ON THE FOUNDATIONS OF COMBINATORIAL THEORY (VI): THE IDEA OF GENERATING FUNCTION

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1. Introduction

Since Laplace discovered the remarkable correspondence between set theoretic operations and operations on formal power series, and put it to use with great success to solve a variety of combinatorial problems, generating functions (and their continuous analogues, namely, characteristic functions) have become an essential probabilistic and combinatorial technique. A unified exposition of their theory, however, is lacking in the literature. This is not surprising, in view of the fact that all too often generating functions have been considered to be simply an application of the current methods of harmonic analysis. From several of the examples discussed in this paper it will appear that this is not the case: in order to extend the theory beyond its present reaches and develop new kinds of algebras of generating functions better suited to combinatorial and probabilistic problems, it seems necessary to abandon the notion of group algebra (or semigroup algebra), so current nowadays, and rely instead on an altogether different approach.

The insufficiency of the notion of semigroup algebra is clearly seen in the example of Dirichlet series. The functions

$$n \to 1/n^s \tag{1.1}$$

defined on the semigroup S of positive integers under multiplication, are characters of S. They are not, however, all the characters of this semigroup, nor does there seem to be a canonical way of separating these characters from the rest (see, for example, Hewitt and Zuckerman [32]). In other words, there does not seem to be a natural way of characterizing the algebra of formal Dirichlet series as a subalgebra of the semigroup algebra (eventually completed under a suitable topology) of the semigroup S. In the present theory, however, the algebra of formal Dirichlet series arises naturally from the incidence algebra (definition below) of the lattice of finite cyclic groups, as we shall see.

The purpose of this work is to begin the development of a theory of generating functions that will not only include all algebras of generating functions used so far (ordinary, exponential, Dirichlet, Eulerian, and so on), but also provide a systematic technique for setting up other algebras of generating functions suited to particular enumerations. Our initial observation is that most families of discrete structures, while often devoid of any algebraic composition laws, are nevertheless often endowed with a natural order structure. The solution of the problem of their enumeration thus turns out to depend more often than not upon associating suitable computational devices to such order structures.

Our starting point is the notion of *incidence algebra*, whose study was briefly begun in a previous paper, and which is discussed anew here. Section 3 contains the main facts on the structure of the incidence algebra of an ordered set; perhaps the most interesting new result is the explicit characterization of the lattice of two sided ideals. It follows from recent results of Aigner, Prins, and Gleason (motivated by the present work) that for an ordered set with a unique minimal element the incidence algebra is uniquely characterized by its lattice of ideals; this assertion is no longer true if the ordered set has no unique minimal element. In particular, the lattice of two sided ideals is distributive, an unusual occurrence in a noncommutative algebra. Our characterization of the radical suggests that a simple axiomatic description of incidence algebras should be possible, and we hope someone will undertake this task.

Section 4 introduces the main working tool, namely, the reduced incidence algebra. This notion naturally arises in endowing the segments of an ordered set with an equivalence relation. Such an equivalence is usually dictated by the problem at hand, and leads to the definition of the *incidence coefficients*, a natural generalization of the classical binomial coefficients. After a brief study of the family of all equivalence relations compatible with the algebra structure, we show by examples that all classical generating functions (and their incidence coefficients) can be obtained as reduced incidence algebras. We believe this is a remarkable fact, and perhaps the most cogent argument for the use of the present techniques.

Section 5 extends the notion of reduced incidence algebras to families of ordered structures. The notion of multiplicative functions on partitions of a set and the isomorphism with the semigroup of formal power series without constant term under functional composition (Theorem 5.1) are perhaps the most important results here. Because of space limitations, we have given only a few applications, which hopefully should indicate the broad range of problems which it can solve (for example, enumeration of solutions of an equation in the symmetric group G_n, as a function of n). Pursuing the same idea, we obtain an algebra of multiplicative functions on a class of ordered structures recently studied by Dowling [19], which were suggested by problems in coding theory. Finally, we obtain the algebra of Philip Hall, arising from the enumeration of abelian groups, as a large incidence algebra.

Section 6 studies the strange phenomenon pointed out in Section 4, that the maximally reduced incidence algebra does not coincide with the algebra obtained by identifying isomorphic segments of an ordered set. The structure of such an algebra is determined.

Sections 7, 8, and 9 make a detailed study of those algebras of generating functions which are closest to the classical cases. Algebras of Dirichlet type are those where all the analogs of classical number theoretic functions can be defined, including the classical product formula for the zeta function. Algebras of binomial type are close to the classical exponential generating functions, and naturally arise in connection with certain block designs. Under mild hypotheses, we give a complete classification of such algebras.

Several applications and a host of other examples could not be treated here. Among them, we mention a general theory of multiplicative functions, and their relation to the coalgebra structure (as sketched in Goldman and Rota [25]), and large incidence algebras arising in the study of classes of combinatorial geometries closed under the operation of taking minors, in particular the *coding geometries* of R. C. Bose and B. Segre, of which the Dowling lattices are special cases.

This work was begun in Los Alamos in the summer of 1966. Since then, the notion of reduced incidence algebra was independently discovered by D. A. Smith and H. Scheid, who developed several interesting properties. The bulk of the material presented here, with the obvious exception of some of the examples, is believed to be new.

2. Notations and terminology

Very little knowledge is required to read this work. Most of the concepts basic enough to be left undefined in the succeeding sections will be introduced here.

A *partial ordering relation* (denoted by $\leqq$) on a set P is one which is reflexive, transitive, and antisymmetric (that is, $a \leqq b$ and $b \leqq a$ imply $a = b$). A set P together with a partial ordering relation is a *partially ordered set*, or simply an *ordered set*. A *segment* $[x, y]$, for x and y in P, is the set of all elements z which satisfy $x \leqq z \leqq y$. A partially ordered set is *locally finite* if every segment is finite. We shall consider locally finite partially ordered sets only.

An ordered set P is said to have a 0 or a 1 if it has a unique minimal or maximal element.

An *order ideal* in an ordered set P is a subset Z of P which has the property that if $x \in Z$ and $y \leqq x$, then $y \in Z$.

The *product* $P \times Q$ of two ordered sets P and Q is the set of all ordered pairs (p, q), where $p \in P$ and $q \in Q$, endowed with the order $(p, q) \geqq (r, s)$ whenever $p \geqq r$ and $q \geqq s$. The product of any number of partially ordered sets is defined similarly. The *direct sum* or *disjoint union* $P + Q$ of two ordered sets P and Q is the set theoretic disjoint union of P and Q, with the ordering $x \leqq y$ if and only if (i) $x, y \in P$ and $x \leqq y$ in P or (ii) $x, y \in Q$ and $x \leqq y$ in Q. Note that if $p \in P$ and $q \in Q$, then p and q are incomparable.

In an ordered set P, an element p *covers* an element q when the segment $[q, p]$ has two elements. An *atom* is an element which covers a minimal element.

A *chain* is an ordered set in which every pair of elements is comparable. A *maximal chain* in a segment $[x, y]$ of an ordered set P is a sequence $(x_0, x_1, \cdots, x_n)$, where $x_0 = x$, $x_n = y$, and x_{i+1} covers x_i for all i. The chain $(x_0, x_1, \cdots, x_n)$ is said to have *length* n. An *antichain* is an ordered set in which no two distinct elements are comparable.

The *dual* P^* of an ordered set P is the ordered set obtained from P by inverting the order.

A *lattice* is an ordered set where max and min of two elements (we call them join and meet, and write them $\vee$ and $\wedge$) are defined. A *complete lattice* is a lattice in which the join and meet of any subset exist. A *sublattice* L' of a lattice L is a subset which is a lattice with the induced order relation and in which join and meet of two elements correspond with the join and meet in L. For the definitions of *distributive*, *modular*, and *semimodular* see Birkhoff.

A *partition* of a set S is a set of disjoint nonempty subsets of S whose union is S. The subsets of S making up the partition are called the *blocks* of the partition. The *lattice of partitions* $\Pi(S)$ of a set S is the set of partitions of S, ordered by *refinement*: a partition π is less than a partition σ (or is a refinement of σ) if every block of π is contained in a block of σ. The 0 of $\Pi(S)$ is the partition whose blocks are the one element subsets of S, and the 1 of $\Pi(S)$ is the partition with one block. There is a natural correspondence between equivalence relations on a set S and partitions of S, since the equivalence classes of an equivalence relation form the blocks of a partition, and hence, there is an induced lattice structure on the family of equivalence relations of S.

At the beginning of Section 3, we define the *incidence algebra* $\mathbf{I}(P, K)$ of a locally finite ordered set P, over a field K. We assume throughout that K has characteristic 0, except for the last paragraph of Section 6 when it is explicitly stated that another characteristic is being considered. We also assume that K is a topological field, and if the topology of K is not specified, we regard K as having the discrete topology.

A certain familiarity is assumed with pp. 342–347 of *Foundations I* ([49]), when the definitions of Möbius function and zeta function are given and some elementary properties of the incidence algebra are derived.

3. Structure of the incidence algebra

3.1 *Basic identifications.* As mentioned in Section 2, we define the *incidence algebra* $\mathbf{I}(P, K)$ of a locally finite ordered set P, over a field K, as follows. The members of $\mathbf{I}(P, K)$ are K valued functions $f(x, y)$ of two variables, with x and y ranging over P and with the sole restriction that $f(x, y) = 0$ unless $x \leqq y$. The sum of two such functions, as well as multiplication by scalars, are defined as usual, and the product $f*g = h$ is defined as follows,

$$h(x, y) = \sum_{z\in P} f(x, z)g(z, y). \tag{3.1}$$

In virtue of the assumption that the ordered set P is locally finite, the variable z in the sum on the right ranges over the finite segment $[x, y]$.

It is immediately verified that this product is associative. It is also easily verified that the incidence algebra is commutative if and only if the order relation of P is trivial, that is, if and only if no two elements of P are comparable. Whenever convenient, we shall omit mention of the field K and briefly write $\mathbf{I}(P)$, with the tacit convention that K is to remain fixed throughout.

The identity element of $\mathbf{I}(P)$ will be denoted by δ, after the Kronecker delta. In addition, we use the following notation for certain elements of $\mathbf{I}(P)$. If $x \in P$, let

$$e_x(u, v) = \begin{cases} 1 & \text{if } u = v = x, \\ 0 & \text{otherwise,} \end{cases} \tag{3.2}$$

and for $x \leqq y$, let

$$\delta_{x,y}(u, v) = \begin{cases} 1 & \text{if } u = x \text{ and } v = y, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

Clearly, the elements e_x are idempotent, and the $\delta_{x,y}$ are analogous to the matrix units of ring theory (see Jacobson [35]). Note that $e_x = \delta_{x,x}$.

The following easily verified identities will be used in the sequel:

$$\text{if } g = \delta_{x,y}*f, \text{ then } g(u, v) = \begin{cases} 0 & \text{if } u \neq x, \\ f(y, v) & \text{if } u = x; \end{cases} \tag{3.4}$$

$$\text{if } g = f*\delta_{z,w}, \text{ then } g(u, v) = \begin{cases} 0 & \text{if } v \neq w, \\ f(u, z) & \text{if } v = w, \end{cases} \tag{3.5}$$

$$\text{if } g = \delta_{x,y}*f*\delta_{z,w}, \text{ then } g(u, v) = \begin{cases} 0 & \text{if } u \neq x \text{ or } v \neq w \\ f(y, z) & \text{if } u = x \text{ and } v = w, \end{cases} \tag{3.6}$$

that is, $\delta_{x,y}*f*\delta_{z,w} = f(y, z)\delta_{x,w}$. In particula , $e_x*f*e_y = f(x, y)\delta_{x,y}$, and $\delta_{x,y}*\delta_{z,w} = \delta(y, z)\delta_{x,w}$.

3.2 *The standard topology.* A topology on $\mathbf{I}(P)$ is defined as follows. A generalized sequence $\{f_n\}$ converges to f in $\mathbf{I}(P)$ if and only if $f_n(x, y)$ converges to $f(x, y)$ in the field K for every x and y. We call this the *standard topology* of $\mathbf{I}(P)$.

PROPOSITION 3.1. *Let P be a locally finite ordered set. Then the incidence algebra $\mathbf{I}(P)$, equipped with the standard topology, is a topological algebra.*

PROOF. In the right side of the definition (3.1) of the product, only a finite number of terms occur for fixed x and y; this implies at once that the product $(f, g) \to f*g$ is continuous in both variables. The verification of all other properties is immediate. *Q.E.D.*

In the sequel, we shall often have occasion to use infinite sums of the form

$$f = \sum_{x, y \in P} f(x, y)\, \delta_{x, y}, \tag{3.7}$$

and we shall presently discuss the meaning that is to be attached to the right side. Let Φ be a directed set of finite subsets of $P \times P$, with the following properties: (i) Φ is ordered by inclusion; (ii) for every pair $x, y \in P$ there exists a member $A \in \Phi$ such that $(x, y) \in A$. We call such a directed set *standard*.

PROPOSITION 3.2. *Let Φ be a standard directed set. Then the set $\{f_A : A \in \Phi)$ defined as*

$$f_A = \sum_{(x, y) \in A} f(x, y)\, \delta_{x, y} \tag{3.8}$$

converges in the standard topology of the incidence algebra $\mathbf{I}(P)$ to the element f.

PROOF. Take $A \in \Phi$ so that $(x, y) \in A$. Then for every $B \in \Phi, B \geqq A$, we have $f_B(x, y) - f(x, y) = 0$. *Q.E.D.*

Speaking in classical language, the preceding proposition states that the "sum" on the right side of (3.7) converges to the element f together with all its "rearrangements". This justifies the use of the summation symbol on the right side of (3.7), and we shall make use of it freely from now on.

3.3. *Ideal structure.* We shall now determine the lattice of (two sided, closed) ideals of the incidence algebra $\mathbf{I}(P)$, endowed with the standard topology. For P finite, all two sided ideals are closed, so Theorem 3.1 below determines the lattice of all ideals.

Let J be a closed ideal in $\mathbf{I}(P)$, and let $\Delta(J)$ be the collection of all elements $\delta_{x, y}$ belonging to J. We call $\Delta(J)$ the *support* of the ideal J. Then, any finite or infinite linear combination of the $\delta_{x, y}$ in $\Delta(J)$ gives a member of J. Conversely, if $f \in J$, then, by 3.6 above,

$$e_x * f * e_y = f(x, y)\, \delta_{x, y}; \tag{3.9}$$

hence, if $f(x, y) \neq 0$, it follows that $\delta_{x, y} \in \Delta(J)$. This proves the following.

LEMMA 3.1. *Every closed ideal J in the incidence algebra $\mathbf{I}(P)$ consists of all functions $f \in \mathbf{I}(P)$ such that $f(x, y) = 0$ whenever $\delta_{x, y} \notin \Delta(J)$.*

Now, let $Z(J)$ be the family of all segments $[x, y]$ such that $f(x, y) = 0$ for all $f \in J$. Then we have

LEMMA 3.2. *If $[x, y] \in Z(J)$ and $x \leqq u \leqq v \leqq y$, then $[u, v] \in Z(J)$.*

The proof is immediate: Let $f \in J$. By (3.6) again,

$$\delta_{x, u} * f * \delta_{v, y} = f(u, v)\, \delta_{x, y}. \tag{3.10}$$

Thus, if $\delta_{x, y} \notin J$, then $f(u, v) = 0$, and $[u, v] \in Z(J)$.

We are now ready to state the main result.

THEOREM 3.1. *In a locally finite ordered set P, let $S(P)$ be the set of all segments of P, ordered by inclusion. Then there is a natural anti-isomorphism between the lattice of closed ideals of the incidence algebra $\mathbf{I}(P)$ and the lattice of order ideals of $S(P)$.*

PROOF. Let J be an ideal of P, and let $Z(J)$ be the family of segments defined above. Lemma 3.1 shows that $Z(J)$ uniquely determines J, and Lemma 3.2 shows that $Z(J)$ is an order ideal in $S(P)$.

Conversely, let Z be an order ideal in $S(P)$, and let J be the set of all $f \in \mathbf{I}(P)$ for which $f(x, y) = 0$, if $[x, y] \in Z$. Then J is an ideal. Indeed, if $g \in \mathbf{I}(P)$ is arbitrarily chosen, if $f \in J$, if $[x, y] \in Z$, and if $h = f * g$, then

$$h(x, y) = \sum_{x \leqq z \leqq y} f(x, z)g(z, y) = 0, \tag{3.11}$$

since all $f(x, z) = 0$ for z between x and y. The case is similar for multiplication on the left. Since we can take arbitrarily infinite sums as in (3.7), it follows that J is closed, and the proof is complete.

COROLLARY 3.1. *The lattice of closed ideals of an incidence algebra is distributive.*

COROLLARY 3.2. *The closed maximal ideals of an incidence algebra* $\mathbf{I}(P)$ *are those of the form*

$$J_x = \{f \in \mathbf{I}(P) \mid f(x, x) = 0\}, \tag{3.12}$$

where $x \in P$.

3.4. *The radical.* We recall the well-known and easily proved fact (see Smith [55], or *Foundations I*) that an element f of the incidence algebra has an inverse if and only if $f(x, x) \neq 0$ for all $x \in P$. From this it follows (Jacobson [35], p. 8, and following) that an element $f \in \mathbf{I}(P)$ is *quasiregular* if and only if $f(x, x) \neq 1$ for all $x \in P$. Hence, an element f has the property that $g * f * h$ is quasiregular for all g and h, if and only if $f(x, x) = 0$ for all $x \in P$. From Proposition 1 on page 9 of Jacobson, we make the following inference.

PROPOSITION 3.3. *The radical* R *of the incidence algebra* $\mathbf{I}(P)$ *of a locally finite ordered set* P *is the set of all* $f \in \mathbf{I}(P)$ *such that* $f(x, x) = 0$ *for all* $x \in P$.

3.5. *The incidence algebra as a functor.* We now determine a class of maps between locally finite ordered sets so that the association of the incidence algebra to such sets can be extended, in a natural way, to a functor into the category of K algebras (where K is the fixed ground ring or field). A function σ from an ordered set P to an ordered set Q will be called a *proper map* if it satisfies the following three conditions:

(a) σ is one to one;

(b) $\sigma(p_1) \leqq \sigma(p_2)$ implies $p_1 \leqq p_2$;

(c) if q_1 and q_2 are in the image of σ, and $q_1 \leqq q_2$, then the whole segment $[q_1, q_2]$ is in the image.

Note that in view of (a) and (b), condition (c) can be replaced by

(c′) if $\sigma(p_1) \leqq \sigma(p_2)$ and $q \in [\sigma(p_1), \sigma(p_2)]$, then there is a unique $p \in [p_1, p_2]$ such that $\sigma(p) = q$.

It is clear that the identity function on any partially ordered set is a proper map, and it is not hard to verify that the composition of proper maps is a proper map. Thus, ordered sets together with proper maps form a category. Let $\mathscr{A}$ be the subcategory of locally finite ordered sets together with proper maps. We then have the following proposition.

PROPOSITION 3.4. (i) *The mapping* $\mathbf{I}$ *from* $\mathscr{A}$ *to the category of* K *algebras, given by* $\mathbf{I}(P)$ = *incidence algebra of* P (*with values in* K) *and*

$$[\mathbf{I}(\sigma)(f)](p_1, p_2) = f(\sigma(p_1), \sigma(p_2)), \tag{3.13}$$

where $\sigma: P \to Q$ *and* $f \in \mathbf{I}(Q)$, *is a contravariant functor.*

(ii) *If* $\rho: P \to Q$ *is a function and* $\mathbf{I}(\rho)$ (*as defined above*) *is a homomorphism from* $\mathbf{I}(Q)$ *to* $\mathbf{I}(P)$, *then* ρ *is a proper map.*

PROOF. (i) If $f \in \mathbf{I}(Q)$ and $p_1, p_2 \in P$, then $[\mathbf{I}(\sigma)(f)](p_1, p_2) \neq 0$ implies that $f(\sigma(p_1), \sigma(p_2)) \neq 0$, which implies that $\sigma(p_1) \leqq \sigma(p_2)$ (since $f \in \mathbf{I}(Q)$) and hence (by condition (b)), that $p_1 \leqq p_2$, and so $\mathbf{I}(\sigma)(f) \in \mathbf{I}(P)$. Thus, $\mathbf{I}(\sigma)$ is a mapping from $\mathbf{I}(Q)$ to $\mathbf{I}(P)$.

It is clearly a linear map. Furthermore, $\mathbf{I}(\sigma)$ takes the identity of $\mathbf{I}(Q)$ to the identity of $\mathbf{I}(P)$, since by condition (a)

$$[\mathbf{I}(\sigma)(\delta_Q)](p_1, p_2) = \delta_Q(\sigma(p_1), \sigma(p_2)) = \delta_P(p_1, p_2). \tag{3.14}$$

Finally, $\mathbf{I}(\sigma)$ preserves multiplication, since

$$\begin{aligned} [\mathbf{I}(\sigma)(f*g)](p_1, p_2) &= f*g(\sigma(p_1), \sigma(p_2)) \\ &= \sum_{q \in [\sigma(p_1), \sigma(p_2)]} f(\sigma(p_1), q) g(q, \sigma(p_2)) \\ &= \sum_{p \in [p_1, p_2]} f(\sigma(p_1), \sigma(p)) g(\sigma(p), \sigma(p_2)) \\ &= \sum_{p \in [p_1, p_2]} [\mathbf{I}(\sigma)(f)](p_1, p) \cdot [\mathbf{I}(\sigma)(g)](p, p_2) \\ &= ([\mathbf{I}(\sigma)(f)] * [\mathbf{I}(\sigma)(g)])(p_1, p_2). \end{aligned} \tag{3.15}$$

(The third equality follows from (c′).)

Thus, $\mathbf{I}(\sigma)$ is an algebra homomorphism from $\mathbf{I}(Q)$ to $\mathbf{I}(P)$. To verify that $\mathbf{I}$ is a functor, it remains to show that $\mathbf{I}(id_P) = id_{\mathbf{I}(P)}$, where id_P is the identity map on P, and where $id_{\mathbf{I}(P)}$ is the identity map on $\mathbf{I}(P)$, and that $\mathbf{I}(\sigma \circ \tau) = \mathbf{I}(\tau) \circ \mathbf{I}(\sigma)$ when the composition is defined; but these are clear.

(ii) Now, let $\rho: P \to Q$ be a function for which $\mathbf{I}(\rho)$ is a homomorphism from $\mathbf{I}(Q)$ to $\mathbf{I}(P)$. Then

$$\begin{aligned} \delta_Q(\rho(p_1), \rho(p_2)) &= [\mathbf{I}(\rho)(\delta_Q)](p_1, p_2) \\ &= \delta_P(p_1, p_2) \end{aligned} \tag{3.16}$$

since $\mathbf{I}(\rho)$ is a homomorphism, so that ρ is one to one, that is, ρ satisfies (a).

That ρ satisfies (b) follows from the fact that if $\rho(p_1) \leqq \rho(p_2)$, then $\zeta_Q(\rho(p_1), \rho(p_2)) = 1$; that is, $[\mathbf{I}(\rho)(\zeta_Q)](p_1, p_2) = 1$, and so $p_1 \leqq p_2$, since $\mathbf{I}(\rho)(\zeta_Q) \in \mathbf{I}(P)$. Finally, let $q_1 = \rho(p_1)$, $q_2 = \rho(p_2)$, $q_1 \leqq q_2$, and $q \in [q_1, q_2]$. Then we have

$$\sum_{p\in[p_1,p_2]} \delta_{q_1,q}(q_1,\rho(p))\cdot\delta_{q,q_2}(\rho(p),q_2) \tag{3.17}$$
$$= \sum_{p\in[p_1,p_2]} [\mathbf{I}(\rho)(\delta_{q_1,q})](p_1,p)\cdot[\mathbf{I}(\rho)(\delta_{q,q_2})](p,p_2)$$
$$= ([\mathbf{I}(\rho)(\delta_{q_1,q})]*[\mathbf{I}(\rho)(\delta_{q,q_2})])(p_1,p_2)$$
$$= [\mathbf{I}(\rho)(\delta_{q_1,q}*\delta_{q,q_2})](p_1,p_2) = \delta_{q_1,q}*\delta_{q,q_2}(q_1,q_2) = 1.$$

Thus, $\rho(p) = q$ for some $p \in [p_1, p_2]$, and so ρ satisfies (c). *Q.E.D.*

We conclude with a number of examples of proper maps.

EXAMPLE 3.1. Any one to one map from an ordered set to an antichain is a proper map.

EXAMPLE 3.2. The proper maps from the integers (with the standard ordering) to themselves are those of the form $f(x) = x + k$, where k is some fixed integer.

EXAMPLE 3.3. If P is any finite or locally finite countable ordered set, a proper map onto P from a chain of integers is obtained by labeling the elements of P with integers so that $p_i < p_j$ only if $i < j$, and then taking the map $\sigma(i) = p_i$. A result of Hinrichs [33] guarantees that such a labeling of P exists.

3.6. *Isomorphic incidence algebras.* In this subsection, we prove the result of Stanley [58] that an ordered set P is uniquely determined by its incidence algebra $\mathbf{I}(P)$.

THEOREM 3.2. *Let P and Q be locally finite ordered sets. If $\mathbf{I}(P)$ and $\mathbf{I}(Q)$ are isomorphic as K algebras (even as rings), then P and Q are isomorphic.*

PROOF. We shall show how the ordered set P can be uniquely recovered from the ring $\mathbf{I}(P)$. If R is the radical of $\mathbf{I}(P)$, then $\mathbf{I}(P)/R$ is isomorphic to a direct product $\Pi_{x\in P}K_x$ of copies of the ground field $K = K_x$, one for each element x of P. The K_x are intrinsically characterized as being the *minimal components* of $\mathbf{I}(P)/R$. Note that the element e_x is an idempotent whose image in $\mathbf{I}(P)/R$ is the identity element of K_x. Moreover, the e_x are orthogonal, that is, $e_xe_y = e_ye_x$ if $x \neq y$.

Define an order relation P' on the e_x as follows: $e_x \leqq e_y$ if and only if $e_x\mathbf{I}(P)e_y \neq \{0\}$. It is clear from equation (3.6) that $e_x \leqq e_y$ if and only if $x \leqq y$ in P. Thus, $P' \simeq P$.

The proof will be complete if we can show that given any set $\{f_x | x \in P\}$ of orthogonal idempotents in $\mathbf{I}(P)$ such that the image of f_x in $\mathbf{I}(P)/R$ is the identity element of K_x, then the order relation defined on the f_x in analogy to the e_x is isomorphic to P'.

It suffices to prove that there is an automorphism σ of $\mathbf{I}(P)$ such that $\sigma(e_x) = f_x$ for all $x \in P$. We will explicitly exhibit an inner automorphism $\sigma(g) = hgh^{-1}$, for some fixed invertible $h \in \mathbf{I}(P)$, with the desired property. Define

$$h = \sum_{x\in P} f_xe_x. \tag{3.18}$$

Clearly, h is a well-defined invertible element of $\mathbf{I}(P)$, since $h(x, y) = f_y(x, y)$.

Now by orthogonality of the e_x and the f_x, we have $he_x = f_x e_x$ and $f_x h = f_x e_x$. Hence, $he_x h^{-1} = f_x$ for all $x \in P$, and the proof is complete.

4. Reduced incidence algebras

4.1. *Order compatible relations.* In most problems of enumeration it is not the full incidence algebra that is required, but only a much smaller subalgebra of it; for example, the algebras of ordinary, exponential, Eulerian and Dirichlet generating functions are obtained by taking subalgebras of suitable incidence algebras (see Examples 4.1 through 4.12). These subalgebras are obtained by taking suitable equivalence relations on segments of a locally finite ordered set P, and then considering functions which take the same values on equivalent segments. We are therefore led to the following.

DEFINITION 4.1. *An equivalence relation $\sim$ defined on the segments of a locally finite ordered set P is said to be order compatible (or simply compatible) when it satisfies the following condition: if f and g belong to the incidence algebra $\mathbf{I}(P)$ and $f(x, y) = f(u, v)$ as well as $g(x, y) = g(u, v)$ for all pairs of segments such that $[x, y] \sim [u, v]$, then $(f*g)(x, y) = (f*g)(u, v)$.*

EXAMPLE 4.1. Set $[x, y] \sim [u, v]$ whenever the two segments are isomorphic; then $\sim$ is an order compatible equivalence relation.

There is in general no simple criterion, expressible in terms of the partial ordering, to decide when an equivalence relation on segments is order compatible. A useful sufficient criterion is the following.

PROPOSITION 4.1 (D. A. Smith). *An equivalence relation $\sim$ on the segments of an ordered set P is order compatible if whenever $[x, y] \sim [u, v]$ there exists a bijection φ, depending in general upon $[x, y]$, of $[x, y]$ onto $[u, v]$ such that $[x_1, y_1] \sim [\varphi(x_1), \varphi(y_1)]$ for all x_1, y_1 such that $x \leqq x_1 \leqq y_1 \leqq y$.*

The easy proof is left to the reader.

We shall be first concerned with the family of all order compatible equivalence relations on P. Its elementary structure is given by the following.

PROPOSITION 4.2. *The family of order compatible equivalence relations on a locally finite ordered set P, ordered by refinement, is a complete lattice $C(P)$, in which joins coincide with joins in the lattice $L(P)$ of all equivalence relations (partitions) on the segments of P.*

PROOF. In proving that joins in $C(P)$ coincide with joins in $L(P)$, it is convenient to use the language of partitions of the set of segments of P. Thus, let $\mathbf{F}$ be a family of partitions each of which defines a compatible equivalence relation. Let π be the join of $\mathbf{F}$, defining an equivalence relation $\sim$. Suppose that $f(x, y) = g(u, v)$ for all pairs of segments such that $[x, y] \sim [u, v]$. Then *a fortiori* for all $\simeq$ in $\mathbf{F}$, we shall have $f(x, y) = g(u, v)$ for all pairs of segments such that $[x, y] \simeq [u, v]$. It follows that $(f*g)(x, y) = (f*g)(u, v)$ for all such pairs of intervals. But, by definition of join of partitions, $[x, y] \sim [u, v]$ if and only if there is a sequence $\simeq_1, \simeq_2, \cdots, \simeq_n$ in $\mathbf{F}$ and segments $[x_i, y_i]$ such that $[x, y] \simeq_1 [x_1, y_1] \cdots \simeq_{n-1} [x_{n-1}, y_{n-1}] \simeq_n [u, v]$. It follows that $f(x, y) =$

$f(x_1, y_1) = \cdots$, similarly for g. Recalling that $\simeq_1$ is order compatible, we have $(f*g)(x, y) = (f*g)(x_1, y_1)$, and so forth, giving finally $(f*g)(x, y) = (f*g)(u, v)$.

The ordered set $C(P)$ has a 0, namely, the equivalence relation where no two distinct segments are equivalent, and therefore arbitrary meets exist by a simple result of lattice theory. *Q.E.D.*

Observe that meets in $C(P)$ do not in general coincide with meets in $L(P)$, so that $C(P)$ is not a sublattice of $L(P)$. Unless P is finite, it follows that $C(P)$ is not locally finite, for it is easy to stretch an infinite chain between 0 and 1 in $C(P)$ by successively "identifying" pairs of segments $[x, x]$ and $[u, u]$.

It is tempting to presume that the maximal element I of $C(P)$ is the equivalence relation described in Example 4.1, where every pair of isomorphic segments is equivalent. Surprisingly, this presumption is not generally true, even for finite ordered sets, as the following example indicates.

EXAMPLE 4.2. Let P be the ordered set obtained by taking the lattices L_1 and L_2 of subspaces of two nonisomorphic finite projective planes of the same order and identifying the top of L_1 with the bottom of L_2. Define $[x, y] \sim [u, v]$ whenever the two segments are isomorphic or whenever $[x, y] \approx L_1$, $[u, v] \approx L_2$.

4.2. *The incidence coefficients.* Let $\sim$ be an order compatible equivalence relation on P, which will remain fixed until further notice. Denote by Greek letters $\alpha, \beta, \cdots$ the nonempty equivalence classes of segments of P relative to $\sim$, and call them *types* (relative to $\sim$) for short.

Consider the set of all functions f defined on the set of types, with addition defined as usual, and multiplication $f*g = h$ defined as follows:

$$h(\alpha) = \sum \begin{bmatrix} \alpha \\ \beta, \gamma \end{bmatrix} f(\beta)g(\gamma). \tag{4.1}$$

The sum ranges over all ordered pairs β, γ of types. The brackets on the right are called the *incidence coefficients*, and are defined as follows:

$$\begin{bmatrix} \alpha \\ \beta, \gamma \end{bmatrix} \tag{4.2}$$

stands for the number of distinct elements z in a segment $[x, y]$ of type α, such that $[x, z]$ is of type β and $[z, y]$ is of type γ.

To see that the incidence coefficients are well defined, define $h_\delta \in \mathbf{I}(P)$ by

$$h_\delta(x, y) = \begin{cases} 1 & \text{if } [x, y] \text{ is of type } \delta, \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

If $[u, v]$ is of type α, then clearly

$$(h_\beta * h_\gamma)(u, v) = \begin{bmatrix} \alpha \\ \beta, \gamma \end{bmatrix}. \tag{4.4}$$

Since $\sim$ is order compatible, the left side of (4.4) is independent of whichever interval $[u, v]$ of type α is chosen, so that the incidence coefficients are well defined. The incidence coefficients are a generalization of the classical *binomial coefficients*, as the examples below will show. The corresponding generalization of the algebra of generating functions is given next.

PROPOSITION 4.3. *Let P be a locally finite ordered set, together with a compatible equivalence relation $\sim$ on the segments of P. Then the set of all functions defined on types forms an associative algebra with identity, with the product defined by* (4.1), *called the* reduced incidence algebra $\mathbf{R}(P, \sim)$ modulo the equivalence relation $\sim$. *The algebra $\mathbf{R}(P, \sim)$ is isomorphic to a subalgebra of the incidence algebra of P.*

To complete the proof (much of which has already been given above), all that needs to be shown is that $\mathbf{R}(P, \sim)$ is isomorphic to a subalgebra of $\mathbf{I}(P)$ which contains δ. This will imply that the algebra $\mathbf{R}(P, \sim)$ is associative.

For $f \in \mathbf{R}(P, \sim)$, define $\hat{f} \in \mathbf{I}(P)$ as follows: $\hat{f}(x, y) = f(\alpha)$ if the segment $[x, y]$ is of type α. The only properties to be checked are that the mapping is an isomorphism and that $\delta = \hat{f}$ for some $f \in \mathbf{R}(P, \sim)$. Since each type is by definition nonempty it follows that $f \to \hat{f}$ is well defined; it is obviously one to one. Furthermore, from the definition of the incidence coefficients, we find immediately that the product is

$$\hat{h}(x, y) = \sum_{x \leqq z \leqq y} \hat{f}(x, z)\hat{g}(z, y), \tag{4.5}$$

and thus coincides with the definition (4.1) of the product in $\mathbf{R}(P, \sim)$. The fact that $\delta = \hat{f}$ for some $f \in \mathbf{R}(P, \sim)$ follows from part (i) of the following lemma.

LEMMA 4.1. *Let $\sim$ be an order compatible equivalence relation on the segments of P, and let $[x, y] \sim [u, v]$. Then*

(i) $\nu([x, y]) = \nu([u, v])$, *where* $\nu([x, y]) =$ *number of* z *in* $[x, y]$;

(ii) *for every n, $[x, y]$ and $[u, v]$ have the same number of maximal chains of length n.*

PROOF. Part (i) follows from the fact that $\nu([x, y]) = \zeta^2(x, y)$ and that ζ is constant on equivalence classes of $\sim$.

From (i), it follows that the function h defined by

$$h(x, y) = \begin{cases} 1 & \text{if } \nu([x, y]) = 2, \text{ that is, } y \text{ covers } x, \\ 0 & \text{otherwise,} \end{cases} \tag{4.6}$$

is constant on equivalence classes of $\sim$; hence, so is h^n for every n, which proves (ii).

COROLLARY 4.1. *If for all types α, β, γ we have $\left[{}_{\beta,\gamma}^{\alpha}\right] = \left[{}_{\gamma,\beta}^{\alpha}\right]$, then the reduced incidence algebra $\mathbf{R}(P, \sim)$ is commutative.*

This follows immediately from definition (4.1) of the product.

Now let $\sim$ and $\simeq$ be two order compatible equivalence relations on the segments of P. Suppose that $[x, y] \sim [u, v]$ implies $[x, y] \simeq [u, v]$. Then, much as in the preceding proposition, $\mathbf{R}(P, \simeq)$ is isomorphic to a subalgebra of $\mathbf{R}(P, \sim)$; the isomorphism is obtained as follows: Let $\hat{\alpha}$ be a type relative to the equivalence relation $\simeq$. For $f \in \mathbf{R}(P, \simeq)$, set $\hat{f} \in \mathbf{R}(P, \sim)$ to be $\hat{f}(\alpha) = f(\hat{\alpha})$, where α is any type in $\mathbf{R}(P, \sim)$ such that the segments of type α are of type $\hat{\alpha}$ in $\mathbf{R}(P, \simeq)$.

Furthermore, $\mathbf{R}(P, \sim)$ strictly contains a natural isomorphic image of $\mathbf{R}(P, \simeq)$ unless $\sim$ equals $\simeq$, as is immediately seen by considering functions equal to one on a given type, and zero elsewhere. Thus, the lattice $C(P)$ is anti-isomorphic to the lattice of reduced incidence algebras, ordered by containment.

If $\sim$ is as in Example 4.1, then we call $\mathbf{R}(P, \sim)$ the *(standard) reduced incidence algebra* $\mathbf{R}(P)$; if $\sim$ is the maximal element of the lattice $C(P)$, we call $\bar{\mathbf{R}}(P) = \mathbf{R}(P, \sim)$ the *maximally reduced* incidence algebra.

PROPOSITION 4.4. *If $\sim$ is a finer order compatible equivalence relation than $\simeq$, and for $f \in \mathbf{R}(P, \simeq)$ the image $\hat{f}$ (as above) in $\mathbf{R}(P, \sim)$ is invertible in $\mathbf{R}(P, \sim)$, then f is invertible in $\mathbf{R}(P, \simeq)$.*

PROOF. Identify both algebras with subalgebras of $\mathbf{I}(P)$, as in the proof of Proposition 4.3, so that $f = \hat{f}$. We must show that f^{-1} is constant on $\simeq$ equivalent segments. Since f is invertible, it takes nonzero values on one point segments. Let $d \in \mathbf{I}(P)$ be the function which equals f on one point segments and is zero elsewhere. Then d is constant on $\simeq$ equivalence classes (by Lemma 4.1 (i)), and d^{-1} is also, since d^{-1} is the inverse of d on one point segments and zero elsewhere. Let $g = f - d$. Then $g \in \mathbf{R}(P, \simeq)$ and

$$\begin{aligned} f^{-1} &= (d + g)^{-1} = \left(1 + (d^{-1} * g)\right)^{-1} * d^{-1} \\ &= \left(1 - (d^{-1} * g) + (d^{-1} * g)^2 - (d^{-1} * g)^3 + \cdots\right) * d^{-1} \end{aligned} \tag{4.7}$$

which is well defined, since $d^{-1} * g$ is zero on one point segments; and hence, $f^{-1} \in \mathbf{R}(P, \simeq)$.

It follows that the *zeta function* and the *Möbius function* belong to all reduced incidence algebras.

We conclude with a simple characterization of reduced incidence algebras. In the finite case, it is purely algebraic, but in the infinite case, topological considerations come in. Recall that the *Schur product* of two elements f, g of $\mathbf{I}(P)$ is the element h defined by

$$h(x, y) = f(x, y) \cdot g(x, y) \tag{4.8}$$

for all x, y in P.

THEOREM 4.1. *Let P be a locally finite ordered set, and A a subalgebra of $\mathbf{I}(P)$ having the same identity as $\mathbf{I}(P)$. If P is finite, then A is a reduced incidence algebra of P if and only if A contains ζ and is closed under Schur multiplication. If P is infinite, then A is a reduced incidence algebra if and only if A contains ζ, is closed under Schur multiplication, and is closed in the standard topology.*

PROOF. The necessity of the conditions is evident, with the possible exception that A must be topologically closed. But if $f \in \mathbf{I}(P)$ is in the topological closure of A then it must clearly be constant on equivalence classes and so $f \in A$.

Now, assume P finite, and let A be a subalgebra of $\mathbf{I}(P)$ containing ζ and closed under Schur multiplication. Let $\sim$ be the equivalence relation on segments of P defined by $[x, y] \sim [u, v]$ if and only if $f(x, y) = f(u, v)$ for all $f \in A$. Once we have shown that the set of all functions constant on the equivalence classes of $\sim$ is precisely A, then it will follow that $\sim$ is order compatible (since A is closed under convolution) and that A is $\mathbf{R}(P, \sim)$. Let $\beta_1, \cdots, \beta_n$ be the equivalence classes of $\sim$. For each $i \neq j$, let $h_{i,j}$ be an element of A such that $h_{i,j}(\beta_i) \neq h_{i,j}(\beta_j)$, and let

$$\bar{h}_{i,j} = \frac{h_{i,j} - c_{i,j}\zeta}{b_{i,j} - c_{i,j}}, \tag{4.9}$$

where $b_{i,j} = h_{i,j}(\beta_i)$, $c_{i,j} = h_{i,j}(\beta_j)$. Then $\delta_i = \Pi_{j \neq i} \bar{h}_{i,j}$ (Schur multiplication) is in A, and δ_i is the indicator function of β_i. Now any function which is constant on equivalence classes is a linear combination of the functions δ_i, and hence is in A, proving our result.

A slight modification proves the infinite case, since δ_i is the limit of finite Schur products of $\bar{h}_{i,j}$ for $j \neq i$, and every function constant on equivalence classes is a limit of finite linear combinations of indicator functions. *Q.E.D.*

The assumption that A be topologically closed in the infinite case is necessary, as the following examples demonstrate.

EXAMPLE 4.3. Let P be an infinite locally finite ordered set in which there is a finite upper bound on the size of segments of P. Then the subset A of $\mathbf{I}(P)$ consisting of all functions which take only finitely many values is a subalgebra closed under Schur multiplication and containing ζ, but is clearly not a reduced incidence algebra, since the equivalence relation it generates is the trivial one, while A is not all of $\mathbf{I}(P)$.

EXAMPLE 4.4. Let P contain chains of arbitrarily large (finite) length, and let A be the closure under the operations of scalar multiplication, addition, convolution, and Schur multiplication of $\{\delta, \zeta\}$ in $\mathbf{I}(P)$. Then A is a subalgebra closed under Schur multiplication and containing ζ, but is not a reduced incidence algebra, since by Lemma 4.1 (ii), any reduced incidence algebra of P must have uncountable vector space dimension over the ground field, while A has countable dimension.

We now consider various examples of reduced incidence algebras and their connection with classical combinatorial theory.

EXAMPLE 4.5. *Formal power series.* Let P be the set of nonnegative integers in their natural ordering. The incidence algebra of P is evidently the algebra of upper triangular infinite matrices. On the other hand, we shall now see that the standard reduced incidence algebra $\mathbf{R}(P)$ is isomorphic to the algebra of formal power series.

Indeed, an element of $\mathbf{R}(P)$ is uniquely determined by a sequence $\{a_n : n = 0, 1, 2, \cdots\}$ of real numbers, by setting $f(i, j) = a_{j-i}$, for $i \leqq j$. The product of such an element by another element $g(i, j) = b_{j-i}$ of the same form is an element h of $\mathbf{I}(P)$ obtained by the convolution rule

$$h(i, j) = \sum_{i \leqq k \leqq j} f(i, k) g(k, j) = \sum_{i \leqq k \leqq j} a_{k-i} b_{j-k}. \tag{4.10}$$

Setting $r = k - i$ and $j - i = n$, we obtain $h(i, j) = \Sigma_{r=0}^{n} a_r b_{n-r} = c_n$. In other words, h is the element of the reduced incidence algebra which is obtained by convoluting the sequences $\{a_n\}$ and $\{b_n\}$. It follows that the map of power series into $\mathbf{R}(P)$ defined by

$$F(x) = \sum_{n=0}^{\infty} a_n x^n \to f(i, j) = a_{j-i}, \qquad j \geqq i \tag{4.11}$$

is an isomorphism. Under this isomorphism, the zeta function corresponds to $1/(1 - x)$, and the Möbius function corresponds to the formal power series $1 - x$. The incidence coefficients equal either 0 or 1.

EXAMPLE 4.6. *Exponential power series.* Let $B(S)$ be the family of all finite subsets of a countable set S, ordered by inclusion. We shall prove that the reduced incidence algebra of $B(S)$ is isomorphic to the algebra of exponential formal power series under formal multiplication, that is, a series of the form

$$F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n, \qquad G(x) = \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n, \qquad H(x) = \sum_{n=0}^{\infty} \frac{c_n}{n!} x^n. \tag{4.12}$$

It is immediately verified that the product $FG = H$ of two such formal power series amounts to taking the *binomial convolution* of their coefficients,

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}. \tag{4.13}$$

We obtain an isomorphism between the algebra of exponential formal power series and the reduced incidence algebra of $B(S)$ by setting

$$F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n \to f(A, B) = a_{\nu(B-A)}, \qquad A \subseteqq B, \tag{4.14}$$

where A and B are finite subsets of S and $\nu(B - A)$ denotes as usual the number of elements of the set $B - A$. The zeta function corresponds to e^x, and the Möbius function to e^{-x}. The Möbius inversion formula reduces to the principle of inclusion–exclusion, that is, to multiplication by e^{-x}.

The incidence coefficients coincide with the binomial coefficients, and the types naturally coincide with the integers.

EXAMPLE 4.7. Let G be the additive group of rational numbers modulo 1, and let $\mathbf{L}(G)$ be the lattice of subgroups excluding G itself. It is well known that every proper subgroup of G is finite cyclic. Let $[X, Y] \sim [U, V]$ in $\mathbf{L}(G)$ when the quotient group Y/X is isomorphic to V/U. The types correspond naturally to the positive integers; the incidence coefficients equal zero or one, and the product in $\mathbf{R}(\mathbf{L}(G), \sim)$ is given by the *Dirichlet convolution* $c_n = \Sigma_{ij=n} a_i b_j$. Thus, $\mathbf{R}(\mathbf{L}(G), \sim)$ is isomorphic to the algebra of *formal Dirichlet series*

$$\sum_{n \geqq 1} \frac{a_n}{n^s} = f(s) \tag{4.15}$$

under ordinary multiplication.

EXAMPLE 4.8. Let P be the set of positive integers, ordered by divisibility, and let $\sim$ be the equivalence relation defined by $[a, b] \sim [m, n]$ if and only if $b/a = n/m$. Then, as in the previous example, $\mathbf{R}(P, \sim)$ is easily seen to be isomorphic to the algebra of formal Dirichlet series. The standard reduced incidence algebra is isomorphic to a subalgebra of the algebra of formal Dirichlet series, namely to those series $\Sigma_{n \geqq 1} a_n/n^s$ in which $a_k = a_n$ if $k = p_1^{a_1} p_2^{a_2} \cdots$, and $n = p_1^{b_1} p_2^{b_2} \cdots$, where $p_1, p_2, \cdots$ are the primes, the a_i and b_i are nonnegative integers, and the b_i are obtained by permuting the a_i.

EXAMPLE 4.9. Let V be a vector space of countable dimension over $GF(q)$, and let $\mathbf{L}(V)$ be the lattice of finite dimensional subspaces. Let $\sim$ be the equivalence relation defined by $[S, T] \sim [X, Y]$ if and only if $T/S \approx Y/X$, that is, $\dim T - \dim S = \dim Y - \dim X$. Then the types are in one to one correspondence with the integers, and multiplication in $\mathbf{R}(\mathbf{L}(V), \sim)$ is given by

$$(4.16) \qquad (f*g)(n) = \sum_{r=0}^{n} \frac{(q^n - 1)(q^n - q) \cdots (q^n - q^{r-1})}{(q^r - 1)(q^r - q) \cdots (q^r - q^{r-1})} f(r)g(n - r)$$

$$= \sum_{r=0}^{n} \frac{(1 - q^n)(1 - q^{n-1}) \cdots (1 - q^{n-r+1})}{(1 - q^r)(1 - q^{r-1}) \cdots (1 - q)} f(r)g(n - r).$$

Hence, $\mathbf{R}(\mathbf{L}(V), \sim)$ is isomorphic to the algebra of *Eulerian power series*, the isomorphism being given by

$$(4.17) \qquad f \rightarrow \sum_{n \geqq 0} \frac{f(n)}{(1 - q)(1 - q^2) \cdots (1 - q^n)} x^n.$$

We now present three examples in which we arrive very simply at previously known results by using the reduced incidence algebra. Let P be a locally finite ordered set, and let $c \in \mathbf{I}(P)$ be the function which assigns to a segment $[x, y]$ the total number of chains, $x = x_0 < x_1 < \cdots < x_m = y$. Since $(\zeta - \delta)^k(x, y)$ is the number of chains, $x = x_0 < x_1 < \cdots < x_k = y$, of length k, we have

$$(4.18) \qquad \begin{aligned} c(x, y) &= \sum_{k=0}^{\infty} (\zeta - \delta)^k(x, y) \\ &= [\delta - (\zeta - \delta)]^{-1}(x, y) \\ &= (2\delta - \zeta)^{-1}(x, y). \end{aligned}$$

EXAMPLE 4.10. Let P be as in Example 4.5. Then $c(x, y)$ is the number c_n of *ordered partitions* (or *compositions*) of $n = y - x$, the chain $x = i_0 < i_1 < \cdots < i_k = y$ corresponding to the composition

$$(4.19) \qquad y - x = (i_1 - i_0) + (i_2 - i_1) + \cdots + (i_k - i_{k-1})$$

Hence,

$$(4.20) \qquad \sum_{n=0}^{\infty} c_n x^n = \frac{1}{2 - (1 - x)^{-1}} = \frac{1 - x}{1 - 2x} = 1 + \sum_{n=1}^{\infty} 2^{n-1} x^n,$$

so $c_n = 2^{n-1}$ if $n > 0$ (a well-known result).

EXAMPLE 4.11. Let P be as in Example 4.6. Then $c(x, y)$ is the number f_n of *ordered set partitions* (or *preferential arrangements*) of the set $y - x$, where n is the number of elements in $y - x$. (See Gross [29].) Hence,

$$\sum_{n=0}^{\infty} \frac{f_n}{n!} x^n = \frac{1}{2 - e^x}, \tag{4.21}$$

a basic result of Gross.

EXAMPLE 4.12. Let P be the positive integers ordered by divisibility, with $[u, v] \sim [x, y]$ if $v/u = y/x$. Then $c(x, y)$ is the number $f(n)$ of *ordered factorizations* of $y/x = n$ (into factors > 1). Hence,

$$\sum_{n=1}^{\infty} f(n) n^{-s} = \frac{1}{2 - \zeta(s)}, \tag{4.22}$$

a result of Titchmarsh ([59], p. 7).

More generally, the theory of *weighted compositions*, as developed by Moser and Whitney [39] and by Hoggart and Lind [34], can be expressed in terms of the reduced incidence algebra of a chain. Thus, this theory can be extended to other ordered sets in the same way that Examples 4.11 and 4.12 extend the usual concept of composition given in Example 4.10.

5. The large incidence algebras

5.1. *Definitions.* Several enumeration problems lead not to a single ordered set, but to a family of ordered sets having some common features; for example, the family of lattices of partitions of finite sets or the family of all lattices of subgroups of finite abelian groups. It then becomes necessary to extend the notions of incidence algebra and reduced incidence algebra to these situations. Recall that we assume the ground field K to have characteristic 0. This avoids complications inherent in dividing by positive integers, such as $n!$ in exponential generating functions. We are now led to the following setup.

Two ordered sets $(P, \sim)$ and $(Q, \sim)$ each with an order compatible equivalence relation (denoted by the same symbol for convenience) are said to be *isomorphic* when there is an isomorphism ϕ of P to Q which preserves the equivalence relation, that is $[x, y] \sim [u, v]$ in P if and only if $[\phi(x), \phi(y)] \sim [\phi(u), \phi(v)]$ in Q. If S is a segment of P, then the equivalence relation $\sim$ induces a compatible equivalence relation on S. (Note that this conclusion does not hold in general if S is only assumed to be an ordered subset of P.)

Now let $\mathbf{C}$ be a category whose objects are pairs $(P, \sim)$ as above, where P is a finite ordered set with 0 and 1, and where morphisms ϕ of $(P, \sim)$ into $(Q, \sim)$ are isomorphisms of $(P, \sim)$ onto a segment of $(Q, \sim)$ with the induced equivalence relation (not all such maps need be included in the category as morphisms). It is further assumed that every segment of an object $(P, \sim)$ is in $\mathbf{C}$, with the induced equivalence relation. Finally, it is assumed that if ϕ and ψ are two morphisms of $(P, \sim)$ into $(Q, \sim)$ having the segments $[x, y]$ and $[u, v]$ as images, then $[x, y] \sim [u, v]$ in Q.

Under these conditions, we can define the *large incidence algebra* $\mathbf{I}(\mathbf{C})$ of $\mathbf{C}$ as follows: the elements of $\mathbf{I}(\mathbf{C})$ are functions f on the isomorphism classes (in the category $\mathbf{C}$) or "types" of the objects of $\mathbf{C}$ such that $f(\alpha) = f(\beta)$, if some object $(P, \sim)$ contains $\sim$-equivalent segments of types α and β (with values, as usual, in a fixed field). The sum of two such functions is defined as usual, and the product is defined by

$$(f*g)(\alpha) = \sum \begin{bmatrix} \alpha \\ \beta, \gamma \end{bmatrix} f(\beta)g(\gamma), \tag{5.1}$$

where the brackets are taken in any object P belonging to the isomorphism class α. Our assumptions imply that the product is well defined; that is, the product remains the same if it is computed in any object of type α, and also that $f*g$ is in $\mathbf{I}(\mathbf{C})$. Thus, we obtain an algebra which is associative by Proposition 4.3. The functions ζ and δ of the ordinary incidence algebra have obvious counterparts in the large incidence algebra, and the result that a function is invertible if and only if it is nonzero on all types containing one point segments (see *Foundations I*) also carries over. Hence, the Möbius function can be defined as the inverse to the zeta function; and clearly, for each object $([0, 1], \sim)$ of the category, the value of the Möbius function on the type containing $[0, 1]$ equals $\mu(0, 1)$.

Most of the classes of incidence algebras (such as binomial type and Dirichlet type) can be trivially extended to large incidence algebras. Also note that we need make no distinction between reduced and nonreduced large incidence algebras, for the degree of reduction is built into the category itself, depending on the equivalence relations in the objects and on the morphisms.

EXAMPLE 5.1. Let L be a locally finite ordered set. Construct a category $\mathbf{C}$ as follows. The objects are all segments of L and the morphisms are the inclusion maps. The equivalence relation is the trivial one (no two distinct segments are isomorphic in $\mathbf{C}$). Note that two isomorphic segments are not in general isomorphic in $\mathbf{C}$. The large incidence algebra $\mathbf{I}(\mathbf{C})$ is isomorphic to the incidence algebra of L.

EXAMPLE 5.2. Let L be as above; let the objects of $\mathbf{C}$ be again all segments of L, but let the morphisms be all isomorphisms; and let $\sim$ be isomorphism. Then $\mathbf{I}(\mathbf{C})$ is isomorphic to the standard reduced incidence algebra of L.

EXAMPLE 5.3. Let the objects of $\mathbf{C}$ all be finite Boolean algebras; let $\sim$ be isomorphism of segments; and let morphisms all be isomorphisms. Then $\mathbf{I}(\mathbf{C})$ is isomorphic to the algebra of exponential power series of Example 5.6.

In the next three subsections, we consider situations which are better looked at from the point of view of the large incidence algebra than from that of the regular incidence algebra.

5.2. *Partition lattices.* The incidence algebra of the family of all partition lattices of finite sets can be studied by taking the lattice $\Pi(S)$ of all partitions of an infinite set S having exactly one infinite block and finitely many finite blocks, ordered by refinement. However, it is more pleasingly done in the context of the large incidence algebra, as follows.

Let the objects of a category $\mathbf{\Pi}$ be all lattices of partitions of finite sets and all segments thereof, and let the equivalence relation be an isomorphism of segments whose top elements have the same number of blocks. Let the morphisms of $\mathbf{\Pi}$ all be isomorphisms onto a segment such that the top element of a segment has the same number of blocks as the top element of the image segment. It is immediate that $\mathbf{\Pi}$ satisfies the required conditions.

The *class* of a segment $[\sigma, \pi]$ is a sequence of nonnegative numbers $(k_1, \cdots, k_n, \cdots)$, where k_i is the number of blocks in π which are the union of precisely i blocks in σ. It is clear that $k_1 + 2k_2 + 3k_3 + \cdots$ equals the number of blocks in σ and $k_1 + k_2 + k_3 + \cdots$ equals the number of blocks in π, and that a segment of class $(k_1, k_2, \cdots)$ is isomorphic to $\Pi_1^{k_1} \times \Pi_2^{k_2} \times \cdots$, where Π_i is the lattice of partitions of an i set, so it follows that two segments have the same class if and only if they are of the same type in $\mathbf{\Pi}$. We denote by $\binom{n}{k_1, \ldots, k_n}$ the number of elements τ in a segment $[\sigma, \pi]$ of type $(\delta_{0,n}, \delta_{1,n}, \cdots)$ (that is, $[\sigma, \pi]$ is isomorphic to Π_n and σ has n blocks) for which $[\sigma, \tau]$ has type $(k_1, k_2, \cdots, k_n, 0, 0, \cdots)$ (and hence, $[\tau, \pi]$ has type $(\delta_{0,m}, \delta_{1,m}, \cdots)$, where $m = k_1 + \cdots + k_n$). To compute $\binom{n}{k_1, \ldots, k_n}$, first note that any object $[\sigma, \pi]$ of type $k_1 = \cdots = k_{n-1} = 0$, $k_n = 1$ is an upper segment of some finite partition lattice; that is, $\pi = 1$ in some finite lattice of partitions. Thus, it is easy to see that

$$\binom{n}{k_1, \cdots, k_n} = \frac{n!}{1!^{k_1} k_1! \, 2!^{k_2} k_2! \cdots n!^{k_n} k_n!} \tag{5.2}$$

when $k_1 + 2k_2 + \cdots + nk_n = n$, and equals 0 when $k_1 + 2k_2 + \cdots + nk_n \neq n$.

For a partition π of some finite set S, we define the class of π to be the class of the segment $[0, \pi]$ of $\Pi(S)$, as defined above.

The fundamental concept associated with the large incidence algebra $\mathbf{I}(\mathbf{\Pi})$ is that of *multiplicative function*. A function f in $\mathbf{I}(\mathbf{\Pi})$ is said to be *multiplicative* when there is a sequence of constants $(a_1, a_2, a_3, \cdots)$ such that

$$f(\pi, \sigma) = a_1^{k_1} a_2^{k_2} a_3^{k_3} \cdots \tag{5.3}$$

when $[\pi, \sigma]$ is a segment of class $(k_1, k_2, k_3, \cdots)$. The function f is said to be determined by the sequence $(a_1, a_2, \cdots)$. Similarly, a function of one variable $F(\sigma)$ for $\sigma \in \Pi(S)$ for some finite set S is said to be multiplicative when

$$F(\sigma) = a_1^{k_1} a_2^{k_2} \cdots, \tag{5.4}$$

where $(k_1, k_2, \cdots)$ is the class of σ.

The following elementary result is fundamental.

PROPOSITION 5.1. *The convolution of two multiplicative functions is multiplicative.*

PROOF. This follows from the fact that if $[\sigma, \pi]$ *is of type* $(k_1, k_2, \cdots)$, *then* $[\sigma, \pi]$ *is isomorphic to* $\Pi_1^{k_1} \times \Pi_2^{k_2} \times \cdots$, *and that if* $f \in \mathbf{I}(P)$ *and* $g \in \mathbf{I}(Q)$ *(where P and Q are any locally finite ordered sets) and if* $f \times g \in \mathbf{I}(P \times Q)$ *is defined by* $f \times g((x, x'), (y, y')) = f(x, y) \cdot g(x', y')$, *then* $(f \times g) * (f' \times g') = (f * f') \times (g * g')$.

COROLLARY 5.1. *If $F(\pi)$ is multiplicative and $f(\pi, \sigma)$ is multiplicative, then so are*

$$G(\sigma) = \sum_{\pi \leqq \sigma} F(\pi) f(\pi, \sigma) \tag{5.5}$$

and

$$H(\sigma) = \sum_{\pi \geqq \sigma} f(\sigma, \pi) F(\pi), \tag{5.6}$$

where the sum is taken in the partition lattice containing σ.

EXAMPLE 5.4. The zeta function of $\mathbf{I}(\mathbf{\Pi})$ is multiplicative and is determined by the sequence $(1, 1, 1, \cdots)$. By Proposition 3 of Section 7 of *Foundations I*, the Möbius function of $\mathbf{I}(\mathbf{\Pi})$ is multiplicative, and determined by the sequence $(a_1, a_2, \cdots)$, where $a_n = (-1)^n(n-1)!$ The delta function δ is multiplicative, determined by $(1, 0, 0, \cdots)$, but $\eta = \zeta - \delta$ is not multiplicative. Hence, the sum of multiplicative functions need not be multiplicative.

Let $\mathbf{M}(\mathbf{\Pi})$ denote the subset of $\mathbf{I}(\mathbf{\Pi})$ consisting of multiplicative functions. By Proposition 5.1, $\mathbf{M}(\mathbf{\Pi})$ is a subsemigroup of the multiplicative semigroup of $\mathbf{I}(\mathbf{\Pi})$. If f is in $\mathbf{M}(\mathbf{\Pi})$, let $f(n)$ denote $f(\Pi_n)$; that is $f(\pi, \sigma)$, where $[\pi, \sigma]$ has class $k_1 = \cdots = k_{n-1} = 0$, $k_n = 1$. Then, for $f, g \in \mathbf{M}(\mathbf{\Pi})$, we get from (5.2) that $(f * g)(n)$ is equal to

$$\sum_{k_1 + 2k_2 + \cdots + nk_n = n} \left(\frac{n!}{1!^{k_1} k_1! 2!^{k_2} k_2! \cdots n!^{k_n} k_n!} \right) f(1)^{k_1} \cdots f(n)^{k_n} g(k_1 + \cdots + k_n). \tag{5.7}$$

THEOREM 5.1. *The semigroup $\mathbf{M}(\mathbf{\Pi})$ is anti-isomorphic to the algebra of all formal exponential power series with zero contant term over K in a variable x, under the operation of composition. The anti-isomorphism is given by $f \to F_f$, where*

$$F_f(x) = \sum_{n=1}^{\infty} \frac{f(n)}{n!} x^n. \tag{5.8}$$

Thus, $F_{f*g}(x) = F_g(F_f(x))$.

PROOF. Clearly, the map defined by (5.8) is a bijection, so we need only check that multiplication is preserved. Now,

$$F_g(F_f(x)) = \sum_{j=1}^{\infty} \frac{g(j)}{j!} \left(\sum_{i=1}^{\infty} \frac{f(i)}{i!} x^i \right)^j. \tag{5.9}$$

The coefficient of x^n in the expansion of $\left(\Sigma_{i=1}^{\infty} (f(i)/i!) x^i\right)^j$ is

$$\sum_{n_1 + n_2 + \cdots + n_j = n} \frac{f(n_1) \cdots f(n_j)}{n_1! \cdots n_j!} = \sum \frac{j!}{k_1! \cdots k_n!} \frac{f(1)^{k_1} \cdots f(n)^{k_n}}{1!^{k_1} \cdots n!^{k_n}}, \tag{5.10}$$

where the summation is taken over $k_1 + 2k_2 + \cdots + nk_n = n$, $k_1 + \cdots + k_n = j$, since there are $j!/k_1! \cdots k_n!$ ways of ordering the partition $k_1 + 2k_2 + \cdots + nk_n = n$. When we multiply (5.10) by $g(j)/j!$ and sum over all j, we get (5.7), and the proof follows.

EXAMPLE 5.5. Under the isomorphism of the proposition, the zeta function corresponds to $e^x - 1$, and the delta function to x, so the Möbius function corresponds to the power series F such that $F(e^x - 1) = x$, that is, to $\log(1 + x)$. Hence, $\mu(0, 1) = (-1)^{n-1}(n - 1)!$ for $[0, 1] = \Pi_n$. This is yet another way of determining the Möbius functions for lattices of partitions.

COROLLARY 5.2. *Let f be a multiplicative function of one variable determined by the sequence $(a_1, a_2, \cdots)$. For every positive integer n, let*

$$b_n = \sum_{\pi \in \Pi_n} f(\pi), \qquad q_n = \sum_{\pi \in \Pi_n} f(\pi)\mu(\pi, 1). \tag{5.11}$$

Then

$$1 + \sum_{n=1}^{\infty} \frac{b_n x^n}{n!} = \exp\left\{a_1 x + \frac{a_2 x^2}{2!} + \frac{a_3 x^3}{3!} + \cdots\right\} \tag{5.12}$$

and

$$\sum_{n=1}^{\infty} \frac{q_n}{n!} x^n = \log\left(1 + a_1 x + \frac{a_2 x^2}{2!} + \frac{a_3 x^3}{3!} + \cdots\right). \tag{5.13}$$

PROOF. For (5.12), let $\bar{f}$ be the function in $\mathbf{M}(\mathbf{\Pi})$ determined by $(a_1, a_2, \cdots)$, and let $b = \bar{f} * \zeta$. Then $b_n = b(n)$ for all $n \geqq 1$, so

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} \frac{b_n x^n}{n!} &= 1 + F_b(x) \\ &= 1 + F_{\bar{f}*\zeta}(x) = 1 + F_\zeta(F_{\bar{f}}(x)) \\ &= \exp\left\{a_1 x + \frac{a_2 x^2}{2!} + \cdots\right\}. \end{aligned} \tag{5.14}$$

For (5.13) let $q = \bar{f} * \mu$, and the proof follows as for (5.12).

We now work out various examples using the above results.

EXAMPLE 5.6 (Waring's formula). Let D and R be finite sets, and label the elements of R by different letters of the alphabet: $x, y, \cdots, z$. To every function $f\colon D \to R$, we associate a monomial $\gamma(f) = x^i y^j \cdots z^k$, where i is the number of elements of D mapped to the element of R labelled x, and so forth; and to every set E of functions from D to R, we associate a polynomial $\gamma(E)$, the sum of $\gamma(f)$ for f ranging over E; $\gamma(E)$ is called the *generating function* of the set E.

For every partition π of the set D, let $A(\pi)$ be the generating function of the set of all functions $f\colon D \to R$ whose kernel (that is, the partition of D whose blocks are the inverse images of elements of R) is π. Let $S(\pi)$ be the generating function of the set of functions whose kernel is some partition $\sigma \geqq \pi$. Clearly, we have $S(\pi) = \Sigma_{\sigma \geqq \pi} A(\sigma)$, from which, by Möbius inversion, we have $A(\pi) = \Sigma_{\sigma \geqq \pi} S(\sigma)\mu(\pi, \sigma)$; and setting $\pi = 0$, we have

$$A(0) = \sum_{\sigma \in \Pi(D)} S(\sigma)\mu(0, \sigma). \tag{5.15}$$

Now assume that D has n elements and that R is larger than D. The polynomial $A(0)$ is the generating function of the set of all one to one functions;

and hence, every term of $A(0)$ is a product of n distinct variables taken among $x, y, \cdots, z$. Furthermore, every product of n distinct variables among $x, y, \cdots, z$ appears $n!$ times as a term in $A(0)$. Thus, $A(0)$ is simply $n! \cdot a_n$, where a_n is the *elementary symmetric function* of degree n in the variables $x, y, \cdots, z$.

Next, if the partition σ has class $(k_1, k_2, \cdots, k_n)$, we claim that

$$S(\sigma) = (x + y + \cdots + z)^{k_1}(x^2 + y^2 + \cdots + z^2)^{k_2} \cdots (x^n + y^n + \cdots + z^n)^{k_n}, \tag{5.16}$$

that is, using the standard notation $s_k = x^k + y^k + \cdots + z^k$,

$$S(\sigma) = s_1^{k_1} s_2^{k_2} \cdots s_n^{k_n}. \tag{5.17}$$

To see this, let $\bar{S}(\sigma)$ be the set of all functions with kernel $\pi \geqq \sigma$, and let $B_1, \cdots, B_k$ be the blocks of the partition σ. Then $\bar{S}(\sigma)$ is the product $U_1 \times \cdots \times U_k$ of the sets U_i, where U_i is the set of all functions from B_i to R taking only one value. It follows that $S(\sigma) = \gamma(\bar{S}(\sigma)) = \gamma(U_1)\gamma(U_2) \cdots \gamma(U_k)$. The generating function $\gamma(U_i)$ is simply $x^k + y^k + \cdots + z^k$ if B_i has k elements, and this completes the verification.

We thus see that (5.15) reduces to the classical formula of Waring, expressing the elementary symmetric functions in terms of sums of powers.

EXAMPLE 5.7. Let V be a finite set of n elements ("vertices"). We count the number C_n of connected graphs whose vertex set is V. To every graph G, we can associate a partition $\pi(G)$ of the set V, the blocks of $\pi(G)$ being connected components of G. A graph is connected if and only if $\pi(G) = 1$, the partition with only one block. For every partition π of V, let $C(\pi)$ be the number of graphs G with $\pi(G) = \pi$, and let $D(\pi)$ be the number of graphs G with $\pi(G) \leqq \pi$. Let a_n be the total number of graphs whose vertex set is V; a simple enumeration gives

$$a_n = 2^{\binom{n}{2}}. \tag{5.18}$$

If $B_1, B_2, \cdots, B_k$ are the blocks of π and $D(B_i)$ is the total number of graphs on the block B_i, then clearly $D(\pi) = D(B_1)D(B_2) \cdots D(B_n)$. Hence, if the class of the partition π is $(k_1, k_2, \cdots, k_n)$, we have

$$D(\pi) = a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n} = t\left(k_1\binom{1}{2} + k_2\binom{2}{2} + k_3\binom{3}{2} + \cdots + k_n\binom{n}{n}\right), \tag{5.19}$$

where $t(x) = 2^x$ and $\binom{1}{2} = 0$ by convention. Furthermore, $D(\pi) = \Sigma_{\sigma \leqq \pi} C(\sigma)$ as follows immediately from the definitions. By the Möbius inversion followed by setting $\pi = 1$, we obtain the identity

$$\begin{aligned} C_n = C(1) &= \sum_{\sigma \in \Pi_n} D(\sigma)\mu(\sigma, 1) \\ &= \sum_{k_1 + 2k_2 + \cdots + nk_n = n} t\left(k_1\binom{1}{2} + k_2\binom{2}{2} + \cdots + k_n\binom{n}{2}\right) \\ &\qquad \cdot (-1)^{k_1 + k_2 + \cdots + k_n - 1}(k_1 + k_2 + \cdots + k_n - 1)! \end{aligned} \tag{5.20}$$

which is an explicit expression for the number of connected graphs. Further, applying (5.13) to (5.20), we get

$$\sum_{n=1}^{\infty} \frac{C_n}{n!} x^n = \log\left(1 + a_1 x + \frac{a_2 x^2}{2!} + \frac{a_3 x^3}{3!} + \cdots\right). \tag{5.21}$$

From this, one can find the values of various probabilistic quantities related to connected graphs, such as the expected number of connected components, expected size of the largest component, asymptotic results, and so forth.

EXAMPLE 5.8. We now determine the number $a(n, k)$ of solutions of the equation $p^k = I$, where p is an element of the group G_n of all permutations of a set S_n of n elements, and I is the identity element of G_n. To every $p \in G_n$, we can associate the partition π of S_n whose blocks are the transitivity classes relative to the subgroup generated by p. Let $F(\pi)$ be the number of permutations p whose associated partition is π and such that $p^k = I$. Clearly, the function F is multiplicative, and so the function G, defined by $G(\sigma) = \sum_{\pi \leqq \sigma} F(\pi)$, is also multiplicative. Further,

$$G(\sigma) = a(1, k)^{k_1}\, a(2, k)^{k_2} \cdots \tag{5.22}$$

if $(k_1, k_2, \cdots)$ is the class of σ. Thus, if $(b_1, b_2, \cdots)$ is the sequence which determines F, then by (5.12), we obtain

$$1 + \sum_{n=1}^{\infty} \frac{a(n, k)}{n!} x^n = \exp\left\{\sum_{n=1}^{\infty} \frac{b_n}{n!} x^n\right\}. \tag{5.23}$$

Now, it is easily seen that $b_n = (n - 1)!$ if n divides k, and $b_n = 0$ otherwise, so we obtain the formula (due to Chowla, Herstein, and Scott [10])

$$\sum_{n=0}^{\infty} \frac{a(n, k)}{n!} x^n = \exp\left(\sum_{n|k} \frac{x^n}{n}\right), \tag{5.24}$$

where we take $a(0, k) = 1$.

EXAMPLE 5.9 (The number of partitions of a set). The number B_n of partitions of a set of n elements is given by $B_n = \sum_{\pi \in \Pi_n} \zeta(\pi)$. Hence, from Corollary (5.12), we get

$$\sum_{n=1}^{\infty} \frac{B_n}{n!} x^n = \exp\{e^x - 1\} - 1, \tag{5.25}$$

which is the classical generating function for B_n.

EXAMPLE 5.10. A set S of n elements splits at time t_1 into a partition π with blocks $B_1, B_2, \cdots$. At a later time $t_2 > t_1$ each block B_i splits into a partition π_i with blocks $B_{i,1}, B_{i,2}, \cdots$, and so on for N steps. Letting $E(x) = e^x - 1$, an argument much like that of the preceding example shows that the exponential generating function for the number of distinct "splittings" is $E[E(\cdots E(x) \cdots)]$, where the iteration is repeated $N + 1$ times.

5.3. *Dowling lattices.* Let F be the field of q elements (q will remain fixed throughout this subsection), and let V be a vector space over F of dimension n, with basis $b_1, \cdots, b_n$. The *Dowling lattice* $Q(V)$ is the lattice of subspaces W of V such that W has a basis whose elements are of the form b_i or $ab_j + a'b_k$, where $a, a' \in F$. Since the lattice $Q(V)$ depends up to isomorphism on the dimension of V, it will generally be denoted by Q_n.

Before attempting to study the combinatorial properties of Q_n, we will define a new lattice D_n, which is isomorphic to Q_n, in which various counting arguments become simpler. First we will state a number of definitions. The concept of *directed graph* is assumed (see Liu [38]), and we will allow loops and multiple edges between vertices. If S is any set, an S *labelled* directed graph is a directed graph $G = (V, E)$ together with a mapping from E to S in which no two edges from v to v' have the same image, for any $v, v' \in V$. The image of an edge e is called its *label*, and $v \xrightarrow{a} v'$ denotes the fact that there is an edge labelled a from v to v'. If G and G' are S labelled graphs, G is a *subgraph* of G', if both graphs have the same vertex set and if $v \xrightarrow{a} v'$ in G implies $v \xrightarrow{a} v'$ in G'. A *totally complete* S labelled directed graph G is one in which $v \xrightarrow{a} v'$ for any pair of vertices v and v' and any $a \in S$. If S consists of the nonzero elements of a field, then an S labelled directed graph G is *inverse symmetric* if $v \xrightarrow{a} v'$ implies $v' \xrightarrow{a^{-1}} v$, and is *antitransitive* if $v \xrightarrow{a} v'$ and $v' \xrightarrow{b} v''$ implies $v \xrightarrow{-ab} v''$. Finally, a *D graph* is an S labelled directed graph G, where S is the set of nonzero elements of a field in which there is at most one distinguished component which is totally complete, and every other component is simple (that is, at most one edge in each direction between two vertices), inverse symmetric, and antitransitive.

Now, let $S = F^*$ (the nonzero elements of F), and let B be a set of n elements ("vertices"). The lattice $D(B)$, or D_n, is the lattice of D graphs with vertex set B (and label set S), with $G \leqq G'$ if and only if G is a subgraph of G' and the distinguished component of G is contained in that of G'. The correspondence with the Dowling lattice Q_n is as follows. Given a Dowling lattice $Q(V)$ and a basis $B = \{b_1, \cdots, b_n\}$, to each subspace W of V in $Q(V)$ associate the graph whose vertex set is B and in which $b_i \xrightarrow{a} b_j$ if and only if $b_i + ab_j$ is in W, and in which the distinguished component is the one whose vertices are those b_i which are in W. The connected components are easily seen to be inverse symmetric and antitransitive, the distinguished component is clearly totally complete, and all other components are simple (for if $b_i \xrightarrow{a} b_j$ and $b_i \xrightarrow{a'} b_j$ with $a \neq a'$, then $b_i + ab_j \in W$, $b_i + a'b_j \in W$; hence $(a - a')b_j \in W$ and so $b_j \in W$ and $b_i \in W$, and thus b_i and b_j are in the distinguished component). This correspondence is easily seen to be a lattice isomorphism, and so D_n and Q_n are isomorphic.

EXAMPLE 5.11. It follows easily from what we have done that $Q_n \simeq \Pi_{n+1}$ if $q = 2$. The following correspondence gives an isomorphism from D_n to Π_{n+1}. Let the vertex set for D_n be $\{1, 2, \cdots, n\}$. To each element G of D_n, we associate the partition of $\{1, 2, \cdots, n, n+1\}$ whose blocks are the nondistinguished components of G as well as the distinguished component with $n + 1$ added.

Now, let $G \in D_n$. Then $[0, G]$ is isomorphic to the product of the lattices of subgraphs of the components of G (where 0 is the trivial graph with no edges and no distinguished component), and the lattice of subgraphs of a nondistinguished (and hence simple) component of G with k vertices is trivially isomorphic to Π_k. Hence, $[0, G]$ is isomorphic to $D_r \times \Pi_1^{k_1} \times \cdots \times \Pi_n^{k_n}$, where r is the size of the distinguished component of G (possibly 0) and k_i is the number of undistinguished components of G of size i. (Note that $r + \Sigma ik_i = n$ and Σk_i equals the number of undistinguished blocks in G.) Let G' be above G in D_n, and let C_1 and C_2 be distinct undistinguished components of G which are in the same undistinguished component of G'. Then all edges between vertices of C_1 and vertices of C_2 can be determined from any one such edge, using the properties of inverse symmetry and antitransitivity. Intuitively, the undistinguished components of G "act like points" in $[G, 1]$, while the distinguished component of G simply "joins with these points as they become distinguished." Using these ideas, it is not difficult to see (or to prove) that $[G, 1]$ is isomorphic to Q_m, where m is the number of undistinguished components of G, that is, to $D_{k_1 + \cdots + k_n}$ (the k_i are introduced earlier in this paragraph).

We are thus led to the following definition corresponding to that in the previous subsection. The *class* of a segment $[G, G']$ of D_n is the sequence $(r; k_1, k_2, \cdots)$, where r is the number of undistinguished components of G which are contained in the distinguished component of G', and k_i is the number of undistinguished components of G' which contain exactly i components of G. (Note that $r + \Sigma ik_i$ equals the number of undistinguished components of G.) It follows from the previous paragraph that $[G, G']$ is isomorphic to $D_r \times \Pi_1^{k_1} \times \Pi_2^{k_2} \times \cdots$. The *class* of an element $G \in D_n$ is defined to be the class of $[0, G]$.

Before going any further, we will put everything preceding in the context of a large incidence algebra in which two segments are of the same type if and only if they have the same class. Let **D** be the category whose objects are the lattices $D(B)$ for all finite sets B, with two segments being equivalent if they are isomorphic and their top elements have the same number of undistinguished components (although one top element may have a distinguished component and the other not). The morphisms of **D** are all isomorphisms into in which the top element of the segment has the same number of undistinguished components as does the top element of the image segment. It is easy to see that **D** satisfies the required conditions, and also that two segments are equivalent if and only if they are of the same class.

Now, a segment $[G, G']$ in some D_n is of type $(r; 0, 0, \cdots)$ if and only if $G' = 1$ and G has r undistinguished components. We denote by $\left[{n \atop r; k_1, k_2, \cdots}\right]$ the number of elements G' in a segment $[G, 1]$ of type $(n; 0, 0, \cdots)$ such that $[G, G']$ has type $(r; k_1, k_2, \cdots)$ (and hence, $[G', 1]$ has type $[k_1 + k_2 + \cdots; 0, 0, \cdots]$). Then

$$\left[{n \atop r; k_1, k_2, \cdots}\right] = \binom{n}{r}\binom{n-r}{k_1, k_2, \cdots}(q-1)^{k_2 + 2k_3 + 3k_4 + \cdots}, \tag{5.26}$$

where $\binom{n-r}{k_1, k_2, \ldots}$ is defined as in the previous subsection, as the following counting argument shows. We may assume that $[G, 1]$ is contained in D_n and that $G = 0$, that is, that $[G, 1]$ is D_n. First, choose r vertices and join edges between all pairs with all labels and distinguish the resulting component. This can be done in $\binom{n}{r}$ ways. Then choose k_1 vertices as the undistinguished one point components. This can be done in $\binom{n-r}{k_1}$ ways. Proceeding in this way, choose k_j distinct j sets of vertices. This can be done in

$$(5.27) \qquad \frac{1}{k_j! j!^{k_j}} \left(n - r - k_1 - 2k_2 - \cdots - (j-1)k_{j-1}\right) \cdot \left(n - r - k_1 - 2k_2 - \cdots - (j-1)k_{j-1}\right) \cdots \left(n - r - k_1 - 2k_2 - \cdots - jk_j + 1\right)$$

ways, and each j set can be made into a labelled, simple, inverse symmetric, antitransitive component in $(q-1)^{j-1}$ ways, since the labelling is completely determined by the labels on a spanning tree, which has $j - 1$ edges (see Liu [38], pp. 185–186). This establishes (5.26).

As for lattices of partitions, the concept of *multiplicative* function is important. A function $f \in I(D)$ is multiplicative, if there is a pair of sequences $(a_1, a_2, a_3, \cdots)$, $(b_0, b_1, b_2, \cdots)$ such that

$$(5.28) \qquad f(G, G') = b_r \cdot a_1^{k_1} a_2^{k_2} \cdots$$

when $[G, G']$ is of type $(r; k_1, k_2, \cdots)$, and f is said to be determined by the pair of sequences. A similar definition holds for multiplicative functions of one variable. The subset $\mathbf{M}(\mathbf{D})$ of $\mathbf{I}(\mathbf{D})$ of all multiplicative functions is closed under convolution (the proof is the same as for $\mathbf{M}(\mathbf{\Pi})$), and hence, $\mathbf{M}(\mathbf{D})$ is a semigroup. Also, if $f \in \mathbf{M}(\mathbf{D})$ and F is a multiplicative function of one variable, then K and L are also multiplicative, where

$$(5.29) \qquad K(G) = \sum_{G' \leqq G} F(G') f(G', G)$$

and

$$(5.30) \qquad L(G) = \sum_{G' \geqq G} f(G, G') F(G').$$

THEOREM 5.2. *The semigroup* $\mathbf{M}(\mathbf{C})$ *is isomorphic to the set of all pairs* $(F(x), G(x))$ *of formal exponential power series in which* $F(x)$ *has zero constant term, with multiplication given by*

$$(5.31) \qquad (F(x), G(x)) \cdot (F'(x), G'(x)) = \left(F'(F(x)),\ G(x) \cdot G'\left(\frac{F((q-1)x)}{q-1}\right)\right).$$

The isomorphism is given by $f \to (F_f^{(1)}(x), F_f^{(2)}(x))$, *where*

$$(5.32) \qquad F_f^{(1)}(x) = \sum_{n=1}^{\infty} \frac{f(\Pi_n)}{n!} x^n,$$

$$F_f^{(2)}(x) = \sum_{n=0}^{\infty} \frac{f(D_n)}{n!} x^n, \tag{5.33}$$

and where $f(\Pi_n)$ *denotes the value of* f *on a segment of type* $r = 0, k_1 = \cdots = k_{n-1} = 0, k_n = 1$, *and* $f(D_n)$ *denotes the value of* f *on a segment of type* $(n; 0, 0, \cdots)$.

PROOF. Clearly, the map defined is a bijection, so we need only check that multiplication is preserved. Let $f, g \in \mathbf{M}(\mathbf{D})$. It follows from Theorem 5.1 that $F_{f*g}^{(1)}(x) = F_g^{(1)}(F_f^{(1)}(x))$. Now, from (5.26) and denoting by Σ^* a summation taken over the set $\{r + k_1 + 2k_2 + \cdots + nk_n = n\}$ and by Σ_r^{**} a summation taken over the set $\{k_1 + 2k_2 + \cdots + nk_n = n - r\}$, we get

$$\begin{aligned}(f*g)(D_n) &\\ &= \sum{}^* \begin{bmatrix} n \\ r; k_1, k_2, \cdots, k_n \end{bmatrix} f(D_r) f(\Pi_1)^{k_1} \cdots f(\Pi_n)^{k_n} g(D_{k_1 + \cdots + k_n}) \\ &= \sum{}^* \binom{n}{r} \binom{n-r}{k_1, k_2, \cdots, k_n} (q-1)^{k_2 + 2k_3 + \cdots + (n-1)k_n} f(D_r) f(\Pi_1)^{k_1} \\ &\qquad \cdots f(\Pi_n)^{k_n} g(D_{k_1 + \cdots + k_n}) \\ &= \sum_{r=0}^{n} \binom{n}{r} f(D_r) \Bigg(\sum{}_r^{**} \binom{n-r}{k_1, k_2, \cdots, k_n} f(\Pi_1)^{k_1} \big(f(\Pi_2)(q-1)\big)^{k_2} \\ &\qquad \cdots \big(f(\Pi_n)(q-1)^{n-1}\big)^{k_n} g(D_{k_1 + \cdots + k_n}) \Bigg).\end{aligned} \tag{5.34}$$

Now, $f(D_r)$ is the coefficient of $x^r/r!$ in $F_f^{(2)}(x)$, and

$$\begin{aligned}\sum{}_r^{**} \binom{n-r}{k_1, k_2, \cdots, k_n} f(\Pi_1)^{k_1} \big(f(\Pi_2)(q-1)\big)^{k_2} \\ \cdots \big(f(\Pi_n)(q-1)^{n-1}\big)^{k_n} g(D_{k_1 + \cdots + k_n})\end{aligned} \tag{5.35}$$

is the coefficient of $x^{n-r}/(n-r)!$ in $F_g^{(2)}\big(F_f^{(1)}((q-1)x)/(q-1)\big)$, and hence the theorem follows.

COROLLARY 5.3. *The Möbius function in* D_n *is given by*

$$\mu(0, 1) = (-1)^n \prod_{i=0}^{n-1} [1 + (q-1)i]. \tag{5.36}$$

PROOF. We have $F_\zeta^{(1)}(x) = e^x - 1$, $F_\zeta^{(2)}(x) = e^x$, $F_\delta^{(1)}(x) = x$, and $F_\delta^{(2)}(x) = 1$. Now, $F_\mu^{(1)}(x) = \Sigma_{n=1}^{\infty} (\mu(\Pi_n)/n!)x^n = \log(1 + x)$ from the previous subsection. Thus,

$$\begin{aligned}1 = F_\delta^{(2)}(x) = F_{\mu*\zeta}^{(2)}(x) &= F_\mu^{(2)}(x) \cdot F_\zeta^{(2)}\left(\frac{F_\mu^{(1)}((q-1)x)}{q-1}\right) \\ &= F_\mu^{(2)}(x) \cdot \exp\left\{\frac{\log(1 + (q-1)x)}{q-1}\right\} \\ &= F_\mu^{(2)}(x) \cdot (1 + (q-1)x)^{1/(q-1)}.\end{aligned} \tag{5.37}$$

Hence,

$$F_\mu^{(2)}(x) = \big(1 + (q - 1)x\big)^{-1/(q-1)}, \tag{5.38}$$

from which the result follows.

COROLLARY 5.4. *Let f be a multiplicative function of one variable. For every nonnegative integer n, let $b_n = \Sigma_{G \in D_n} f(G)$ and $q_n = \Sigma_{G \in D_n} f(G)\mu(G, 1)$. Then*

$$\sum_{n=0}^{\infty} \frac{b_n}{n!} x^n = F_f^{(2)}(x) \cdot \exp\left\{\frac{F_f^{(1)}((q-1)x)}{q-1}\right\}, \tag{5.39}$$

$$\sum_{n=0}^{\infty} \frac{q_n}{n!} x^n = F_f^{(2)}(x) \cdot [1 + F_f^{(1)}((q-1)x)]^{-1/(q-1)}. \tag{5.40}$$

PROOF. The proof follows from Theorem 5.2 and Corollary 5.3 in the same way as Corollary 5.2 is proved.

Let us now return to the lattice Q_n to get an idea of what the class of a segment means in terms of the corresponding segment of vector spaces. Let $W \in Q(V)$, that is, W is a subspace of V which has a basis whose elements are of the form b_i or $ab_j + a'b_k$ (where $a, a' \in F^*$). Then it is not difficult to see that W has a basis of the form

$$\begin{aligned}\{b_{i_1}, b_{i_2}, \cdots, b_{i_r}, b_{j_1} + a_1 b_{j_2}, b_{j_2} + a_2 b_{j_3}, \cdots, b_{j_s} + a_s b_{j_{s+1}}, b_{k_1} + a_1' b_{k^2}, \\ b_{k_2} + a_2' b_{k_3}, \cdots, b_{k_t} + a_t' b_{k_{t+1}}, \cdots, b_{\ell_1} + a_1'' b_{\ell_2}, \cdots, b_{\ell_u} + a_u'' b_{\ell_{u+1}}\},\end{aligned} \tag{5.41}$$

where the a_i are nonzero and no b_i appears twice. Such a basis can be obtained by taking any basis and noting that if $ab_i + a'b_j$ and $\bar{a}b_i + \bar{a}'b_j$ both appear (and hence $a/a' \neq \bar{a}/\bar{a}'$), then b_i and b_j are in W and can replace $ab_i + a'b_j$ and $\bar{a}b_i + \bar{a}'b_j$ in the basis. The collection $\{b_{j_1} + a_1 b_{j_2}, b_{j_2} + a_2 b_{j_3}, \cdots, b_{j_s} + a_s b_{j_{s+1}}\}$ in the above basis is called an $(s + 1)$ cycle of the basis. Let k_1 equal the number of basis elements $\{b_1, b_2, \cdots, b_n\}$ which do not appear in the above basis (that is, are not among $\{b_{i_1}, b_{i_2}, \cdots, b_{i_r}, b_{j_1}, \cdots, b_{j_{s+1}}, \cdots, b_{\ell_1}, \cdots, b_{\ell_{u+1}}\}$), and for $i > 1$, let k_i be the number of i cycles in the basis. Then $(r; k_1, k_2, \cdots, k_n)$ is the class of the segment $[0, G]$ in D_n (where G is the graph corresponding to W), and $[G, 1]$ has class $[k_1 + k_2 + \cdots + k_n; 0, 0, \cdots]$. Note that it follows from this that r and $k_1, k_2, \cdots, k_n$ do not depend on the basis (of the proper type) chosen for W. The class of a general segment $[W, W']$ could also be determined from bases of the proper form chosen for W and W'. Thus, the class of a segment of Q_n could have been defined without resorting to the lattice D_n, but it then becomes necessary to prove that the class does not depend on the bases chosen.

5.4. *Abelian groups.* Let $\mathbf{C}(p)$ be the category whose objects are lattices of subgroups of finite abelian p groups (where p is a prime, fixed throughout) and all segments thereof, with the equivalence relation in each object being given by $[A, B] \simeq [G, H]$ if and only if $B/A \simeq H/G$. Morphisms in $\mathbf{C}(p)$ are all isomorphisms into such that if $[A, B]$ is the domain and $[G, H]$ the image of the isomorphism, then $B/A \simeq H/G$.

A *partition of an integer* n is a sequence $\lambda = (\lambda_1, \lambda_2, \cdots)$ of nonnegative integers arranged in decreasing order, whose sum is n. The types in the category $\mathbf{C}(p)$ above are in one to one correspondence with partitions, the type of a segment $[A, B]$ being $\lambda = (\lambda_1, \lambda_2, \cdots)$, where $B/A \simeq Z(p^{\lambda_1}) \oplus Z(p^{\lambda_2}) \oplus \cdots$. The type of a group G is defined to be the type of $[0, G]$. The incidence coefficient $\binom{\lambda}{\alpha, \beta}$ is equal to the number of members $G \in [A, B]$ (where $[A, B]$ is of type λ) such that $[A, G]$ has type α and $[G, B]$ has type β, or equivalently the number of subgroups G of a group H (where H is of type λ) such that G has type α and H/G has type β. This is precisely the "Hall polynomial" $g^{\lambda}_{\alpha,\beta}(p)$ defined in Hall [31], p. 156, and further studied by Green [27] and Klein [37]. (The Hall polynomials $g^{\rho}_{\lambda\mu\cdots\nu}(p)$ are simply the coefficients in the expression $(f*g*\cdots*h)(\rho) = \Sigma g^{\rho}_{\lambda\mu\cdots\nu}(p) f(\lambda) \cdots h(\nu)$.) Hall's algebra $A(p)$ is isomorphic to the subalgebra of $\mathbf{I}(\mathbf{C}(p))$ consisting of functions which are nonzero on only finitely many types, the isomorphism being given by linearly extending the map $\delta_\lambda \to G_\lambda(p)$, where δ_λ is the indicator function of the type λ in $\mathbf{I}(\mathbf{C}(p))$ and where $G_\lambda(p)$ is as in Hall's paper. The incidence coefficients $g^{\lambda}_{\alpha,\beta}(p)$ satisfy $g^{\lambda}_{\alpha,\beta}(p) = g^{\lambda}_{\beta,\alpha}(p)$, which follows from the well-known fact that the lattice of subgroups of a finite abelian group is self dual, and hence by Corollary 4.1, $\mathbf{I}(\mathbf{C}(p))$ is commutative. Various properties of the incidence coefficients $g^{\lambda}_{\alpha,\beta}(p)$ are worked out by Hall and extended by Klein and Green, the most basic being that $g^{\lambda}_{\alpha,\beta}(p)$ is a polynomial in p with integer coefficients. A condition for this polynomial to be identically zero, that is, for $g^{\lambda}_{\alpha,\beta}(p)$ to equal zero for all p, is given by Hall in terms of multiplication of Schur functions (see [31], p. 157).

EXAMPLE 5.12. Let $(r_1, r_2, \cdots, r_m)$ be an ordered partition of n. Then it follows from the commutativity of $\mathbf{I}(\mathbf{C}(p))$ that given any partition λ of n, the number of towers $1 \leqq H_1 \leqq H_2 \leqq \cdots \leqq H_m = G$ (where G has type λ) in which H_i/H_{i-1} has order p^{r_i} is independent of the arrangement of $(r_1, r_2, \cdots, r_m)$. This is because the number of such chains is given by $(h_{r_1} * h_{r_2} * \cdots * h_{r_m})(\lambda)$, where h_r is the function which takes the value 1 on segments $[A, B]$ in which B/A has order p^r, and is zero elsewhere (h_r is clearly constant on each type).

6. Residual isomorphism

In this section we are mainly concerned with the problem of determining when two segments are equivalent in the maximally reduced incidence algebra $\bar{\mathbf{R}}(P)$. As has been seen in Section 4, the two segments need not be isomorphic, that is, the standard reduced incidence algebra need not equal the maximally reduced incidence algebra. Until further notice, we will assume *the ground field K has characteristic* 0.

First, we give a criterion when two segments of P are equivalent in $\bar{\mathbf{R}}(P)$. Let $P = [0, 1]$ and $P' = [0', 1']$ be two finite ordered sets with unique minimal elements 0 and $0'$, and unique maximal elements 1 and $1'$, respectively. We say that P and P' are 1-*equivalent*, without imposing any other conditions on them. Define inductively P and P' to be $(n + 1)$-*equivalent* (written $P \overset{n+1}{\sim} P'$) if there exists a bijection $x \leftrightarrow x'$ between P and P' such that $[0, x] \overset{n}{\sim} [0', x']$ and $[x, 1] \overset{n}{\sim} [x', 1']$. Note that $P \overset{n}{\sim} P'$ implies $P \overset{m}{\sim} P'$ for $1 \leqq m \leqq n$. Note also that $P \overset{2}{\sim} P'$ if and only if P and P' have the same number of elements.

PROPOSITION 6.1. *Two segments $[x, y]$ and $[u, v]$ of a locally finite ordered set P are equivalent in $\bar{\mathbf{R}}(P)$ if and only if they are n equivalent for all positive integers n.*

Before proving Proposition 6.1, we show that the apparently infinite sequence of conditions that must be satisfied in order that $P \overset{n}{\sim} P'$ for every positive integer n reduces to a finite number of conditions for any given choice of P, P'.

PROPOSITION 6.2. *Let ℓ be the length of the longest chain of the two finite ordered sets $P = [0, 1]$ and $P' = [0', 1']$. Then $P \overset{n}{\sim} P'$ for every $n \geqq 1$ if and only if $P \overset{\ell}{\sim} P'$.*

PROOF. The proof is by induction on ℓ. The conclusion clearly holds when $\ell = 1$ and $\ell = 2$, since then P and P' are isomorphic. Now assume that the conclusion holds for $\ell \geqq 2$ and that the longest chain of P and P' has length $\ell + 1$ and that $P \overset{\ell+1}{\sim} P'$. Assume that $P \overset{n}{\sim} P'$ for some $n \geqq \ell + 1$. We will be done if we show $P \overset{n+1}{\sim} P'$. Since $P \overset{n}{\sim} P'$, there exists a bijection $x \leftrightarrow x'$ with $[0, x] \overset{n-1}{\sim} [0', x']$ and $[x, 1] \overset{n-1}{\sim} [x', 1']$. Clearly, $0 \leftrightarrow 0'$ and $1 \leftrightarrow 1'$, since $n \geqq 3$. If $x \neq 0, 1$, then $[0, x]$ and $[0', x']$ have no chain of length $\geqq \ell + 1$, so by the induction hypothesis $[0, x] \overset{n}{\sim} [0', x']$. Similarly, $[x, 1] \overset{n}{\sim} [x', 1']$. Hence, the bijection $x \leftrightarrow x'$ defines an $n + 1$ equivalence between P and P', and the proof is complete.

We conjecture that the following converse to Proposition 6.2 holds: for every $\ell \geqq 1$, there exist finite ordered sets $P = [0, 1]$ and $P' = [0', 1']$ with longest chain of length ℓ such that $P \overset{\ell-1}{\sim} P'$ but *not* $P \overset{\ell}{\sim} P'$. Figure 1 illustrates the validity of this conjecture for $\ell = 4$.

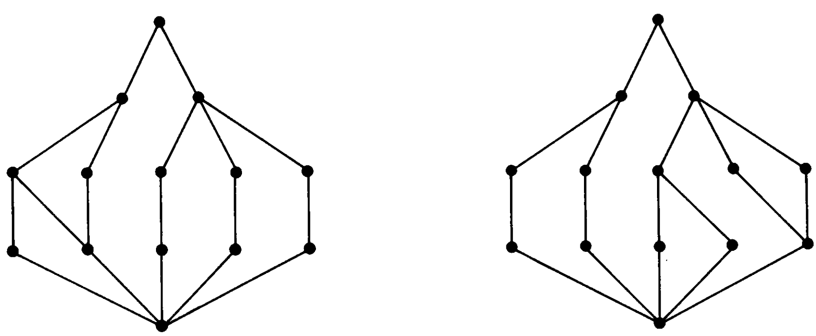

FIGURE 1
Ordered sets of length 4 which are 3-equivalent, but not 4-equivalent.

PROOF OF PROPOSITION 6.1. Define $[x, y] \sim [x', y']$ in P if and only if $[x, y] \overset{n}{\sim} [x', y']$ for all positive integers n. To prove the "if" part, we need to show that the equivalence relation $\sim$ is order compatible. It suffices to show that the coefficient $\left[\begin{smallmatrix}\alpha\\ \beta,\gamma\end{smallmatrix}\right]$ is well defined for any equivalence classes (types) α, β, γ. Let $[x, y]$ and $[x', y']$ be two segments of P of type α. Define $r(x, y, n)$ to be the number of points $z \in [x, y]$ such that $[x, z]$ is n equivalent to a segment of type β and $[z, y]$ is n equivalent to a segment of type γ. The number $r(x, y, n)$ is independent of the particular choice of segments of type β and γ, since all such segments are n equivalent. Since $[x, y] \overset{n+1}{\sim} [x', y']$, we have $r(x, y, n) = r(x', y', n)$. But then

$$\left[\begin{matrix}\alpha\\ \beta, \gamma\end{matrix}\right] = \lim_{n\to\infty} f(x, y, n) = \lim_{n\to\infty} r(x', y', n), \tag{6.1}$$

so $\left[\begin{smallmatrix}\alpha\\ \beta,\gamma\end{smallmatrix}\right]$ is well defined.

Conversely, suppose $[x, y] \sim [x', y']$ in $\bar{\mathbf{R}}(P)$. We prove by induction on n that $[x, y] \overset{n}{\sim} [x', y']$ for all n. Trivially $[x, y] \overset{1}{\sim} [x', y']$ for all $[x, y] \sim [x', y']$ (indeed, for *any* pair $[x, y]$, $[x', y']$). Assume $[x, y] \overset{n}{\sim} [x', y']$ for all $[x, y] \sim [x', y']$. Given any segment $[u, v]$ of P, define $f_{u,v,n} \in \mathbf{I}(P)$ by

$$f_{u,v,n}(x, y) = \begin{cases} 1 & \text{if } [x, y] \overset{n}{\sim} [u, v], \\ 0 & \text{otherwise.} \end{cases} \tag{6.2}$$

By the induction hypothesis $f_{u,v,n} \in \bar{\mathbf{R}}(P)$. Hence, $f_{u,v,n} * f_{u',v',n} \in \bar{\mathbf{R}}(P)$. But $f_{u,v,n} * f_{u',v',n}(x, y)$ is just the number of elements $z \in [x, y]$ such that $[x, z] \overset{n}{\sim} [u, v]$ and $[z, y] \overset{n}{\sim} [u', v']$. (This is where the assumption that K has characteristic 0 is needed.) Since $f_{u,v,n} * f_{u',v',n} \in \bar{\mathbf{R}}(P)$,

$$f_{u,v,n} * f_{u',v',n}(x, y) = f_{u,v,n} * f_{u',v',n}(x', y'). \tag{6.3}$$

Hence, $[x, y] \overset{n+1}{\sim} [x', y']$, and the proof is complete.

The proof of Proposition 6.1 allows us to characterize the form of functions in $\bar{\mathbf{R}}(P)$, at least when the characteristic of the ground field K is 0. If $f \in \mathbf{I}(P)$, define $\chi_f \in \mathbf{I}(P)$ by

$$\chi_f(x, y) = \begin{cases} 1 & \text{if } f(x, y) = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{6.4}$$

COROLLARY 6.1. The algebra $\bar{R}(P)$ *consists of those functions which can be obtained from* ζ *by a sequence of operations of the following three types:*

(i) *linear combination (possible infinite),*

(ii) *convolution,*

(iii) *the operation* $f \to \chi_f$.

PROOF. Clearly, all functions of the type described are in $\bar{\mathbf{R}}(P)$. The proof of Proposition 6.1 shows that for any segment $[u, v]$ of P, the function $f_{u,v,n} \in \bar{\mathbf{R}}(P)$. Proposition 6.2 shows that the sequence $f_{u,v,1}, f_{u,v,2}, \cdots$ is eventually constant, and that its limit (namely, $f_{u,v,n}$, where n is the greater of 2 and the number of elements in $[u, v]$, as is easily verified) is the characteristic function for the type of $[u, v]$ in $\bar{\mathbf{R}}(P)$. All functions in $\bar{\mathbf{R}}(P)$ are linear combinations (infinite if $\bar{\mathbf{R}}(P)$ has infinitely many types) of such characteristic functions.

Finally, it is not difficult to show (by induction on n) that $f_{u,v,n}$ is in the class of functions described for every segment $[u, v]$ and every n, so the proof is complete.

Define two locally finite ordered sets P and Q to be *residually isomorphic* (r isomorphic for short) if there is a bijection between the types of P relative to $\bar{\mathbf{R}}(P)$ and the types of Q relative to $\bar{\mathbf{R}}(Q)$ (over the same ground field K, which we still assume to have characteristic 0) inducing an isomorphism of $\bar{\mathbf{R}}(P)$ and $\bar{\mathbf{R}}(Q)$.

NOTE. It is possible for $\bar{\mathbf{R}}(P)$ and $\bar{\mathbf{R}}(Q)$ to be isomorphic as K algebras, and yet P and Q are not r isomorphic.

PROPOSITION 6.3. *Two finite ordered sets P and P', each with 0 and 1, are r isomorphic if and only if $P \overset{n}{\sim} P'$ for all $n \geqq 1$. Equivalently, two segments of a locally finite ordered set P are equivalent in $\bar{\mathbf{R}}(P)$ if and only if those segments are r isomorphic. Furthermore, α and α' are corresponding types in the isomorphism $\bar{\mathbf{R}}(P) \cong \bar{\mathbf{R}}(P')$ if and only if the segments in P of type α are r isomorphic to the segments in P' of type α'.*

PROOF. Assume P and P' are r isomorphic, and that a type α relative to $\bar{\mathbf{R}}(P)$ corresponds to a type α' relative to $\bar{\mathbf{R}}(P')$. Let Q be the disjoint union (direct sum) $P + P'$. Define an equivalence relation on segments of Q by $[x, y] \sim [x', y']$ if either (1) $[x, y] \sim [x', y']$ in $\bar{\mathbf{R}}(P)$, (2) $[x, y] \simeq [x', y']$ in $\bar{\mathbf{R}}(P')$, (3) $[x, y]$ is of type α in P and $[x', y']$ of type α' in P', or (4) $[x, y]$ is of type α' in P' and $[x', y']$ of type α in P. Clearly, this equivalence relation is order compatible. Hence, by Proposition 6.1 segments of type α are n equivalent to segments of type α' for all $n \geqq 1$, and in particular $P \overset{n}{\sim} P'$.

Conversely, if $P \overset{n}{\sim} P'$ for all $n \geqq 1$, define a bijection $\alpha \leftrightarrow \alpha'$ between types α relative to $\bar{\mathbf{R}}(P)$ and types α' relative to $\bar{\mathbf{R}}(P')$ by requiring that segments of type α be n equivalent to segments of type α' for all $n \geqq 1$. It follows easily from Proposition 6.1 that this bijection induces an isomorphism between $\bar{\mathbf{R}}(P)$ and $\bar{\mathbf{R}}(P')$, and the proof is complete.

COROLLARY 6.2. *Two finite r isomorphic ordered sets $P = [0, 1]$ and $P' = [0', 1']$ have the following properties in common:*

(i) *number of maximal chains of a given length,*

(ii) *number of elements a given minimum length from the bottom (or top); consequently, total number of elements, number of atoms, and number of dual atoms.*

PROOF. It follows from Corollary 6.1 that the function $\eta = \chi_{\zeta^2 - \zeta}$ is in $\bar{\mathbf{R}}(P)$ and $\bar{\mathbf{R}}(P')$. Note that

$$\eta(x, y) = \begin{cases} 1 & \text{if } \; y \text{ covers } x, \\ 0 & \text{otherwise,} \end{cases} \tag{6.5}$$

so that $\eta^r(x, y)$ is the number of maximal chains of $[x, y]$ of length r. By Proposition 6.3, $\eta^r(0, 1) = \eta^r(0', 1')$, since P and P' are r isomorphic. This proves (i).

Similarly one can find functions in $\bar{\mathbf{R}}(P)$ and $\bar{\mathbf{R}}(P')$, explicitly expressed in the form given by Corollary 6.1, which enumerate the quantities in (ii). The details we omit.

PROPOSITION 6.4. *Let P be an ordered set with 0 and 1 with $\leqq 7$ elements, and let Q be any finite ordered set with 0 and 1. Then P and Q are r isomorphic if and only if they are isomorphic.*

The proof is essentially by inspection of all possibilities, and will be omitted. Figure 2 shows two r isomorphic nonisomorphic ordered sets with eight elements. Another example of r isomorphic nonisomorphic ordered sets is the lattice of subspaces of two nonisomorphic finite projective planes of the same order.

FIGURE 2
Residually isomorphic nonisomorphic ordered sets.

We say that a finite ordered set P with 0 and 1 is *residually self dual* (r self dual for short) if it is r isomorphic to its dual. The next proposition uses this concept to characterize those P for which $\bar{\mathbf{R}}(P)$ is commutative.

PROPOSITION 6.5. *Let P be a locally finite ordered set. Then $\bar{\mathbf{R}}(P)$ is commutative if and only if every segment of P is r self dual.*

PROOF. Suppose $\bar{\mathbf{R}}(P)$ is commutative. This means that $\left[{\alpha \atop \beta,\gamma}\right] = \left[{\alpha \atop \gamma,\beta}\right]$ for all types α, β, γ. If δ is the type of a segment, let δ^* be the type of its dual. If $[x, y]$ is a segment of type α, consider the bijection $\delta \leftrightarrow \delta^*$ between types of segments in $[x, y]$ and types in the dual $[x, y]^*$. Then

$$\left[\begin{matrix} \alpha^* \\ \beta^*, \gamma^* \end{matrix}\right] = \left[\begin{matrix} \alpha \\ \gamma, \beta \end{matrix}\right] = \left[\begin{matrix} \alpha \\ \beta, \gamma \end{matrix}\right], \tag{6.6}$$

so the bijection $\delta \leftrightarrow \delta^*$ induces an isomorphism between $\bar{\mathbf{R}}([x, y])$ and $\bar{\mathbf{R}}([x, y]^*)$, that is, $[x, y]$ is r self dual.

Conversely, suppose every segment $[x, y]$ of P is r self dual. Since $[x, y]$ is r self dual, the number of elements $z \in [x, y]$ such that $[x, z]$ is of type β and $[z, y]$ of type γ is equal to the number of elements $z' \in [x, y]$ such that $[x, z']$ is of type γ^* and $[z', y]$ of type β^*. But $\beta = \beta^*$ and $\gamma = \gamma^*$, since every segment of these types is r self dual. Hence, if $[x, y]$ is of type α, then $\left[{\alpha \atop \beta,\gamma}\right] = \left[{\alpha \atop \gamma,\beta}\right]$ and $\bar{\mathbf{R}}(P)$ is commutative. This completes the proof.

Figure 3 illustrates an r self dual ordered set P which is not self dual. For this ordered set, $\bar{\mathbf{R}}(P)$ is equal to the standard reduced incidence algebra. This answers a question of Smith ([55], p. 632) on the existence of such ordered sets.

REMARKS. *On characteristic p.* Proposition 6.1 and its consequences are false if the characteristic of the ground field is not 0. For example, whenever the

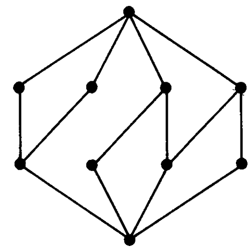

FIGURE 3
A residually self dual ordered set which is not self dual.

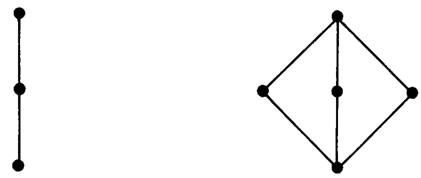

FIGURE 4
Equivalent segments in characteristic 2 which are not residually isomorphic.

two ordered sets of Figure 4 occur as segments of a locally finite ordered set P, then they are equivalent in $\bar{\mathbf{R}}(P)$ over a ground field of characteristic 2. It is not difficult, however, to modify the results of this section to get corresponding results for characteristic p, basically by replacing all concepts by the corresponding concepts modulo p. We will not go into the details here.

7. Algebras of Dirichlet type

7.1. *Definitions.* Let P be a locally finite ordered set, having a unique minimal element 0. Let $\mathbf{R}(P, \sim)$ be a reduced incidence algebra whose types are in one to one correspondence with a subset of the positive integers, the type of a segment $[x, y]$ being denoted by $O(x, y)$. Suppose the function O satisfies the following property: if $x \leqq y \leqq z$ in P, then $O(x, z) = O(x, y)O(y, z)$.

We then call $\mathbf{R}(P, \sim)$ an *algebra of Dirichlet type*. The bracket $\left[{n \atop k, \ell}\right]$ stands for the number of points y in a segment $[x, z]$ of type n such that $O(x, y) = k$ and $O(y, z) = \ell$. Clearly, $\left[{n \atop k, \ell}\right] = 0$ unless $n = k\ell$. Hence, it makes sense to define the *brace* $\left\{{n \atop k}\right\} = \left[{n \atop k, n/k}\right]$. The reduced incidence algebra $\mathbf{R}(P, \sim)$ is isomorphic to the algebra of all sequences a_n, $n = 1, 2, \cdots$, where $a_n \neq 0$ only if there is a segment of type n in P. The convolution of two such sequences is

$$c_n = \sum_{k|n} \left\{{n \atop k}\right\} a_k b_{n/k}. \tag{7.1}$$

EXAMPLE 7.1. Let P be the set of all positive integers, ordered by divisibility. Set $O(k, n) = n/k$, for $k, n \in P$. This gives the reduced incidence algebra mentioned at the beginning of Example 4.8. The braces are identically equal to one, the convolution is commutative, and it reduces to the classical *Dirichlet convolution*

$$c_n = \sum_{k|n} a_k b_{n/k}. \tag{7.2}$$

The reduced incidence algebra $\mathbf{R}(P, \sim)$ is isomorphic to the algebra of formal Dirichlet series. The zeta function is mapped into the *Riemann zeta function*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{7.3}$$

and the Möbius function goes into the function

$$\zeta(s)^{-1} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \tag{7.4}$$

where $\mu(n)$ is the classical Möbius function, as has already been sketched in *Foundations I*.

Algebras of Dirichlet type satisfy the following fundamental recursion:

$$\begin{Bmatrix} n \\ m \end{Bmatrix} \begin{Bmatrix} m \\ k \end{Bmatrix} = \begin{Bmatrix} n \\ k \end{Bmatrix} \begin{Bmatrix} n/k \\ m/k \end{Bmatrix}. \tag{7.5}$$

This is obtained by counting in two ways the number of subsegments $[x_1, y_1]$ of a segment $[x, y]$ of type n such that $O(x, x_1) = k$, $O(x, y_1) = m$. There are $\{{n \atop m}\}$ ways of choosing y_1, and for each such choice there are $\{{m \atop k}\}$ ways of choosing x_1 below it. On the other hand, there are $\{{n \atop k}\}$ ways of choosing x_1, and for each such choice there are $\{{n/k \atop m/k}\}$ ways of choosing y_1 above it. This establishes (7.5).

There are three kinds of algebras $\mathbf{R}(P, \sim)$ of Dirichlet type of special importance.

(A) The algebra $\mathbf{R}(P, \sim)$ is *commutative* if and only if $\{{n \atop k}\} = \{{n \atop n/k}\}$ for all types n and all $k|n$.

(B) The algebra $\mathbf{R}(P, \sim)$ is said to be *full*, if whenever n is a type and $k|n$, then $\{{n \atop k}\} \neq 0$.

(C) The algebra $\mathbf{R}(P, \sim)$ is said to be of *binomial type*, if there is a prime p such that all types are powers of p. We then write

$$\begin{Bmatrix} p^b \\ p^a \end{Bmatrix} = \begin{bmatrix} b \\ a \end{bmatrix}. \tag{7.6}$$

The recursion (7.5) becomes

$$\begin{bmatrix} c \\ b \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} c \\ a \end{bmatrix} \begin{bmatrix} c - a \\ b - a \end{bmatrix}. \tag{7.7}$$

An algebra of binomial type is simply the additive analogue of an algebra of Dirichlet type. We shall always speak of algebras of binomial type in an additive sense, so a segment of type n in an algebra $\mathbf{R}(P, \sim)$ of binomial type is of type p^n when $\mathbf{R}(P, \sim)$ is regarded as an algebra of Dirichlet type.

There are, *a priori*, eight kinds of algebras of Dirichlet type obtained by specifying which of (A), (B), (C) hold or do not hold. It is easy to construct examples of seven of these kinds; in the next section, we shall see that every algebra of full binomial type is commutative.

7.2. *Full commutative algebras of Dirichlet type.* In this section, we show that if $\mathbf{R}(P, \sim)$ is a full commutative algebra of Dirichlet type, then there is an isomorphism of $\mathbf{R}(P, \sim)$ into formal Dirichlet series.

LEMMA 7.1. *Let $\mathbf{R}(P, \sim)$ be a full commutative algebra of Dirichlet type. Then the segments of P of type 1 are precisely the one point segments, and a segment has a prime type if and only if it is a two point segment. Further, P satisfies the Jordan–Dedekind chain condition, that is, in all segments of P, all maximal chains have the same length.*

PROOF. If $[x, x]$ has type k, then $k^2 = k$, so $k = 1$. Conversely, if $[x, y]$ has type 1, it follows from Lemma 4.1 that $x = y$.

If $[x, y]$ has prime type p, then $x \neq y$ (by the above), and if $[x, y]$ contained a third point z, then $p = O(x, z) \cdot O(z, y)$, which is impossible. Conversely, if $[x, y]$ is a two point segment and has type n, then n must be prime, for if it had a nontrivial factor k, then since $\mathbf{R}(P, \sim)$ is full there would be an element $z \in [x, y]$ such that $[x, z]$ would have type k. Finally, it follows from this that for any segment $[x, y]$, the length of any maximal chain is the number of primes in the prime decomposition of $O(x, y)$. This completes the proof.

Let $[x, y]$ be a segment of P of type n, and let C be a maximal chain of $[x, y]$, say $x = x_0 < x_1 < x_2 < \cdots < x_m = y$. If p_i is the type of $[x_{i-1}, x_i]$, then $n = p_1 p_2 \cdots p_m$ is an ordered factorization of n into primes; we call it the *factorization of n induced by C*, or more briefly, the *factorization of C*.

LEMMA 7.2. *Let $\mathbf{R}(P, \sim)$ be a full commutative algebra of Dirichlet type and $[x, y]$ a segment of type n. Let $n = p_1 p_2 \cdots p_m$ be any ordered factorization of n into primes. The number of maximal chains of $[x, y]$ with factorization $p_1 p_2 \cdots p_m$ is given by*

$$B(n) = \begin{Bmatrix} n \\ p_1 \end{Bmatrix} \begin{Bmatrix} n/p_1 \\ p_2 \end{Bmatrix} \begin{Bmatrix} n/p_1 p_2 \\ p_3 \end{Bmatrix} \cdots \begin{Bmatrix} n/p_1 \cdots p_{m-1} \\ p_m \end{Bmatrix} \tag{7.8}$$

and this number depends only on n, not on the factorization chosen.

Hence, if $n = q_1^{a_1} q_2^{a_2} \cdots q_r^{a_r}$ is the canonical factorization of n, then

$$B(n) = \frac{M(n) a_1! \, a_2! \cdots a_r!}{(a_1 + a_2 + \cdots + a_r)!}, \tag{7.9}$$

where $M(n)$ is the number of maximal chains in $[x, y]$.

PROOF. The number of maximal chains with factorization $p_1 p_2 \cdots p_m$ is obviously the expression on the right side of (7.8). By the commutativity relation

$\left\{{s \atop r}\right\} = \left\{{s \atop s/r}\right\}$ and the recursion (7.5),

$$\left\{{n/p_1p_2\cdots p_{k-1} \atop p_k}\right\}\cdot\left\{{n/p_1p_2\cdots p_k \atop p_{k+1}}\right\} \tag{7.10}$$
$$= \left\{{n/p_1p_2\cdots p_{k-1} \atop n/p_1p_2\cdots p_k}\right\}\cdot\left\{{n/p_1p_2\cdots p_k \atop p_{k+1}}\right\}$$
$$= \left\{{n/p_1p_2\cdots p_{k-1} \atop p_{k+1}}\right\}\cdot\left\{{n/p_1p_2\cdots p_{k-1}p_{k+1} \atop n/p_1p_2\cdots p_kp_{k+1}}\right\}$$
$$= \left\{{n/p_1p_2\cdots p_{k-1} \atop p_{k+1}}\right\}\cdot\left\{{n/p_1p_2\cdots p_{k-1}p_{k+1} \atop p_k}\right\}.$$

Hence, $B(n)$ is not changed when p_k and p_{k+1} are interchanged. Since all permutations of $p_1, \cdots p_m$ are generated by such interchanges, the proof follows.

PROPOSITION 7.1. *Let* $\mathbf{R}(P, \sim)$ *be a full algebra of Dirichlet type with types* $n_1 = 1, n_2, \cdots$. *If* $f \in \mathbf{R}(P, \sim)$, *then the map*

$$f \to \sum_k \frac{f(n_k)}{B(n_k)n_k^s} \tag{7.11}$$

of $\mathbf{R}(P, \sim)$ *into formal Dirichlet series is an isomorphism, if when we multiply Dirichlet series we ignore all* β^{-s} *terms when* β *is not some* n_k.

PROOF. Let $[x, y]$ be of type n. For any type $\ell | n$, let $n = p_1 \cdots p_m$ be any factorization with $p_1p_2\cdots p_k = \ell$. Exactly $B(\ell)$ maximal chains with the factorization $p_1p_2\cdots p_k$ connect x with a fixed point z such that $[x, z]$ is of type ℓ. Exactly $B(n/\ell)$ maximal chains with the factorization $p_{k+1}\cdots p_m$ connect z with y. Thus, the number of such z is

$$\left\{{n \atop \ell}\right\} = \frac{B(n)}{B(\ell)B(n/\ell)}, \tag{7.12}$$

and the isomorphism follows.

REMARK. As we will see in the next section, when $\mathbf{R}(P, \sim)$ is of full binomial type we know that we can write $B(n) = A(1)A(2)\cdots A(n)$, where $A(n) = \left\{{n \atop 1}\right\}$ is the number of points covered by y in an interval $[x, y]$ of length n. The analogy for full commutative algebras of Dirichlet type is formula (7.8). Here

$$A(k) = \left\{{n/p_1p_2\cdots p_{m-k} \atop p_{m-k+1}}\right\} \tag{7.13}$$

depends on the particular ordered factorization of n into the primes chosen. A canonical choice of $A(k)$ can be specified by the requirement $p_1 \leqq p_2 \leqq \cdots \leqq p_m$.

In certain cases it is possible to know considerably more about the structure of P and $\mathbf{R}(P, \sim)$.

PROPOSITION 7.2. *Let* $\mathbf{R}(P, \sim)$ *be a full commutative algebra of Dirichlet type. Suppose the function* B *is "multiplicative if defined," that is, if* $(m, n) = 1$ *and if* mn *is a type, then* $B(mn) = B(m)B(n)$. *Let* $[x, y]$ *be a segment of type*

$n = p_1^{a_1}p_1^{a_2} \cdots p_m^{a_m}$ and let $[x, x_1], \cdots, [x, x_m]$ be segments of $[x, y]$ of types $p_1^{a_1}, \cdots, p_m^{a_m}$, respectively. Then $[x, y]$ is the product, $[x, y] = [x, x_1] \times [x, x_2] \times \cdots \times [x, x_m]$, and $\mathbf{R}(P, \sim)$ *restricted to $[x, y]$ is given by the tensor product (over κ),* $\mathbf{R}([x, y], \sim) = \mathbf{R}([x, x_1], \sim) \otimes \cdots \otimes \mathbf{R}([x, x_m], \sim)$. *Each of the algebras* $\mathbf{R}([x, x_i], \sim)$ *is of full binomial type.*

PROOF. If $1 \leqq i \leqq m$, we have

$$\left\{ \begin{matrix} n \\ p_i^{a_i} \end{matrix} \right\} = \frac{B(n)}{B(p_i^{a_i})B(n/p_i^{a_i})} = \frac{B(n)}{B(n)} = 1. \tag{7.14}$$

Thus, the segments $[x, x_i]$ are unique. If $z \in [x, y]$ and $[x, z]$ is of type $\ell = p_1^{b_1}p_2^{b_2} \cdots p_m^{b_m}$, then as above $\{ {\ell \atop p_i^{b_i}} \} = 1$, and z lies above a unique point $z_i \in [x, x_i]$ with $[x, z_i]$ of type $p_i^{b_i}$. Hence, we have a mapping $z \to (z_1, \cdots, z_m)$. Now, the number of $z \in [x, y]$ such that $[x, z]$ is of type $p_1^{b_1} \cdots p_m^{b_m}$ equals

$$\begin{aligned} \left\{ \begin{matrix} p_1^{a_1} \cdots p_m^{a_m} \\ p_1^{b_1} \cdots p_m^{b_m} \end{matrix} \right\} &= \frac{B(p_1^{a_1} \cdots p_m^{a_m})}{B(p_1^{b_1} \cdots p_m^{b_m})B(p_1^{a_1 - b_1} \cdots p_m^{a_m - b_m})} \\ &= \frac{B(p_1^{a_1}) \cdots B(p_m^{a_m})}{B(p_1^{b_1}) \cdots B(p_m^{b_m})B(p_1^{a_1 - b_1}) \cdots B(p_m^{a_m - b_m})} \\ &= \left\{ \begin{matrix} p_1^{a_1} \\ p_1^{b_1} \end{matrix} \right\} \cdot \left\{ \begin{matrix} p_2^{a_2} \\ p_2^{b_2} \end{matrix} \right\} \cdots \left\{ \begin{matrix} p_m^{a_m} \\ p_m^{b_m} \end{matrix} \right\} \end{aligned} \tag{7.15}$$

which is the number of m-tuples $(z_1, \cdots, z_m)$ with $[x, z_i]$ of type $p_i^{b_i}$. Further, the mapping is injective, as the following argument shows. Suppose z and $\bar{z}$ are distinct elements of $[x, y]$ with $[x, z]$ and $[x, \bar{z}]$ of type $p_1^{b_1} \cdots p_m^{b_m}$, and suppose both z and $\bar{z}$ lie over $z_1, \cdots, z_m$, where $O(x, z_i) = p_i^{b_i}$. Take $w_1 \in [z, y]$ with $O(z, w_1) = p_1^{a_1 - b_1}$, $w_2 \in [w_1, y]$ with $O(w_1, w_2) = p_2^{a_2 - b_2}, \cdots, w_n \in [w_{n-1}, y]$ with $O(w_{n-1}, w_n) = p_m^{a_m - b_m}$, and similarly take elements $\bar{w}_1, \cdots, \bar{w}_m$ above $\bar{z}$. Note that $w_n = \bar{w}_n = y$, since

$$\begin{aligned} O(x, w_n) &= O(x, z)O(z, w_1)O(w_1, w_2) \cdots O(w_{n-1}, w_n) \\ &= p_1^{a_1} \cdots p_m^{a_m}. \end{aligned} \tag{7.16}$$

However, we show by induction that $w_j \neq \bar{w}_j$ for $1 \leqq j \leqq m$, which gives the desired contradiction.

For $j = 1$, if $w_1 = \bar{w}_1$, then

$$\left\{ \begin{matrix} p_1^{b_1 - a_1}p_2^{b_2}p_3^{b_3} \cdots p_m^{b_m} \\ p_2^{b_2}p_3^{b_3} \cdots p_m^{b_m} \end{matrix} \right\} > 1 \tag{7.17}$$

(since $z, \bar{z} \in [z_1, w_1]$), which is not the case as B is multiplicative. Assume $w_{j-1} \neq \bar{w}_{j-1}$ for $j \leqq m$. If $w_j = \bar{w}_j$, then

$$\left\{ \begin{matrix} p_1^{b_1} \cdots p_{j-1}^{b_{j-1}} p_{j+1}^{b_{j+1}} \cdots p_m^{b_m} p_1^{a_1 - b_1} \cdots p_{j-1}^{a_{j-1} - b_{j-1}} p_j^{a_j - b_j} \\ p_1^{b_1} \cdots p_{j-1}^{b_{j-1}} p_{j+1}^{b_{j+1}} \cdots p_m^{b_m} p_1^{a_1 - b_1} \cdots p_{j-1}^{a_{j-1} - b_{j-1}} \end{matrix} \right\} > 1 \tag{7.18}$$

(since $w_{j-1}, \bar{w}_{j-1} \in [z_j, w_j]$) which is not the case, as B is multiplicative.

Thus the mapping $z \to (z_1, \cdots, z_m)$ is the desired isomorphism $[x, y] \simeq [x, x_1] \times \cdots \times [x, x_m]$, and the rest of the proof follows easily.

As a converse to the above proposition, suppose $\bar{\mathbf{R}}(P_1)$, $\bar{\mathbf{R}}(P_2), \cdots$ are full algebras of binomial type. Let $p_1, p_2, \cdots$ be distinct primes, and let $[x, y] = [(x_1, x_2, \cdots), (y_1, y_2, \cdots)]$ be a segment of $P_1 \times P_2 \times \cdots$, where $[x_i, y_i]$ is of type a_i in $\bar{\mathbf{R}}(P_i)$. Then defining $O(x, y) = p_1^{a_1} p_2^{a_2} \cdots$ gives a full commutative algebra of Dirichlet type such that B is multiplicative if defined.

REMARK. The condition that B is "multiplicative if defined" is equivalent to saying that the Dirichlet series corresponding to the zeta function $\zeta \in \mathbf{R}(P, \sim)$ has an Euler product in the sense that the Dirichlet series

$$\sum_k \frac{1}{B(n_k) n_k^s} - \prod_p \sum_a \frac{1}{B(p^a) p^{as}} \tag{7.19}$$

for some $n_k = p^a$ vanishes at all terms m^{-s} whenever m is a type.

All the usual number theoretic functions such as the Euler totient function ϕ, the number of divisors d, the sum of the divisors σ, and so forth, have analogues in full commutative algebras of Dirichlet type (even in any algebra of Dirichlet type, although some of their properties do not carry over). For instance, if $O(x, y) = n$, we define

$$\begin{aligned} \phi(n) &= \mu * O(n) = \sum_{z \in [x, y]} \mu(x, z) O(z, y), \\ d(n) &= \zeta^2(n) \quad = \sum_{z \in [x, y]} 1, \\ \sigma(n) &= O * \zeta(n) = \sum_{z \in [x, y]} O(x, z), \end{aligned} \tag{7.20}$$

and so on.

These functions, along with μ, will be "multiplicative if defined" if and only if B is also.

PROBLEM. It is easy to construct examples of infinite noncommutative Dirichlet algebras. For instance, let P be the lattice of positive integers under $\leqq$ (a discrete chain). If $m \leqq n$, define

$$O(m, n) = \begin{cases} 2^{n-m} & \text{if } 1 < m, \\ 3 \cdot 2^{n-m-1} & \text{if } 1 = m < n, \\ 1 & \text{if } m = n = 1. \end{cases} \tag{7.21}$$

The corresponding Dirichlet algebra $\mathbf{R}(P, \sim)$ is infinite, that is, there are infinitely many values of $O(m, n)$, and noncommutative.

Suppose, however, we require $\mathbf{R}(P, \sim)$ to have the following properties:

(a) $\mathbf{R}(P, \sim)$ is a full algebra of Dirichlet type, and

(b) any two elements of P have an upper bound. We know of no infinite noncommutative algebras $\mathbf{R}(P, \sim)$ satisfying (a) and (b).

7.3. *Abelian groups.* Suppose G is an abelian group whose lattice P of subgroups gives a Dirichlet algebra $\mathbf{R}(P, \sim)$ if we take $O(x, y)$ to be the order of the quotient group y/x. Then G is finite (since P must be locally finite and every infinite group has infinitely many subgroups), and every Sylow subgroup of G is either cyclic or elementary abelian. Conversely, any such G gives rise to such a Dirichlet algebra, which in fact is of full Dirichlet type whose zeta function has an Euler product.

PROOF. Suppose a Sylow p subgroup of G is not cyclic or elementary abelian. Then it contains a subgroup isomorphic to $Z(p) \oplus Z(p^2)$, where $Z(n)$ denotes the cyclic group of order n. The segments $[0, Z(p^2)]$ and $[0, Z(p) \otimes Z(p)]$ both have type p^2 but are not residually isomorphic, so $\mathbf{R}(P, \sim)$ cannot be of Dirichlet type.

That the converse is true is a straightforward verification.

8. Algebras of full binomial type

8.1. *Structure.* Recall from the previous section that $\mathbf{R}(P, \sim)$ is an algebra of full binomial type if the types are in one to one correspondence with a subset of the nonnegative integers, the type of a segment $[x, y]$ being denoted $O(x, y)$, satisfying:

(A) if $x \leqq z \leqq y$, then $O(x, y) = O(x, z) + O(z, y)$;

(*B*) if n is a type and if $k \leqq n$, then $\left[{n \atop k}\right] \neq 0$, where $\left[{n \atop k}\right]$ is the number of elements z in a segment $[x, y]$ of type n for which $O(x, z) = k$ (and hence, $O(z, y) = n - k$). We then had the following relation

$$\begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} m \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n - k \\ m - k \end{bmatrix}. \tag{8.1}$$

PROPOSITION 8.1. *Every algebra of full binomial type is commutative.*

PROOF. Suppose $\mathbf{R}(P, \sim)$ is a full algebra of binomial type. We prove by induction on n that $\left[{n \atop m}\right] = \left[{n \atop n-m}\right]$ when $0 < m < n$. Since $\left[{n \atop 0}\right] = \left[{n \atop n}\right] = 1$, it will follow that $\mathbf{R}(P, \sim)$ is commutative.

The statement is clear for $n = 0, 1, 2$. Assume it is true for all $n_0 < n$. Suppose $0 < m < n \leqq 2m$. From the relation (8.1), we have

$$\begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} m \\ n - m \end{bmatrix} = \begin{bmatrix} n \\ n - m \end{bmatrix} \begin{bmatrix} m \\ 2m - n \end{bmatrix}. \tag{8.2}$$

Since $0 < m < n \leqq 2m$ and $\mathbf{R}(P, \sim)$ is full, we have $\left[{m \atop n-m}\right] \neq 0$, $\left[{m \atop 2m-n}\right] \neq 0$. By induction $\left[{m \atop n-m}\right] = \left[{m \atop 2m-n}\right]$. Hence, $\left[{n \atop m}\right] = \left[{n \atop n-m}\right]$. If $0 < m < n$ but $2m < n$, then $0 < n - m < n$ and $n \leqq 2(n - m)$, so again $\left[{n \atop n-m}\right] = \left[{n \atop m}\right]$, and the proof is complete.

LEMMA 8.1. *Let $\mathbf{R}(P, \sim)$ be an algebra of binomial type. Then the segments of type 0 are precisely the points of P. Moreover, if $\mathbf{R}(P, \sim)$ is of full binomial type, then the segments of type 1 are those segments of P which contain exactly two points.*

PROOF. If $[x, x]$ is of type n, then $n + n = n$, so $n = 0$. Conversely, if $[x, y]$ is of type 0, it follows from Lemma 4.1 that $x = y$.

If $\mathbf{R}(P, \sim)$ is full and $[x, y]$ is a two point segment of type $n > 0$, then since $\left[{n \atop 1}\right] \neq 0$, we must have $n = 1$. Conversely, by Lemma 4.1 any segment of type 1 contains exactly two points. This completes the proof.

We assume for the rest of this subsection that $\mathbf{R}(P, \sim)$ is a full algebra of binomial type. Let N be the largest type of any segment of P (or $N = \infty$ if there is no largest type). Since $\mathbf{R}(P, \sim)$ is full, we have

$$(8.3) \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix} = A(0) = 0, \begin{bmatrix} n \\ 1 \end{bmatrix} = A(n) \neq 0, \qquad 1 \leqq n \leqq N \text{ (except } n = \infty).$$

Define $B(n) = A(1)A(2) \cdots A(n)$, with $B(0) = 1$.

Setting $k = 1$ in (8.1) and iterating, we find

$$(8.4) \qquad \begin{bmatrix} n \\ m \end{bmatrix} = \frac{A(n)A(n-1)\cdots A(n-m+1)}{A(m)A(m-1)\cdots A(1)} = \frac{B(n)}{B(m)B(n-m)}, \qquad 0 \leqq m \leqq n \leqq N \text{ (except } n = \infty),$$

where we have used the obvious fact that

$$(8.5) \qquad \begin{bmatrix} n \\ 0 \end{bmatrix} = 1, \qquad 0 \leqq n \leqq N, n \neq \infty.$$

We have therefore shown that a full algebra of binomial type is isomorphic to an algebra of formal power series, taken modulo z^{N+1}, the isomorphism being given by

$$(8.6) \qquad f \to \sum_{n=0}^{N} \frac{f(n)}{B(n)} z^n \pmod{z^{N+1}},$$

where $f(n)$ denotes the value $f \in \mathbf{R}(P, \sim)$ takes at any segment of type n. The converse to this statement, and a characterization of full algebras of binomial type, is provided by the next theorem.

THEOREM 8.1. *Suppose P is a locally finite ordered set and $\mathbf{R}(P, \sim)$ a reduced incidence algebra of P with types labelled* $0, 1, 2, \cdots, N$, $1 \leqq N \leqq \infty$, *such that* (8.6) *is an isomorphism of $\mathbf{R}(P, \sim)$ onto formal power series modulo z^{N+1}. The isomorphism* (8.6) *can be "normalized" by setting $z' = (1/B(1))z$, so we can assume $B(1) = 1$. Then $\mathbf{R}(P, \sim)$ is a full algebra of binomial type and the following hold:*

(i) *P satisfies the Jordan–Dedekind chain condition;*

(ii) *all segments of P of length n have the same number of maximal chains;*

(iii) *a segment of length n is of type n;*

(iv) *in the isomorphism* (8.6) *(normalized to $B(1) = 1$), $B(n)$ is the number of maximal chains in a segment of length n and N is the length of P;*

(v) $\mathbf{R}(P, \sim) = \bar{\mathbf{R}}(P)$.

Conversely, if P is a locally finite ordered set satisfying (i) *and* (ii), *then $\bar{\mathbf{R}}(P)$ is a full algebra of binomial type given by* (iii) *and* (iv).

PROOF. Suppose $\mathbf{R}(P, \sim)$ satisfies the hypothesis of the theorem. Then (A) follows from the isomorphism (8.6) using the law of exponents $z^{m+n} = z^m z^n$. Hence, $\mathbf{R}(P, \sim)$ is of binomial type. By the hypothesis that the isomorphism (8.6) is onto, it follows that $\mathbf{R}(P, \sim)$ is a full algebra of binomial type.

Define $g \in \mathbf{R}(P, \sim)$ by

$$g(x, y) = \begin{cases} 1 & \text{if } [x, y] \text{ is of type } 1, \\ 0 & \text{otherwise.} \end{cases} \tag{8.7}$$

Using Lemma 8.1, we see $g^n(x, y)$ is the number of maximal chains of $[x, y]$ of length n. By (8.6), $g^n(x, y) \neq 0$ if and only if $[x, y]$ is of type n. Hence, every maximal chain of $[x, y]$ has length n. Since

$$\left(\frac{z}{B(1)}\right)^n = \frac{B(n)}{B(1)^n} \frac{z^n}{B(n)}, \tag{8.8}$$

then by (8.6), $B(n)$ is the number of maximal chains in an interval of type n when we take $B(1) = 1$. Clearly, N is the length of P. By Lemma 4.1 (ii), $\mathbf{R}(P, \sim) = \bar{\mathbf{R}}(P)$. This proves (i), (ii), (iii), (iv), (v) of the theorem.

Conversely, suppose P satisfies (i) and (ii). (Actually, (i) and (ii) follow easily from the slightly weaker condition that all segments of P of the same minimum length contain the same number of maximal chains.) Let $B(n)$ be the number of maximal chains in a segment of length n. Then each segment $[x, y]$ of length n contains $B(n)/B(k)B(n - k)$ points of height k, since $B(k)B(n - k)$ maximal chains in $[x, y]$ pass through a point of height k. Thus, if $f, g \in \mathbf{I}(P)$ depend only on the length n of any segment $[x, y]$, we have

$$(f*g)(n) = (f*g)(x, y) = \sum_{k=0}^{n} \frac{B(n)}{B(k)B(n - k)} f(k)g(n - k), \tag{8.9}$$

which is a function of n only. Thus, specifying all segments of the same length to be of the same type gives a reduced incidence algebra $\mathbf{R}(P, \sim)$, which by Lemma 4.1 (ii) must be $\bar{\mathbf{R}}(P)$. The isomorphism (8.6) now follows immediately from (8.9). We have proved that if (8.6) holds, then $\mathbf{R}(P, \sim)$ is of full binomial type, so the proof is complete.

COROLLARY 8.1. *If P is a locally finite ordered set and if every segment of P of the same minimum length is of the same type in $\bar{\mathbf{R}}(P)$, then $\bar{\mathbf{R}}(P)$ is a full algebra of binomial type.*

PROOF. By Lemma 4.1, all segments of P of the same minimum length contain the same number of maximal chains, since they are of the same type. We have already remarked that it is easy to prove from this that P satisfies the Jordan-Dedekind chain condition. The proof now follows from Theorem 8.1.

REMARK. Suppose $\mathbf{R}(P, \sim)$ is of full binomial type. By the previous theorem any two segments of P of the same length are r isomorphic. Moreover, any segment of P is "r self dual", that is, is r isomorphic to its dual, since $\mathbf{R}(P, \sim) \simeq \mathbf{R}(P^*, \sim)$, when P is of full binomial type and P^* is the dual of P.

A further characterization of full algebras $\mathbf{R}(P, \sim)$ of binomial type, at least when P does not have arbitrarily large chains, is given by the next proposition.

PROPOSITION 8.2. *Suppose $\mathbf{R}(P, \sim)$ is a reduced incidence algebra of a locally finite ordered set P with 0 which when considered as an algebra with identity over the ground field (which we have been assuming has characteristic 0) is generated* *by a single function f. Then* $\mathbf{R}(P, \sim)$ *is a full algebra of binomial type, and there is an integer N such that the longest chain in P has length N.*

Conversely, if $\mathbf{R}(P, \sim)$ *is a full algebra of binomial type and if the longest chain in P has finite length N, then* $\mathbf{R}(P, \sim)$ *is generated by any function* $f \in \mathbf{R}(P, \sim)$ *not vanishing on segments of length* 1 *(for example,* $f = \zeta$*).*

PROOF. Suppose f generates $\mathbf{R}(P, \sim)$. We first show that all points of P belong to the same equivalence class relative to $\sim$. Otherwise, since P has a 0, there is a two point segment $[x, y]$ of P such that $[x, x]$ and $[y, y]$ are not equivalent. Hence $\mathbf{R}(P, \sim)$, when restricted to $[x, y]$, has dimension three as a vector space. But if $f(x, x) = a$ and $f(y, y) = b$, then $(f - a)(f - b)$ vanishes on all three subsegments of $[x, y]$ and hence f generates, together with the identity, a vector space of dimension $\leqq$ two when restricted to $[x, y]$. This contradiction shows $[x, x] \sim [y, y]$ and hence all points of P are equivalent.

If P contains arbitrarily long chains, then $\mathbf{R}(P, \sim)$ has uncountable dimension as a vector space, while f generates a vector space of countable dimension. Hence, there is an integer N such that the longest chain in P has length N. The preceding paragraph shows that f is constant on points, say $f(x, x) = a$. Then $(f - a)^{N+1} = 0$. Hence, f, together with the identity, generates a vector space of dimension $\leqq N + 1$. Since two segments of different maximum lengths must be of different types, it follows that any two segments of the same maximum length are of the same type (because the dimension of $\mathbf{R}(P, \sim)$ is equal to the number of types). It then follows from Corollary 8.1 that $\mathbf{R}(P, \sim)$ is a full algebra of binomial type.

The converse is a trivial consequence of the isomorphism (8.6), and the proof is complete.

8.2. *Lattices of full binomial type.* An ordered set P is said to be of full binomial type if it satisfies (i) and (ii) of Theorem 8.1.

Examples of ordered sets P of full binomial type are discrete chains with 0, lattices of finite subsets of a set, and lattices of finite subspaces of a projective space. Various other examples are given in Figure 5.

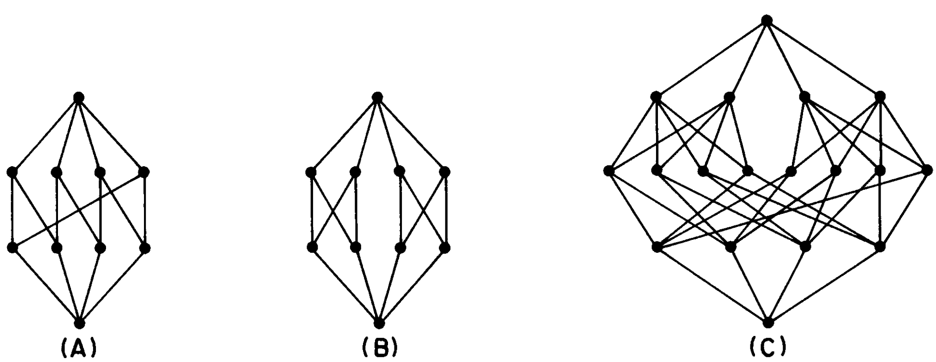

FIGURE 5
Ordered sets of full binomial type.

The ordered sets (A) and (B) have isomorphic reduced incidence algebras of full binomial type, although they are not isomorphic as ordered sets. In fact, (A) is a lattice and (B) is not. The ordered set (C) has two interesting properties: not all its segments of the same length are isomorphic (it has 3 segments isomorphic to (A) and (B)), and its Möbius function (see *Foundations I*) does not alternate in sign.

We now prove some results relating the structure of P to the numbers $B(1)$, $B(2), \cdots$.

PROPOSITION 8.3. *Let P be of full binomial type. An n segment of P is a chain if and only if $B(n) = 1$.*

The proof is obvious.

PROPOSITION 8.4. *Let L be a* lattice *of full binomial type. Every element of L is the join of atoms (that is, L is* atomic*) if and only if $A(2) > 1$.*

PROOF. If $A(2) = 1$, then a 2 segment is a chain; hence any element of L of height 2 is not the join of atoms.

Conversely, suppose L is not atomic and let y be an element of L of minimum height $n > 1$ which is not the join of atoms. Let x be an element of height $n - 2$ lying below y. Then $[x, y]$ is a chain of length 2, so $A(2) = 1$.

PROPOSITION 8.5. *Let L be a lattice of full binomial type and $[x, y]$ an n segment of L. The join of any two distinct atoms of $[x, y]$ is of height 2 if and only if*

$$A(k) = 1 + (A(2) - 1) + (A(2) - 1)^2 + \cdots + (A(2) - 1)^{k-1} \tag{8.10}$$

when $1 \leqq k \leqq n$.

PROOF. Suppose the join of any two distinct atoms of $[x, y]$ has height 2. Then the same is true for $[x, y']$, where y' is any point of $[x, y]$ of height $k \leqq n$. Now any element of $[x, y']$ of height 2 lies above $A(2)$ atoms and $\binom{A(2)}{2}$ pairs of atoms. But $[x, y']$ contains $\left[\begin{smallmatrix}k\\2\end{smallmatrix}\right]$ elements of height 2 and $\binom{A(k)}{2}$ pairs of atoms. Hence,

$$\binom{A(k)}{2} = \begin{bmatrix}k\\2\end{bmatrix}\binom{A(2)}{2}, \tag{8.11}$$

which implies $A(k) = A(k - 1)(A(2) - 1) + 1$. By induction

$$A(k) = 1 + (A(2) - 1) + (A(2) - 1)^2 + \cdots + (A(2) - 1)^{k-1}, \quad 1 \leqq k \leqq n. \tag{8.12}$$

Conversely, if two atoms of $[x, y']$ have join of height > 2, then the above argument yields

$$\binom{A(k)}{2} > \begin{bmatrix}k\\2\end{bmatrix}\binom{A(2)}{2}. \tag{8.13}$$

Consequently,

$$A(k) > 1 + (A(2) - 1) + \cdots + (A(2) - 1)^{k-1}, \tag{8.14}$$

and the proof is complete.

LEMMA 8.2. *Let L be a lattice of full binomial type such that the join of any two distinct atoms of L has height 2. Then L is modular.*

PROOF. Let x, y be two elements of L such that x and y cover $x \wedge y$. (If no such x, y exist, then L is a chain and hence modular.) Let n be the length of $[x \wedge y, x \vee y] = L'$. Then L' is a lattice of full binomial type whose invariants $B(1), B(2), \cdots B(n)$ are the same as those for L. Hence, by Proposition 8.5, the join of any two distinct atoms of L' has height 2; in particular, $x \vee y$ has height 2 and thus covers x and y. This means L is upper semimodular. Dually, if x and y are covered by $x \vee y$, then the same argument applied to the dual of $[x \wedge y, x \vee y]$ shows that L is lower semimodular. Hence, L is modular and the proof is complete.

Finally, we come to the main theorem of this subsection.

THEOREM 8.2. *Let L be a lattice of full binomial type, such that the join of any two atoms of L has height 2. Then L is isomorphic to either:*

(i) *a chain;*

(ii) *the lattice of finite subsets of a set; or to*

(iii) *the lattice of finite subspaces of a projective space.*

PROOF. Suppose L is not a chain. Then L, being of binomial type, has two distinct atoms whose join has height 2; hence, $A(2) > 1$. By Proposition 8.4, L is atomic. By Lemma 8.2, L is modular. Thus, every segment $[x, y]$ of L is a modular geometric lattice. By Birkhoff (Theorem IV–10), $[x, y]$ is the product of a Boolean algebra with projective geometries. The only such products which are of full binomial type are the single factor ones, that is, $[x, y]$ is of the type (ii) or (iii). Since every segment $[0, x]$ of L is of the type (ii) or (iii), so is L, and the proof is complete.

9. Algebras of triangular type

In this section, we investigate locally finite ordered sets P with 0 which have a reduced incidence algebra $\mathbf{R}(P, \sim)$ which is isomorphic, in a natural way, to the algebra of all upper triangular $N \times N$ matrices (possibly $N = \infty$) over the ground field of $\mathbf{R}(P, \sim)$. First we describe a class of such P. Let P be a locally finite ordered set with 0 satisfying the Jordan–Dedekind chain condition. If $[x, y]$ is a segment of P with x of height m and y of height n, we call $[x, y]$ an (m, n) segment. Suppose that for all m, n any two (m, n) segments contain the same number $B(m, n)$ of maximal chains. (By convention $B(n, n) = 1$ if P contains an element of height n.) We then call P an ordered set of *triangular type*. Geometric lattices of triangular type are considered by Edmonds, Murty, and Young [20] under a different name.

PROPOSITION 9.1. *The equivalence relation on the segments of an ordered set P of triangular type defined by $[x, y] \sim [x', y']$, if and only if $[x, y]$ and $[x', y']$ are both (m, n) segments for some m, n, gives a reduced incidence algebra $\mathbf{R}(P, \sim)$. If $f(m, n)$ denotes the value that $f \in \mathbf{R}(P, \sim)$ takes on an (m, n) segment, then the* *mapping*

$$(9.1)\qquad f \to \begin{pmatrix} \frac{f(0,0)}{B(0,0)} & \frac{f(0,1)}{B(0,1)} & \frac{f(0,2)}{B(0,2)} & \cdots \\ 0 & \frac{f(1,1)}{B(1,1)} & \frac{f(1,2)}{B(1,2)} & \cdots \\ 0 & 0 & \frac{f(2,2)}{B(2,2)} & \cdots \\ 0 & 0 & & \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

is an isomorphism of $\mathbf{R}(P, \sim)$ *onto the algebra of all upper triangular* $N \times N$ *matrices, where* N *is the height of* P *(possibly* ∞*).*

PROOF. Let $[x, y]$ be an (m, n) segment. The number of points $z \in [x, y]$ such that $[x, z]$ is an (m, m') segment and $[z, y]$ is an (m', n) segment is given by $B(m, n)/B(m, m')B(m', n)$. Thus, if f, g are constant on equivalence classes relative to $\sim$, we have

$$(9.2)\qquad (f*g)(x, y) = \sum_{m'=m}^{n} \frac{B(m, n)}{B(m, m')B(m', n)} f(m, m')g(m', n),$$

which is a function of m, n only. Hence, $\sim$ gives a reduced incidence algebra, and (9.2) is the condition for (9.1) to be an isomorphism.

The converse of Proposition 9.1 is provided by the next proposition.

PROPOSITION 9.2. *Let* P *be a locally finite ordered set with* 0 *and* $\mathbf{R}(P, \sim)$ *a reduced incidence algebra whose types can be labeled by ordered pairs* (m, n), $0 \leqq m \leqq n$, *such that whenever* (m, n) *is a type and* $0 \leqq m' \leqq n' \leqq n$, *then* (m', n') *is a type. Suppose there are numbers* $B(m, n)$ *for every type* (m, n) *such that the mapping* (9.1) *is an isomorphism of* $\mathbf{R}(P, \sim)$ *onto the algebra of all upper triangular* $N \times N$ *matrices for some* $N \leqq \infty$. *Then the following hold:*

(i) *P satisfies the Jordan–Dedekind chain condition;*

(ii) $B(n, n) = 1$ *whenever* (n, n) *is a type;*

(iii) *we can take new values of* $B(m, n)$ *preserving the isomorphism* (9.1) *such that* $B(n, n + 1) = 1$ *whenever* $(n, n + 1)$ *is a type;*

(iv) *every* (m, n) *segment of* P *contains the same number of maximal chains, and when the isomorphism* (9.1) *is normalized by* (iii), *then* $B(m, n)$ *is the number of maximal chains in an* (m, n) *segment.*

PROOF. Define $h_{m,n} \in \mathbf{R}(P, \sim)$ by

$$(9.3)\qquad h_{m,n}(x, y) = \begin{cases} 1 & \text{if } [x, y] \text{ is an } (m, n) \text{ segment,} \\ 0 & \text{otherwise.} \end{cases}$$

It follows from (9.1) that

$$(9.4)\qquad h_{m,k}h_{k,n} = \frac{B(m, n)}{B(m, k)B(k, n)} h_{m,n}.$$

Thus, if $[x, y]$ is an (m, k) segment and $[y, z]$ a (k, n) segment, then $[x, z]$ is an (m, n) segment. Conversely, if $[x, z]$ is an (m, n) segment and $m \leqq k \leqq n$, then there is a point $y \in [x, z]$ such that $[x, y]$ is an (m, k) segment and $[y, z]$ a (k, n) segment. It follows that points are (n, n) segments for some n and that two point segments are $(n, n + 1)$ segments for some n. Moreover, every maximal chain in an (m, n) segment has length $n - m$, which proves (i).

Since the identity of $\mathbf{R}(P, \sim)$ goes into the identity matrix under (9.1), we have $B(n, n) = 1$ whenever (n, n) is a type, proving (ii).

If $B(m, n)$ is replaced by

$$\frac{B(m, n)}{B(m, m + 1)B(m + 1, m + 2) \cdots B(n - 1, n)}, \tag{9.5}$$

then the isomorphism (9.1) is preserved and $B(n, n + 1)$ is replaced by 1. This proves (iii).

Hence, suppose each $B(n, n + 1) = 1$ whenever $(n, n + 1)$ is a type. Let $\eta \in \mathbf{R}(P, \sim)$ be the function which is 1 on two point segments and 0 elsewhere, that is,

$$\eta(x, y) = \begin{cases} 1 & \text{if } [x, y] \text{ is an } (n, n + 1) \text{ segment for some } n, \\ 0 & \text{otherwise.} \end{cases} \tag{9.6}$$

If $[x, y]$ is an (m, n) segment, then $\eta^{n-m}(x, y)$ is the number of maximal chains in $[x, y]$, so that this number depends only on m and n. Using (9.1), $\eta^{n-m}(x, y) = B(m, n)$, so (iv) is proved.

Propositions 9.1 and 9.2 give a characterization of ordered sets P which have a reduced incidence algebra isomorphic to the algebra of all upper triangular $N \times N$ matrices, namely P is of triangular type. If we assume P is a lattice, then some of the structure of P can be inferred from the numbers $B(m, n)$.

PROPOSITION 9.3. *Let L be a lattice of triangular type. Set $T(n) = B(n, n + 2) - 1$.*

(i) *If $T(n) \neq 0$ for every type $(n, n + 2)$, then L is atomic (that is, every element of L is the join of atoms); the converse is true if L is semimodular;*

(ii) *L is upper semimodular if and only if for all types (m, n),*

$$\frac{B(m, n)}{B(m + 1, n)} = 1 + T(m) + T(m)T(m + 1) + T(m)T(m + 1)T(m + 2) + \cdots + T(m)T(m + 1) \cdots T(n - 2); \tag{9.7}$$

(iii) *L is lower semimodular if and only if for all types (m, n),*

$$\frac{B(m, n)}{B(m, n - 1)} = 1 + T(n - 2) + T(n - 2)T(n - 3) + \cdots + T(n - 2)T(n - 3) \cdots T(m). \tag{9.8}$$

PROOF. For (i), suppose L is not atomic, and let $y \in L$ be a join irreducible of L of height $n + 2 > 1$. If x is any element of height n lying below y, then $[x, y]$ is a chain, so $T(n) = 0$. The converse will be proved after (ii) and (iii).

For (ii), L is upper semimodular if and only if whenever x and y cover $x \wedge y$, then $x \vee y$ covers x and y; that is, if and only if in every (m, n) segment, the join of any two distinct atoms has height 2. Now an (m, n) segment has $B(m, n)/B(m + 1, n) = A(m, n)$ atoms and $\binom{A(m,n)}{2}$ pairs of distinct atoms. Moreover, each element of height 2 in an (m, n) segment covers $B(m, m + 2)$ atoms, and hence $\binom{B(m,m+2)}{2}$ pairs of distinct atoms. Since an (m, n) segment has $B(m, n)/B(m, m + 2)B(m + 2, n)$ elements of height 2, we see that L is upper semimodular if and only if

$$\binom{A(m, n)}{2} = \frac{B(m, n)}{B(m, m + 2)B(m + 2, n)} \binom{B(m, m + 2)}{2} \tag{9.9}$$

for all types (m, n). Simplifying (9.9) gives

$$\begin{aligned} \frac{B(m, n)}{B(m + 1, n)} &= 1 + T(m) \frac{B(m + 1, n)}{B(m + 2, n)} \\ &= 1 + T(m)\big(1 + T(m + 1)\big) \frac{B(m + 2, n)}{B(m + 3, n)} \\ &\ \ \vdots \\ &= 1 + T(m) + T(m)T(m + 1) + \cdots \\ &\qquad + T(m)T(m + 1) \cdots T(n - 2). \end{aligned} \tag{9.10}$$

Case (iii) is the dual of (ii).

We now prove the second part of (i); that is, if L is semimodular and $T(m) = 0$ for some type $(m, m + 2)$, then L is not atomic. Say L is upper semimodular. (The dual argument works when L is lower semimodular.) We show that there is only one element of L of height $m + 1$. Suppose there are two elements of L of height $m + 1$, and let $n > m + 1$ be the height of their join. We prove by "descending induction" on k that

$$\frac{B(k, n)}{B(k, m + 1)B(m + 1, n)} = 1 \tag{9.11}$$

when $0 \leqq k \leqq m + 1$. The case $k = 0$ asserts that a $(0, n)$ segment has only one element of height $m + 1$, a contradiction.

Clearly, (9.11) holds for $k = m + 1$. Assume it holds for $k + 1$ with $0 < k + 1 \leqq m + 1$. By (ii),

$$\begin{aligned} &\frac{B(k, n)}{B(k, m + 1)B(m + 1, n)} \\ &= \frac{B(k + 1, n)}{B(k + 1, m + 1)B(m + 1, n)} \\ &\quad \cdot \frac{\big(1 + T(k) + T(k)T(k + 1) + \cdots + T(k) \cdots T(n - 2)\big)}{\big(1 + T(k) + T(k)T(k + 1) + \cdots + T(k) \cdots T(m - 1)\big)}. \end{aligned} \tag{9.12}$$

By assumption,

$$\frac{B(k+1, n)}{B(k+1, m+1)B(m+1, n)} = 1. \tag{9.13}$$

Since $T(m) = 0$,

$$1 + T(k) + \cdots + T(k) \cdots T(n-2) = 1 + T(k) + \cdots + T(k) \cdots T(m-1). \tag{9.14}$$

Hence, $B(k, n)/B(k, m + 1)B(m + 1, n) = 1$, and the proof follows.

If L is a, say, upper semimodular lattice of triangular type, then Proposition 9.3 (ii) expresses $B(m, n)$ in terms of $B(m + 1, n)$ and the $T(k)$. By iteration, we can in fact express $B(m, n)$ in terms of the $T(k)$ only, namely

$$B(m, n) = \prod_{i=0}^{n-m-2} [1 + T(m+i) + T(m+i)T(m+i+1) + \cdots + T(m+i) \cdots T(n-2)], \quad n \geqq m+2. \tag{9.15}$$

A dual formula holds for lower semimodularity.

The proof of the second part of Proposition 9.3(i) reduces the theory of semimodular lattices of triangular type to that of atomic semimodular lattices. In fact, we have the following theorems.

THEOREM 9.1. *Let L be an upper semimodular lattice of triangular type. Then there are geometric lattices (that is, upper semimodular atomic lattices of finite length) $L_1, L_2, \cdots, L_r$ of triangular type and an upper semimodular atomic lattice L_{r+1} of triangular type, such that L is isomorphic to the lattice obtained by identifying the top of L_i with the bottom of L_{i+1} for $1 \leqq i \leqq r$.*

THEOREM 9.2. *If L is a modular lattice of triangular type, then the lattices $L_1, \cdots, L_{r+1}$ of Theorem 9.1 are either Boolean algebras or projective geometries.*

Theorem 9.2 follows from the well-known structure theorem for modular geometric lattices (Birkhoff, Theorem IV–10). Any such lattice is the product of a Boolean algebra and projective geometries, and it is easily seen that this is of triangular type if and only if the product has only one factor.

EXAMPLE 9.1. *Chains.* Discrete chains with 0 are modular lattices of triangular type. Each lattice L_i of Theorem 9.1 consists of two points. Here $B(m, n) = 1$ whenever (m, n) is a type, or equivalently $T(n) = 0$ whenever $(n, n + 2)$ is a type.

EXAMPLE 9.2. *Projective geometries.* The lattice of finite subspaces of a projective geometry with $q + 1$ points on a line is a modular lattice of triangular type with $T(n) = q$ whenever $(n, n + 2)$ is a type.

EXAMPLE 9.3. *Boolean algebras.* These are modular lattices of triangular type with $T(n) = 1$ whenever $(n, n + 2)$ is a type.

Examples 9.1, 9.2, and 9.3 all have the property that $B(m, n)$ depends only on $n - m$ when (m, n) is a type. Such ordered sets are of full binomial type defined in the previous section. It is proved there that a semimodular lattice of full binomial type is one of Example 9.1, 9.2, 9.3.

EXAMPLE 9.4. *Affine geometries.* The lattice of finite affine subspaces of an affine space with q points on a line is an upper semimodular (but not modular unless there is only one line) lattice of triangular type with $T(0) = q - 1$, $T(n) = q$, $n > 0$, when $(n, n + 2)$ is a type.

EXAMPLE 9.5. Various ways of putting together and taking apart ordered sets of triangular type give other ordered sets of triangular type. The simplest examples are (a) segments, (b) identifying all elements above or below a certain level to a single element (called upper or lower truncation), and (c) identifying the top of an ordered set of triangular type with 1 with the bottom of an ordered set of triangular type. All of these operations except lower truncation preserve upper semimodularity.

EXAMPLE 9.6. *Block designs.* Let L be a geometric lattice of triangular type of height 3. If we regard the atoms of L as objects and the co-atoms as blocks containing the atoms they cover, then L determines a balanced incomplete block design with parameters

$$\begin{aligned} v &= 1 + T(0) + T(0)T(1) = 1 - B(1, 3) + B(0, 2)B(1, 3), \\ b &= \frac{B(1, 3)}{B(0, 2)} v, \\ k &= B(0, 2), \\ r &= B(1, 3), \\ \lambda &= 1. \end{aligned} \tag{9.16}$$

Conversely, any balanced incomplete blocks design with $\lambda = 1$ determines a geometric lattice of triangular type of height 3. Thus, geometric lattices of triangular type can be regarded as generalizations of $\lambda = 1$ block designs. Proposition 9.3 is then the generalization of the well-known relations $bk = vr$ and $r(k - 1) = v - 1$ holding in any block design with $\lambda = 1$ (see Hall, [30], Chapter 10).

EXAMPLE 9.7. Miscellaneous other examples and a method for classifying them, can be found in the paper of Edmonds, Murty, and Young [20].

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