligned} P(X(t + \Delta t) = j + 1 | X(t) = j) &= c_j(t)\Delta t + o(\Delta t), \\ P(X(t + \Delta t) = j | X(t) = j) &= 1 - c_j(t)\Delta t + o(\Delta t). \end{aligned}$$

Also for the function f of the integral (1.1), let

$$(3.24)\qquad f(j, t) = f_j(t), \qquad j = i, i + 1, \cdots.$$

Here the functions $c_j(t)$ and $f_j(t)$ are assumed to be nonnegative, continuous and integrable over $(0, t)$ for every finite t. Furthermore, let the functions $c_j(t)$ be such that the process $X(t)$ has, with probability one, a finite number of jumps in any finite time interval. Then using (1.2) and (3.23), one obtains in a standard manner the forward differential equations for the probabilities $\tilde{P}_{ij}(t)$ as given by

$$\frac{d\tilde{P}_{ij}(t)}{dt} = -\big(c_j(t) + \delta f_j(t)\big)\tilde{P}_{ij}(t) + c_{j-1}(t)\tilde{P}_{i,j-1}(t), \qquad j = i, i+1, \cdots, \tag{3.25}$$

where $\tilde{P}_{ij}(t) \equiv 0$ for all $j < i$. Recursively, these equations can be solved subject to $\tilde{P}_{ij}(0) = \delta_{ij}$ to yield

$$\tilde{P}_{ii}(t) = \exp\left\{-\int_0^t \big(c_i(\tau) + \delta f_i(\tau)\big)\, d\tau\right\} \tag{3.26}$$

and

$$\tilde{P}_{ij}(t) = \int_0^t \exp\left\{-\int_\tau^t \big(c_j(u) + \delta f_j(u)\big)\, du\right\} c_{j-1}(\tau)\tilde{P}_{i,j-1}(\tau)\, d\tau, \qquad j = i+1, i+2, \cdots. \tag{3.27}$$

Unfortunately, in general, there appears to be no way of obtaining the expression for $\tilde{P}_{ij}(t)$ in a closed form. Instead, in the rest of this section, we restrict ourselves to the case of linear birth processes with

$$c_j(t) = \alpha(t) + j\nu(t), \qquad j = i, i+1, \cdots, \tag{3.28}$$

and with $f_j(t) = j\beta(t)$. Thus, here we are interested in the integral of the form

$$Y(t) = \int_0^t \beta(\tau)X(\tau)\, d\tau. \tag{3.29}$$

For this case, the equations (3.25) take the form

$$\frac{d\tilde{P}_{ij}(t)}{dt} = -\big[\alpha(t) + j\big(\delta\beta(t) + \nu(t)\big)\big]\tilde{P}_{ij}(t) + \big[\alpha(t) + (j-1)\nu(t)\big]\tilde{P}_{i,j-1}(t), \tag{3.30}$$

for $j = i, i+1, \cdots$. Let

$$\tilde{G}(s, t) = \sum_{j=i}^{\infty} s^j \tilde{P}_{ij}(t), \qquad |s| \leqq 1. \tag{3.31}$$

Then multiplying both sides of (3.30) by s^j and adding over j we obtain, after some simplification, the equation

$$\tilde{G}_t - [\nu(t)s - \nu(t) - \delta\beta(t)]s\tilde{G}_s = -\alpha(t)(1-s)\tilde{G}, \tag{3.32}$$

where here and elsewhere $\tilde{G}_t$ and $\tilde{G}_s$ denote the respective partial derivatives of $\tilde{G}$. Equation (3.32) is subject to the initial condition $\tilde{G}(s, 0) = s^i$ and can be solved by standard methods yielding

$$\tilde{G}(s, t) = [\phi(s; 0, t)]^i \exp\left\{-\int_0^t \alpha(\tau)[1 - \phi(s; \tau, t)]\, d\tau\right\}, \tag{3.33}$$

where, for $0 \leqq \tau \leqq t$,

$$[\phi(s; \tau, t)]^{-1} = \frac{1}{s} \exp\left\{\int_\tau^t \big(\nu(u) + \delta\beta(u)\big)\, du\right\} - \int_\tau^t \nu(u) \exp\left\{\int_\tau^u \big(\nu(v) + \delta\beta(v)\big)\, dv\right\} du. \tag{3.34}$$

Since in the present case of (3.28), it is known that in any finite time interval, with probability one, there are only a finite number of jumps of the process $X(t)$, (3.33) with $s = 1$ yields

$$E\left(\exp\left\{-\delta \int_0^t \beta(\tau)X(\tau)\, d\tau\right\}\middle| X(0) = i\right) = \tilde{G}(1, t). \tag{3.35}$$

In the next subsection, we specialize the above results to Pólya process which arises very often in various live situations such as the theory of accident proneness (see Bates and Neyman [2]).

3.2.1. *Pólya process.* This is a special case of the linear birth process discussed above with

$$\alpha(t) = \lambda(1 + \rho\lambda t)^{-1}, \qquad \nu(t) = \lambda\rho(1 + \rho\lambda t)^{-1}, \qquad \lambda > 0, \qquad \rho > 0. \tag{3.36}$$

Also, we consider the special case where $\beta(t) \equiv 1$, so that we are interested in the integral $Y(t) = \int_0^t X(\tau)\, d\tau$. For this case, expression (3.33) simplifies to

$$\tilde{G}(s, t) = s^i \exp\{-i\delta t\}\left[(1 + \lambda\rho t) - \frac{\lambda\rho}{\delta}(1 - e^{-\delta t})s\right]^{-(i+\rho^{-1})}. \tag{3.37}$$

This then gives the joint distribution of $X(t)$ and $Y(t)$. In particular, from this one easily obtains, for $n = 0; 1, 2, \cdots$,

$$\tilde{P}_{i,n+i}(t) = (1 + \lambda\rho t)^{-(i+\rho^{-1})} e^{-i\delta t}\left[\frac{(i + n - 1 + \rho^{-1}) \cdots (i + \rho^{-1})}{n!}\right] \cdot \left(\frac{\lambda\rho t}{1 + \lambda\rho t}\right)^n \left(\frac{1 - e^{-\delta t}}{\delta t}\right)^n. \tag{3.38}$$

Again, one can easily invert (3.37) with $s = 1$ to yield the distribution function $H_t(y)$ of the integral $Y(t)$, given by

$$(3.39)\quad H_t(y) = (1 + \lambda\rho t)^{-(i+\rho^{-1})} \cdot \sum_{n=0}^{\infty} \frac{(i + n - 1 + \rho^{-1}) \cdots (i + \rho^{-1})}{n!} \left(\frac{\lambda\rho t}{1 + \lambda\rho t}\right)^n U_t * F_t^{*(n)}(y)$$

for $y > it$, and for $y \leqq it$, $H_t(y) = 0$. Here the operation $*$ denotes the convolution between two distribution functions, U_t is the distribution function corresponding to a degenerate random variable taking value as (ti), and $F_t^{*(n)}$ is as defined in Section 3.1.2. Finally, if we let $i = 0$ and $\rho \to 0$ in (3.37), we obtain the expression (3.21) for the Poisson process as expected.

4. Birth and death processes

We now consider the case of a time homogeneous birth and death process (b-d process) $X(t)$ where, in notation of Section 2, $c_k = (\lambda_k + \mu_k)$,

$$(4.1)\qquad p_{kj} = \begin{cases} \lambda_k(\lambda_k + \mu_k)^{-1} & \text{if } j = k + 1, \\ \mu_k(\lambda_k + \mu_k)^{-1} & \text{if } j = k - 1, \\ 0 & \text{elsewhere,} \end{cases}$$

and λ_k and μ_k are nonnegative constants with $\mu_0 = 0$. Let $X(0) = i$. The backward and forward equations, analogues of (2.6) and (2.7) but converted into differential equations, are given by

$$(4.2)\qquad \frac{d\tilde{P}_{ik}(t)}{dt} = -(\lambda_i + \mu_i + \delta f_i)\tilde{P}_{ik}(t) + \lambda_i \tilde{P}_{i+1,k}(t) + \mu_i \tilde{P}_{i-1,k}(t)$$

and

$$(4.3)\qquad \frac{d\tilde{P}_{ik}(t)}{dt} = -(\lambda_k + \mu_k + \delta f_k)\tilde{P}_{ik}(t) + \lambda_{k-1}\tilde{P}_{i,k-1}(t) + \mu_{k+1}\tilde{P}_{i,k+1}(t),$$

respectively. Unfortunately, we shall not consider these here in this generality. Instead, we shall consider certain special cases which arise in several practical situations. For this, we shall particularly be making use of the forward system (4.3). Since $X(0) = i$ will be kept fixed, we shall write for brevity $\tilde{P}_{ik}(t) = \tilde{P}_k(t)$.

4.1. *Linear birth and death processes with immigration.* Consider a b-d process with $\lambda_k = k\lambda + v$ and $\mu_k = k\mu$ for $k = 0, 1, 2, \cdots$, where λ, v, and μ are positive constants. Such processes arise in the study of population dynamics and also with $\lambda = 0$ in the queueing theory of $M/M/\infty$ queues. Here we wish to obtain the joint distribution of $X(t)$, $\int_0^t X(\tau)\, d\tau$, and $T(t)$, the last one being the length of time during $(0, t)$ the process remains in nonzero states. In $M/M/\infty$

queue, $T(t)$ represents the time for which at least one channel remains busy during $(0, t)$; $\int_0^t X(\tau)\,d\tau$ represents the cumulative time the customers spend during $(0, t)$ while they are being served. In the study of response of host to injection of virulent bacteria, $\int_0^t X(\tau)\,d\tau$ could be regarded as a measure of the total amount of toxins produced by the live bacteria during $(0, t)$, assuming a constant toxin excretion rate per bacterium (see Puri [16], [17], and [18]).

In order to accomplish our purpose, it is sufficient to take

$$\delta f_k = \beta_1(1 - \delta_{k0}) + k\beta_2, \qquad k = 0, 1, 2, \cdots, \tag{4.4}$$

so that

$$\begin{aligned}\tilde{P}_k(t) &= E\left(\exp\left\{-\delta\int_0^t f(X(\tau))\,d\tau\right\} I_k(t)\right)\\ &= E\left(\exp\left\{-\beta_1 T(t) - \beta_2\int_0^t X(\tau)\,d\tau\right\} I_k(t)\right).\end{aligned} \tag{4.5}$$

Furthermore, for the present case, the system (4.3) takes the form

$$\frac{d\tilde{P}_k(t)}{dt} = \begin{cases} -[\nu + \beta_1 + k(\mu + \lambda + \beta_2)]\tilde{P}_k \\ \qquad + (k+1)\mu\tilde{P}_{k+1} + [\nu + (k-1)\lambda]\tilde{P}_{k-1}, & k \geqq 1, \\ -\nu\tilde{P}_0 + \mu\tilde{P}_1, & k = 0. \end{cases} \tag{4.6}$$

Let $\tilde{G}(s, t)$ be as defined in (3.31). Then from (4.6) we have

$$\tilde{G}_t - \lambda(s - r_1)(s - r_2)\tilde{G}_s = -[\nu(1 - s) + \beta_1]\tilde{G} + \beta_1\tilde{P}_0, \tag{4.7}$$

where r_1 and r_2 denote with plus and minus signs, respectively,

$$\frac{1}{2\lambda}[(\mu + \lambda + \beta_2) \pm \{(\mu + \lambda + \beta_2)^2 - 4\mu\lambda\}^{1/2}]. \tag{4.8}$$

The problem now is to solve (4.7) for $\tilde{G}$ subject to the side condition $\tilde{G}(s, 0) = s^i$. This can be accomplished by standard methods. We give here only the final answer in terms of its L.t. Let

$$\psi_1(s, t) = [h_1(s, t)]^i[h_2(s, t)]^{-\nu/\lambda}\exp\{-(\nu + \beta_1 - \nu r_2)t\} \tag{4.9}$$

and

$$\psi_2(s, t) = [h_2(s, t)]^{-\nu/\lambda}\exp\{-(\nu + \beta_1 - \nu r_2)t\}, \tag{4.10}$$

where

$$h_1(s, t) = \left\{\frac{r_2(r_1 - s) + r_1(s - r_2)\exp\{-\lambda(r_1 - r_2)t\}}{(r_1 - s) + (s - r_2)\exp\{-\lambda(r_1 - r_2)t\}}\right\} \tag{4.11}$$

and

$$h_2(s, t) = \{(s - r_2)\exp\{-\lambda(r_1 - r_2)t\} + (r_1 - s)\}(r_1 - r_2)^{-1}. \tag{4.12}$$

Also, let $\psi_1^*(s, \alpha)$, $\psi_2^*(s, \alpha)$, $\tilde{G}^*(s, \alpha)$, and $\tilde{P}_0^*(\alpha)$ be the Laplace transforms over time t of ψ_1, ψ_2, $\tilde{G}$, and $\tilde{P}_0$, respectively, defined for $\alpha > 0$. From (4.7), it is easy to show that

$$\tilde{G}(s, t) = \psi_1(s, t) + \beta_1 \int_0^t \tilde{P}_0(t - \tau)\psi_2(s, \tau)\, d\tau, \tag{4.13}$$

from which it follows that

$$\tilde{G}^*(s, \alpha) = \psi_1^*(s, \alpha) + \beta_1 \tilde{P}_0^*(\alpha)\psi_2^*(s, \alpha). \tag{4.14}$$

With $s = 0$, (4.14) yields

$$\tilde{P}_0^*(\alpha) = \psi_1^*(0, \alpha)[1 - \beta_1\psi_2^*(0, \alpha)]^{-1}. \tag{4.15}$$

Finally, by using this in (4.14), we obtain

$$\tilde{G}^*(s, \alpha) = \psi_1^*(s, \alpha) + \beta_1\psi_2^*(s, \alpha)\psi_1^*(0, \alpha)[1 - \beta_1\psi_2^*(0, \alpha)]^{-1}. \tag{4.16}$$

We now consider briefly a special case with no immigration, that is, with $\nu = 0$, where we explicitly obtain

$$\tilde{P}_0(t) = \left(\frac{\mu}{\lambda}\right)^i \{\exp\{-\beta_1 t\}[J(t)]^i + \beta_1 \int_0^t \exp\{-\beta_1\tau\}[J(\tau)]^i\, d\tau\}, \tag{4.17}$$

and

$$\tilde{G}(s, t) = \exp\{-\beta_1 t\}\left\{\beta_1 \int_0^t \exp\{\beta_1\tau\}\tilde{P}_0(\tau)\, d\tau + [h_1(s, t)]^i\right\}, \tag{4.18}$$

with

$$J(t) = [1 - \exp\{-\lambda(r_1 - r_2)t\}][r_1 - r_2\exp\{-\lambda(r_1 - r_2)t\}]^{-1}. \tag{4.19}$$

The case without the random variable $T(t)$ has previously been considered elsewhere by the author [15]. For the present case ($\nu = 0$), it is known that $P(X(t) \to 0 \text{ or } \infty) = 1$ and that $P(X(t) \to 0) = \min(1, \mu/\lambda)$. Also, $T(t)$ being a nondecreasing function of t tends almost surely to a random variable T as $t \to \infty$. Here T is the first passage time of the process to the state zero. Also $P(T < \infty) = \min(1, \mu/\lambda)$. Thus by using (4.17), we have

$$\begin{aligned} \lim_{t\to\infty} \tilde{G}(s, t) &= \lim_{t\to\infty} \tilde{P}_0(t) \\ &= E\left(\exp\left\{-\beta_1 T - \beta_2 \int_0^T X(\tau)\, d\tau\right\}\right) \\ &= \begin{cases} r_2^i & \text{if } \beta_1 = 0, \\ \left(\dfrac{\mu}{\lambda}\right)^i \beta_1 \displaystyle\int_0^t \exp\{-\beta_1 t\}[J(t)]^i\, dt & \text{if } \beta_1 > 0. \end{cases} \end{aligned} \tag{4.20}$$

On the other hand, in the presence of immigration ($\nu > 0$), if we wish to find the joint distribution of T and $\int_0^T X(\tau)\,d\tau$, we first modify our process (see also [21]) by taking $\lambda_0 = 0$, $\lambda_k = k\lambda + \nu$ for $k \geqq 1$, so that zero is an absorption state of the process. Here we allow immigration only as long as the process has not touched the state zero. The analogue of equation (4.7) for the modified process is then given by

$$\tilde{G}_t - \lambda(s - r_1)(s - r_2)\tilde{G}_s = -[\beta_1 + \nu(1 - s)](\tilde{G} - \tilde{P}_0). \tag{4.21}$$

The solution of (4.21) subject to $\tilde{G}(s, 0) = s^i$, $i \geqq 1$, is given by (4.13) through (4.16) with ψ_2 replaced by

$$\begin{aligned}\hat{\psi}_2(s, t) &= \frac{1}{\beta_1}\psi_2(s, t)\left[(\nu + \beta_1 - \nu r_1) + \frac{\nu(r_1 - r_2)(r_1 - s)}{(r_1 - s) + (s - r_2)\exp\{-\lambda(r_1 - r_2)t\}}\right],\end{aligned} \tag{4.22}$$

and ψ_2^* by $\hat{\psi}_2^*$. We thus have from the new (4.16), while using a Tauberian argument,

$$\begin{aligned} E\left(\exp\left\{-\beta_1 T - \beta_2\int_0^T X(\tau)\,d\tau\right\}\middle| X(0) = i\right) &= \lim_{t\to\infty}\tilde{G}(1, t) = \lim_{\alpha\to 0}\alpha\tilde{G}^*(1, \alpha)\\ &= \lim_{\alpha\to 0}\left[\alpha\psi_1^*(1, \alpha) + \beta_1\frac{\alpha\hat{\psi}_2^*(1, \alpha)\psi_1^*(0, \alpha)}{1 - \beta_1\hat{\psi}_2^*(0, \alpha)}\right].\end{aligned} \tag{4.23}$$

On the other hand, it can be easily shown that

$$\begin{aligned}&\lim_{\alpha\to 0}\alpha\psi_1^*(1, \alpha) = \lim_{t\to\infty}\psi_1(1, t) = 0,\\ &\lim_{\alpha\to 0}\hat{\psi}_2^*(1, \alpha) = \int_0^\infty\hat{\psi}_2(1, t)\,dt,\end{aligned} \tag{4.24}$$

and

$$\begin{aligned}\lim_{\alpha\to 0}\alpha\psi_1^*(0, 1)[1 - \beta_1\hat{\psi}_2^*(0, \alpha)]^{-1} &= \lim_{t\to\infty}\tilde{P}_0(t) = \tilde{P}_0(\infty)\\ &= \frac{\int_0^\infty \exp\{-(\nu + \beta_1 - \nu r_2)t\}[h_1(0, t)]^i[h_2(0, t)]^{-\nu/\lambda}\,dt}{\int_0^\infty \exp\{-(\nu + \beta_1 - \nu r_2)t\}[h_2(0, t)]^{-\nu/\lambda}\,dt}.\end{aligned} \tag{4.25}$$

Thus, we finally have from (4.23),

$$E\left(\exp\left\{-\beta_1 T - \beta_2\int_0^T X(\tau)\,d\tau\right\}\middle| X(0) = i\right) = \beta_1\tilde{P}_0(\infty)\int_0^\infty\hat{\psi}_2(1, t)\,dt. \tag{4.26}$$

4.2. *M/M/I queue.* This corresponds to the case with $\lambda_k = \lambda$ for $k \geqq 0$ and $\mu_k = \mu$ for $k \geqq 1$ with $\mu_0 = 0$. This is the case which has recently been explored by Gaver [9], Daley [4], Daley and Jacobs [5], and McNeil [12]. Most of these authors have used the backward system analogues, while we, based on our method, shall use the forward system. Also, this section will apparently have some relevance to the paper presented by Professor Gani at this Symposium. In the case of *M/M/I* queue, $T(t)$ as defined in the previous section represents the period for which the channel remains busy during $(0, t)$ and $[\int_0^t X(\tau)\, d\tau - T(t)]$ represents the total time wasted by the customers during $(0, t)$ while standing in the queue and waiting for their turn for service. Although these random variables are of some practical importance, to the best of author's knowledge, their distributions have not been considered before. The integrals studied by Gaver, Daley, and others were restricted only to a busy period of the queue, where "zero" acts as an absorption state. We shall touch this case briefly later.

As before, we are interested in obtaining the joint distribution of $X(t)$, $\int_0^t X(\tau)\, d\tau$, and $T(t)$. The analogue of equation (4.7) for the present case is given by

$$s\tilde{G}_s + \beta_2 s^2 \tilde{G}_s = \lambda(s - \tilde{r}_1)(s - \tilde{r}_2)\tilde{G} + [(\mu + \beta_1)s - \mu]\tilde{P}_0, \tag{4.27}$$

which is to be solved subject to $\tilde{G}(s, 0) = s^i$. Here $\tilde{r}_1$ and $\tilde{r}_2$, with positive and negative signs, respectively, are given by

$$\frac{1}{2\lambda}[(\mu + \lambda + \beta_1) \pm \{(\mu + \lambda + \beta_1)^2 - 4\mu\lambda\}^{1/2}]. \tag{4.28}$$

Unfortunately, the solution of (4.27) appears quite complex and involves Bessel functions. The author did not succeed in obtaining an explicit solution of (4.27). However, one can easily solve it when $\beta_2 = 0$, giving only the joint distribution of $X(t)$ and $T(t)$. Taking L.t. of (4.27) (with $\beta_2 = 0$) with respect to t, we obtain for $\alpha > 0$,

$$\tilde{G}^*(s, \alpha) = [\{\mu - (\mu + \beta_1)s\}\tilde{P}_0^*(\alpha) - s^{i+1}][\lambda(s - r_1^*)(s - r_2^*)]^{-1}, \tag{4.29}$$

where r_1^* and r_2^*, with positive and negative signs, respectively, are

$$\frac{1}{2\lambda}[(\mu + \lambda + \beta_1 + \alpha) \pm \{(\mu + \lambda + \beta_1 + \alpha)^2 - 4\mu\lambda\}^{1/2}], \tag{4.30}$$

and they satisfy the relation $0 < r_2^* < 1 < r_1^*$. Since $\tilde{G}^*$ is analytic for $|s| < 1$, the first of the two expressions on the right side of (4.29) must vanish at $s = r_2^*$. This fact yields

$$\tilde{P}_0^*(\alpha) = (r_2^*)^{i+1}[\mu - (\mu + \beta_1)r_2^*]^{-1}. \tag{4.31}$$

On substitution of this in (4.29), we obtain

$$(4.32)\qquad \tilde{G}^*(s, \alpha) = \frac{(r_2^*)^{i+1}[\mu - (\mu + \beta_1)s] - s^{i+1}[\mu - (\mu + \beta_1)r_2^*]}{\lambda(s - r_1^*)(s - r_2^*)[\mu - (\mu + \beta_1)r_2^*]}.$$

Finally, on putting $s = 1$ in (4.32) and inverting the resultant L.t. by lengthy yet standard methods (see Saaty [22]), we obtain

$$(4.33)\qquad \tilde{G}(1, t) = E(\exp\{-\beta_1 T(t)\} | X(0) = i) = \sum_{n=0}^{\infty} \tilde{P}_n(t),$$

where

$$(4.34)\ \tilde{P}_n(t) = \exp\{-(\lambda + \mu + \beta_1)t\}\Bigg\{\left(\frac{\mu}{\lambda}\right)^{(i-n)/2} B_{n-i}(\xi) + \frac{\mu + \beta_1}{\lambda}\left(\frac{\mu}{\lambda}\right)^{(i-n-1)/2} B_{n+i+1}(\xi) + \left(\frac{\mu}{\lambda}\right)^{(i-n)/2}(1 - \lambda\mu(\mu + \beta_1)^{-2}) \sum_{k=2}^{\infty} [(\mu + \beta_1)(\lambda\mu)^{-1/2}]^k B_{n+i+k}(\xi)\Bigg\},$$

and $B_n(u)$ denotes the Bessel function of the first kind and $\xi = 2(\lambda\mu)^{1/2}t$.

We now consider only the busy period of the $M/M/I$ queue started with $X(0) = i$. For this we take $\lambda_k = \lambda$ for $k \geqq 1$, and $\lambda_0 = 0$, so that "zero" is an absorption state of the process $X(t)$. As before $T(t) \uparrow T$ a.s. as $t \to \infty$, where T is the length of the busy period. If $\mu \geqq \lambda$, it is known that $X(t) \to 0$ with probability one, so that $P(T < \infty) = 1$. On the other hand if $\mu < \lambda$, $P(T < \infty) = P(X(t) \to 0) = \mu/\lambda$, and $P(T = \infty) = P(X(t) \to \infty) = 1 - (\mu/\lambda)$. The analogue of (4.27) now takes the form

$$(4.35)\qquad s\tilde{G}_t + \beta_2 s^2 \tilde{G}_s = [\lambda s^2 - (\lambda + \mu + \beta_1)s + \mu](\tilde{G} - \tilde{P}_0),$$

to be solved subject to $\tilde{G}(s, 0) = s^i$. Unfortunately, the solution of this presents similar difficulties as of equation (4.27). However, once it is solved we have the desired result given by

$$(4.36)\qquad \lim_{t\to\infty} \tilde{G}(1, t) = E\left(\exp\left\{-\beta_1 T - \beta_2 \int_0^T X(\tau)\, d\tau\right\}\Bigg| X(0) = i\right).$$

This result has been studied through other methods by Daley [4] and Daley and Jacobs [5]. Again, the equation (4.35) can be easily solved like (4.27) when $\beta_2 = 0$. However, since the distribution of the length of the busy period T is already known (see Saaty [22]), we shall not pursue this further here.

5. Illness-death processes

These processes have been extensively studied by Fix and Neyman [8] and more recently by Chiang [3]. Briefly, these are finite Markov chains with two sets of states; S_i, $i = 1, 2, \cdots, s$, are the illness states and R_θ, $\theta = 1, 2, \cdots, r$, are the death states. (In this section, symbols i, j, and k will stand for S_i, S_j, and S_k, and θ for R_θ.) Here all the death states are absorption states. Also, in terms of the notation of Section 2, for the transitions among the various states we have on adopting Chiang's notation,

$$\begin{aligned} c_i p_{ij} &= \nu_{ij}, & i \neq j, i, j &= 1, 2, \cdots, s, \\ c_i p_{i\theta} &= \mu_{i\theta}, & i = 1, 2, \cdots, s, \theta &= 1, 2, \cdots, r, \\ -c_i &= \nu_{ii}, & i &= 1, 2, \cdots, s. \end{aligned} \tag{5.1}$$

Consider a typical person moving from one state to another according to the above M.c. until he is absorbed into one of the death states. Chiang ([3], pp. 81 and 160) has considered the lengths of this person's stay in various states within a period of length t, and has given expressions for their expected values only; while our method leads easily to their joint distribution. For this, take

$$\delta f_\ell = \begin{cases} \delta_i & \text{if } \ell \text{ is the state } S_i, \\ \sigma_\theta & \text{if } \ell \text{ is the state } R_\theta. \end{cases} \tag{5.2}$$

Analogous to Chiang's notation (see [3], p. 152), let

$$\tilde{P}_{ij}(t) = P(Z(t) = 1, X(t) = S_j | X(0) = S_i, Z(0) = 1) \tag{5.3}$$

and

$$\tilde{Q}_{i\theta}(t) = P(Z(t) = 1, X(t) = R_\theta | X(0) = S_i, Z(0) = 1), \tag{5.4}$$

where $Z(t)$ represents the "quantal response process" as defined in Section 1. Then the forward system of equations for $\tilde{P}$ are given by

$$\frac{d\tilde{P}_{ij}(t)}{dt} = -\Big(\sum_{k \neq j} \nu_{jk} + \sum_{\theta} \mu_{j\theta} + \delta_i\Big)\tilde{P}_{ij}(t) + \sum_{k \neq j} \tilde{P}_{ik}(t)\nu_{kj} \tag{5.5}$$

for $i, j = 1, 2, \cdots, s$. Similarly, one could write down the backward system. Either of these systems can be uniquely solved for $\tilde{P}_{ij}(t)$, which then can be used to obtain $\tilde{Q}_{i\theta}(t)$ by noticing that

$$\tilde{Q}_{i\theta}(t) = \sum_{j=1}^{s} \mu_{j\theta} \int_0^t \tilde{P}_{ij}(\tau) \exp\{-\sigma_\theta(t - \tau)\}\, d\tau, \qquad \theta = 1, 2, \cdots, r. \tag{5.6}$$

Here the factor $\exp\{-\sigma_\theta(t-\tau)\}$ under the integral sign denotes the probability that our hypothetical animal of the "quantal response process" $Z(t)$ does not die during (τ, t), once the process has touched the state R_θ at moment τ. Let $X(0) = S_i$, and

$$\begin{aligned} T_{ij}(t) &= \text{length of stay in } S_j \text{ during } (0, t), \qquad j = 1, 2, \cdots, s, \\ \tilde{T}_{i\theta}(t) &= \text{length of stay in } R_\theta \text{ during } (0, t), \qquad \theta = 1, 2, \cdots, r. \end{aligned} \tag{5.7}$$

Having obtained $\tilde{P}_{ij}$ and $\tilde{Q}_{i\theta}$ in the above manner, the L.t. of the joint distribution of $T_{ij}(t)$ and $\tilde{T}_{i\theta}(t)$ is then given by

$$E\left(\exp\left\{-\sum_{j=1}^{s}\delta_j T_{ij}(t) - \sum_{\theta=1}^{r}\sigma_\theta \tilde{T}_{i\theta}(t)\right\}\right) = \sum_{j=1}^{s}\tilde{P}_{ij}(t) + \sum_{\theta=1}^{r}\tilde{Q}_{i\theta}(t), \tag{5.8}$$

where δ_j and σ_θ act as the dummy variables for the L.t. The L.t. (5.8), in general, is a rational function of δ's and σ's and can be inverted by standard methods to yield the desired distribution. We shall illustrate the above approach through an example, where we take $s = 2$, $r = 1$, and $X(0) = S_1$. Since

$$T_{11}(t) + T_{12}(t) + \tilde{T}_{11}(t) = t \qquad \text{a.s.,} \tag{5.9}$$

it is sufficient to study the joint distribution of $T_{11}(t)$ and $T_{12}(t)$ only, in which case we may take $\sigma_1 = 0$. The equations (5.5) are now given by

$$\begin{aligned} \frac{d\tilde{P}_{11}}{dt} &= -(\nu_{12} + \mu_{11} + \delta_1)\tilde{P}_{11} + \nu_{21}\tilde{P}_{12}, \\ \frac{d\tilde{P}_{12}}{dt} &= -(\nu_{21} + \mu_{21} + \delta_2)\tilde{P}_{12} + \nu_{12}\tilde{P}_{11}. \end{aligned} \tag{5.10}$$

Let

$$A = (\delta_1 + \nu_{12} + \mu_{11}), \qquad B = (\delta_2 + \nu_{21} + \mu_{21}), \tag{5.11}$$

and a_1 and a_2, with positive and negative signs, respectively, be given by

$$\tfrac{1}{2}[-(A + B) \pm \{(A - B)^2 + 4\nu_{12}\nu_{21}\}^{1/2}]. \tag{5.12}$$

The solution of (5.10) is given by

$$\begin{aligned} \tilde{P}_{11}(t) &= [(a_1 + B)\exp\{a_1 t\} - (B + a_2)\exp\{a_2 t\}](a_1 - a_2)^{-1}, \\ \tilde{P}_{12}(t) &= \nu_{12}[\exp\{a_1 t\} - \exp\{a_2 t\}](a_1 - a_2)^{-1}. \end{aligned} \tag{5.13}$$

Now using (5.6) with $\sigma_\theta = 0$, we can obtain $\tilde{Q}_{11}(t)$. Finally, omitting details, we have by using (5.8),

$$\begin{aligned} &E(\exp\{-\delta_1 T_{11}(t) - \delta_2 T_{12}(t)\}) \\ &\quad = \tilde{P}_{11}(t) + \tilde{P}_{12}(t) + \tilde{Q}_{11}(t) \\ &\quad = \Big\{[(a_1 + B + \nu_{12} + \mu_{11}) + (\mu_{11}B + \mu_{21}\nu_{12})a_1^{-1}]\exp\{a_1 t\} \\ &\qquad - [(B + a_2 + \nu_{12} + \mu_{11}) + (\mu_{11}B + \mu_{21}\nu_{12})a_2^{-1}] \\ &\qquad \cdot \exp\{a_2 t\}\Big\}(a_1 - a_2)^{-1} + \{(\mu_{11}B + \nu_{12}\mu_{21})\}(AB - \nu_{12}\nu_{21})^{-1}. \end{aligned} \tag{5.14}$$

Since $T_{11}(t)$ and $T_{12}(t)$ are monotone nondecreasing functions of t, $T_{11}(t) \uparrow T_{11}$ and $T_{12}(t) \uparrow T_{12}$ almost surely as $t \to \infty$. Here the random variables T_{11} and T_{12} represent the lengths of time the person spends in S_1 and S_2, respectively, before he finally dies. Letting $t \to \infty$ and using the fact that a_1 and a_2 are negative, we obtain from (5.14)

$$(5.15)\quad E(\exp\{-\delta T_{11} - \delta_2 T_{12}\}) = \frac{\mu_{11}(\nu_{21} + \mu_{21} + \delta_2) + \nu_{12}\mu_{21}}{(\delta_1 + \nu_{12} + \mu_{11})(\delta_2 + \nu_{21} + \mu_{21}) - \nu_{12}\nu_{21}}.$$

One can easily invert this transform to give explicitly the joint distribution of T_{11} and T_{12}. However, we shall not venture into this here. Instead, we refer the reader to [21] for further details concerning the sojourn times of the type T_{11}, T_{12}, and so forth, and close with the remark that T_{11} and T_{12}, in the present case, are positively correlated. Furthermore, marginally each one of them is (negative) exponentially distributed with a positive probability mass at zero only in the case of T_{12}.

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