ON THE EXISTENCE OF A LIFTING COMMUTING WITH THE LEFT TRANSLATIONS OF AN ARBITRARY LOCALLY COMPACT GROUP

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1. Introduction

The main purpose of this paper is to establish the existence of a lifting commuting with the left translations of an arbitrary locally compact group. The material is divided into nine sections and two appendices. The second section contains the notations and terminology used throughout the paper. The third one contains several preliminary results and remarks. In sections 4 and 5 we define and study the conditional expectation P_H arising from a quotient group. In sections 6 and 7 we give various results concerning liftings, and in particular, we study the problem of extending a lifting "from a quotient group to the group." The main results of this paper are given in sections 8 and 9. Appendix I contains various remarks on adequate families of measures. In appendix II we prove a maximal ergodic theorem.

2. Notations and terminology

Let Z be a locally compact space. As usual, we denote by $C^\infty(Z)$ the algebra of all bounded real-valued continuous functions on Z and by $\mathcal{K}(Z)$, the subalgebra of $C^\infty(Z)$ consisting of all $f \in C^\infty(Z)$ having compact support. We use the notation $\mathfrak{M}(Z)$ for the vector space of all real Radon measures on Z and the notation $\mathfrak{M}_+(Z)$ for the cone of all positive Radon measures on Z.

Now let $\mu \in \mathfrak{M}_+(Z)$, $\mu \neq 0$. As usual, we denote by $\mathfrak{L}(Z, \mu)$ the algebra of all real-valued μ-measurable functions on Z and by $\mathfrak{N}(Z, \mu)$, the ideal of all $f \in \mathfrak{L}(Z, \mu)$ which are locally μ-negligible. For $f, g \in \mathfrak{L}(Z, \mu)$ we write $f \equiv g(\mu)$, if f and g coincide locally almost everywhere with respect to μ, that is, if $f - g \in \mathfrak{N}(Z, \mu)$. We denote by $f \to \tilde{f}$ the canonical mapping of $\mathfrak{L}(Z, \mu)$ onto the quotient algebra $\mathfrak{L}(Z, \mu)/\mathfrak{N}(Z, \mu)$.

For a real-valued function g which is defined on the complement of a locally μ-negligible set and is μ-measurable, we agree to call equivalence class of g and

Research supported by the U. S. Army Research Office (Durham), under contract DA-31-124-ARO-D-288.

use the notation $\tilde{g}$ for the equivalence class of any function belonging to $\mathcal{L}(Z, \mu)$ and coinciding with $g(z)$ at the points z where g is defined. We use a similar convention and notation in the case when g takes values in $\overline{R}$, but is finite-valued locally almost everywhere with respect to μ.

As usual we denote by $\mathfrak{L}^1(Z, \mu)$ the vector space of all real-valued essentially μ-integrable functions on Z, and by $L^1(Z, \mu)$ the image of $\mathfrak{L}^1(Z, \mu)$ under the canonical mapping $f \to \tilde{f}$.

We denote by $M^\infty(Z, \mu)$ the algebra of all bounded real-valued μ-measurable functions on Z. In what follows we shall often consider $M^\infty(Z, \mu)$ endowed with the supremum norm $f \to \|f\|_\infty = \sup_{z \in Z} |f(z)|$.

We denote by $\mathfrak{N}^\infty(Z, \mu)$ the ideal of all $f \in M^\infty(Z, \mu)$ which are locally μ-negligible, that is, $\mathfrak{N}^\infty(Z, \mu) = \mathfrak{N}(Z, \mu) \cap M^\infty(Z, \mu)$, and by $L^\infty(Z, \mu)$ the image of $M^\infty(Z, \mu)$ under the mapping $f \to \tilde{f}$; obviously, $L^\infty(Z, \mu)$ can be identified with the quotient algebra $M^\infty(Z, \mu)/\mathfrak{N}^\infty(Z, \mu)$. Finally, we denote by N_∞ the essential supremum norm on $L^\infty(Z, \mu)$.

Let us recall that a mapping ρ: $M^\infty(Z, \mu) \to M^\infty(Z, \mu)$ is called a *lifting of* $M^\infty(Z, \mu)$ (*linear lifting of* $M^\infty(Z, \mu)$, respectively), if it satisfies the axioms (I)–(VI) ((I)–(V), respectively) below:

(I) $\rho(f) \equiv f$;

(II) $f \equiv g$ implies $\rho(f) = \rho(g)$;

(III) $f \geq 0$ implies $\rho(f) \geq 0$;

(IV) $\rho(1) = 1$;

(V) $\rho(af + bg) = a\rho(f) + b\rho(g)$;

(VI) $\rho(fg) = \rho(f)\rho(g)$.

Let us recall that if ρ is a lifting (linear lifting) of $M^\infty(Z, \mu)$, then

$$\|\rho(f)\|_\infty \leq \|f\|_\infty \qquad \text{for every} \quad f \in M^\infty(Z, \mu); \tag{2.1}$$

hence, ρ: $M^\infty(Z, \mu) \to M^\infty(Z, \mu)$ is *continuous* when $M^\infty(Z, \mu)$ is endowed with the supremum norm (see also [10]).

Let us also recall that a lifting (linear lifting) ρ of $M^\infty(Z, \mu)$ is called *strong* if:

(VII) $\rho(f) = f$ for every $f \in C^\infty(Z)$

(see also [12] and [13]).

Suppose now that Z is a locally compact group and that μ is a left Haar measure on Z. If $s \in Z$ and f: $Z \to R$, we define the left translate $\gamma(s)f$ of f by $(\gamma(s)f)(z) = f(s^{-1}z)$ for all $z \in Z$.

We shall say that a lifting (linear lifting) ρ of $M^\infty(Z, \mu)$ *commutes with the left translations of* Z if:

(VIII) $\rho(\gamma(s)f) = \gamma(s)\rho(f)$ for all $s \in Z$ and $f \in M^\infty(Z, \mu)$

(see also [8], [9]).

We shall need a few more notations which will be consistently used in what follows.

If X is a locally compact group, we shall denote by μ_X a left Haar measure on X. Whenever X is compact, we shall assume μ_X normalized, so that $\mu_X(1) = 1$.

If X is a locally compact group and $H \subset X$ a compact distinguished subgroup of X, we denote by $\Pi_H: x \to \Pi_H(x) = \dot{x}$ the canonical mapping of X onto the quotient group X/H. Since $f \in \mathcal{K}(X/H)$ implies $f \circ \Pi_H \in \mathcal{K}(X)$, $\Pi_H(\mu_X)$ is a well-defined Radon measure on X/H (see [2], chapters V and VII); in fact $\Pi_H(\mu_X)$ is a left Haar measure on X/H, which we shall denote in what follows by $\mu_{X/H}$. Now for each $x \in X$, let u_x be the mapping of H into X defined by $u_x(t) = xt$ for $t \in H$. Since $f \in \mathcal{K}(X)$ implies $f \circ u_x \in \mathcal{K}(H) = C^\infty(H)$, $u_x(\mu_H)$ is a well-defined positive Radon measure on X which we shall denote by β_x; hence β_x is defined by the equations

$$\beta_x(f) = \int_H f(xt)\, d\mu_H(t) \qquad \text{for} \quad f \in \mathcal{K}(X). \tag{2.2}$$

It is clear that if $x \equiv y \pmod H$, then $\beta_x = \beta_y$; hence, we may *unambiguously* define $\beta_{\dot{x}}$ by the equation

$$\beta_{\dot{x}} = \beta_x. \tag{2.3}$$

It is obvious that for each $\dot{x} \in X/H$, $\beta_{\dot{x}}$ is a positive Radon measure on X, that

$$\operatorname{Supp} \beta_{\dot{x}} = u_x(H) = xH = \Pi_H^{-1}(\dot{x}), \tag{2.4}$$

and that $\beta_{\dot{x}}(1) = 1$ (see [2], chapter V).

Finally we note that for each $f \in \mathcal{K}(X)$, the mapping $\dot{x} \to \beta_{\dot{x}}(f)$ belongs to $\mathcal{K}(X/H)$; in particular, the mapping $\dot{x} \to \beta_{\dot{x}}$ of X/H into $\mathfrak{M}(X)$ is vaguely continuous (here $\mathfrak{M}(X)$ is endowed with the topology $\sigma(\mathfrak{M}(X), \mathcal{K}(X))$). Hence $\dot{x} \to \beta_{\dot{x}}$ is an *adequate family with respect to* $\mu_{X/H}$ (see [2], chapter V, pp. 17–18), and therefore, we may define $\int_{X/H} \beta_{\dot{x}}\, d\mu_{X/H}(\dot{x})$ as a positive Radon measure on X. It is well known that this leads back to the original Haar measure on X (see [2], chapter VII, paragraph 2),

$$\mu_X = \int_{X/H} \beta_{\dot{x}}\, d\mu_{X/H}(\dot{x}). \tag{2.5}$$

Let us also remark here that *the adequate family* $\dot{x} \to \beta_{\dot{x}}$ *satisfies condition* (C) *of appendix* I (for $K \subset X/H$ compact we take $K(1) = \Pi_H^{-1}(K)$).

We now make use of the theory of adequate families (see [2], chapter V) and of the results in appendix I:

If $f \in \bar{\mathfrak{L}}^1(X, \mu_X)$, then $f \in \mathfrak{L}^1(X, \beta_{\dot{x}})$ for locally almost every $\dot{x} \in X/H$ (with respect to $\mu_{X/H}$); the function $f^{\flat(H)}$, defined locally almost everywhere on X/H with respect to $\mu_{X/H}$) by

$$f^{\flat(H)}(\dot{x}) = \int_X f\, d\beta_{\dot{x}}, \tag{2.6}$$

belongs to $\bar{\mathfrak{L}}^1(X/H, \mu_{X/H})$ and satisfies the equation

$$\int_X f\, d\mu_X = \int_{X/H} f^{\flat(H)}(\dot{x})\, d\mu_{X/H}(\dot{x}). \tag{2.7}$$

From (2.6) and (2.7) we easily deduce

$$\int_{X/H} |f^{\flat(H)}(\dot{x})|\, d\mu_{X/H}(\dot{x}) \le \int_X |f|\, d\mu_X. \tag{2.8}$$

3. Some preliminary results and remarks

Throughout this section, G will be a *locally compact group with left Haar measure* μ_G.

PROPOSITION 1. *Let η be a linear lifting of $M^\infty(G, \mu_G)$ commuting with the left translations of G. Then η is strong.*

PROOF. To prove that η is strong, it is enough to verify that $\eta(f) = f$ for every $f \in \mathcal{K}(G)$ (see the method of proof of theorem 1 in ([13], pp. 447–448)).

Then let $f \in \mathcal{K}(G)$ be fixed. The mappings $s \to s^{-1}$ of G into G and $t \to \gamma(t)f$ of G into $M^\infty(G, \mu_G)$ (here $M^\infty(G, \mu_G)$ is endowed with the topology defined by the supremum norm) are continuous. Combining this with (2.1), we deduce that $s \to \eta(\gamma(s^{-1})f)$ is a continuous mapping of G into $M^\infty(G, \mu_G)$; in particular, $s \to \eta(\gamma(s^{-1})f)(e) = \eta(f)(s)$ is continuous. Since f and $\eta(f)$ are continuous and coincide locally almost everywhere (with respect to μ_G), it follows that $\eta(f) = f$. Hence η is strong, and the proposition is proved.

If $G_0 \subset G$ is an *open subgroup* of G and μ_{G_0} a left Haar measure on G_0, then μ_{G_0} is equivalent with the restriction of μ_G to G_0.

PROPOSITION 2. *Let $G_0 \subset G$ be an open subgroup of G and let η be a lifting of $M^\infty(G_0, \mu_{G_0})$ commuting with the left translations of G_0. For $f \in M^\infty(G, \mu_G)$, define $\delta(f)$ on G by the equations*

$$\delta(f)(x) = \eta(\gamma(x^{-1})f|G_0)(e), \qquad x \in G. \tag{3.1}$$

Then the mapping δ: $f \to \delta(f)$ is a lifting of $M^\infty(G, \mu_G)$ commuting with the left translations of G.

PROOF. It is clear that δ is well-defined and that δ verifies the axioms (II)–(VI) of a lifting.

Let us show that δ verifies axiom (I). Let $f \in M^\infty(G, \mu_G)$. For each $s \in G$, denote by N_s the locally μ_G-negligible set consisting of all $y \in G_0$ for which

$$\eta(\gamma(s^{-1})f|G_0)(y) \neq (\gamma(s^{-1})f|G_0)(y). \tag{3.2}$$

Let now $x \in sG_0$ such that $\delta(f)(x) \neq f(x)$. Note that

$$f(x) = (\gamma(s^{-1})f)(s^{-1}x) = (\gamma(s^{-1})f|G_0)(s^{-1}x) \tag{3.3}$$

and that

$$\begin{aligned}\delta(f)(x) &= \eta(\gamma(x^{-1})f|G_0)(e) = \eta(\gamma(x^{-1}s)\gamma(s^{-1})f|G_0(e) \\ &= (\gamma(x^{-1}s)\eta(\gamma(s^{-1})f|G_0))(e) = \eta(\gamma(s^{-1})f|G_0)(s^{-1}x),\end{aligned} \tag{3.4}$$

whence $x \in sN_s$. It follows that

$$\{x|\delta(f)(x) \neq f(x)\} \cap sG_0 \subset sN_s, \tag{3.5}$$

and therefore the set $\{x|\delta(f)(x) \neq f(x)\} \cap sG_0$ is locally μ_G-negligible. Since $s \in G$ was arbitrary, we deduce that the set $\{x|\delta(f) \neq f(x)\}$ is locally μ_G-negligible. Hence δ is a lifting of $M^\infty(G, \mu_G)$.

It remains to show that δ commutes with the left translations of G (axiom (VIII)). For this let $f \in M^\infty(G, \mu_G)$, $u \in G$, and $x \in G$. We have

$$(3.6)\qquad \begin{aligned}\delta(\gamma(u)f)(x) &= \eta(\gamma(x^{-1})(\gamma(u)f)|G_0)(e)\\ &= \eta(\gamma(x^{-1}u)f|G_0)(e) = \delta(f)(u^{-1}x) = (\gamma(u)\delta(f))(x),\end{aligned}$$

and hence, the assertion is proved. This completes the proof of proposition 2.

Using the classical terminology (see [17]), we shall say that a locally compact group X can be *approximated by Lie groups* if given any neighborhood V of the identity element e, there is a compact distinguished subgroup H of X such that $H \subset V$ and X/H is a Lie group.

REMARK. The *main approximation theorem by Lie groups* (see [17], chapter IV) tells us that if G is an arbitrary locally compact group, then there is an *open subgroup* G_0 of G (which is in fact generated by a compact symmetric neighborhood of e, and therefore is *countable at infinity* ($= \sigma$-compact)), such that G_0 *can be approximated by Lie groups*. We deduce from proposition 2 above, that in order *to prove the existence of a lifting commuting with the left translations for an arbitrary locally compact group, it is enough to consider the case of a locally compact group which is countable at infinity and can be approximated by Lie groups.*

If G' is a locally compact group and $u\colon G \to G'$ an *isomorphism* of G *onto* G', then it is clear that $u(\mu_G) = \mu_{G'}$ is a left Haar measure on G' and that $f \to f \circ u$ is an isomorphism of the algebra $M^\infty(G', \mu_{G'})$ onto the algebra $M^\infty(G, \mu_G)$.

PROPOSITION 3. *Let $u\colon G \to G'$ be an isomorphism of the locally compact group G onto the locally compact group G'. If ρ is a lifting (linear lifting, respectively) of $M^\infty(G, \mu_G)$ commuting with the left translations of G, then the mapping δ defined on $M^\infty(G', \mu_{G'})$ by the equations*

$$(3.7)\qquad \delta(f) = \rho(f \circ u) \circ u^{-1}, \qquad f \in M^\infty(G', \mu_{G'})$$

is a lifting (linear lifting, respectively) of $M^\infty(G', \mu_{G'})$ commuting with the left translations of G'.

PROOF. The proof is straightforward.

4. The conditional expectation P_H

Let X be a locally compact group and $H \subset X$ a *compact distinguished subgroup* of X.

For $f \in \mathfrak{L}^1(X, \mu_X)$ define (see also formulas (2.6) and (2.7))

$$(4.1)\qquad P_H\tilde{f} = \tilde{f}^{\flat(H)} \circ \Pi_H$$

(in the right-hand side of this formula, the symbol $\sim$ was written on f instead of on $f^{\flat(H)} \circ \Pi_H$, *for the printer's convenience;* we shall continue this convention throughout the rest of the paper). The definition is meaningful; if $g \in \mathfrak{L}^1(X, \mu_X)$ and $\tilde{g} = \tilde{f}$, then $f^{\flat(H)}$ and $g^{\flat(H)}$ coincide locally almost everywhere with respect to $\mu_{X/H}$, and hence $f^{\flat(H)} \circ \Pi_H$ and $g^{\flat(H)} \circ \Pi_H$ coincide locally almost everywhere with respect to μ_X.

It is clear that P_H maps $L^1(X, \mu_X)$ *into* $L^1(X, \mu_X)$, that P_H is a *positive linear operator*, and that $\|P_H\|_1 \leq 1$ (see also formula (2.8)).

PROPOSITION 4. (i) *If H and L are compact distinguished subgroups of X and if $H \supset L$, then $P_H P_L = P_L P_H = P_H$.*

(ii) *For $\tilde{g} \in L^1(X/H, \mu_{X/H})$, $P_H(\tilde{g} \circ \Pi_H) = \tilde{g} \circ \Pi_H$.*

PROOF. (i) Since $\mathcal{K}(X)$ is dense in $\bar{\mathcal{L}}^1(X, \mu_X)$, it is enough to verify the equalities in (i) for functions belonging to $\mathcal{K}(X)$. Let $f \in \mathcal{K}(X)$: for $g = f^{\flat(H)} \circ \Pi_H$, we have

$$g^{\flat(L)}(\Pi_L(x)) = \int_L g(xu)\, d\mu_L(u) = \int_L f^{\flat(H)}(\Pi_H(xu))\, d\mu_L(u)$$
$$= \int_L f^{\flat(H)}(\Pi_H(x))\, d\mu_L(u) = f^{\flat(H)}(\Pi_H(x)) = g(x) \tag{4.2}$$

for each $x \in X$. We deduce $P_L(P_H\tilde{f}) = P_L\tilde{g} = \tilde{g} = P_H\tilde{f}$. For $k = f^{\flat(L)} \circ \Pi_L$ we have

$$k^{\flat(H)}(\Pi_H(x)) = \int_H k(xs)\, d\mu_H(s) = \int_H d\mu_H(s) \int_L f(xsu)\, d\mu_L(u)$$
$$= \int_L d\mu_L(u) \int_H f(xsu)\, d\mu_H(s)$$
$$= \int_L d\mu_L(u) \int_H (\gamma(x^{-1})f)(su)\, d\mu_H(s)$$
$$= \int_L d\mu_L(u) \int_H (\gamma(x^{-1})f)(s)\, d\mu_H(s)$$
$$= \int_L d\mu_L(u) \int_H f(xs)\, d\mu_H(s)$$
$$= \int_H f(xs)\, d\mu_H(s) = f^{\flat(H)}(\Pi_H(x)) \tag{4.3}$$

for each $x \in X$. We deduce $P_H(P_L\tilde{f}) = P_H\tilde{k} = P_H\tilde{f}$.

(ii) To prove this assertion, we remark first that there is a set $A \subset X/H$, locally $\mu_{X/H}$-negligible, such that $g \circ \Pi_H$ belongs to $\mathcal{L}^1(X, \beta_{\dot{x}})$ if $\dot{x} \notin A$. We then use the formula $\beta_{\dot{x}} = u_x(\mu_H)$ (see (2.2)). Hence, the proposition is proved.

Below we denote by $L^1(X/H, \mu_{X/H}) \circ \Pi_H$ the set

$$\{\tilde{f} \circ \Pi_H | \tilde{f} \in L^1(X/H, \mu_{X/H})\}. \tag{4.4}$$

COROLLARY 1. *If H is a compact distinguished subgroup of X, then $P_H P_H = P_H$; hence, P_H is a positive projection of $L^1(X, \mu_X)$ into itself with range*

$$L^1(X/H, \mu_{X/H}) \circ \Pi_H$$

and norm equal to one.

PROOF. The corollary is an immediate consequence of proposition 4, the definition of P_H, and formula (2.8).

Let again $H \subset X$ be a *compact distinguished subgroup* of X. We shall now extend the definition of P_H to functions that are locally μ_X-integrable. Let f: $X \to R$ be a locally μ_X-integrable function. By proposition (I.5) in the appendix, $f \in \mathcal{L}^1(X, \beta_{\dot{x}})$ for locally almost every $\dot{x} \in X/H$ with respect to $\mu_{X/H}$, and the function $f^{\flat(H)}$ *defined locally almost everywhere on* X/H with respect to $\mu_{X/H}$ by

$$f^{\flat(H)}(\dot{x}) = \int_X f \, d\beta_{\dot{x}} \tag{4.5}$$

is locally $\mu_{X/H}$-integrable. Whenever $f \geq 0$, we shall (sometimes) define $f^{\flat(H)}$ everywhere on X/H by the equations

$$f^{\flat(H)}(\dot{x}) = \int_X^* f \, d\beta_{\dot{x}}. \tag{4.6}$$

By proposition (I.2) in the appendix, if $g \equiv f(\mu_X)$, then $f^{\flat(H)} \equiv g^{\flat(H)}(\mu_{X/H})$, and hence, $f^{\flat(H)} \circ \Pi_H \equiv g^{\flat(H)} \circ \Pi_H(\mu_X)$. We may then *unambiguously* define

$$P_H\tilde{f} = \tilde{f}^{\flat(H)} \circ \Pi_H. \tag{4.7}$$

Remarks. (1) It is clear that for $f \in M^\infty(X, \mu_X)$, $P_H\tilde{f} \in L^\infty(X, \mu_X)$. Hence $\tilde{f} \to P_H\tilde{f}$ is a positive linear mapping of $L^\infty(X, \mu_X)$ into $L^\infty(X, \mu_X)$ taking $\tilde{1}$ onto $\tilde{1}$.

(2) From corollary 1 and remark (1) above, it follows that for each $1 < p < \infty$, $\tilde{f} \to P_H\tilde{f}$ is a positive linear mapping of $L^p(X, \mu_X)$ into $L^p(X, \mu_X)$ of norm equal to one.

(3) If $g\colon X/H \to R$ is locally $\mu_{X/H}$-integrable, then $P_H(\tilde{g} \circ \Pi_H) = \tilde{g} \circ \Pi_H$.

(4) If $g\colon X \to R$ is locally μ_X-integrable and if $E \subset X/H$ is a $\mu_{X/H}$-measurable set, then $P_H((\tilde{\varphi}_E \circ \Pi_H)\tilde{g}) = (\tilde{\varphi}_E \circ \Pi_H)P_H\tilde{g}$.

From remark (4) above and formula (2.7), we deduce, in particular, that if $\tilde{g} \in L^1(X, \mu_X)$ and if $E \subset X/H$ is a $\mu_{X/H}$-measurable set, then

$$\int_{\Pi_H^{-1}(E)} P_H\tilde{g} \, d\mu_X = \int_{\Pi_H^{-1}(E)} \tilde{g} \, d\mu_X. \tag{4.8}$$

This shows that P_H is in fact the "*conditional expectation* ([6], [16]) *with respect to the tribe* (= σ-algebra) *of all μ_X-measurable subsets of X that are cosets of H*," and thus justifies the terminology used in the title of section 4.

Before stating our next result, let us note that for an *arbitrary* mapping g: $X/H \to \bar{R}$ and *any* $s \in X$ we have

$$(\gamma(\dot{s})g) \circ \Pi_H = \gamma(s)(g \circ \Pi_H). \tag{4.9}$$

In fact, for each $x \in X$ we have

$$\begin{aligned} (\gamma(\dot{s})g) \circ \Pi_H(x) &= (\gamma(\dot{s})g)(\dot{x}) = g(\dot{s}^{-1}\dot{x}) = g(\Pi_H(s)^{-1}\Pi_H(x)) \\ &= g(\Pi_H(s^{-1}x)) = (g \circ \Pi_H)(s^{-1}x) = (\gamma(s)(g \circ \Pi_H))(x), \end{aligned} \tag{4.10}$$

and (4.9) is proved.

Proposition 5. *Let $H \subset X$ be a compact distinguished subgroup of X. If f: $X \to R$ is a locally μ_X-integrable function, then for every $s \in X$ we have $P_H(\gamma(s)\tilde{f}) = \gamma(s)(P_H\tilde{f})$.*

Proof. It is obviously enough to consider the case $f \geq 0$. Let now $s \in X$. For each $\dot{x} \in X/H$ we have

$$\begin{aligned} (\gamma(s)f)^{\flat(H)}(\dot{x}) &= \int_H^* (\gamma(s)f)(xu) \, d\mu_H(u) = \int_H^* f(s^{-1}xu) \, d\mu_H(u) \\ &= f^{\flat(H)}(\dot{s}^{-1}\dot{x}) = (\gamma(\dot{s})f^{\flat(H)})(\dot{x}), \end{aligned} \tag{4.11}$$

whence (combining with formula (4.9)),

$$(\gamma(s)f)^{\flat(H)} \circ \Pi_H = (\gamma(\dot{s})f^{\flat(H)}) \circ \Pi_H = \gamma(s)(f^{\flat(H)} \circ \Pi_H). \tag{4.12}$$

We deduce $(\gamma(s)\tilde{f}))^{\flat(H)} \circ \Pi_H = \gamma(s)(\tilde{f}^{\flat(H)} \circ \Pi_H)$, that is, $P_H(\gamma(s)\tilde{f}) = \gamma(s)(P_H\tilde{f})$. Thus the proposition is proved.

5. Convergence properties of the conditional expectations P_H

In this section we present several results due (essentially) to M. Jerison and G. Rabson (see [14]); these results lead to theorem 2 which will be used later on. Since these results were given in [14] in a different setting, and since we want to preserve the unity of this paper, we shall sketch their proofs below (see also [2], chapter VII, p. 113, and [7], paragraph 4).

Except for several propositions previously given, the main part of the proof of the existence of a lifting commuting with the left translations of an arbitrary locally compact group starts in the next section.

Throughout this section X will be a *locally compact group* and $(H_j)_{j\in J}$ a *directed family* ($j' \leq j''$ implies $H_{j'} \supset H_{j''}$) *of compact distinguished subgroups* of X. Let

$$H_\infty = \bigcap_{j\in J} H_j. \tag{5.1}$$

Let us remark here that if E is a *compact* subset of X/H, then

$$\Pi_{H_\infty}^{-1}(E) = \bigcap_{j\in J} \Pi_{H_j}^{-1}(\Pi_{H_j}(\Pi_{H_\infty}^{-1}(E))). \tag{5.2}$$

Let A and B be two sets and $p\colon A \to B$. If $\mathfrak{F}$ is a set of functions on B to $\overline{R}$, we denote by $\mathfrak{F} \circ p$ the set $\{f \circ p | f \in \mathfrak{F}\}$.

Let now

$$\mathfrak{A} = \bigcup_{j\in J} \mathcal{K}(X/H_j) \circ \Pi_{H_j}. \tag{5.3}$$

With these notations we have the following proposition.

PROPOSITION 6. *Let $f \in \mathcal{K}(X/H_\infty) \circ \Pi_{H_\infty}$ and U an open set containing* Supp f. *There is then a sequence (u_n) of functions belonging to $\mathfrak{A}$, which converges uniformly to f and satisfies* Supp $u_n \subset U$ *for each n.*

PROOF. We first prove the existence of a function $h \in \mathfrak{A}$ such that $h(x) = 1$ for $x \in \operatorname{Supp} f$ and $\operatorname{Supp} h \subset U$. This can be obtained as follows, since $f \in \mathcal{K}(X/H_\infty) \circ \Pi_{H_\infty}$, f has the form $f_1 \circ \Pi_{H_\infty}$ for some $f_1 \in \mathcal{K}(X/H_\infty)$. Let $C = \operatorname{Supp} f_1$; then $\Pi_{H_\infty}^{-1}(C) = \operatorname{Supp} f$. There is then a compact neighborhood E of C such that $\Pi_{H_\infty}^{-1}(E) \subset U$. Using (5.2) above, we deduce the existence of a $j \in J$ such that

$$\Pi_{H_j}^{-1}(\Pi_{H_j}(\Pi_{H_\infty}^{-1}(E))) \subset U. \tag{5.4}$$

By Urysohn's theorem, there is then $h_1 \in \mathcal{K}(X/H_j)$ such that $h_1(\dot{x}) = 1$ if $\dot{x} \in \Pi_{H_j}(\operatorname{Supp} f)$ and $\operatorname{Supp} h_1 \subset \Pi_{H_j}(\Pi_{H_\infty}^{-1}(E))$. It follows that $h = h_1 \circ \Pi_{H_j}$ is the desired function.

Once the function h is obtained, the proof is completed by showing the existence of a sequence (v_n) of functions in $\mathfrak{A}$ converging uniformly to f. For this we reason as follows: to every $u \in \mathfrak{A}$ there corresponds a unique function $u_\infty \in \mathcal{K}(X/H_\infty)$ such that $u = u_\infty \circ \Pi_{H_\infty}$. Clearly, $\mathfrak{A}_\infty = \{u_\infty | u \in \mathfrak{A}\}$ is a subalgebra of $\mathcal{K}(X/H_\infty)$; it is easily seen that $\mathfrak{A}_\infty$ separates the points of X/H_∞, and that for each point in X/H_∞ there is a function in $\mathfrak{A}_\infty$ which does not vanish at that point. An application of the Weierstrass-Stone theorem yields the sequence (v_n); hence the proof is completed (see also [2], chapter VII, p. 24).

COROLLARY 2. *For each* $1 \leq p < \infty$ *the set* $\mathfrak{A}$ *is dense in* $\overline{\mathfrak{L}}^p(X/H_\infty, \mu_{X/H_\infty}) \circ \Pi_{H_\infty}$.

PROOF. Let $f = f_1 \circ \Pi_{H_\infty}$, with $f_1 \in \overline{\mathfrak{L}}^p(X/H_\infty, \mu_{X/H_\infty})$ and let $\epsilon > 0$. There is then $g_1 \in \mathcal{K}(X/H_\infty)$ such that

$$\int_{X/H_\infty} |f_1 - g_1|^p \, d\mu_{X/H_\infty} \leq (\epsilon/2)^p. \tag{5.5}$$

By proposition 6, there is $g_2 \in \mathfrak{A}$ such that

$$\int_X |g_1 \circ \Pi_{H_\infty} - g_2|^p \, d\mu_X \leq (\epsilon/2)^p. \tag{5.6}$$

Combining these two inequalities, we get

$$\left(\int_X |f - g_2|^p \, d\mu_X\right)^{1/p} = \left(\int_X |f_1 \circ \Pi_{H_\infty} - g_2|^p \, d\mu_X\right)^{1/p} \leq \epsilon. \tag{5.7}$$

Thus the corollary is proved.

REMARK. For H and L compact distinguished subgroups of X with $H \supset L$, and $1 \leq p < \infty$, we have

$$\overline{\mathfrak{L}}^p(X/H, \mu_{X/H}) \circ \Pi_H \subset \overline{\mathfrak{L}}^p(X/L, \mu_{X/L}) \circ \Pi_L. \tag{5.8}$$

THEOREM 1. *For each* $1 \leq p < \infty$, *the directed family* $(P_{H_j})_{j \in J}$ *converges strongly in* $L^p(X, \mu_X)$ *to* P_{H_∞}.

PROOF. If $f \in \mathfrak{A}$, then it is clear that $P_{H_\infty}\tilde{f} = \tilde{f} = P_{H_j}\tilde{f}$ for all $j \in J$ large enough (use the remark preceding theorem 1 and remark (3) in section 4). By remark (2) in section 4 and corollary 2 above, we deduce that $(P_{H_j}\tilde{f})_{j \in J}$ converges to $P_{H_\infty}\tilde{f}$ for every

$$f \in \overline{\mathfrak{L}}^p(X/H_\infty, \mu_{X/H_\infty}) \circ \Pi_{H_\infty}. \tag{5.9}$$

For an arbitrary element $\tilde{f} \in L^p(X, \mu_X)$, we may write

$$\tilde{f} = P_{H_\infty}\tilde{f} + (I - P_{H_\infty})\tilde{f}. \tag{5.10}$$

The proof is concluded by noting that

$$P_{H_\infty}\tilde{f} \in L^p(X/H_\infty, \mu_{X/H_\infty}) \circ \Pi_{H_\infty} \tag{5.11}$$

and that $P_{H_j}((I - P_{H_\infty})\tilde{f}) = 0$ for all $j \in J$ (make use of proposition 4).

Assume now that $J = N = \{0, 1, 2, \cdots\}$. In what follows the notation $P_H f$ will be used to designate a representative of the equivalence class $P_H\tilde{f}$.

THEOREM 2. *For every* $f\colon X \to R$ *locally* μ_X*-integrable, the sequence* $(P_{H_n}f)_{n \in N}$ *converges to* $P_{H_\infty}f$ *locally almost everywhere with respect to* μ_X.

PROOF. Consider a *compact* set K in X; K is contained in the compact set

$K_0 = \Pi_{H_0}^{-1}(\Pi_{H_0}(K))$, and hence it is enough to show that $(P_{H_n}f(x))_{n\in N}$ converges to $P_{H_\infty}f(x)$ almost everywhere on K_0 with respect to μ_X.

Consider now the *bounded* measure $\nu = \varphi_{K_0}\cdot\mu_X$. During this proof only, we shall use the notation $\dot{g}$ for the equivalence class of an element $g \in \mathcal{L}^1(X, \nu)$. For each $n \in N$ define the operator P_n on $L^1(X, \nu)$ as follows:

$$(5.12) \qquad P_n\dot{g} = \dot{\widehat{\varphi_{K_0}}}\cdot\dot{\widehat{P_{H_n}(\varphi_{K_0}g)}}, \qquad \text{for} \quad \dot{g} \in L^1(X, \nu);$$

define also

$$(5.13) \qquad P_\infty\dot{g} = \dot{\widehat{\varphi_{K_0}}}\cdot\dot{\widehat{P_{H_\infty}(\varphi_{K_0}g)}}, \qquad \text{for} \quad \dot{g} \in L^1(X, \nu).$$

It is clear that $P_n(P_\infty$, respectively) is well-defined, maps $L^1(X, \nu)$ into $L^1(X, \nu)$, $L^\infty(X, \nu)$ into $L^\infty(X, \nu)$, and that as a linear operator in each one of these spaces it has norm inferior to one; hence, $P_n(P_\infty$, respectively) is a Dunford-Schwartz operator. Finally it is easily verified that $(P_n)_{n\in N}$ is "an increasing sequence of projections" (use (i), proposition 1 and remark (4) in section 4) and that $(P_n)_{n\in N}$ converges *strongly* in $L^1(X, \nu)$ to P_∞ (use theorem 1 above). By the pointwise convergence theorem for increasing sequences of Dunford-Schwartz projections (see [11] and [18]), we know that for each $g \in \mathcal{L}^1(X, \nu)$, the sequence $(P_ng(x))_{n\in N}$ converges to $P_\infty g(x)$ almost everywhere on X with respect to ν.

Let now f: $X \to R$ be a locally μ_X-integrable function; the function $g = \varphi_{K_0}f$ belongs to $\mathcal{L}^1(X, \nu)$, and we have (use remark (4) in section 4)

$$(5.14) \qquad P_n(\varphi_{K_0}f) = \varphi_{K_0}\cdot P_{H_n}f \qquad \text{and} \qquad P_\infty(\varphi_{K_0}f) = \varphi_{K_0}\cdot P_{H_\infty}f.$$

We deduce that the set of all $x \in K_0$ for which $(P_{H_n}f(x))_{n\in N}$ does not converge to $P_{H_\infty}f(x)$ is μ_X-negligible. This completes the proof of the theorem.

6. Liftings of $M^\infty(X/H, \mu_{X/H})$ and of $M^\infty(X,H, \mu_X)$

Let X be a locally compact group and $D \subset M^\infty(X, \mu_X)$ a subalgebra of $M^\infty(X, \mu_X)$ which contains 1 and is "saturated" for the equivalence relation "$\equiv (\mu_X)$" in $M^\infty(X, \mu_X)$. In what follows we shall use a *relativization* of the notion of lifting (linear lifting, respectively). We shall say that a mapping δ: $D \to D$ is a lifting (linear lifting) of D if δ satisfies the axioms (I)–(VI) ((I)–(V)) of section 2. Moreover, if D is invariant under the left translations of X (that is, the relations $f \in D$ and $s \in X$ imply $\gamma(s)f \in D$), then we say that the lifting (linear lifting) δ of D commutes with the left translations of X if δ satisfies also axiom (VIII) of section 1.

Let now $H \subset X$ be a compact distinguished subgroup of X. It is clear that

$$(6.1) \qquad M^\infty(X/H, \mu_{X/H}) \circ \Pi_H$$

is a subalgebra of $M^\infty(X, \mu_X)$ and that this subalgebra contains 1 and is closed for the pointwise convergence of bounded sequences.

Denote by $M^\infty(X, H, \mu_X)$ the "saturated" of (6.1) for the equivalence relation "$\equiv (\mu_X)$" in $M^\infty(X, \mu_X)$. Then

$$(6.2) \qquad M^\infty(X, H, \mu_X) = M^\infty(X/H, \mu_{X/H}) \circ \Pi_H + \mathfrak{N}^\infty(X, \mu_X).$$

Note that $M^\infty(X, H, \mu_X)$ is also a subalgebra of $M^\infty(X, \mu_X)$ containing 1 and closed for the pointwise convergence of bounded sequences. The first assertion is obvious. The second one can be proved as follows. Let $(f_n)_{n\in N}$ be a sequence of functions in $M^\infty(X, H, \mu_X)$; then $f_n = g_n + h_n$ with $g_n \in M^\infty(X/H, \mu_{X/H}) \circ \Pi_H$ and $h_n \in \mathfrak{N}^\infty(X, \mu_X)$ for each $n \in N$. Suppose that $(f_n)_{n\in N}$ converges pointwise to f_∞ and that $\sup_{n\in N} \|f_n\|_\infty = L < \infty$. Denote by A the set of all $x \in X$ such that $\sup_{n\in N} |g_n(x)| \leq L$ and such that the sequence $(g_n(x))_{n\in N}$ is convergent. Since the functions in $\mathfrak{N}^\infty(X, \mu_X)$ are locally μ_X-negligible we deduce that the complement of A is locally μ_X-negligible. Since the functions in the sequence $(g_n)_{n\in N}$ belong to $M^\infty(X/H, \mu_{X/H}) \circ \Pi_H$ we deduce that $A = \Pi_H^{-1}(A')$ for some $A' \subset X/H$, whence $\varphi_A g_n \in M^\infty(X/H, \mu_{X/H}) \circ \Pi_H$ for all $n \in N$. The sequence $(\varphi_A g_n)_{n\in N}$ is bounded and pointwise convergent to some function

$$(6.3) \qquad u_\infty \in M^\infty(X/H, \mu_{X/H}) \circ \Pi_H.$$

For every $n \in N$, let $k_n = f_n - \varphi_A g_n$. Then $k_n \in \mathfrak{N}^\infty(X, \mu_X)$ for every $n \in N$, the sequence $(k_n)_{n\in N}$ is bounded and pointwise convergent to some function $v_\infty \in \mathfrak{N}^\infty(X, \mu_X)$. Since $f_\infty = u_\infty + v_\infty$, we conclude that the function $f_\infty \in M^\infty(X, H, \mu_X)$.

Let us remark here that if K and L are compact distinguished subgroups of X and $K \supset L$, then

$$(6.4) \qquad \begin{cases} \mathcal{K}(X/K) \circ \Pi_K \subset \mathcal{K}(X/L) \circ \Pi_L, \\ M^\infty(X/K, \mu_{X/K}) \circ \Pi_K \subset M^\infty(X/L, \mu_{X/L}) \circ \Pi_L, \\ M^\infty(X, K, \mu_X) \subset M^\infty(X, L, \mu_X). \end{cases}$$

Finally let us note that *each* of the algebras $M^\infty(X/H, \mu_{X/H}) \circ \Pi_H$ and $M^\infty(X, H, \mu_X)$ is *invariant under the left translations of* X. It is obviously enough to verify the assertion for the algebra $M^\infty(X/H, \mu_{X/H}) \circ \Pi_H$. Let then $f \in M^\infty(X/H, \mu_{X/H}) \circ \Pi_H$ and $s \in X$; we have $f = g \circ \Pi_H$ with $g \in M^\infty(X/H, \mu_{X/H})$, and hence (use formula (4.9)),

$$(6.5) \qquad \gamma(s)f = \gamma(s)(g \circ \Pi_H) = (\gamma(\dot{s})g) \circ \Pi_H \in M^\infty(X/H, \mu_{X/H}) \circ \Pi_H.$$

Let now ρ be a *linear lifting* of $M^\infty(X/H, \mu_{X/H})$ *commuting with the left translations of* X/H. Define the mapping ω: $M^\infty(X,H, \mu_X) \rightarrow M^\infty(X, H, \mu_X)$ as follows: let $f \in M^\infty(X, H, \mu_X)$. Then $f \equiv g \circ \Pi_H(\mu_X)$ for some $g \in M^\infty(X/H, \mu_{X/H})$; write

$$(6.6) \qquad \omega(f) = \rho(g) \circ \Pi_H.$$

It is clear that ω is well-defined and that

$$(6.7) \qquad \omega\colon M^\infty(X, H, \mu_X) \rightarrow M^\infty(X/H, \mu_{X/H}) \circ \Pi_H.$$

It is also easily verified that ω is *a linear lifting of* $M^\infty(X, H, \mu_X)$ and that ω *commutes with the left translations of* X. We shall only verify this last assertion: let $f \in M^\infty(X, H, \mu_X)$ and let $s \in X$. Then $f \equiv g \circ \Pi_H(\mu_X)$ with

$$(6.8) \qquad g \in M^\infty(X/H, \mu_{X/H}),$$

and we have (use formula (4.9))

(6.9)
$$\gamma(s)(\omega(f)) = \gamma(s)(\rho(g) \circ \Pi_H) = (\gamma(\dot{s})\rho(g)) \circ \Pi_H$$
$$= \rho(\gamma(\dot{s})g) \circ \Pi_H = \omega(\gamma(s)f).$$

Hence, to every linear lifting ρ of $M^\infty(X/H, \mu_{X/H})$ commuting with the left translations of X/H, we can associate a linear lifting ω of $M^\infty(X, H, \mu_X)$ satisfying (6.7) and commuting with the left translations of X, *via formula* (6.6); it is clear that the mapping $\rho \to \omega$ which we just defined is *injective*.

Conversely, let ω be a linear lifting of $M^\infty(X, H, \mu_X)$ satisfying (6.7) and commuting with the left translations of X. For each $g \in M^\infty(X/H, \mu_{X/H})$ define $\rho(g)$ by the equation $\rho(g) \circ \Pi_H = \omega(g \circ \Pi_H)$. It is easily seen that ρ is a linear lifting of $M^\infty(X/H, \mu_{X/H})$ commuting with the left translations of X/H; it is clear that the mapping $\omega \to \rho$ defined here is the inverse of the mapping $\rho \to \omega$ defined above.

Thus the mapping $\rho \to \omega$ is a *bijection* of the set of all linear liftings of $M^\infty(X/H, \mu_{X/H})$ commuting with the left translations of X/H *onto* the set of all linear liftings of $M^\infty(X, H, \mu_X)$ satisfying (6.7) and commuting with the left translations of X. Clearly, ω is "multiplicative" (that is, it satisfies (VI)) if and only if ρ is.

Before we state our next result we need several remarks.

REMARKS. (1) Let f: $X/H \to R$; then f is a characteristic function if and only if $f \circ \Pi_H$ is a characteristic function.

(2) Let $\mathcal{E} \subset M^\infty(X, H, \mu_X)$ be a subalgebra containing 1 and closed for the pointwise convergence of bounded sequences. Let $\mathfrak{T} = \{A | \varphi_A \in \mathcal{E}\}$. Then $\mathfrak{T}$ is a tribe (use the fact that $\mathcal{E}$ is closed for the pointwise convergence of bounded sequences and see ([2], chapter IV, p. 160), and $f \in$ if and only if f is $\mathfrak{T}$-measurable. Moreover, if η_1 and η_2 are linear liftings of $M^\infty(X, H, \mu_X)$ satisfying $\eta_1(\varphi_A) = \eta_2(\varphi_A)$ for every $A \in \mathfrak{T}$, then $\eta_1|\mathcal{E} = \eta_2|\mathcal{E}$. To prove this last assertion, we remark first that η_1 and η_2 coincide on the simple functions constructed with sets in $\mathfrak{T}$, and then use the fact that these functions are dense in $\mathcal{E}$ for the topology defined by the supremum norm.

(3) Let ρ' be a linear lifting of $M^\infty(X/H, \mu_{X/H})$ commuting with the left translations of X/H. For every $\mu_{X/H}$-measurable set $B \subset X/H$, define

(6.10)
$$\theta'(B) = \{\dot{x}|\rho'(\varphi_B)(\dot{x}) = 1\} \quad \text{and} \quad \theta''(B) = \{\dot{x}|\rho'(\varphi_B)(\dot{x}) > 0\}.$$

Let $\mathfrak{D}$ be the set of all linear liftings ρ of $M^\infty(X/H, \mu_{X/H})$ *commuting with the left translations of* X/H and satisfying

(6.11)
$$\varphi_{\theta'(B)} \leq \rho(\varphi_B) \leq \varphi_{\theta''(B)}$$

for every $\mu_{X/H}$-measurable set $B \subset X/H$. Then $\mathfrak{D}$ is convex, compact and every *extremal element* of $\mathfrak{D}$ is *multiplicative* (see remarks (2) and (3) following theorem 1 in [8]).

In the proposition below, ω' and ω_H correspond to ρ' and ρ_H respectively under the mapping $\rho \to \omega$ (given by formula (6.6)).

PROPOSITION 7. *Let ρ' be a linear lifting of $M^\infty(X/H, \mu_{X/H})$ commuting with the left translations of X/H. Let $\mathfrak{D}$ be the convex set corresponding to ρ' (defined in* *remark* (3) *above), and let* ρ_H *be an extremal element of* $\mathfrak{D}$. *Then* (i) ρ_H *is a lifting of* $M^\infty(X/H, \mu_{X/H})$ *commuting with the left translations of* X/H; (ii) *if* $\mathcal{E} \subset M^\infty(X, H, \mu_X)$ *is a subalgebra containing* 1, *closed for the pointwise convergence of bounded sequences and such that* $\omega'|\mathcal{E}$ *is multiplicative, then* $\omega_H|\mathcal{E} = \omega'|\mathcal{E}$.

PROOF. (i) The fact that ρ_H is a lifting of $M^\infty(X/H, \mu_{X/H})$ commuting with the left translations of X/H follows from remark (3) above.

(ii) Let $\mathcal{E} \subset M^\infty(X, H, \mu_X)$ be a subalgebra containing 1, closed for the pointwise convergence of bounded sequences and such that $\omega'|\mathcal{E}$ is multiplicative. Let $\mathfrak{J} = \{A|\varphi_A \in \mathcal{E}\}$. By remark (2) above, to prove that $\omega_H|\mathcal{E} = \omega'|\mathcal{E}$, it is enough to show that $\omega_H(\varphi_A) = \omega'(\varphi_A)$ for every $A \in \mathfrak{J}$.

Let $A \in \mathfrak{J}$; it is clear that $\omega'(\varphi_A)$ is a characteristic function and that $\omega'(\varphi_A) \in M^\infty(X/H, \mu_{X/H}) \circ \Pi_H$. There is then (use remark (1) above) a $\mu_{X/H}$-measurable set $B \subset X/H$ such that $\varphi_A \equiv \varphi_B \circ \Pi_H(\mu_X)$. Now $\omega'(\varphi_A) = \rho'(\varphi_B) \circ \Pi_H$, and therefore, $\rho'(\varphi_B)$ is a characteristic function. We deduce (with the notations of remark (3) above) that $\theta'(B) = \theta''(B)$, and hence that $\rho_H(\varphi_B) = \rho'(\varphi_B)$. Consequently,

$$\omega_H(\varphi_A) = \rho_H(\varphi_B) \circ \Pi_H = \rho'(\varphi_B) \circ \Pi_H = \omega'(\varphi_A). \tag{6.12}$$

This completes the proof of the proposition.

For further reference, we state here (with somewhat modified notations) the following corollary.

COROLLARY 3. *Let* $S: f \to S_f$ *be a linear lifting of* $M^\infty(X, \mu_X)$ *commuting with the left translations of* X. *There is then a lifting* η *of* $M^\infty(X, \mu_X)$ *with the following properties:* (i) η *commutes with the left translations of* X; (ii) *if* $\mathcal{E} \subset M^\infty(X, \mu_X)$ *is a subalgebra containing* 1, *closed for the pointwise convergence of bounded sequences, and such that* $S|\mathcal{E}$ *is multiplicative, then* $\eta|\mathcal{E} = S|\mathcal{E}$.

7. Extension of a lifting "from a quotient group to the group"

Let Z be a locally compact space and let $\mu \in \mathfrak{M}_+(Z)$, $\mu \neq 0$. We shall denote by $M_0^\infty(Z, \mu)$ the set of all functions f which are defined on the complement of a locally μ-negligible set and coincide where defined with a function belonging to $M^\infty(Z, \mu)$. It is clear that a lifting (linear lifting) of $M^\infty(Z, \mu)$ can be extended in a natural way to a lifting (linear lifting) of $M_0^\infty(Z, \mu)$. In fact, if $f \in M_0^\infty(Z, \mu)$, then f coincides where defined with some function $g \in M^\infty(Z, \mu)$ and we may define $\rho(f) = \rho(g)$. We shall also define $\rho(\tilde{f})$ by the equation $\rho(\tilde{f}) = \rho(f)$ for $f \in M_0^\infty(Z, \mu)$.

Let us also remark that if Z is a locally compact group and $\mu = \mu_Z$ a left Haar measure on Z, then $M_0^\infty(Z, \mu_Z)$ is "invariant" under the left translations of Z (let $z \in Z$ and $f \in M_0^\infty(Z, \mu_Z)$; if f is defined on the set E, then $\gamma(z)f$ is defined on zE, and thus $\gamma(z)f$ belongs also to $M_0^\infty(Z, \mu_Z)$).

Throughout this section, G will be a *locally compact group which is countable at infinity*, $F \subset G$ will be a *compact distinguished subgroup* of G which is also a *Lie group*, and δ will be a *lifting of* $M^\infty(G/F, \mu_{G/F})$ *commuting with the left translations of* G/F.

For every couple $(g, h) \in M^\infty(G, \mu_G) \times \mathcal{K}(G)$ we denote by $[g, h]$ the mapping $\dot{x} \to \int_G gh\, d\beta_{\dot{x}}$; it is clear that $[g, h]$ is well-defined with the exception of a $\mu_{G/F}$-negligible set, and that for each $\dot{x} \in G/F$ for which $[g, h]$ is defined we have:

$$(7.1) \qquad |[g, h](\dot{x})| \leq \|g\|_\infty \beta_{\dot{x}}(|h|) \leq \|g\|_\infty \|h\|_\infty.$$

When $g \geq 0$, $h \geq 0$ we shall (sometimes) set $[g, h](\dot{x}) = \int_G^* gh\, d\beta_{\dot{x}}$ for *every* $\dot{x} \in G/F$. In any case, formula (7.1) shows that $[g, h] \in M_0^\infty(G/F, \mu_{G/F})$. Let us also note that if $g_1 \in M^\infty(G, \mu_G)$, $g_2 \in M^\infty(G, \mu_G)$, $g_1 \equiv g_2(\mu_G)$ and $h \in \mathcal{K}(G)$, then (with obvious notations) we have $[g_1, h] \equiv [g_2, h](\mu_{G/F})$.

For each couple $(g, h) \in M^\infty(G, \mu_G) \times \mathcal{K}(G)$, we may then define

$$(7.2) \qquad B(\tilde{g}, h) = \delta([g, h]).$$

We shall now establish several important properties of the mapping $(\tilde{g}, h) \to B(\tilde{g}, h)$.

(7.3) *The mapping $(\tilde{g}, h) \to B(\tilde{g}, h)$ is a bilinear mapping of $L^\infty(G, \mu_G) \times \mathcal{K}(G)$ into $M^\infty(G/F, \mu_{G/F})$;*

$$(7.4) \qquad B(\tilde{g}, h) \geq 0 \quad \textit{if} \quad g \geq 0, \qquad h \geq 0;$$

$$(7.5) \qquad B(\gamma(s)\tilde{g}, \gamma(s)h) = \gamma(\dot{s})B(\tilde{g}, h) \quad \textit{for all} \quad s \in G.$$

To prove (7.5) we may without loss of generality assume that $g \geq 0$, $h \geq 0$. Let $s \in G$; for each $\dot{x} \in G/F$ we then have

$$\begin{aligned}(7.6) \qquad (\gamma(\dot{s})[g, h])(\dot{x}) &= [g, h](\dot{s}^{-1}\dot{x}) \\ &= \int_G^* gh\, d\beta_{\dot{s}^{-1}\dot{x}} = \int_F^* g(s^{-1}xy)h(s^{-1}xy)\, d\mu_F(y) \\ &= \int_F^* (\gamma(s)g)(xy)(\gamma(s)h)(xy)\, d\mu_F(y) = [\gamma(s)g, \gamma(s)h](\dot{x});\end{aligned}$$

hence,

$$(7.7) \qquad \gamma(\dot{s})[g, h] = [\gamma(s)g, \gamma(s)h].$$

Applying δ to both sides of this equation and using the fact that the lifting δ commutes with the left translations of G/F, we obtain (7.5).

(7.8) *If g and h belong to $\mathcal{K}(G)$ then $B(\tilde{g}, h)(\dot{x}) = \int_G gh\, d\beta_{\dot{x}}$ for every $\dot{x} \in G/F$.*

To prove this assertion it is enough to remark that the mapping $\dot{x} \to \int_G gh\, d\beta_{\dot{x}}$ belongs to $\mathcal{K}(G/F)$ and that δ is strong (see proposition 1).

(7.9) *For every $\dot{x} \in G/F$, $g \in M^\infty(G, \mu_G)$, $h \in \mathcal{K}(G)$ we have*

$$|B(\tilde{g}, h)(\dot{x})| \leq N_\infty(\tilde{g})\beta_{\dot{x}}(|h|);$$

in particular, for each $\dot{x} \in G/F$ and each $\tilde{g} \in L^\infty(G, \mu_G)$, $h \to B(\tilde{g}, h)(\dot{x})$ is a continuous linear form on $\mathcal{K}(G) \subset \mathcal{L}^1(G, \beta_{\dot{x}})$.

To prove (7.9), let $g \in M^\infty(G, \mu_G)$. For each $h \in \mathcal{K}(G)$ consider $|h|^{\flat(F)}$. Let us recall that $|h|^{\flat(F)}(\dot{x}) = \beta_{\dot{x}}(|h|)$ for each $\dot{x} \in G/F$ and that $|h|^{\flat(F)} \in \mathcal{K}(G/F)$. By (7.1) we have almost everywhere on G/F (with respect to $\mu_{G/F}$)

$$(7.10) \qquad |[g, h](\dot{x})| \leq \|g\|_\infty |h|^{\flat(F)}(\dot{x}).$$

Now we apply the lifting δ to this inequality and use the fact that δ is strong; we obtain

$$(7.11) \qquad |\delta([g, h])| = \delta(|[g, h]|) \leq \delta(\|g\|_\infty |h|^{\flat(F)}) = \|g\|_\infty |h|^{\flat(F)},$$

whence

$$(7.12) \qquad |B(\tilde{g}, h)(\dot{x})| \leq \|g\|_\infty \beta_{\dot{x}}(|h|)$$

for each $\dot{x} \in G/F$. Thus the assertion (7.9) is proved.

(7.13) *Let $g \in M^\infty(G, \mu_G)$. There exists then a $\mu_{G/F}$-negligible set $A(g) \subset G/F$ such that if $\dot{x} \notin A(g)$ then $g \in \mathcal{L}^1(G, \beta_{\dot{x}})$ and $|B(\tilde{g}, h)(\dot{x})| \leq \|h\|_\infty \int_G |g|\, d\beta_{\dot{x}}$ for all $h \in \mathcal{K}(G)$.*

Let $A_1(g)$ be the $\mu_{G/F}$-negligible set of all $\dot{x} \in G/F$ such that g is not $\beta_{\dot{x}}$-measurable. If $\dot{x} \notin A_1(g)$, we have

$$(7.14) \qquad |[g, h](\dot{x})| = \left|\int_G gh\, d\beta_{\dot{x}}\right| \leq \|h\|_\infty \int_G |g|\, d\beta_{\dot{x}}$$

for all $h \in \mathcal{K}(G)$. Obviously the mapping $u\colon \dot{x} \to \int_G |g|\, d\beta_{\dot{x}}$ (defined on the complement of the set $A_1(g)$) belongs to $M_0^\infty(G/F, \mu_{G/F})$. Hence, there is a $\mu_{G/F}$-negligible set $A(g) \supset A_1(g)$ such that

$$(7.15) \qquad \delta(u)(\dot{x}) = \int_G |g|\, d\beta_{\dot{x}} \qquad \text{for} \quad \dot{x} \notin A(g).$$

Applying δ to both sides of (7.14) we obtain

$$(7.16) \qquad |B(\tilde{g}, h)(\dot{x})| \leq \|h\|_\infty \delta(u)(\dot{x}) = \|h\|_\infty \int_G |g|\, d\beta_{\dot{x}}$$

for all $\dot{x} \notin A(g)$ and all $h \in \mathcal{K}(G)$. Hence, (7.13) is proved.

(7.17) *Let $g \in M^\infty(G, \mu_G)$. There is then a $\mu_{G/F}$-negligible set $A'(g) \supset A(g)$ such that if $\dot{x} \notin A'(g)$ we have*

$$(7.18) \qquad B(\tilde{g}, h)(\dot{x}) = \int_G gh\, d\beta_{\dot{x}} \qquad \text{for all} \quad h \in \mathcal{K}(G).$$

Let $(g_n)_{n \in N}$ be a bounded sequence of functions belonging to $\mathcal{K}(G)$ which converges almost everywhere to g, with respect to μ_G. There is then a $\mu_{G/F}$-negligible set $A_1 \subset G/F$ such that

$$(7.19) \qquad \lim_{n \in N} \int_G^* |g - g_n|\, d\beta_{\dot{x}} = 0 \qquad \text{if} \quad \dot{x} \notin A_1.$$

By (7.13) there is a $\mu_{G/F}$-negligible set $A_2 \subset G/F$ such that

$$(7.20) \qquad |B(\tilde{g}, h)(\dot{x}) - B(\tilde{g}_n, h)(\dot{x})| = |B(\tilde{g} - \tilde{g}_n, h)(\dot{x})| \leq \|h\|_\infty \int_G^* |g - g_n|\, d\beta_{\dot{x}}$$

for all $\dot{x} \notin A_2$, all $h \in \mathcal{K}(G)$, and all $n \in N$.

Let $A'(g) = A_1 \cup A_2 \cup A(g)$ ($A(g)$ is the set introduced in (7.13)). We deduce, for all $\dot{x} \notin A'(g)$ and $h \in \mathcal{K}(G)$ (use also (7.8)),

$$B(\tilde{g}, h)(\dot{x}) = \lim_{n \in N} B(\tilde{g}_n, h)(\dot{x}) = \lim_{n \in N} \int_G g_n h \, d\beta_{\dot{x}} = \int_G gh \, d\beta_{\dot{x}}; \tag{7.21}$$

hence (7.17) is completely proved.

Let $\dot{x} \in G/F$ and $\tilde{g} \in L^\infty(G, \mu_G)$; let us recall (see (7.9)) that $h \to B(\tilde{g}, h)(\dot{x})$ *is a continuous linear form on* $\mathcal{K}(G) \subset \mathcal{L}^1(G, \beta_{\dot{x}})$. Hence, there is

$$V(\dot{x}, \tilde{g}) \in M^\infty(G, \beta_{\dot{x}}), \tag{7.22}$$

uniquely determined modulo $\beta_{\dot{x}}$, such that

$$B(\tilde{g}, h)(\dot{x}) = \int_G V(\dot{x}, \tilde{g})(y) h(y) \, d\beta_{\dot{x}}(y) \tag{7.23}$$

for all $h \in \mathcal{K}(G)$. By the inequality in (7.9) we may obviously suppose that $|V(\dot{x}, \tilde{g})| \leq N_\infty(\tilde{g})$.

PROPOSITION 8. (i) *For every* $g \in M^\infty(G, \mu_G)$ *there is a* $\mu_{G/F}$*-negligible set* $A'(g) \subset G/F$ *(here* $A'(g)$ *is the set introduced in* (7.17)) *such that for each* $\dot{x} \notin A'(g)$,

$$V(\dot{x}, \tilde{g}) \equiv g(\beta_{\dot{x}}). \tag{7.24}$$

(ii) *Let* $\dot{x} \in G/F$. *Then*

$$\begin{aligned} &V(\dot{x}, \tilde{1}) \equiv 1(\beta_{\dot{x}}), \\ &g \geq 0 \quad \text{implies} \quad V(\dot{x}, \tilde{g}) \geq 0(\beta_{\dot{x}}), \\ &V(\dot{x}, a_1\tilde{g}_1 + a_2\tilde{g}_2) \equiv a_1 V(\dot{x}, \tilde{g}_1) + a_2 V(\dot{x}, \tilde{g}_2)(\beta_{\dot{x}}), \end{aligned} \tag{7.25}$$

for every $a_1, a_2 \in R$, *and* $g_1, g_2 \in M^\infty(G, \mu_G)$.

(iii) *If* $g = g_1 \circ \Pi_F$ *with* $g_1 \in M^\infty(G/F, \mu_{G/F})$ *and* $\delta(g_1) = g_1$, *then for each* $\dot{x} \in G/F$ *we may take*

$$V(\dot{x}, \tilde{g}) = \text{constant} = g_1(\dot{x})(\beta_{\dot{x}}). \tag{7.26}$$

PROOF. (i) follows from (7.17), (7.18), (7.22), and (7.23).

(ii) Fix $\dot{x} \in G/F$.

For each $h \in \mathcal{K}(G)$ we have (use the fact that δ is strong)

$$\int_G h \, d\beta_{\dot{x}} = \int_G 1.h \, d\beta_{\dot{x}} = B(\tilde{1}, h)(\dot{x}) = \int_G V(\dot{x}, \tilde{1})(y) h(y) \, d\beta_{\dot{x}}(y); \tag{7.27}$$

since $h \in \mathcal{K}(G)$ is arbitrary, we deduce $V(\dot{x}, \tilde{1}) \equiv 1(\beta_{\dot{x}})$.

Let now $g \in M^\infty_+(G, \mu_G)$. For each $h \in \mathcal{K}_+(G)$ we have (use (7.4))

$$0 \leq B(\tilde{g}, h)(\dot{x}) = \int_G V(\dot{x}, \tilde{g})(y) h(y) \, d\beta_{\dot{x}}(y); \tag{7.28}$$

this implies $V(\dot{x}, \tilde{g}) \geq 0(\beta_{\dot{x}})$.

Finally, let $a_1, a_2 \in R$, and $g_1, g_2 \in M^\infty(G, \mu_G)$. For each $h \in \mathcal{K}(G)$ we have

$$(7.29)\qquad \int_G V(\dot{x}, a_1\tilde{g}_1 + a_2\tilde{g}_2)(y)h(y)\,d\beta_{\dot{x}}(y) = B(a_1\tilde{g}_1 + a_2\tilde{g}_2, h)(\dot{x})$$
$$= a_1B(\tilde{g}_1, h)(\dot{x}) + a_2B(\tilde{g}_2, h)(\dot{x})$$
$$= a_1 \int_G V(\dot{x}, \tilde{g}_1)(y)h(y)\,d\beta_{\dot{x}}(y) + a_2 \int_G V(\dot{x}, \tilde{g}_2)(y)h(y)\,d\beta_{\dot{x}}(y)$$
$$= \int_G (a_1V(\dot{x}, \tilde{g}_1) + a_2V(\dot{x}, \tilde{g}_2))(y)h(y)\,d\beta_{\dot{x}}(y);$$

since $h \in \mathcal{K}(G)$ was arbitrary, we deduce

$$(7.30)\qquad V(\dot{x}, a_1\tilde{g}_1 + a_2\tilde{g}_2) \equiv a_1V(\dot{x}, \tilde{g}_1) + a_2V(\dot{x}, \tilde{g}_2)(\beta_{\dot{x}}).$$

(iii) Let $g = g_1 \circ \Pi_F$ with $g_1 \in M^\infty(G/F, \mu_{G/F})$ and $\delta(g_1) = g_1$. For $\dot{x} \notin A'(g)$ and every $h \in \mathcal{K}(G)$ we have

$$(7.31)\qquad \int_G gh\,d\beta_{\dot{x}} = \int_F g_1(\Pi_F(xy))h(xy)\,d\mu_F(y) = g_1(\dot{x}) \int_G h\,d\beta_{\dot{x}}.$$

On the other hand, by (7.17), for every $\dot{x} \notin A'(g)$ and $h \in \mathcal{K}(G)$ we have $B(\tilde{g}, h)(\dot{x}) = \int_G gh\,d\beta_{\dot{x}}$. Since $\dot{x} \to B(\tilde{g}, h)(\dot{x})$ and $\dot{x} \to g_1(\dot{x}) \int_G h\,d\beta_{\dot{x}}$ are both invariant under δ, we deduce $B(\tilde{g}, h)(\dot{x}) = g_1(\dot{x}) \int_G h\,d\beta_{\dot{x}}$ for *all* $\dot{x} \in G/F$ and all $h \in \mathcal{K}(G)$. It follows that for each fixed $\dot{x} \in G/F$,

$$(7.32)\qquad \int_G V(\dot{x}, \tilde{g})(y)h(y)\,d\beta_{\dot{x}}(y) = g_1(\dot{x}) \int_G h(y)\,d\beta_{\dot{x}}(y)$$

for all $h \in \mathcal{K}(G)$; this shows that we may take

$$(7.33)\qquad V(\dot{x}, \tilde{g}) = \text{constant} = g_1(\dot{x})(\beta_{\dot{x}}).$$

This completes the proof of proposition 8.

(7.34) *Let $U \subset F$ be a compact set with $\mu_F(U) > 0$. For each $f \in M^\infty(G, \mu_G)$ the mapping $x \to \int_G f(y)\varphi_{xU}(y)\,d\beta_{\dot{x}}(y)$ is defined almost everywhere on G with respect to μ_G and belongs to $M_0^\infty(G, \mu_G)$.*

In fact, suppose first that $f \in \mathcal{K}(G)$. Then

$$(7.35)\qquad F(x) = \int_G f(y)\varphi_{xU}(y)\,d\beta_{\dot{x}}(y)$$

exists for every $x \in G$; note also that

$$(7.36)\qquad F(x) = \int_F f(xy)\varphi_{xU}(xy)\,d\mu_F(y) = \int_F f(xy)\varphi_U(y)\,d\mu_F(y)$$

for every $x \in G$. Given $\epsilon > 0$, there is a neighborhood V of e in G such that the relations $s \in G$, $t \in G$, $st^{-1} \in V$ imply $|f(st) - f(ty)| \leq \epsilon/\mu_F(U)$ for all $y \in G$. We deduce that $s, t \in G$, $st^{-1} \in V$ imply

$$(7.37)\qquad |F(s) - F(t)| \leq (\epsilon/\mu_F(U))\mu_F(U) = \epsilon.$$

Hence $x \to F(x)$ is continuous and the assertion is proved in the case when $f \in \mathcal{K}(G)$.

Let now $f \in M^\infty(G, \mu_G)$. There is then a bounded sequence (f_n) of functions belonging to $\mathcal{K}(G)$, which converges to f almost everywhere with respect to μ_G. Denote by A the set of all $x \in G$ for which $(f_n(x))$ does not converge to $f(x)$; then $\mu_G(A) = 0$, and there is a set $B \subset G/F$, $\mu_{G/F}$-negligible such that for each $\dot{x} \notin B$, the set A is $\beta_{\dot{x}}$-negligible and f is $\beta_{\dot{x}}$-measurable. It follows that $\Pi_F^{-1}(B)$ is μ_G-negligible and that for *each* $x \notin \Pi_F^{-1}(B)$ we have

$$\int_G f(y)\varphi_{xU}(y)\, d\beta_{\dot{x}}(y) = \lim_n \int_G f_n(y)\varphi_{xU}(y)\, d\beta_{\dot{x}}(y). \tag{7.38}$$

Hence the mapping $x \to \int_G f(y)\varphi_{xU}(y)\, d\beta_{\dot{x}}(y)$ defined on the complement of the set $\Pi_F^{-1}(B)$ is μ_G-measurable; as it is obviously bounded, it belongs to $M_0^\infty(G, \mu_G)$.

Let us recall that a sequence (U_n) of parts of F is a *compact D'-sequence in* F if:

(i) (U_n) is a decreasing sequence of compact sets;

(ii) every neighborhood of e contains some set U_k;

(iii) there exists a constant $C > 0$ such that

$$0 < \mu_F(U_n U_n^{-1}) \le C\mu_F(U_n) \qquad \text{for all } n. \tag{7.39}$$

Since F is a *Lie group,* there exists a compact D'-sequence in F (see [7], theorem 2.10; see also [4]).

Let now (U_n) be a *fixed compact D'-sequence in F.* By the *main derivation theorem,* for every $f \in \mathcal{L}^1(F, \mu_F)$ (in particular, for every $f \in M^\infty(F, \mu_F)$) there is a set $C(f) \subset F$ with $\mu_F(C(f)) = 0$ such that

$$\lim_n \frac{1}{\mu_F(U_n)} \int_{sU_n} f\, d\mu_F = f(s) \tag{7.40}$$

for each $s \notin C(f)$ (see [7], theorem 2.5, and appendix II).

Let now $g \in M^\infty(G, \mu_G)$. Since g is μ_G-measurable, there is a set $A_1 \subset G/F$, $\mu_{G/F}$-negligible, such that for each $\dot{x} \notin A_1$, g is $\beta_{\dot{x}}$-measurable. The set $\Pi_F^{-1}(A_1)$ is then μ_G-negligible. For each n define the function $F_g^{(n)}$ on the complement of $\Pi_F^{-1}(A_1)$ by the equation

$$F_g^{(n)}(x) = \frac{1}{\mu_F(U_n)} \int_G g(y)\varphi_{xU_n}(y)\, d\beta_{\dot{x}}(y). \tag{7.41}$$

Let

$$A_g = \complement(\{z \in G \mid \lim_n F_g^{(n)}(z) \text{ exists and } = g(z)\}). \tag{7.42}$$

With the notations introduced in (7.41) and (7.42), we have the following proposition.

PROPOSITION 9. *The set A_g is μ_G-measurable and $\mu_G(A_g) = 0$.*

PROOF. Since $F_g^{(n)}$ is μ_G-measurable for each n (see (7.34)), the measurability of the set A_g is obvious.

Let us recall that g is $\beta_{\dot{x}}$-measurable if and only if the mapping $y \to g(xy)$ of F into R is μ_F-measurable. It follows that:

(α) For each $x \notin \Pi_F^{-1}(A_1)$, the mapping $y \to g(xy)$ of F into R is μ_F-measurable.

Let now $x \notin \Pi_F^{-1}(A_1)$. Define:

$$B_x = \complement_F\left(\left\{s \in F \middle| \lim_n \frac{1}{\mu_F(U_n)} \int_{sU_n} g(xy)\, d\mu_F(y) \text{ exists and } = g(xs)\right\}\right). \tag{7.43}$$

Since the mapping $y \to g(xy)$ belongs to $M^\infty(F, \mu_F)$ by (α), the derivation theorem implies that B_x is μ_F-measurable and $\mu_F(B_x) = 0$ (note that $yB_y = xB_x$ if $\dot{y} = \dot{x}$). Since

$$\begin{aligned} \beta_{\dot{x}}^*(xB_x) &= \int_G^* \varphi_{xB_x}\, d\beta_{\dot{x}} = \int_F^* \varphi_{xB_x}(xy)\, d\mu_F(y) \\ &= \int_F^* \varphi_{B_x}(y)\, d\mu_F(y) = \mu_F^*(B_x), \end{aligned} \tag{7.44}$$

it follows that for each $x \notin \Pi_F^{-1}(A_1)$,

(β) $\beta_{\dot{x}}(xB_x) = 0$.

We shall show now that for *each* $\dot{x} \notin A_1$ we have

(γ) $A_g \cap \Pi_F^{-1}(\dot{x}) \subset xB_x$.

Let $z \in A_g \cap \Pi_F^{-1}(\dot{x})$. Since $z \in \Pi_F^{-1}(\dot{x})$, $z = xt$ for some $t \in F$. On the other hand, since $z = xt \in A_g$ and $z \notin \Pi_F^{-1}(A_1)$, the sequence $(F_g^{(n)}(xt))$ *does not converge to* $g(xt)$. But we have

$$\begin{aligned} F_g^{(n)}(xt) &= \frac{1}{\mu_F(U_n)} \int_G g(y)\varphi_{xtU_n}(y)\, d\beta_{\dot{x}\dot{t}}(y) \\ &= \frac{1}{\mu_F(U_n)} \int_G g(y)\varphi_{xtU_n}(y)\, d\beta_{\dot{x}}(y) \\ &= \frac{1}{\mu_F(U_n)} \int_F g(xy)\varphi_{xtU_n}(xy)\, d\mu_F(y) \\ &= \frac{1}{\mu_F(U_n)} \int_F g(xy)\varphi_{tU_n}(y)\, d\mu_F(y) \\ &= \frac{1}{\mu_F(U_n)} \int_{tU_n} g(xy)\, d\mu_F(y); \end{aligned} \tag{7.45}$$

we deduce that $t \in B_x$ and thus $z = xt \in xB_x$. Hence the inclusion (γ) is proved.

Since $\operatorname{Supp} \beta_{\dot{x}} \subset \Pi_F^{-1}(\dot{x})$ for each $\dot{x} \in G/F$ (see (2.4)), the relations (β) and (γ) above show that for each $\dot{x} \notin A_1$,

(δ) $\beta_{\dot{x}}(A_g) = 0$.

Since we already know that the set A_g is μ_G-measurable, the relations (δ) and (2.7) imply that $\mu_G(A_g) = 0$. Therefore, proposition 9 is proved.

Let $g \in M^\infty(G, \mu_G)$. For each $n \in N$ define the function $T_g^{(n)}$ on G by the equations (see also formula (7.23)),

$$T_g^{(n)}(x) = \frac{1}{\mu_F(U_n)} \int_G V(\dot{x}, \tilde{g})(y)\varphi_{xU_n}(y)\, d\beta_{\dot{x}}(y), \qquad x \in G. \tag{7.46}$$

Since $V(\dot{x}, \tilde{g}) \in M^\infty(G, \beta_{\dot{x}})$ and $|V(\dot{x}, \tilde{g})| \leq N_\infty(\tilde{g})$ for each $\dot{x} \in G/F$, $T_g^{(n)}$ is well-defined everywhere on G and $\|T_g^{(n)}\|_\infty \leq N_\infty(\tilde{g})$.

We shall establish below several important properties of the mapping $T^{(n)}$: $g \to T_g^{(n)}$. We shall first list them completely, and afterwards we shall proceed to prove them.

(7.47) *For each* $g \in M^\infty(G, \mu_G)$, $T_g^{(n)} \equiv F_g^{(n)}(\mu_G)$. *In particular*
$$T^{(n)}: M^\infty(G, \mu_G) \to M^\infty(G, \mu_G);$$

$$f \equiv g(\mu_G) \quad \textit{implies} \quad T_f^{(n)} = T_g^{(n)}; \tag{7.48}$$

$$g \geq 0 \quad \textit{implies} \quad T_g^{(n)} \geq 0; \tag{7.49}$$

$$T_1^{(n)} = 1; \tag{7.50}$$

$$T_{af+bg}^{(n)} = aT_f^{(n)} + bT_g^{(n)}; \tag{7.51}$$

$$T_{\gamma(s)g}^{(n)} = \gamma(s)T_g^{(n)} \quad \textit{for all} \quad s \in G; \tag{7.52}$$

$$T_{f\circ\Pi_F}^{(n)} = \delta(f) \circ \Pi_F \quad \textit{for every} \quad f \in M^\infty(G/F, \mu_{G/F}). \tag{7.53}$$

PROOFS. First, we will prove (7.47). By (i), proposition 8, there is a set $A'(g) \subset G/F$, $\mu_{G/F}$-negligible, such that for each $\dot{x} \notin A'(g)$, g is $\beta_{\dot{x}}$-measurable and $g \equiv V(\dot{x}, \tilde{g})(\beta_{\dot{x}})$. The set $\Pi_F^{-1}(A'(g))$ is μ_G-negligible and for $x \notin \Pi_F^{-1}(A'(g))$, we obviously have

$$\begin{aligned} T_g^{(n)}(x) &= \frac{1}{\mu_F(U_n)} \int_G V(\dot{x}, \tilde{g})(y)\varphi_{xU_n}(y)\, d\beta_{\dot{x}}(y) \\ &= \frac{1}{\mu_F(U_n)} \int_G g(y)\varphi_{xU_n}(y)\, d\beta_{\dot{x}}(y) = F_g^{(n)}(x). \end{aligned} \tag{7.54}$$

Hence (7.47) is proved.

Statement (7.48) is obvious from the definition of $T^{(n)}$, and statements (7.49), (7.50), and (7.51) are immediate consequences of the corresponding properties of the kernel $V(\dot{x}, \tilde{g})$ (see (ii), proposition 8).

To prove (7.52), it is obviously enough to consider the case $g \geq 0$. Let, therefore, $g \in M_+^\infty(G, \mu_G)$, $s \in G$, and $x \in G$. By (7.5) we have for each $h \in \mathcal{K}(G)$,

$$\begin{aligned} \int_G V(\dot{x}, \gamma(s)\tilde{g})(y)h(y)\, d\beta_{\dot{x}}(y) &= B(\gamma(s)\tilde{g}, h)(\dot{x}) \\ &= B(\gamma(s)\tilde{g}, \gamma(s)(\gamma(s^{-1})h))(\dot{x}) = (\gamma(\dot{s})B(\tilde{g}, \gamma(s^{-1})h))(\dot{x}) \\ &= B(\tilde{g}, \gamma(s^{-1})h)(\dot{s}^{-1}\dot{x}) = \int_G V(\dot{s}^{-1}\dot{x}, \tilde{g})(y)(\gamma(s^{-1})h)(y)\, d\beta_{\dot{s}^{-1}\dot{x}}(y). \end{aligned} \tag{7.55}$$

This shows that the positive Radon measures

$$\sigma: h \to \int_G V(\dot{x}, \gamma(s)\tilde{g})(y)h(y)\, d\beta_{\dot{x}}(y) \tag{7.56}$$

and

$$\theta: h \to \int_G V(\dot{s}^{-1}\dot{x}, \tilde{g})(y)(\gamma(s^{-1})h(y)\, d\beta_{\dot{s}^{-1}\dot{x}}(y) \tag{7.57}$$

on G are identical. It follows that for every bounded "universally measurable" function $h: G \to R$,

$$\int_G h\, d\sigma = \int_G h\, d\theta. \tag{7.58}$$

In particular, for $h = \varphi_{xU_n}$ we obtain

$$\int_G V(\dot{x}, \gamma(s)\tilde{g})(y)\varphi_{xU_n}(y)\, d\beta_{\dot{x}}(y) = \int_G V(\dot{s}^{-1}\dot{x}, \tilde{g})(y)\varphi_{xU_n}(sy)\, d\beta_{\dot{s}^{-1}\dot{x}}(y) \tag{7.59}$$
$$= \int_G V(\dot{s}^{-1}\dot{x}, \tilde{g})(y)\varphi_{s^{-1}xU_n}(y)\, d\beta_{\dot{s}^{-1}\dot{x}};$$

we deduce

$$T^{(n)}_{\gamma(s)g}(x) = T^{(n)}_g(s^{-1}x) = (\gamma(s)T^{(n)}_g)(x), \tag{7.60}$$

and hence (7.52) is proved.

For (7.53), let $f \in M^\infty(G/F, \mu_{G/F})$. Then $f \circ \Pi_F \equiv \delta(f) \circ \Pi_F(\mu_G)$, and hence, by (7.48), and taking into account (iii), proposition 8, we have for each $x \in G$,

$$T^{(n)}_{f\circ\Pi_F}(x) = T^{(n)}_{\delta(f)\circ\Pi_F}(x) = \frac{1}{\mu_F(U_n)}\int_G V(\dot{x}, \widetilde{\delta(f)} \circ \Pi_F)(y)\varphi_{xU_n}(y)\, d\beta_{\dot{x}}(y) \tag{7.61}$$
$$= \frac{1}{\mu_F(U_n)}\delta(f)(\dot{x})\int_G \varphi_{xU_n}(y)\, d\beta_{\dot{x}}(y) = \delta(f)(\dot{x}) = (\delta(f) \circ \Pi_F)(x).$$

Therefore, (7.53) is also proved.

Let now $\mathfrak{U}$ be an *ultrafilter* on N finer than the Fréchet filter. Let $g \in M^\infty(G, \mu_G)$ and $x \in G$. Let us recall that $\sup_{n\in N} |T^{(n)}_g(x)| \le N_\infty(\tilde{g})$ (see the remark following formula (7.46)); hence we may define (the use of the ultrafilter $\mathfrak{U}$ was suggested by [5])

$$T_g(x) = \lim_{\mathfrak{U}} T^{(n)}_g(x) \qquad \text{for} \quad x \in G. \tag{7.62}$$

On the basis of (7.46)–(7.53) and (7.62), we may then state the following proposition.

PROPOSITION 10. (i) *The mapping $T: g \to T_g$ is a linear lifting of $M^\infty(G, \mu_G)$.*

(ii) *The linear lifting T commutes with the left translations of G.*

(iii) *Further, $T_{f\circ\Pi_F} = \delta(f) \circ \Pi_F$ for every $f \in M^\infty(G/F, \mu_{G/F})$.*

PROOF. (i) We shall only verify axiom (I). The verification of the other axioms of a linear lifting (namely (II)–(V)) follows from the corresponding properties ((7.48)–(7.51)) of $T^{(n)}$. Let then $g \in M^\infty(G, \mu_G)$. Since $T^{(n)}_g \equiv F^{(n)}_g$ (μ_G), for each $n \in N$ (see (7.47)), and since the sequence $(F^{(n)}_g(z))$ converges to $g(z)$ almost everywhere with respect to μ_G (see (7.42) and proposition 9), we conclude that $T_g \equiv g(\mu_G)$.

(ii) Let $g \in M^\infty(G, \mu_G)$, $s \in G$, and $x \in G$. We have

$$T_{\gamma(s)g}(x) = \lim_{\mathfrak{U}} T^{(n)}_{\gamma(s)g}(x) \tag{7.63}$$
$$= \lim_{\mathfrak{U}} T^{(n)}_g(s^{-1}x) \qquad \text{(by (7.52))}$$
$$= T_g(s^{-1}x) = (\gamma(s)T_g)(x).$$

Hence (ii) is proved.

(iii) Let $f \in M^\infty(G/F, \mu_{G/F})$ and $x \in G$. Using (7.53), we can write

(7.64) $$T_{f \circ \Pi_F}(x) = \lim_{\mathfrak{U}} T^{(n)}_{f \circ \Pi_F}(x) = (\delta(f) \circ \Pi_F)(x).$$

This completes the proof of proposition 10.

Combining proposition 10 with corollary 3 of section 6, we obtain the following theorem (which we state with complete hypotheses in order to facilitate further references).

THEOREM 3. *Let G be a locally compact group which is countable at infinity, $F \subset G$ a compact distinguished subgroup of G which is also a Lie group, and δ a lifting of $M^\infty(G/F, \mu_{G/F})$ commuting with the left translations of G/F. There is then a lifting η of $M^\infty(G, \mu_G)$ commuting with the left translations of G and such that $\eta(f \circ \Pi_F) = \delta(f) \circ \Pi_F$ for every $f \in M^\infty(G/F, \mu_{G/F})$.*

PROOF. Consider the linear lifting T of $M^\infty(G, \mu_G)$ given by proposition 10. Since T commutes with the left translations of G we may apply corollary 3 (with $X = G$ and $S = T$). There is then a lifting η of $M^\infty(G, \mu_G)$ with the following properties: the lifting η commutes with the left translations of G, and if $\mathcal{E} \subset M^\infty(G, \mu_G)$ is a subalgebra containing 1, closed for the pointwise convergence of bounded sequences, and such that $T|\mathcal{E}$ is multiplicative, then $\eta|\mathcal{E} = T|\mathcal{E}$. Now the subalgebra $M^\infty(G/F, \mu_{G/F}) \circ \Pi_F$ of $M^\infty(G, \mu_G)$ has the required properties. It follows that for each $f \in M^\infty(G/F, \mu_{G/F})$,

(7.65) $$\eta(f \circ \Pi_F) = T_{f \circ \Pi_F} = \delta(f) \circ \Pi_F.$$

Henceforth, theorem 3 is completely proved.

8. The set $\mathcal{J}$; proof of the main theorem

As we already remarked in section 3 (see the remark following proposition 2), in order to prove the existence of a lifting commuting with the left translations of an arbitrary locally compact group, it is sufficient to consider the case of a locally compact group which is countable at infinity and which can be approximated by Lie groups.

Throughout this section we shall assume therefore that *X is a locally compact group which is countable at infinity and which can be approximated by Lie groups.*

Denote by $\mathcal{J}$ the set of all couples (H, ρ_H) where

(8.1) *H is a compact distinguished subgroup of X;*

(8.2) *ρ_H is a lifting of $M^\infty(X/H, \mu_{X/H})$ commuting with the left translations of X/H.*

We shall *order* the set $\mathcal{J}$ as follows: we write $(H, \rho_H) \leq (K, \rho_K)$ if (we use here the correspondence $\rho \to \omega$ defined in section 6)

(8.3) $$H \supset K;$$

(8.4) $$\omega_K | M^\infty(X, H, \mu_X) = \omega_H.$$

In connection with (8.4), see also formulas (6.4).

The relation "$\leq$" is obviously reflexive. If $(H, \rho_H) \leq (K, \rho_K)$ and $(K, \rho_K) \leq (H, \rho_H)$, then of course $H = K$; therefore $M^\infty(X, H, \mu_X) = M^\infty(X, K, \mu_X)$, $\omega_K = \omega_H$ and finally $\rho_K = \rho_H$. Hence, the relation "$\leq$" is antisymmetric.

If $(H, \rho_H) \leq (K, \rho_K)$ and $(K, \rho_K) \leq (L, \rho_L)$, then $H \supset K$, $K \supset L$, and hence $H \supset L$. Further,

$$(8.5) \qquad \omega_L | M^\infty(X, H, \mu_X) = (\omega_L | M^\infty(X, K, \mu_X)) | M^\infty(X, H, \mu_X) = \omega_K | M^\infty(X, H, \mu_X) = \omega_H.$$

It follows that $(H, \rho_H) \leq (L, \rho_L)$, and therefore the relation "$\leq$" is transitive. Hence "$\leq$" defines an *order relation* in $\mathfrak{J}$.

REMARKS. (1) For two couples $(H, \rho_H) \in \mathfrak{J}$ and $(K, \rho_K) \in \mathfrak{J}$ with $H \supset K$, the equation (8.4) is equivalent with

$$(8.4') \qquad \omega_K(f \circ \Pi_H) = \omega_H(f \circ \Pi_H) \quad \text{for every} \quad f \in M^\infty(X/H, \mu_{X/H}).$$

This is an immediate consequence of the definition of $M^\infty(X, H, \mu_X)$ as the "saturated" of the algebra $M^\infty(X/H, \mu_{X/H}) \circ \Pi_H$ (see (6.2)).

(2) Let $H \subset X$, $H \neq \{e\}$ be a compact distinguished subgroup of X. There is then a compact distinguished subgroup K of X such that

(8.6) $K \subset H$, $K \neq H$ and H/K is a Lie group;

(8.7) H/K is isomorphic with $A = \Pi_K(H) \subset \Pi_K(X) = X/K$;

(8.8) the groups X/H and $(X/K)/A$ are isomorphic. The canonical isomorphism of X/H onto $(X/K)/A$ is given by u: $\Pi_H(x) \rightarrow \Pi_A(\Pi_K(x))$. The inverse of u is denoted by v.

In fact, let V be a neighborhood of e that does *not* contain H. Take $L \subset V$, a compact distinguished subgroup of X, such that X/L is a Lie group. Let $K = H \cap L$; then K is a compact distinguished subgroup of X, $K \subset H$, $K \subset V$, and hence $K \neq H$. Also H/K is isomorphic with $\Pi_L(H) \subset \Pi_L(X) = X/L$ (see [3], chapter 3, p. 48). Since $\Pi_L(H)$ is a compact subgroup of the Lie group X/L, it follows that (see [17], p. 186) $\Pi_L(H)$, and therefore, also H/K are Lie groups. In connection with the isomorphisms in (8.7) and (8.8), see ([3], chapter 3, p. 48) and ([3], chapter 3, pp. 28–29).

THEOREM 4. (i) *The set $\mathfrak{J}$ is inductive for the order relation "$\leq$" defined above.*

(ii) If $(H, \rho_H) \in \mathfrak{J}$ *is a maximal element, then* $H = \{e\}$.

PROOF. (i) Let $((H_j, \rho_{H_j}))_{j \in J}$ be a totally ordered family of elements of $\mathfrak{J}$ (we suppose that $j' < j''$ is equivalent with $(H_{j'}, \rho_{H_{j'}}) < (H_{j''}, \rho_{H_{j''}})$. We may assume without loss of generality that *all H_j are contained in some compact distinguished subgroup H of X.* Let $H_\infty = \bigcap_{j \in J} H_j$; then H_∞ is a compact distinguished subgroup of X.

We shall now construct a lifting ρ_{H_∞} of $M^\infty(X/H_\infty, \mu_{X/H_\infty})$ such that $(H_\infty, \rho_{H_\infty}) \geq (H_j, \rho_{H_j})$ for all $j \in J$.

Let $f \in M^\infty(X, \mu_X)$, and consider the family $(P_{H_j}\tilde{f})_{j \in J}$. For each $j \in J$ we shall choose a definite representative of the equivalence class $P_{H_j}\tilde{f}$, by applying the lifting ω_{H_j}. In fact, for each $j \in J$ we may define

(8.9) $$f_j = \omega_{H_j}(P_{H_j}\tilde{f})$$

(it is clear what the notation $\omega_{H_j}(\tilde{g})$ means).

We shall list below several properties of the mapping $f \to f_j$. Suppose $j \in J$ fixed; then we have

(8.10) $$f_j \in M^\infty(X/H_j, \mu_{X/H_j}) \circ \Pi_{H_j} \quad \text{and} \quad |f_j| \leq N_\infty(\tilde{f})$$

(see (6.6) and (6.7) in section 6, and remark (1) in section 4, respectively);

(8.11) *$f \equiv g(\mu_X)$ implies $f_j = g_j$;*

(8.12) *$f \geq 0$ implies $f_j \geq 0$ (since $P_{H_j}\tilde{f} \geq 0$);*

(8.13) *$1_j = 1$ (since $P_{H_j}(\tilde{1}) = \tilde{1}$);*

(8.14) *the mapping $f \to f_j$ is linear;*

(8.15) *the mapping $f \to f_j$ commutes with the left translations of X.*

We shall verify here only (8.15). In fact, let $f \in M^\infty(X, \mu_X)$ and $s \in X$. Since P_{H_j} commutes with the left translations of X (see proposition 5 in section 4) and ω_{H_j} commutes with the left translations of X (see section 6), we have

(8.16) $$\begin{aligned}(\gamma(s)f)_j &= \omega_{H_j}(P_{H_j}(\gamma(s)\tilde{f})) = \omega_{H_j}(\gamma(s)P_{H_j}\tilde{f}) \\ &= \gamma(s)\omega_{H_j}(P_{H_j}\tilde{f}) = \gamma(s)(f_j).\end{aligned}$$

There are now *two possibilities:*

(A) there is a countable cofinal sequence $(j(n))_{n\in N}$ in J;

(B) there is no countable cofinal sequence in J.

Case (A). It is obvious that $H_\infty = \bigcap_{n\in N} H_{j(n)}$. Let now $f \in M^\infty(X, H_\infty, \mu_X)$; then $f \equiv f_1 \circ \Pi_{H_\infty}(\mu_X)$, with $f_1 \in M^\infty(X/H_\infty, \mu_{X/H_\infty})$. By theorem 2 in section 5 and remark (3) in section 4, the sequence $(P_{H_{j(n)}} f)_{n\in N}$ converges to $P_{H_\infty} f = f_1 \circ \Pi_{H_\infty}$ almost everywhere with respect to μ_X. We deduce that

(8.17) *the sequence $(f_{j(n)}(x))_{n\in N}$ converges to $f(x)$ almost everywhere with respect to μ_X.*

Let us also remark that (use (8.10) and (6.4)) we have

(8.18) $$f_{j(n)} \in M^\infty(X/H_\infty, \mu_{X/H_\infty}) \circ \Pi_{H_\infty} \qquad \text{for every} \quad n \in N.$$

Let now $\mathfrak{U}$ be an ultrafilter on N finer than the Fréchet filter on N; since $\sup_{n\in N} |f_{j(n)}(x)| \leq N_\infty(\tilde{f})$ for each $x \in X$ (see (8.10)), we may define

(8.19) $$f_\infty(x) = \lim_{\mathfrak{U}} f_{j(n)}(x), \qquad \text{for} \quad x \in X.$$

It is clear that $f_\infty \in M^\infty(X, H_\infty, \mu_X)$. Actually, we have

(8.20) $$f_\infty \in M^\infty(X/H_\infty, \mu_{X/H_\infty}) \circ \Pi_{H_\infty};$$

in fact, if $\Pi_{H_\infty}(x) = \Pi_{H_\infty}(y)$, then $f_{j(n)}(x) = f_{j(n)}(y)$ for all $n \in N$ (use (8.18)), whence $f_\infty(x) = f_\infty(y)$. It is also clear (use (8.17)) that $f_\infty \equiv f(\mu_X)$.

Denote by ω' the mapping $f \to f_\infty$; hence $\omega'(f) = f_\infty$. Then

(8.21) $$\omega' \colon M^\infty(X, H_\infty, \mu_X) \to M^\infty(X/H_\infty, \mu_{X/H_\infty}) \circ \Pi_{H_\infty},$$

whence ω' *satisfies condition* (6.7) of section 6. It is easily verified that ω' *is a linear lifting of* $M^\infty(X, H_\infty, \mu_X)$ *commuting with the left translations of* X (use (8.11)–(8.15)). Let us also remark that

$$\omega'|M^\infty(X, H_{j(n)}, \mu_X) = \omega_{H_{j(n)}} \qquad \text{for each} \quad n \in N. \tag{8.22}$$

In fact, fix $n \in N$. Let $f \in M^\infty(X, H_{j(n)}, \mu_X)$. For $m \geq n$ we have (use the fact that $\tilde{f} = P_{H_{j(n)}}\tilde{f}$ (see remark (3) in section 4) and the equation

$$P_{H_{j(m)}}P_{H_{j(n)}} = P_{H_{j(n)}} \tag{8.23}$$

(see (i), proposition 4))

$$\begin{aligned} f_{j(m)} &= \omega_{H_{j(m)}}(P_{H_{j(m)}}\tilde{f}) = \omega_{H_{j(m)}}(P_{H_{j(m)}}(P_{H_{j(n)}}\tilde{f})) \\ &= \omega_{H_{j(m)}}(P_{H_{j(n)}}\tilde{f}) = \omega_{H_{j(n)}}(P_{H_{j(n)}}\tilde{f}) = f_{j(n)}. \end{aligned} \tag{8.24}$$

Since this is true for all $m \geq n$, we deduce that $f_\infty = f_{j(n)} = \omega_{H_{j(n)}}(f)$, that is $\omega'(f) = \omega_{H_{j(n)}}(f)$ and hence (8.22) is proved.

Let ρ' be the linear lifting of $M^\infty(X/H_\infty, \mu_{X/H_\infty})$ corresponding to ω' (under the correspondence $\omega \longrightarrow \rho$); ρ' commutes with the left translations of X/H_∞. Applying proposition 7 (with $H = H_\infty$), we get a lifting ρ_{H_∞} of $M^\infty(X/H_\infty, \mu_{X/H_\infty})$ with the following properties: the lifting ρ_{H_∞} commutes with the left translations of X/H_∞, and if $\mathcal{E} \subset M^\infty(X, H_\infty, \mu_X)$ is a subalgebra containing 1, closed for the pointwise convergence of bounded sequences and such that $\omega'|\mathcal{E}$ is multiplicative, then $\omega_{H_\infty}|\mathcal{E} = \omega'|\mathcal{E}$. For fixed $n \in N$, $M^\infty(X, H_{j(n)}, \mu_X)$ is a subalgebra of $M^\infty(X, H_\infty, \mu_X)$ with the required properties (use (8.22)); we deduce

$$\omega_{H_\infty}|M^\infty(X, H_{j(n)}, \mu_X) = \omega'|M^\infty(X, H_{j(n)}, \mu_X) = \omega_{H_{j(n)}}. \tag{8.25}$$

Formula (8.25) shows that

$$(H_\infty, \rho_{H_\infty}) \geq (H_{j(n)}, \rho_{H_{j(n)}}) \tag{8.26}$$

for each $n \in N$. Since the sequence $(j(n))_{n \in N}$ is cofinal in J, this completes the proof of case (A).

Case (B). We shall show first that

$$M^\infty(X, H_\infty, \mu_X) = \bigcup_{j \in J} M^\infty(X, H_j, \mu_X). \tag{8.27}$$

For this purpose, let $(K_n)_{n \in N}$ be an increasing sequence of compact subsets of X/H such that $\bigcup_{n \in N} K_n = X/H$. Let now $L_n = \Pi_H^{-1}(K_n)$; then $(L_n)_{n \in N}$ is an *increasing* sequence of *compact* subsets of X such that $\bigcup_{n \in N} L_n = X$. Since

$$\varphi_{L_n} = \varphi_{K_n} \circ \Pi_H \in M^\infty(X/H, \mu_{X/H}) \circ \Pi_H \tag{8.28}$$

for every $n \in N$, and since $H \supset H_\infty$, we deduce

$$\varphi_{L_n} \in M^\infty(X/H_\infty, \mu_{X/H_\infty}) \circ \Pi_{H_\infty} \qquad \text{for all} \quad n \in N. \tag{8.29}$$

To prove the equality (8.27) it is obviously enough to show that every function in the algebra $M^\infty(X/H_\infty, \mu_{X/H_\infty}) \circ \Pi_{H_\infty}$ belongs to some $M^\infty(X, H_j, \mu_X)$ (for a suitable $j \in J$). Let then

$$f \in M^\infty(X/H_\infty, \mu_{X/H_\infty}) \circ \Pi_{H_\infty}; \tag{8.30}$$

we have

$$\varphi_{L_n} f \in \mathcal{L}^1(X/H_\infty, \mu_{X/H_\infty}) \circ \Pi_{H_\infty} \qquad \text{for every} \quad n \in N. \tag{8.31}$$

By corollary 2 in section 5, for each $n \in N$ there is a sequence $(u_{n,p})_{p \in N}$ of functions belonging to

$$\mathcal{A} = \bigcup_{j \in J} \mathcal{K}(X/H_j) \circ \Pi_H \tag{8.32}$$

such that

$$\varphi_{L_n} f = \lim_{p \in N} u_{n,p} \tag{8.33}$$

almost everywhere with respect to μ_X; we may also assume that

$$|u_{n,p}| \leq \|f\|_\infty \qquad \text{for all} \quad n \in N, \quad p \in N. \tag{8.34}$$

Define now $u_n = \limsup_{p \in N} u_{n,p}$ for each $n \in N$, and $u = \limsup_{n \in N} u_n$; we obviously have $u_n \equiv \varphi_{L_n} f(\mu_X)$ for all $n \in N$, whence

$$u \equiv f(\mu_X). \tag{8.35}$$

But for each $(n, p) \in N \times N$, there is $j(n, p) \in J$ such that

$$u_{n,p} \in \mathcal{K}(X/H_{j(n,p)}) \circ \Pi_{H_{j(n,p)}} \subset M^\infty(X/H_{j(n,p)}, \mu_{X/H_{j(n,p)}}) \circ \Pi_{H_{j(n,p)}}. \tag{8.36}$$

The set $\{j(n, p) | (n, p) \in N \times N\}$ is *countable*. Hence, there is $j_0 \in J$ such that j_0 is a majorant of the set $\{j(n, p) | (n, p) \in N \times N\}$. Since $j_0 \geq j(n, p)$ for all $(n, p) \in N \times N$, we deduce

$$u_{n,p} \in M^\infty(X/H_{j_0}, \mu_{X/H_{j_0}}) \circ \Pi_{H_{j_0}} \qquad \text{for all} \quad (n, p) \in N \times N. \tag{8.37}$$

But

$$M^\infty(X/H_{j_0}, \mu_{X/H_{j_0}}) \circ \Pi_{H_{j_0}} \tag{8.38}$$

is closed under taking lim sup of bounded sequences (this follows from the fact that (8.38) is a lattice and is closed under limits of bounded pointwise convergent sequences); it follows that

$$u \in M^\infty(X/H_{j_0}, \mu_{X/H_{j_0}}) \circ \Pi_{H_{j_0}}. \tag{8.39}$$

From (8.35) and (8.39) we obtain $f \in M^\infty(X, H_{j_0}, \mu_X)$, and thus formula (8.27) is proved.

We shall now define ω_{H_∞} on $M^\infty(X, H_\infty, \mu_X)$ as follows. Let $f \in M^\infty(X, H_\infty, \mu_X)$. There is then $j \in J$ such that $f \in M^\infty(X, H_j, \mu_X)$; define

$$\omega_{H_\infty}(f) = \omega_{H_j}(f). \tag{8.40}$$

Note that $\omega_{H_\infty}(f)$ is well-defined by the equation (8.40). In fact, suppose $f \in M^\infty(X, H_i, \mu_X)$ for some other $i \in J$. Since the set J is totally ordered, we have either $i \leq j$ (hence $(H_i, \rho_{H_i}) \leq (H_j, \rho_{H_j})$), or $j \leq i$ (hence $(H_j, \rho_{H_j}) \leq (H_i, \rho_{H_i})$). In either case we deduce

$$\omega_{H_i}(f) = \omega_{H_j}(f); \tag{8.41}$$

thus ω_{H_∞} is well-defined. Formula (8.40) also shows that

(8.42) $\omega_{H_\infty}(f) = \omega_{H_i}(f) \in M^\infty(X/H_j, \mu_{X/H_i}) \circ \Pi_{H_i} \subset M^\infty(X/H_\infty, \mu_{X/H_\infty}) \circ \Pi_{H_\infty}$,

and hence, ω_{H_∞} *satisfies condition* (6.7):

(8.43) $$\omega_{H_\infty}: M^\infty(X, H_\infty, \mu_X) \to M^\infty(X/H_\infty, \mu_{X/H_\infty}) \circ \Pi_{H_\infty}.$$

It is also easily verified that ω_{H_∞} is a *lifting of* $M^\infty(X, H_\infty, \mu_X)$ *commuting with the left translations of* X (make use of the corresponding properties of the liftings ω_{H_j}, and the fact that the set J is totally ordered).

Let ρ_{H_∞} be the *lifting of* $M^\infty(X/H_\infty, \mu_{X/H_\infty})$ *corresponding* to ω_{H_∞} (under the correspondence $\omega \to \rho$). The lifting ρ_{H_∞} of $M^\infty(X/H_\infty, \mu_{X/H_\infty})$ *commutes with the left translations of* X/H_∞. Moreover, the formula (8.40) of definition of ω_{H_∞} shows that

(8.44) $$\omega_{H_\infty}|M^\infty(X, H_j, \mu_X) = \omega_{H_i} \qquad \text{for each} \quad j \in J.$$

It follows that $(H_\infty, \rho_{H_\infty}) \geq (H_j, \rho_{H_j})$ for each $j \in J$; this completes the proof for case (B).

(ii) Let (H, ρ_H) be a maximal element of $\mathcal{G}$ and suppose $H \neq \{e\}$. By remark (2) preceding theorem 4, there is a compact distinguished subgroup K of X satisfying the conditions (8.6), (8.7) and (8.8). We shall use below the notations of these conditions; in particular, we shall use the isomorphisms u and v. Define δ on $M^\infty((X/K)/A, \mu_{(X/K)/A})$ by the equations

(8.45) $$\delta(f) = \rho_H(f \circ u) \circ v \qquad \text{for} \quad f \in M^\infty((X/K)/A, \mu_{(X/K)/A}).$$

By proposition 3 in section 3, δ is a lifting of $M^\infty((X/K)/A, \mu_{(X/K)/A})$, commuting with the left translations of $(X/K)/A$. Since X/K is countable at infinity, and since the compact distinguished subgroup A of X/K is a Lie group (see (8.6), (8.7)), we may apply theorem 3 (with $G = X/K$ and $F = A$, of course). We deduce the existence of a lifting ρ_K of $M^\infty(X/K, \mu_{X/K})$ commuting with the left translations of X/K and such that

(8.46) $$\rho_K(f \circ \Pi_A) = \delta(f) \circ \Pi_A \qquad \text{for} \quad f \in M^\infty((X/K)/A, \mu_{(X/K)/A}).$$

We shall show that

(8.47) $$\omega_K(g \circ \Pi_H) = \omega_H(g \circ \Pi_H) \qquad \text{for every} \quad g \in M^\infty(X/H, \mu_{X/H}).$$

In fact, let $g \in M^\infty(X/H, \mu_{X/H})$. Using (8.46), (8.45), and the explicit form of the canonical isomorphisms u and v (see (8.8)), we can write the following:

(8.48)
$$\begin{aligned}
\omega_K(g \circ \Pi_H) &= \omega_K((g \circ v) \circ u \circ \Pi_H) = \omega_K((g \circ v) \circ \Pi_A \circ \Pi_K) \\
&= \rho_K((g \circ v) \circ \Pi_A) \circ \Pi_K \quad (\text{since } (g \circ v) \circ \Pi_A \in M^\infty(X/K, \mu_{X/K})) \\
&= (\delta(g \circ v) \circ \Pi_A) \circ \Pi_K = ((\rho_H((g \circ v) \circ u) \circ v) \circ \Pi_A) \circ \Pi_K \\
&= (\rho_H(g) \circ v) \circ \Pi_A \circ \Pi_K = (\rho_H(g) \circ v) \circ u \circ \Pi_H \\
&= \rho_H(g) \circ \Pi_H = \omega_H(g \circ \Pi_H).
\end{aligned}$$

Hence (8.47) is verified. This shows that (see also remark (1) preceding theorem 4) $(K, \rho_K) \geq (H, \rho_H)$; since $K \neq H$, this means $(K, \rho_K) > (H, \rho_H)$. The inequality contradicts the maximality of the element (H, ρ_H) and thus completes the proof of (ii).

Therefore, theorem 4 is completely proved.

9. Conclusions

As an immediate consequence of theorem 4 we deduce (see also the remark following proposition 2 in section 3, or the comment at the beginning of section 8).

THEOREM 5. *Let X be an arbitrary locally compact group. There exists a lifting of $M^\infty(X, \mu_X)$ commuting with the left translations of X.*

Before ending we wish to make one more remark. Again let X be a locally compact group with left Haar measure μ_X. Denote by $\mathfrak{I}$ the tribe of all μ_X-measurable subsets of X. Let us recall that a mapping $\theta: \mathfrak{I} \to \mathfrak{I}$ is called a *lower density* of $\mathfrak{I}$ if it satisfies the following axioms (for $A \in \mathfrak{I}$, $B \in \mathfrak{I}$ we write $A \equiv B$ if $A \Delta B = (A - B) \cup (B - A)$ is locally μ_X-negligible):

(I′) $\theta(A) \equiv A$;

(II′) $A \equiv B$ implies $\theta(A) = \theta(B)$;

(III′) $\theta(\emptyset) = \emptyset$ and $\theta(X) = X$;

(IV′) $\theta(A \cap B) = \theta(A) \cap \theta(B)$.

The lower density θ of $\mathfrak{I}$ is said to commute with the group of left translations of X if

$$\text{(V}'\text{)}\quad \theta(sA) = s\theta(A) \qquad \text{for every} \quad s \in X \quad \text{and} \quad A \in \mathfrak{I}. \tag{9.1}$$

It was already known (see [8], theorem 1, p. 823) that for an arbitrary locally compact group X, the existence of a lifting of $M^\infty(X, \mu_X)$ commuting with the left translations of X is equivalent with the existence of a lower density of $\mathfrak{I}$ commuting with the left translations of X (the implication "$\Rightarrow$" is obvious; the implication "$\Leftarrow$" was proved in [8] using the "topology on X induced by the lower density" and considerations of maximal ideals of L^∞).

We now know (by theorem 5 above) that for an arbitrary locally compact group X there always exists a lower density of $\mathfrak{I}$ commuting with the left translations of X: we shall call such a lower density of $\mathfrak{I}$ (for obvious reasons) *a Haar lower density.*

It should be pointed out here that, in contrast to the Haar measure, a Haar lower density is by no means unique, not even in the case of the real line. In fact, let $I_n^{(l)}(x) = [x - 1/n, x]$, $I_n^{(r)}(x) = [x, x + 1/n]$ for each $x \in R$ and each $n \in N^*$. For every $A \in \mathfrak{I}$ define

$$D_A^{(l)}(x) = \lim_{n \to \infty} (\mu_R(A \cap I_n^{(l)}(x))/\mu_R(I_n^{(l)}(x)) \tag{9.2}$$

whenever this limit exists; similarly define

$$D_A^{(r)}(x) = \lim_{n \to \infty} (\mu_R(A \cap I_n^{(r)}(x))/\mu_R(I_n^{(r)}(x)). \tag{9.3}$$

Let now

$$\theta^{(l)}(A) = \{x \in R | D_A^{(l)}(x) \quad \text{exists and } = 1\} \tag{9.4}$$

and

$$\theta^{(r)}(A) = \{x \in R | D_A^{(r)}(x) \quad \text{exists and } = 1\}. \tag{9.5}$$

It is clear that $\theta^{(l)}$ and $\theta^{(r)}$ are lower densities commuting with the translations of R. However $\theta^{(l)}([0, 1]) = (0, 1]$ and $\theta^{(r)}([0, 1]) = [0, 1)$; hence $\theta^{(l)} \neq \theta^{(r)}$. In the case of the real line, or more generally in R^n, we do have, however, a canonical Haar lower density—namely the classical "Lebesgue lower density"; this is defined at a given point $x \in R^n$ using all intervals containing x (see [19]).

For further remarks connected with the subject of this paper, see ([9], section 4).

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APPENDIX I. REMARKS ON ADEQUATE FAMILIES

Let T and X be locally compact spaces and $\mu \in \mathfrak{M}_+(T)$. Let $\lambda: t \to \lambda_t$ be a μ-adequate family of Radon measures on X and $\nu = \int_T \lambda_t \, d\mu(t)$ (for the terminology and results used in this appendix, (see [2], chapter V, paragraph 3, pp. 17–24). We suppose that the following condition (C) is satisfied.

(C) *For every compact $K \subset T$ there is a compact set $K(1) \subset X$ such that, for each $t \in K$, we have* Supp $\lambda_t \subset K(1)$.

The following assertions are then valid.

PROPOSITION (I.1). *For every numerical function $f \geq 0$ on X we have*

$$\overline{\int_X^*} f \, d\nu \geq \overline{\int_T^*} d\mu(t) \int_X^* f \, d\lambda_t. \tag{1}$$

Let $K \subset T$ be a compact set. Then

(2)

$$\overline{\int_T^*} \varphi_K(t) \, d\mu(t) \int_X^* f \, d\lambda_t = \overline{\int_T^*} \varphi_K(t) \, d\mu(t) \int_X^* \varphi_{K(1)} f \, d\lambda_t$$
$$\leq \overline{\int_T^*} d\mu(t) \int_X^* \varphi_{K(1)} f \, d\lambda_t \leq \int_X^* \varphi_{K(1)} f \, d\nu \leq \overline{\int_X^*} f \, d\nu;$$

since $K \subset T$ was arbitrary, the proposition is proved.

From proposition (I.1) we immediately deduce the following.

PROPOSITION (I.2). *If $f: X \to \overline{R}$ is locally ν-negligible, then the set of all $t \in T$ for which f is not λ_t-negligible is locally μ-negligible.*

PROPOSITION (I.3). *If $f: X \to \overline{R}$ is ν-measurable, then the set of all $t \in T$ for which f is not λ_t-measurable is locally μ-negligible.*

Let $K \subset T$ be a compact set. Then the set of all $t \in K$ for which $\varphi_{K(1)} f$ is not λ_t-measurable is μ-negligible. But $\varphi_{K(1)} f$ and f coincide on Supp λ_t for $t \in K$; we deduce that the set of all $t \in K$ for which f is not λ_t-measurable is μ-negligible. Since K was arbitrary, proposition (I.3) is proved.

PROPOSITION (I.4). *For every numerical ν-measurable function $f \geq 0$ on X, the mapping $t \to \int_X^* f\, d\lambda_t$ is μ-measurable and*

$$\overline{\int_X^*} f\, d\nu = \overline{\int_T^*} d\mu(t) \int_X^* f\, d\lambda_t. \tag{3}$$

Let $K \subset T$ be a compact set; then $t \to \int_X^* \varphi_{K(1)} f\, d\lambda_t$ is μ-measurable. Since

$$\int_X^* \varphi_{K(1)} f\, d\lambda_t = \int_X^* f\, d\lambda_t \qquad \text{for} \quad t \in K \tag{4}$$

and since K was arbitrary, the measurability of the mapping $t \to \int_X^* f\, d\lambda_t$ follows. To prove the equation in proposition (I.4), consider a compact $L \subset X$; then

$$\int_X^* \varphi_L f\, d\nu = \overline{\int_T^*} d\mu(t) \int_X^* \varphi_L f\, d\lambda_t \leq \overline{\int_T^*} d\mu(t) \int_X^* f\, d\lambda_t. \tag{5}$$

Since L was arbitrary, we deduce

$$\overline{\int_X^*} f\, d\nu \leq \overline{\int_T^*} d\mu(t) \int_X^* f\, d\lambda_t. \tag{6}$$

Combining this with the inequality in proposition (I.1), we obtain the equation in proposition (I.4).

PROPOSITION (I.5). *If $f\colon X \to \overline{R}$ is locally ν-integrable, then the set of all $t \in T$ for which f is not λ_t-integrable is locally μ-negligible, and the mapping $t \to \int_X f\, d\lambda_t$ (defined locally almost everywhere with respect to μ) is locally μ-integrable.*

Let $K \subset T$ be a compact set. Then $\varphi_{K(1)} f$ is ν-integrable, whence the set of all $t \in K$ for which $\varphi_{K(1)} f$ is not λ_t-integrable is μ-negligible. It follows that the set of all $t \in K$ for which f is not λ_t-integrable is μ-negligible. Since K was arbitrary, we deduce that the set of all $t \in T$ for which f is not λ_t-integrable is locally μ-negligible. Let again $K \subset T$ be a compact set. Then

$$\varphi_K(t) \int_X f\, d\lambda_t = \int_X \varphi_{K(1)} f\, d\lambda_t \tag{7}$$

for almost every $t \in K$ (with respect to μ). Since $\varphi_{K(1)} f$ is ν-integrable, we deduce that $t \to \varphi_K(t) \int_X f\, d\lambda_t$ is μ-integrable; therefore, $t \to \int_X f\, d\lambda_t$ (which is defined only locally almost everywhere with respect to μ) is locally μ-integrable.

REMARK. From the above follows that if $f\colon X \to \overline{R}$ is locally ν-integrable (in particular, if f is essentially ν-integrable), then the mapping $t \to \int_X f\, d\lambda_t$ (defined only locally almost everywhere with respect to μ) is μ-measurable.

PROPOSITION (I.6). *If $f\colon X \to \overline{R}$ is essentially ν-integrable, then the set of all $t \in T$ for which f is not λ_t-integrable is locally μ-negligible, the mapping $t \to \int_X f\, d\lambda_t$ (defined only locally almost everywhere with respect to μ) is essentially μ-integrable and*

$$\int_X f\, d\nu = \int_T d\mu(t) \int_X f\, d\lambda_t. \tag{8}$$

Since $f: X \to \overline{R}$ is essentially ν-integrable if and only if $\sup(f, 0)$ and $\sup(-f, 0)$ are, it is enough to consider the case $f \geq 0$. The assertion then follows from proposition 7 in ([2], chapter V, paragraph 2), and from propositions (I.4) and (I.5) above.

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APPENDIX II. A MAXIMAL THEOREM

(II.1) Let $(X, \mathfrak{Z}, \mu)$ be an abstract measure space, and let ν be a second measure on $\mathfrak{Z}$. We suppose that there is a clan $\mathfrak{Z}_0 \subset \mathfrak{Z}$ consisting of sets having μ and ν finite measure, such that

(i) $A \in \mathfrak{Z}$ and $B \in \mathfrak{Z}_0$ implies $A \cap B \in \mathfrak{Z}_0$;

(ii) for every $A \in \mathfrak{Z}$ we have

$$\mu(A) = \sup_{B \in T_0} \mu(A \cap B) \quad \text{and} \quad \nu(A) = \sup_{B \in T_0} \nu(A \cap B).$$

THE MAXIMAL THEOREM. *Let $(A_n)_{n \in N}$ be a sequence of sets belonging to $\mathfrak{Z}_0$. For every $n \in N$ and $x \in A_n$ let $U(n, x)$ and $U^*(n, x)$ be two sets, the first of which belongs to $\mathfrak{Z}$, such that*

(1) $\mu^*(\bigcup_{x \in A_n} U(n, x)) < \infty$ *for each* $n \in N$;

(2) $\nu(U(n, x)) \geq \mu(U(n, x))$ *for every* $n \in N$ *and* $x \in A_n$;

(3) $\inf_{x \in A_n} \mu(U(n, x)) > 0$ *for every* $n \in N$ *for which* $A_n \neq \varnothing$;

(4) *for every* $n \in N$ *and* $x \in A_n$ *we have* $\mu^*(U^*(n, x)) \leq C\mu(U(n, x))$, *where* C *is a constant independent of* $n \in N$ *and* $x \in A_n$;

(5) *let* $p \in N$, $q \in N$, $p \leq q$; *if* $x \in A_p$ *and* $y \in A_q \cap \complement U^*(p, x)$ *we have*

$$U(p, x) \cap U(q, y) = \varnothing.$$

There is then a sequence $(U(u(j), x(j)))_{j \in I}$ (with $I \subset N$ and $u: I \to N$) consisting of disjoint sets such that

(6) $A_\infty = \bigcup_{n \in N} A_n \subset \bigcup_{j \in I} U^*(u(j), x(j))$;

(7) $\mu(A_\infty) \leq C\nu(\bigcup_{j \in I} U(u(j), x(j)))$.

REMARKS. (a) Once (6) is proved, (7) follows immediately from (4) and (2), respectively. (b) We may of course suppose during the proof below that $A_\infty \neq \varnothing$. (c) From (7) we deduce $\mu(A_\infty) \leq C\nu(X)$; in many applications $\nu(X) < \infty$, and then this inequality becomes relevant.

PROOF. Let α be the first integer $n \in N$ for which $A_n \neq \varnothing$. Let $\mathfrak{U}$ be the set of all sequences $(U(u(j), x(j)))_{j \in I}$, with I an interval of N and $u: I \to N$ an increasing mapping such that

(8) $I \ni 0$, $u(0) = \alpha$, and $x(0) \in A_{u(0)}$;

(9) if $j \in N$ and $j + 1 \in I$, then

$$x(j+1) \in A_{u(j+1)} - \bigcup_{0 \leq s \leq j} U^*(u(s), x(s));$$

(10) if $j \in N$ and $j + 1 \in I$, then

$$A_h - \bigcup_{0 \leq s \leq j} U^*(u(s), x(s)) = \emptyset$$

for all $0 \leq h < u(j+1)$ (if such an h exists).

Let now $(U(u(j), x(j)))_{j \in J} \in \mathfrak{U}$. It is clear then that $x(j) \in A_{u(j)}$ for all $j \in I$. Also the sets in the sequence are disjoint (if there are more than one). In fact, let $i \in I$, $j \in I$ with $i < j$; let $p = u(i) \leq q = u(j)$. Then $x(i) \in A_p$ and $x(j) \in A_q - \bigcup_{0 \leq s \leq j-1} U^*(u(s), x(s))$. By (5) the sets $U(u(i), x(i))$ and $U(u(j), x(j))$ are disjoint, whence the assertion is proved. Clearly, $\mathfrak{U} \neq \emptyset$ since $(U(u(0), x(0)))$, with $u(0) = \alpha$ and $x(0)$ some element in A_α, belongs to $\mathfrak{U}$.

By remark (1) above, in order to prove the theorem it is enough to show the existence of a sequence $(U(u(j), x(j)))_{j \in I}$ in $\mathfrak{U}$ satisfying (6). We shall divide the proof into two parts:

(I) we shall show here that if $(U(u(j), x(j)))_{j \in N} \in \mathfrak{U}$, that is, I is infinite, then this family satisfies (6). Note first that

$$\lim_{j \in N} u(j) = \infty. \tag{11}$$

In fact, otherwise $u(j) \leq q$ for some $q \in N$ and all $j \in N$. Then $x(j) \in A_0 \cup A_1 \cup \cdots \cup A_q$ for all $j \in N$. Let now $\tilde{B}$ be a set with $\mu^*(\tilde{B}) < \infty$ containing all $U(n, x)$ with $0 \leq n \leq q$ and $x \in A_n$ (use (1)). Since the sets $(U(u(j), x(j)))_{j \in N}$ are disjoint, we obtain $\lim_{j \in N} \mu(U(u(j), x(j))) = 0$; since $u(j)$ must be constant for j large enough, this contradicts (3) and therefore (11) is proved. Let now $h \in N$. By (11), there is $j \in N$ such that $h < u(j+1)$; by (10),

$$A_h \subset \bigcup_{0 \leq s \leq j} U^*(u(s), x(s)) \subset \bigcup_{s \in N} U^*(u(s), x(s)). \tag{12}$$

Since $h \in N$ was arbitrary, (6) is proved in this case.

(II) If there is a *finite* family in $\mathfrak{U}$ satisfying (6), then clearly the proof is completed. Otherwise, let $(U(u(j), x(j)))_{j \in I} \in \mathfrak{U}$ be a finite family; then $A_\infty - \bigcup_{j \in I} U^*(u(j), x(j)) \neq 0$. Let $p = \sup I$ and let $I' = I \cup \{p+1\}$. Let q be the first integer in N for which

$$A_q - \bigcup_{j \in I} U^*(u(j), x(j)) \neq \emptyset; \tag{13}$$

define $u(p+1) = q$. If $p = 0$, we obviously have $q = u(p+1) \geq u(p) = \alpha$, since $A_q \neq \emptyset$. If $p > 0$, then by (10), $A_h - \bigcup_{0 \leq s \leq p-1} U^*(u(s), x(s)) = \emptyset$, and hence $A_h - \bigcup_{0 \leq s \leq p} U^*(u(s), x(s)) = \emptyset$ for every $0 \leq h < u(p)$; since q is the first integer in N satisfying (13), we deduce $q = u(p+1) \geq u(p)$. Let $x(p+1)$ be an (arbitrary) element in the set (13). Clearly the sequence $(U(u(j), x(j)))_{j \in I'} \in \mathfrak{U}$. This inductive argument shows how to construct a family $(U(u(j), x(j)))_{j \in N} \in \mathfrak{U}$ and hence completes the proof of the theorem.

A VARIANT OF THE MAXIMAL THEOREM. *Let $(A_n)_{n \in N}$ be a sequence of sets belonging to $\mathfrak{Z}$. For every $n \in N$ and $x \in A_n$ let $U(n, x)$ and $U^*(n, x)$ be two sets, the first of which belongs to $\mathfrak{Z}$. Suppose that*

(14) *for each $B \in \mathfrak{Z}_0$ and each $n \in N$ we have*

$$\mu^*\Big(\bigcup_{x \in B \cap A_n} U(n, x)\Big) < \infty;$$

(15) *the conditions* (2), (3), (4), (5) *of the maximal theorem are satisfied. Then*

$$\mu(A_\infty) \leq C\nu(X) \quad \text{if} \quad A_\infty = \bigcup_{n\in N} A_n. \tag{16}$$

PROOF. For every $n \in N$ let $A'_n \subset A_n$, $A'_n \in \mathfrak{J}_0$. The previous theorem gives then

$$\mu(\bigcup_{n\in N} A'_n) \leq C\nu(X); \tag{17}$$

from this (16) follows immediately.

(II.2) Here $(X, \mathfrak{J}, \mu)$ is as in (II.1), the clan $\mathfrak{J}_0$ being the set of all $A \in \mathfrak{J}$ for which $\mu(A) < \infty$. Let β be a measure on $\mathfrak{J}$ with $\beta(X) < \infty$. Before proceeding further, we want to remark that the following is also a consequence of the maximal theorem.

THEOREM 1. *Let $(\mathfrak{F}_n)_{n\in N}$ be an increasing sequence of tribes contained in $\mathfrak{J}$. For each $n \in N$ let $f_n \in \mathfrak{L}^1(X, \mathfrak{J}, \mu)$ be a $\mathfrak{F}_n$-measurable function satisfying*

$$\beta(A) \geq \int_A f_n\, d\mu \qquad \text{for all} \quad A \in \mathfrak{F}_n. \tag{18}$$

For every $a > 0$ we then have

$$a\mu(\{x \mid \sup_{n\in N} f_n(x) > a\}) \leq \beta(X). \tag{19}$$

PROOF. Define $\nu = (1/a)\beta$; define $A'_0 = \{x \mid f_0(x) > a\}$ and

$$A'_n = \{x \mid \sup_{0\leq j\leq n-1} f_j(x) \leq a, f_n(x) > a\} \qquad \text{for} \quad n \geq 1. \tag{20}$$

Clearly the set $\{x \mid \sup_{n\in N} f_n(x) > a\}$ equals the union of $(A'_n)_{n\in N}$. Let $A_n = A'_n$ if $\mu(A'_n) \neq 0$, and $A_n = \emptyset$ if $\mu(A'_n) = 0$. Finally let $U(n, x) = U^*(n, x) = A_n$ for all $x \in A_n$ and all $n \in N$ for which $A_n \neq \emptyset$. It is then easily seen that the conditions (1)–(5) of the maximal theorem are satisfied. For instance, (5) can be verified as follows: let $p \in N$, $q \in N$, $p \leq q$. If $x \in A_p$ and $y \in A_q \cap \complement A_p$, then $A_q \neq A_p$; hence, $p \neq q$, and therefore, $A_p \cap A_q = \emptyset$; that is, $U(p, x) \cap U(q, y) = \emptyset$. Hence the proof is completed.

(II.3) Let X be a locally compact group, λ a *left Haar measure*, $\mathfrak{J}$ the tribe of all λ-measurable parts of X, and $\mathfrak{J}_0$ the clan of all $A \in \mathfrak{J}$ which are relatively compact. Let μ be the *essential measure* corresponding to λ; note that λ and μ coincide on $\mathfrak{J}_0$.

Let $U \in \mathfrak{J}_0$ with $\lambda(U) > 0$. For f *locally λ-integrable* define

$$f_U(x) = \frac{1}{\lambda(U)} \int_{xU} f\, d\lambda. \tag{21}$$

It is clear that f_U is a continuous function; moreover, if $f \in \mathfrak{L}^1(X, \lambda)$, then it is easily verified that $f_U \in \mathfrak{L}^1(X, \lambda)$ and

$$\|f_U\|_1 \leq (\sup_{s\in U} \Delta(s^{-1}))\|f\|_1 \qquad (\Delta = \text{the modular function}). \tag{22}$$

Suppose now that $(U_n)_{n\in N}$ is a sequence of sets belonging to $\mathfrak{J}_0$ and satisfying the following conditions:

(j) the sequence $(U_n)_{n \in N}$ is decreasing;

(jj) every neighborhood of e contains some set U_k with $k \in N$;

(jjj) $0 < \lambda^*(U_n U_n^{-1}) \leq C\lambda(U_n)$ for all $n \in N$ and some constant C.

REMARK. In theorem 2 below we do not make use of condition (jj) above.

THEOREM 2 (R. E. Edwards and E. Hewitt). *Let $f \in \mathcal{L}^1(X, \lambda)$. For each $n \in N$, let $f_n = f_{U_n}$ (see (21) above). For every $a > 0$ we then have*

$$a\lambda(\{x | \sup_{n \in N} f_n(x) > a\}) \leq C \int_X |f| \, d\lambda. \tag{23}$$

PROOF. Let $\nu = (1/a)|f|.\lambda$. Since the Radon measure ν is absolutely continuous with respect to λ, the sets in $\mathfrak{I}$ are *all* ν-measurable, whence ν is naturally defined on $\mathfrak{I}$. Let now $A_n = \{x | f_n(x) > a\}$ for $n \in N$. Clearly,

$$\{x | \sup_{n \in N} f_n(x) > a\} = \bigcup_{n \in N} A_n. \tag{24}$$

Finally let $U(n, x) = xU_n$, $U^*(n, x) = xU_n U_n^{-1}$ for all $n \in N$ for which $A_n \neq \emptyset$. Then the conditions (14) and (15) of the variant of the maximal theorem are satisfied (use also the left invariance of λ, and note that *on the sets we consider here, λ and μ coincide*. To verify (5), for instance, we reason as follows: let $p \in N$, $q \in N$, $p \leq q$. If $x \in A_p$ and $y \in A_q \cap \complement xU_p U_p^{-1}$, then $y \notin xU_p U_p^{-1}$, and hence $y \notin xU_p U_q^{-1}$ (we have $U_p \supset U_q$, and thus $U_p^{-1} \supset U_q^{-1}$). We deduce $xU_p \cap yU_q = \emptyset$ (otherwise $xu_p = yu_q$ for some $u_p \in U_p$, $u_q \in U_q$, which would imply $y = xu_p u_q^{-1} \in xU_p U_q^{-1}$). The conclusion then follows from the variant of the maximal theorem.

REMARK. Let f: $X \to R$ be locally λ-integrable and $K \subset X$ a compact set. There is then a compact set $L \subset X$ such that $xU_0 \subset L$ for all $x \in K$. We deduce $f_n | K = (f\varphi_L)_n | K$ for each $n \in N$ (it is clear that we use here notations similar to those of theorem 2; see also (21)).

COROLLARY 1. *Let f: $X \to R$ be locally λ-integrable. The sequence $(f_n)_{n \in N}$ converges to f locally almost everywhere with respect to λ.*

PROOF. The remark preceding this corollary shows that it is enough to consider the case when $f \in \mathcal{L}^1(X, \lambda)$. Since the assertion is obvious for $f \in \mathcal{K}(X)$, and since $\mathcal{K}(X)$ is dense in $\mathcal{L}^1(X, \lambda)$, the corollary follows from theorem 2 above, formula (22) and the classical (almost everywhere) convergence theorem of Banach.

(II.4) Concerning the subject of this appendix, see also [1], [4], [7], and [15].

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