ON SOME STOCHASTIC PROBLEMS OF RELIABILITY THEORY

YU. K. BELYAEV, B. V. GNEDENKO and A. D. SOLOVIEV
MOSCOW UNIVERSITY

1. Introduction

The new direction in scientific investigations, associated with modern tendencies in engineering and called "reliability theory," has imposed enormous demands on the theory of probability and mathematical statistics. Naturally, in the present short paper we are forced to restrict our selection to just a small number of problems.

Without the ideas and concepts of the theory of probability, even the fundamental concepts of reliability theory cannot be clearly defined. Therefore, the theory of probability is not only a computational apparatus, but also a methodological basis for reliability theory. An inadequate perception of the nature of those phenomena which must be encountered in problems of reliability theory often leads engineers to certain misunderstandings. An attempt for greater clarity naturally obliges us to turn to the elaboration of general initial concepts.

We understand the reliability of an object to be the capacity to retain the properties determining its quality unscathed. All possible states of the object which are equivalent from the viewpoint of its reliability, will be combined in the class x. The set of all possible classes x generates a phase space of the states of the object E.

For example, if the object consists of n units each of which may be either in the in-service or out-of-service state, the phase space E is generated by points $(\epsilon_1, \epsilon_2, \cdots, \epsilon_n)$ whose coordinates may take only two values: $\epsilon_i = 0$ if the i-th unit is in service, and $\epsilon_i = 1$ if the i-th block is not in service. Under the assumption that the out-of-service blocks have been repaired and work serviceably during random time intervals with an exponential distribution, the state of the object is described entirely by the fact that the object is at some point $(\epsilon_1, \epsilon_2, \cdots, \epsilon_n)$ of the phase space E at the time t. If the out-of-service units are subject to repair which lasts a time distributed according to $F(t) \neq 1 - e^{-\mu t}$, we must also indicate the time τ_i during which the failing unit with number i is repaired, in order to describe the state of the system. Here we assume that each unit goes to repair immediately after going out of service. Therefore, in this case we should examine a more complex phase space E consisting of points of the form $(\epsilon_1, \tau_1, \cdots, \epsilon_n, \tau_n)$. If $\epsilon_i = 0$, then τ_i is also assumed to be zero. If $\epsilon_1 = 1$, then $0 < \tau_i < \infty$.

Various changes associated with wear (in mechanical systems), or with aging (in radio electronic units), occur in the component parts of an apparatus as time passes. Hence, if the state of the object were x_1 at time t_1, the state x_2 of the object at time $t_2 > t_1$ might not coincide with x_1. If x_t denotes the state of the object at time t, the sequence of states x_t for $t > 0$ may be considered a random process. The establishment of the stochastic structure of the process x_t, describing the evolution of the unit under study, is one of the most important problems. In substance, this is one of the fundamental problems of the statistics of stochastic processes.

After the phase space E has been determined and the stochastic process x_t has been given, the problem of selecting the numerical characteristics of reliability arises. Such a selection may not be unique since it depends on the purpose for which the unit has been manufactured. Sometimes one single number will yield a thoroughly inadequate representation of the reliability, and several numerical indices of reliability are necessary. Any numerical characteristic of reliability may be considered the mathematical expectation of some functional Φ defined on the trajectory of the process x_t:

$$\alpha = M\Phi[x_t]. \tag{1}$$

Such an approach actually means that a certain weight is attributed to each trajectory in the trajectory space of x_t.

Thus the mathematical model describing the behavior of an object consists of three elements: the phase space E, the stochastic process $x_t \in E$, and the system of functionals $\Phi_i[x_t]$.

The concept of failure is defined as follows. In the space of states E some subset $Q \subset E$ is specified. Loss of operational capability of the item is represented by a trajectory of x_t falling into this subset. The form of the set Q is determined by starting from specific conditions. When the trajectory x_t falls into the set Q, a failure is said to occur. Failures are defined as sudden (catastrophic), gradual (wearout), and intermittent. If the change in trajectory should occur as a jump, with the trajectory dropping into the domain Q, the failure is called sudden. If the trajectory should change gradually, as a result of changes in the values of the parameters (gradual wear of the surfaces of rubbing parts, aging of radio electronic units), the failure will be gradual. If the trajectory, having fallen into the set Q, should be capable of leaving this set (without relying on the repair unit), the failures will be called intermittent.

In such an approach all known numerical indices of reliability are easily obtained. Define the functional Φ_1 as follows: $\Phi_1[x_t] = 1$ when the trajectory x_s is not in the set Q for any $s < t$, and $\Phi_1[x_t] = 0$ otherwise. Evidently,

$$\alpha_1 = M\Phi_1[x_t] = R(t) \tag{2}$$

where $R(t)$ is the probability of faultless operation of the unit for the length of time t. Usually $R(t)$ is called the reliability of the unit.

Now, let the functional Φ_2 be the length of the time interval up to the time of first entry into the set Q. The constant α_2 is a useful numerical characteristic of reliability, the mean time of faultless operation.

The simultaneous examination of a large set of mathematical models of reliability theory $\{\epsilon_\delta, x_{t,\delta}, \Phi_\delta\}$, $\delta \in \Delta$ is characteristic of a number of problems of reliability theory. The choice of the optimum model in the sense of the numerical characteristic Φ_δ is a fundamental problem here. Included in such a scheme are problems of synthesis of reliable systems, particularly, the selection of the optimum number of standby blocks. In this latter case, a mathematical model corresponds to each set of numbers of standby blocks.

The proposed system of exposition is customary for modern probability theory. We consider it natural and necessary in reliability theory.

Important questions are associated with the class of random processes which is of particular interest in reliability theory. First of all, it is clear from the above that a sufficiently complete approach requires the analysis of general stochastic processes whose theory is expounded, for example, in Doob's book.

At present, in many monographs and journal articles devoted to reliability theory, the customary approach is restricted to the examination of the case in which all the distributions involved are assumed to be exponential. This is unnecessarily restrictive and not in keeping with the requirements of the field. Actually, the restriction to this simplest class of homogeneous Markov processes radically oversimplifies the true situation in the overwhelming majority of problems.

This fact was noted long ago in connection with queueing theory problems. Imbedded Markov chains (in the terminology of D. Kendall) were examined by A. Ya. Khinchin [2]. Later W. Smith [3] introduced semi-Markov processes. Stochastic Markov processes which are specific for reliability and queueing theory problems, were studied by D. Cox [4], B. A. Sevastyanov [5], and Yu. K. Belyaev [6].

Many years ago A. N. Kolmogorov proposed the examination of a class of so-called Markov processes with discrete occurrence of events. The behavior of the trajectory of these processes in the phase space ϵ is described as follows. In some parts the trajectory is defined in a deterministic manner (as the solution of differential equations, say). Then at random times both the location in phase space and the parameters of the governing motion change by a jump. The motion proceeds deterministically up to a new occurrence of an event, and so on.

2. Approximate formulas for a general stochastic model

In this sketch, we consider a general mathematical model of a system with standbys. The system consists of m different kinds of blocks $b_1, \cdots, b_m$. There are k_i blocks of the i-th kind, of which ℓ_i operate, ℓ_i' are out of service, and $\ell_i'' = k_i - (\ell_i + \ell_i')$ are standbys. The system has n repair units $R_1, R_2, \cdots, R_n$. The repair unit R_i may repair blocks of $b_{1,j_1}, \cdots, b_{i,j_i}$ types. Moreover, a definite priority system for the orders of repairs is given when out-of-service blocks accumulate and there are queues for the repair units. Blocks of the i-th kind

$i = 1, \cdots, m$ may get out of order after random time intervals which are distributed according to the law $F_i(t)$, when the block of type B_i operates, and according to the law $\tilde{F}_i(t)$ when it is in standby. The time to repair the j-th block by the i-th repair unit is also considered stochastic with the distribution law $F_{i,j}(t)$.

In such an approach the phase space $\mathcal{E}$ consists of the points

$$\{\ell_1, z_{i,j_1}, \cdots, z_{i,j_i}, \ell'_i, z'_{i,j'_1}, \cdots, z'_{i,j_i}, \ell''_i, z''_{i,j_1}, \cdots, z''_{i,j''_i}\}, \tag{3}$$

$i = 1, \cdots, m; j, j', j'' = 1, 2, \cdots, \sum_{i=1}^{m} k_i$. For blocks of the i-th type, ℓ_i is the number in service, ℓ'_i is the number out of service, ℓ''_i is the number in standby; $z_{i,j}, z'_{i,j}, z''_{i,j}$ are quantities numerically equal to the time during which the blocks are, respectively, in service and operating, in service and in standby, out of service and being repaired. If it is assumed that the in-service and in-repair times are mutually independent random variables, then the process of the change of state in the phase space $\mathcal{E}$ will be Markovian with discrete occurrences of events.

Actually, a change of state during the time Δt may consist only of translations along half-lines. In this case all the positive coordinates $z_{i,j}, z'_{i,j}, z''_{i,j}$ are increased by Δt during the time $(t, t + \Delta t)$. Also, jumps may lead into states which differ from the states preceding the jump, in that one of the coordinates $z_{i,j}, z'_{i,j}, z''_{i,j}$ either vanishes (which corresponds, say, to failure or repair), or starts to increase from zero. Thus, the appropriate numbers $\ell_i, \ell'_i, \ell''_i$ change by one.

One may derive integro-differential equations for the stochastic behavior of this system. However, a solution in analytic form has apparently been obtained, so far, in only two cases. In the first case, there is an unlimited number of repair units; in the second, only a single repair unit.

The solution of the corresponding systems of integro-differential equations has successfully been found in simple explicit form in a number of works devoted to the study of stationary distributions. The most general model was considered by I. N. Kovalenko [7].

Let us assume that the unit consists of S groups of elements; the j-th group containing N_j elements. The elements may fail according to the following stochastic law. Let the event $(k_1, \cdots, k_s, t)$ be that k_j elements of the j-th group are out of order at time t. Then, the hazard rate of an element of the j-th group equals $\lambda_j(k_1, \cdots, k_s)$. The probability of failure of two or more elements in the time Δt is $o(\Delta t)$. It is assumed that the number of repair units is $N_1 + \cdots + N_s$, that is, each element which has gone out of order, immediately starts to recover. The recovery lasts a random time with the distribution law $\Phi_j(x)$, with the mean $\tau_j < \infty$. Let $p(k_1, \cdots, k_s, t)$ be the probability of being in the $(k_1, \cdots, k_s, t)$ state at the time t. Let $p_{k_1, \cdots, k_s}$ denote the limit $\lim p(k_1, \cdots, k_s, t)$ when $t \to \infty$.

THEOREM 1. *In order that for all $k_1, \cdots, k_s$ there exist probabilities $p_{k_1, \cdots, k_s}$,* *defined by the set of parameters* $\lambda_j(k_1, \cdots, k_s)$ *and* τ_j, *and independent of the form of* $\Phi_j(x)$, *it is necessary and sufficient that the following condition hold: for every set* $(k_1, \cdots, k_s)$ *and for every pair of sets* $(i_1, \cdots, i_{k_1+\cdots+k_s})$, $(i'_1, \cdots, i'_{k_1+\cdots+k_s})$ *containing exactly* k_ν *elements equal to* ν, $1 \leqq \nu \leqq s$, *the equality*

$$\prod_{\ell=1}^{k_1+\cdots+k_s} \lambda_{i_\ell}\left(\sum_{m=1}^{\ell-1} \delta_{1,i_m}, \cdots, \sum_{m=1}^{\ell-1} \delta_{s,i_m}\right) = \prod_{\ell=1}^{k_1+\cdots+k_s} \lambda_{i'_\ell}\left(\sum_{m=1}^{\ell-1} \delta_{1,i'_m}, \cdots, \sum_{m=1}^{\ell-1} \delta_{s,i'_m}\right) \tag{4}$$

is satisfied with $\delta_{i,j} = 1$ *for* $i = j$, $\delta_{i,j} = 0$, $i \neq j$. *In this case the* $p_{k_1 \cdots k_s}$ *are determined by the formula.*

$$p_{k_1 \cdots k_s} = \frac{\tau_1^{k_1}}{k_1!} \cdots \frac{\tau_s^{k_s}}{k_s!} \prod_{\ell=1}^{k_1+\cdots+k_s} \lambda_{i_\ell}\left(\sum_{m=1}^{\ell-1} \delta_{1,i_m}, \cdots, \sum_{m=1}^{\ell-1} \delta_{s,i_m}\right) p_{0,0\cdots 0}, \tag{5}$$

where $(i_1, \cdots, i_{k_1+\cdots+k_s})$ *is any set for which exactly* $k\nu$ *elements equal* ν, $1 \leqq \nu \leqq s$.

Somewhat earlier, T. P. Maryanovich [8], [9] considered specific problems with standbys. He obtained stationary probabilities whose form was determined entirely by the mean repair times τ_i.

In order to solve a number of reliability problems with one repair unit, a special class of processes with discrete occurrence of events, the linear Markov processes [6], may turn out to be useful. The stochastic process x_t is called a linear Markov process if the phase space $\mathcal{E}$ consists of groups of points $\mathcal{E}_0$ and sets of half-lines $\{\ell, j, z)$, $z > 0$, $(\ell, j) \in \mathcal{E}_\ell$. If $x_t = i \in \mathcal{E}_0$ at time t, then during the time $(t, t + \Delta t)$ the trajectory will go over into the state $j \in \mathcal{E}_\ell$ with probability $\lambda_{i,j}\Delta t + {}_o(\Delta t)$, or into the beginning of the half-line (ℓ, j, z) with probability $\lambda_{i,j}(\ell, j)\, \Delta t + o(\Delta t)$ if $j \in \mathcal{E}_\ell$. If $x_t = (\ell, j, z)$, $j \notin \Delta_0$, then jumps of two kinds may occur mutually independently.

A jump of the first kind will occur when the process skips from the state $(\ell, i, z) \in \mathcal{E}_\ell$ into the state $(\ell, j, z + \Delta t)$ during the time with $(t, t + \Delta t)$ with probability $\lambda_{i,j}\Delta t + o(\Delta t)$. A jump of the second kind occurs when the process skips from the state (ℓ, i, z) either into the state $j \in \mathcal{E}_0$ or into the beginning of the half-line $(\ell', j, 0)$ with probability

$$\mu_{i,j} \frac{F_\ell(z + \Delta t) - F_\ell(x)}{1 - F_\ell(z)}. \tag{6}$$

Some general properties and examples of the utilization of this class of processes are presented in [6]. Let us note that processes of this class are called semi-Markov processes.

The study of a class of Markov processes with discrete intervention by an event is important both for the development of general reliability theory and queueing theory.

A number of problems in reliability theory, in both formulation and in methods of solution, are similar to those occurring in queueing theory. Sometimes this difference is purely verbal, and the comparison of problems of the two areas may be accomplished with a small dictionary.

RELIABILITY THEORY	QUEUEING THEORY
Failure	Call, Demand
Time to repair	Duration of service, conversation
Repair unit	Servicing apparatus

The occurrences of events of a definite kind, unit failures in reliability theory, and service demands in queueing theory, play a central part in both queueing theory and reliability theory. The question is, what are the characteristics of the distribution in time of demands for service? Many investigations have been devoted to this question as far back as the time of the classical work of K. Erlang. In particular, the appropriate results of A. Ya. Khinchin are well known from his book [2]. Within recent years we have obtained new results in this area and we present here a formulation of some of them.

A system of random variables $\eta(\Delta)$, given for intervals Δ, is called a random flow if, $\eta(\Delta) = 0, 1, 2, \cdots$.

$$\eta(\Delta_1 \cup \Delta_2) = \eta(\Delta_1) + \eta(\Delta_2), \qquad \text{for } \Delta_1 \cap \Delta_2 = \phi, \tag{7}$$

Let $\eta_n(\Delta) = \sum_{r=1}^{k_n} \eta_{n,r}(\Delta)$, where $\eta_{n,r}(\Delta)$ are mutually independent random flows. Let us say that the sequence of flows $\eta_n(\Delta)$ converges to the flow $\eta(\Delta)$ as $n \to \infty$ if for any set of intervals $\Delta_1, \cdots, \Delta_k$ the distribution functions of the vectors $\{\eta_n(\Delta_1), \cdots, \eta_n(\Delta_k)\}$ converge to the distribution function of the vector $\{\eta(\Delta_1), \cdots, \eta(\Delta_k)\}$ at points of continuity. Let $\Lambda(\Delta)$ be a measure on the line. We call the flow $\eta(\Delta)$ Poisson with fundamental measure $\Lambda(\Delta)$ if the values of $\eta(\Delta)$ are mutually independent for nonintersecting intervals and

$$P\{\eta(\Delta) = k\} = \frac{[\Lambda(\Delta)]^k}{k!} e^{-\Lambda(\Delta)}. \tag{8}$$

Let us introduce the following notations:

$$\begin{aligned} P_{n,r}(k; t, s) &= P\{\eta_{n,r}(s, t) = k\},\ s < t,\ k = 0, 1, \cdots \\ \Lambda_n(t, s) &= \sum_{r=1}^{k_n} p_{n,r}(1; t, s) \\ B_n(t, s) &= \sum_{r=1}^{k_n} [1 - p_{n,r}(0; t, s) - p_{n,r}(1; t, s)]. \end{aligned} \tag{9}$$

The flows $\eta_{n,r}(\Delta)$ are called infinitesimal if for any fixed interval $\Delta = (0, t)$,

$$\lim_{n\to\infty} \max_{1 \leq r \leq k_n} [1 - p_{n,r}(0; t, s)] = 0. \tag{10}$$

In other words, the flows $\eta_{n,r}(\Delta)$ are infinitesimal if for any $\epsilon > 0$ and an arbitrary but fixed interval $\Delta = (0, t)$ it is possible to give an n such that for all r, $P\{\eta_{n,r}(\Delta) > 0\} > \epsilon$.

B. I. Grigelionis [10] proved the following assertion.

THEOREM 2. *For the convergence of sums of mutually independent infinitesimal* *random flows* $\eta_{n,r}(\Delta)$ *to a Poisson flow with fundamental measure* $\Lambda(\Delta)$, *it is necessary and sufficient that for any fixed* s *and* t, $(s < t)$, *there be satisfied the relationships*

$$\text{(a)} \qquad \lim_{n\to\infty} \Lambda_n(t, s) = \Lambda[(t, s)],$$

$$\text{(b)} \qquad \lim_{n\to\infty} B_n(t, 0) = 0.$$

The proof of the necessity of the conditions of the theorem is a simple corollary of a theorem of B. V. Gnedenko and Marcinkiewicz (see [17], p. 133). The proof of the sufficiency is also based on this theorem, but here one must establish the asymptotic mutual independence of the values of the total flow $\eta_n(\Delta)$ on nonintersecting time intervals.

Several earlier results are contained in this theorem, including the theorems of A. Ya. Khinchin [2] and G. A. Ososkov [12].

It is natural to pose the question of the rapidity of convergence of the sum to the limiting process. B. I. Grigelionis [10] also obtained very general results in this area. In a similar vein, for renewal processes which are constructed as sequences of sums of identically distributed mutually independent random variables, an expansion has been found in powers of n^{-1} on the basis of Charlier series of type B [14].

If the beginning time of failure is considered as time of occurrence, in a random manner, of a change in the value of the parameter x_t, we then arrive at a classical problem of studying the number of level intersections by a random process. Yu. K. Belyaev has obtained new results in this area. It turned out that, under very broad assumptions, the k-th factorial moment of the number $\eta(\Delta, a_t)$ of up-crossings of the continuously differentiable function a_t by a nonstationary continuously differentiable random process x_t is given by the formula

$$\begin{aligned}(11)\quad J_{(k)} &= M\eta(\Delta, a_t)[\eta(\Delta, a_t) - 1] \cdots [\eta(\Delta, a_t) - k + 1] \\ &= \int_\Gamma M\left\{\prod_{i=1}^{k} (\dot{x}_t - \dot{a}_t)^+ | x_{t_i} = a_{t_i}, i = 1, \cdots, k\right\} p_{t_1\cdots t_k}(a_{t_1},\dots, a_{t_k})\, dt_1 \cdots dt_k,\end{aligned}$$

where $\Gamma = \{(t_1, \cdots t_k) : t_i \neq t_j, t_i \in \Delta; \quad i, j = 1, \cdots, k\}$, $\dot{x}_t = (dx_t/dt)$, and $p_{t_1\cdots t_k}(a_{t_1}, \cdots, a_{t_k})$ is the value of the probability density of $x_{t_1}, \cdots, x_{t_k}$ at the point $(a_{t_1}, \cdots, a_{t_k})$.

If x_t is a Gaussian k-times differentiable stochastic process, then the k-th factorial moment $J_{(k)}$ is finite. The investigation of the asymptotic behavior of $J_{(k)}$ when $a_t \equiv a \uparrow \infty$ and the length of the interval $|\Delta| \uparrow \infty$ through substitution of the accompanying Gaussian processes $\tilde{x}_t$ obtained from x_t by curtailment of the spectrum, has shown that under very general assumptions, the process generated by the intersections of a high level by a stationary Gaussian process, converges to a Poisson process.

A wide range of problems is associated with standbys [15], [16]. By "standby" we understand any redundancy in the system, such that the system functions normally even in the case of failure of some of its elements. In a narrower sense, standby is understood to be used when several identical units are associated with certain elements, or parts, of a system in such a way that basic elements are replaced in sequence in case of failure.

Depending on the states of the standbys prior to insertion in operation, three kinds of standbys are distinguished: hot, cold, and warm. In the case of a hot standby, all the elements, fundamental and reserve, are in the same regime, and consequently, may fail with the same probability. It is also assumed that the failure of some of the elements does not change the failure probabilities of the other elements. For a cold standby it is assumed that an element in reserve does not age and cannot fail. The term warm standby refers to the intermediate case when the elements in reserve may fail, but with a lower probability.

The probability of faultless operation of a standby group is expressed by explicit formulas in these cases; however, many of them are not convenient in practice. Hence, the problem arises of finding approximate formulas and estimates for the different reliability characteristics of the standby group. Thus, in the case of a hot standby the mean lifetime T_n of a standby group of n elements is expressed by the formula

$$T_n = \int_0^\infty [1 - q^n(t)]\, dt \tag{12}$$

where $q(t)$ is the probability of failure of one element.

If it is assumed that tangents to the graphs of $q(t)$ and $\ln (1/1 - q(t))$, at $t = T_n$ lie below these graphs, then T_n satisfies the inequality

$$\frac{0.56}{n+1} \leq \frac{e^{-C}}{n+1} \leqq 1 - q(T_n) \leqq \frac{1}{n+1} \tag{13}$$

(C is the Euler constant). It follows that the mean time T_n may be estimated with sufficient accuracy by means of the formula $q(T_n) = n/(n+1)$.

For a cold standby the probability of failure of a standby group of n elements $Q_n(t)$ is expressed by using several convolutions. If it is assumed that the elements are aging, that is, if $\lambda(t) = q'(t)/(1 - q(t))$ increases monotonely, then one obtains the simple estimate

$$Q_n(t) \leqq 1 - \left[1 + \Lambda(t) + \tfrac{1}{2}\Lambda^2(t) + \cdots + \frac{1}{(n-1)!}\Lambda^{n-1}(t)\right]e^{-\Lambda(t)} \tag{14}$$

with $\Lambda(t) = \ln (1/1 - q(t))$. In particular, this estimate is convenient in that the right side depends only on the value of the function at the time t.

If the lifetime of the elements in the standby group is subject to an exponential law, the flow of failures of the elements in the group is described in almost all cases by a pure death process:

$$P_k'(t) = \lambda_{k-1}P_{k-1}(t) - \lambda_k P_k(t), \qquad k = 0, 1, 2, \cdots \tag{15}$$

where $P_k(t)$ is the probability that k failures will occur at time t.

A very simple estimate

(16) $$\frac{\lambda_0 \cdots \lambda_{k-1}}{k!} t^k \left[1 - \frac{\lambda_0 + \cdots + \lambda_k}{k+1}\right] \leqq P_k(t) \leqq \frac{\lambda_0 \cdots \lambda_{k-1}}{k!} t^k$$

may be obtained for the probability $P_k(t)$; from which follows that the approximate formula

(17) $$P_k(t) \approx \frac{\lambda_0\lambda_1 \cdots \lambda_{k-1}}{k!} t$$

whose relative error does not exceed $(\lambda_0 + \cdots + \lambda_k/k + 1)t$ is valid. This formula is convenient to estimate the reliability of a standby group when the elements are highly reliable.

In a redundant system we may have standbys for either the individual elements, or for blocks or for the system as a whole. Reliability of the system always drops when the unit size of standby is enlarged. Let us note first that it is sufficient to prove this for the case in which there are two elements and one standby element for each, and in which the reliability of the standby elements is generally different from the reliability of the corresponding fundamental elements.

Let us compute the reliability of such a group of four elements for two cases: group standbys and individual standbys. For brevity, we shall limit ourselves to the case of hot standbys.

Let $\tau_1, \tau_1', \tau_2, \tau_2'$ be the random lifetimes of our elements, and let T_1 and T_2 be the random lifetimes of the whole group for the individual and group standbys, respectively. Then

(18) $$T_1 = \min\,[\max\,(\tau_1, \tau_1'), \max\,(\tau_2, \tau_2')],$$

and

(19) $$T_2 = \max\,[\min\,(\tau_1, \tau_2), \min\,(\tau_1', \tau_2')].$$

However,

(20) $$T_2 \leqq \max\,(\tau_1', \tau_1), \qquad T_2 \leqq \max\,(\tau_2, \tau_2'),$$

and therefore,

(21) $$T_2 \leqq \min\,[\max\,(\tau_1, \tau_1'), \max\,(\tau_2, \tau_2')].$$

Hence, the individual standby is always more reliable.

If it is assumed that the lifetime and the repair time of all the elements in a system are subject to an exponential law, and the elements are independent, the state of such a system is always described by a homogeneous Markov process with a finite number of states, where the number of states in the general case is 2^n. Hence, even for a system with a small number of elements, such an approach to the description of the system operation is useless because of insuperable computational difficulties. However, in a number of important cases the transition probabilities are independent of the particular elements which failed at a given time, but depend only on the number of failing elements.

Under these conditions the behavior of the system is described by a birth and death process. The probabilities of states of this process are found from the system of equations

$$p'_k = \lambda_{k-1}p_{k-1} - (\lambda_k + \mu_k)p_k + \mu_{k+1}p_{k+1} \tag{22}$$

where $p_k(t)$ is the probability that k elements are out of service at time t.

The following kind of problem often occurs in reliability theory: to find the probability that our process will never exceed the level m, $(m < n)$, during the length of time t. Let $\Pi_m(t)$ denote this probability. The following assertions may be proved by algebraic methods: let

$$T_m = \sum_{k=0}^{m} \frac{\sum_{s=0}^{k} \theta_s}{\lambda_k \theta_k} \quad \text{and} \quad \theta_k = \frac{\lambda_0 \lambda_1 \cdots \lambda_{k-1}}{\mu_1 \mu_2 \cdots \mu_k}; \tag{23}$$

then T_m is the mean time to the first excursion to the level $(m + 1)$, under the condition that the system was in the state at the initial instant (the latter is inessential, however). Then in order for $\lim \Pi_m(T_m x) = e^{-x}$, it is necessary and sufficient that

$$\lim a_m = \lim \sum_{k=0}^{m} \frac{\sum_{s=0}^{k} (T_s - T_{s-1})\theta_s}{\lambda_k \theta_k} = 0. \tag{24}$$

Here both the level m and the parameters λ_k and μ_k may change arbitrarily in this passage to the limit, assuming only that $a_m \to 0$. It can be shown that

$$\sup_{0 \leqq t \leqq \infty} |\Pi_m(t) - e^{-t/T_m}| < C \cdot a_m. \tag{25}$$

A rough estimate yields the value $C = 10$ for C. Thus, the approximate equality $\Pi_m(t) \approx e^{-t/T_m}$ is valid with an error of the order of a_m.

Corrections which take into account the principal part of the error may be introduced into this approximate formula. Then we arrive at the new approximate formula

$$\Pi_m(t) \approx \exp\left[-\frac{(t/T_m) - a_m}{1 - a_m}\right] \tag{26}$$

whose error is of the order of a_m^2.

In classical mathematical statistics the original statistical data have the form of samples $(x_1, \cdots, x_N)$, all of whose components are mutually independent, identically distributed random variables. In reliability theory the failure times t_i play the part of statistical data. However, in the majority of cases it turns out to be advantageous to cease testing long in advance of failure of all the elements. It is sometimes necessary to lengthen or alternatively, to shorten the test program. Hence, the observed random times t_i will not form mutually independent values. Even in the case of single plans, various cases are possible. To distinguish them we shall use the notation in which the first letter indicates the number of elements undergoing test. The second letter is Б if the failing elements are not replaced, and B otherwise. The third component of the notation describes the cutting-off of the test. It is T if the tests are cut off at time T and r if they are cut off at the time of the r-th failure. It is (r, T) if the tests are cut off at the time of either the r-th failure or at time T, when $t_r > T$. Hence, the plan $[N, Б(r, T)]$ means that N elements are tested, the failures are not replaced, the tests are cut off at time $t^* = \min(t_r, T)$, where t_r is the time of the r-th failure. Also possible is a plan in which the time of stopping the test is determined by the value of the total test time $S(t) = \int_0^t N(s)\,ds$, where $N(s)$ is the number of elements tested in the time s. Many problems arise in the processing of the statistical data obtained as a result of testing according to different plans. For each plan it is necessary to give a method of construction for point and interval estimates of the parameters and to give criteria for testing the various hypotheses.

In conclusion, let us note the problem of constructing confidence bounds for a function $f(\theta)$ of unknown values of the parameters θ. The presence of a large number of unknown components in the vector $\theta = (\theta_1, \cdots, \theta_m)$ is characteristic of problems of reliability theory. An analysis of the problem of a series-parallel circuit leads to the necessity of constructing lower confidence limits for the function

$$R = \exp\left\{-\sum_{i=1}^{m}\sum_{j=1}^{l_i} a_{i,j}\,\lambda_i^{n_{i,j}}\right\} \tag{27}$$

where $a_{i,j} > 0$, $n_{i,j}$ are integers, and $\lambda_i > 0$ are unknown parameters of the Poisson variables ζ_i, for which the values $\xi_i = d_i$ are known. In this case the confidence limit is given in conformity with the following theorem.

THEOREM. *A lower confidence bound with a coefficient of confidence not less than γ is given by*

$$\underline{R} = \exp\left\{-\max_{(\lambda_1,\cdots,\lambda_m)\in G}\left[\sum_{i=1}^{m}\sum_{j=1}^{l_i} a_{i,j}\,\lambda_i^{n_{i,j}}\right]\right\} \tag{28}$$

where

$$G = \left\{(\lambda_1, \cdots, \lambda_m) : \sum_{(k_1,\cdots,k_m)\in D_i} \frac{\lambda_1^{k_1}}{k_1!}\cdots\frac{\lambda_m^{k_m}}{k_m!}\,e^{-(\lambda_1+\cdots+\lambda_m)} \geqq 1-\gamma\right\} \tag{29}$$

and the set D_i consists of all possible sets of integers $(k_1, \cdots, k_m)$ such that

$$\begin{aligned}\sum_{i=1}^{m}\sum_{j=1}^{l_i} a_{i,j}\,k_i(k_i-1)\cdots(k_i-n_{i,j}+1)&\\ \leqq \sum_{i=1}^{m}\sum_{j=1}^{l_i} a_{i,j}&d_i(d_i-1)\cdots(d_i-n_{i,j}+1).\end{aligned} \tag{30}$$

In the particular case where $d_i < n_{i,j}$, the lower confidence limit is given by the formula

$$\underline{R} = \exp\left\{\max_{1\leqq i\leqq m}\left[\sum_{j=1}^{l_i} a_{i,j}\,\Delta_{1-\gamma}^{n_{i,j}}(d_i)\right]\right\} \tag{31}$$

where $\Delta_\alpha(d)$ is the solution of the transcendental equation

$$\sum_{k=0}^{d}\frac{[\Delta_\alpha(d)]^k}{k!}\,e^{-\Delta_\alpha(d)} = \alpha. \tag{32}$$

REFERENCES

[1] J. L. Doob, *Stochastic Processes*, New York, Wiley, 1953.

[2] A. Ya. Khinchin, "Mathematical methods in queueing theory," *Trudy Mat. Inst. Steklov*, Vol. 49 (1955), p. 122. (English translation, Pergamon Press.)

[3] W. L. Smith, "Renewal theory and its ramifications," *J. Roy. Statist. Soc., Ser. B*, Vol. 20 (1958), pp. 243–284.

[4] D. R. Cox, "The analysis of non-Markovian stochastic processes by inclusion of supplementary variables," *Proc. Cambridge Philos. Soc.*, Vol. 51, Part 3 (1955), pp. 433–441.

[5] B. A. Sevastyanov, "Ergodic theorem for Markov processes and its application to telephone systems with delays," *Teor. Verojatnost. i Primenen.*, Vol. 2 (1957), pp. 106–116.

[6] Yu. K. Belyaev, "Linear Markov processes and their application to problems of reliability theory," *Transactions of the All-Union Conference on Probability Theory*, Vilnius (1962), pp. 309–323.

[7] I. N. Kovalenko, "On conditions of independence of probabilities of states of a service system from the kind of service-time distribution," *Problems of Information Transmission*, Vol. 11 (1961), pp. 147–151.

[8] T. P. Maryanovich, "Reliability of a system in the presence of standbys," *Dopovidi Akad. Nauk Ukrain. RSR*, No. 7 (1961), pp. 850–853.

[9] ———, "Reliability of a system with mixed standby," *Dopovidi Akad. Nauk Ukrain. RSR*, No. 8 (1961), pp. 994–997.

[10] B. I. Grigelionis, "On the convergence of sums of random processes," *Teor. Verojatnost. i Primenen.*, Vol. 2 (1963), pp. 189–194.

[11] B. V. Gnedenko, *A Course in Probability Theory*, Fizmatgiz, 1962. (English translation, Chelsea Publ. Co.)

[12] G. A. Ososkov, "A limit theorem for flows of homogeneous events," *Teor. Verojatnost. i Primenen.*, Vol. 1 (1956), pp. 274–282.

[13] B. I. Grigelionis, "On the asymptotic expansion of the remainder term in the case of convergence to a Poisson law," *Liet. Mat. Sb.*, Vol. 1 (1962), pp. 35–48.

[14] P. Franken, "Refinement of the limit theorem for superposition of independent renewal processes," *Teor. Verojatnost. i Primenen.*, Vol. 8 (1963), pp. 341–349.

[15] A. D. Soloviev, "Reserve without repair," *Kibernetika na Sluzhbu Kommunizmu* (*Cybernetics at the Service of Communism*), Energiia Press, Vol. 2 (1964), pp. 83–121.

[16] ———, "On the determination of reserves of multiple operation systems," *Izv. Akad. Nauk SSSR Ser. Energ. and Automat.*, Vol. 2 (1962), pp. 124–129.

[17] B. V. Gnedenko and A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables*, Cambridge, Mass., Addison Wesley, 1949.