A CLASS OF OPTIMAL STOPPING PROBLEMS

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1. Introduction and summary

Let $x_1, x_2, \cdots$, be independent random variables uniformly distributed on the interval $[0, 1]$. We observe them sequentially, and must stop with some x_i, $1 \leq i < \infty$; the decision whether to stop with any x_i must be a function of the values $x_1, \cdots, x_i$ only. (For a general discussion of optimal stopping problems we refer to [1], [3].) If we stop with x_i we lose the amount $i^\alpha x_i$, where $\alpha \geq 0$ is a given constant. What is the minimal expected loss we can achieve by the proper choice of a stopping rule?

Let C denote the class of all possible stopping rules t; then we wish to evaluate the function

$$v(\alpha) = \inf_{t \in C} E(t^\alpha x_t). \tag{1}$$

If there exists a t in C such that $E(t^\alpha x_t) = v(\alpha)$, we say that t is optimal for that value of α. Let C^N for $N \geq 1$ denote the class of all t in C such that $P[t \leq N] = 1$; then $C^1 \subset C^2 \subset \cdots \subset C$, and hence, defining

$$v^N(\alpha) = \inf_{t \in C^N} E(t^\alpha x_t), \tag{2}$$

we have

$$\tfrac{1}{2} = v^1(\alpha) \geq v^2(\alpha) \geq \cdots \geq v(\alpha) \geq 0. \tag{3}$$

We shall show that as $N \to \infty$,

$$v^N(\alpha) \sim \begin{cases} 2(1-\alpha)/N^{1-\alpha} & \text{for } 0 \leq \alpha < 1, \\ 2/\log N & \text{for } \alpha = 1, \end{cases} \tag{4}$$

from which it follows that

$$v(\alpha) = 0, \qquad \text{for } 0 \leq \alpha \leq 1. \tag{5}$$

(For $\alpha = 0$, J. P. Gilbert and F. Mosteller [4] give the expression $v^N(0) \approx 2/(N + \log(N+1) + 1.767)$; this case is closely related to a problem of optimal selection considered in [2]. It can be shown that $Nv^N(0) \uparrow 2$ as $N \to \infty$.)

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We shall show, moreover, that

$$\begin{cases} 0 < v(\alpha) < \tfrac{1}{2}, & \text{for } 1 < \alpha \leq 1.4, \\ \quad v(\alpha) = \tfrac{1}{2}, & \text{for } \alpha \geq 1.5, \end{cases} \tag{6}$$

and that the relation

$$\lim_{N \to \infty} v^N(\alpha) = v(\alpha) \tag{7}$$

holds for all $\alpha \geq 0$. No optimal rule exists for $0 \leq \alpha \leq 1$ by (5), since $E(t^\alpha x_t) > 0$ for every t in C. We shall show that an optimal rule does exist for every $\alpha > 1$; when $v(\alpha) = \frac{1}{2}$ the optimal rule is $t = 1$, but for any α such that $0 < v(\alpha) < \frac{1}{2}$ the optimal rule t is such that $Et = \infty$. The function $v(\alpha)$ is continuous for all $\alpha \geq 0$.

2. Proof of (4)

For any fixed $\alpha \geq 0$ and $N \geq 1$, set $v_{N+1}^N = \infty$ and define

$$v_i^N = E\{\min(i^\alpha x_i, v_{i+1}^N)\} = \int_0^1 \min(i^\alpha x, v_{i+1}^N)\, dx \qquad (i = N, \cdots, 1). \tag{8}$$

The constants v_i^N can be computed recursively from (8), and by a familiar argument it follows that

$$v^N(\alpha) = v_1^N = E(t^\alpha x_t), \tag{9}$$

where

$$t = \text{first} \quad i \geq 1 \quad \text{such that} \quad i^\alpha x_i \leq v_{i+1}^N. \tag{10}$$

For the remainder of this section we shall regard N as a fixed positive integer and α as a fixed constant such that $0 \leq \alpha \leq 1$; for brevity we shall write v_i for v_i^N. Then from (8),

$$v_i \leq E(i^\alpha x_i) = i^\alpha/2, \qquad (i = 1, \cdots, N), \tag{11}$$

so that

$$v_{i+1} i^{-\alpha} \leq \tfrac{1}{2}\left(\frac{i+1}{i}\right)^\alpha \leq \tfrac{1}{2} \cdot 2^\alpha \leq 1, \qquad (i = 1, \cdots, N-1). \tag{12}$$

Hence from (8),

$$\begin{aligned} v_i &= \int_0^{v_{i+1} i^{-\alpha}} i^\alpha x\, dx + (1 - v_{i+1} i^{-\alpha}) v_{i+1} \\ &= v_{i+1}\left(1 - \frac{v_{i+1}}{2i^\alpha}\right), \qquad (i = 1, \cdots, N-1). \end{aligned} \tag{13}$$

Noting that $v_i > 0$ for $i = 1, \cdots, N$, we can rewrite (13) as

$$\frac{1}{v_i} = \frac{1}{v_{i+1}} + \frac{1}{2i^\alpha - v_{i+1}} = \frac{1}{v_{i+1}} + \frac{1}{2i^\alpha} + \frac{v_{i+1}}{2i^\alpha(2i^\alpha - v_{i+1})}, \qquad (i = 1, \cdots, N-1). \tag{14}$$

Summing (14) for $i = 1, \cdots, N-1$ and noting that from (8)

$$v_N = \frac{N^\alpha}{2}, \tag{15}$$

we obtain the formula

$$\frac{1}{v_1} = \frac{2}{N^\alpha} + \frac{1}{2}\sum_1^{N-1}\frac{1}{i^\alpha} + \frac{1}{2}\sum_1^{N-1}\frac{v_{i+1}}{i^\alpha(2i^\alpha - v_{i+1})}. \tag{16}$$

We shall show at the end of this section that, setting

$$I_N = \frac{1}{2}\sum_1^{N-1}\frac{1}{i^\alpha}, \qquad J_N = \frac{1}{2}\sum_1^{N-1}\frac{v_{i+1}}{i^\alpha(2i^\alpha - v_{i+1})}, \tag{17}$$

we have as $N \to \infty$

$$J_N = o(I_N), \qquad I_N \sim \begin{cases} N^{1-\alpha}/2(1-\alpha), & (0 \le \alpha < 1), \\ \log N/2, & (\alpha = 1). \end{cases} \tag{18}$$

Relations (4) follow from (9), (16), and (18).

Proof of (18). The second part of (18) follows from the relation

$$I_N \sim \frac{1}{2}\int_1^N \frac{dt}{t^\alpha}. \tag{19}$$

The first part of (18) follows from two lemmas.

Lemma 1. *The following inequality holds*:

$$v_i \le \frac{2N^\alpha}{N - i + 1}, \qquad (i = 1, \cdots, N). \tag{20}$$

Proof. Equation (20) holds for $i = N$ by (15). Suppose it holds for some $i + 1 = 2, \cdots, N$; we shall show that it holds for i also.

(a). If $2N^\alpha/(N - i) > i^\alpha$, then by (11),

$$v_i \le \frac{i^\alpha}{2} \le \frac{N^\alpha}{N-i} \le \frac{2N^\alpha}{N-i+1}. \tag{21}$$

(b). If $2N^\alpha/(N - i) \le i^\alpha$, then setting

$$f(x) = x\left(1 - \frac{x}{2i^\alpha}\right), \quad f'(x) = 1 - \frac{x}{i^\alpha} \ge 0, \qquad \text{for} \quad x \le i^\alpha, \tag{22}$$

so by (13)

$$v_i = f(v_{i+1}) \le f\left(\frac{2N^\alpha}{N-i}\right) = \frac{2N^\alpha}{N-i}\left(1 - \frac{N^\alpha}{i^\alpha(N-i)}\right) \le \frac{2N^\alpha}{N-i+1}, \tag{23}$$

which completes the proof.

From (12) and (20) we have

$$J_N = \frac{1}{2}\sum_1^{N-1}\frac{v_{i+1}}{i^\alpha(2i^\alpha - v_{i+1})} \le N^\alpha \sum_1^{N-1}\frac{1}{(N-i)i^{2\alpha}}. \tag{24}$$

To prove the first part of (18), in view of the second part, it will suffice to show the following.

Lemma 2. *As* $N \to \infty$,

$$N^\alpha \sum_1^{N-1}\frac{1}{(N-i)i^{2\alpha}} = \begin{cases} o(N^{1-\alpha}), & (0 \le \alpha \le 1), \\ 0(1), & (\alpha = 1). \end{cases} \tag{25}$$

PROOF. (a). Assume $0 \leq \alpha < 1$. For any $0 < \delta < 1$, the left side of (25) can be written as

$$N^{\alpha}\left(\sum_{1}^{[\delta N]} + \sum_{[\delta N]+1}^{N-1}\right)\frac{1}{(N-i)i^{2\alpha}} \tag{26}$$
$$\leq N^{\alpha}\left(\frac{1}{N(1-\delta)}\sum_{1}^{N-1}\frac{1}{i^{\alpha}} + N(1-\delta)(\delta N)^{-2\alpha}\right)$$
$$\sim N^{\alpha}\left(\frac{1}{N(1-\delta)}\frac{N^{1-\alpha}}{1-\alpha} + N(1-\delta)(\delta N)^{-2\alpha}\right) \sim \frac{(1-\delta)N^{1-\alpha}}{\delta^{2\alpha}}.$$

Hence,

$$\overline{\lim_{N\to\infty}}\frac{J_N}{N^{1-\alpha}} \leq \frac{1-\delta}{\delta^{2\alpha}}. \tag{27}$$

Since δ can be arbitrarily near 1, the left-hand side of (27) must be 0.

(b). Assume $\alpha = 1$. We have for the left-hand side of (25), setting $M = [N/2]$,

$$N\sum_{1}^{N-1}\frac{1}{(N-i)i^2} = N\left(\sum_{1}^{M} + \sum_{M+1}^{N-1}\right)\frac{1}{(N-i)i^2} \leq 2\sum_{1}^{M} i^{-2} + N\sum_{M+1}^{N-1} i^{-2} \tag{28}$$
$$\leq 2\int_{1/2}^{\infty}\frac{dt}{t^2} + N\left(\frac{N}{2}\right)\left(\frac{2}{N}\right)^2 = 0(1).$$

3. An optimal rule exists for $\alpha > 1$ and $v(\alpha) > 0$

Define $z_n = \inf_{i\geq n}(i^{\alpha}x_i)$. Then for any constant $0 \leq A \leq n^{\alpha}$, we have

$$P[z_n \geq A] = P[i^{\alpha}x_i \geq A; i \geq n] = \prod_{n}^{\infty}\left(1 - \frac{A}{i^{\alpha}}\right). \tag{29}$$

Hence,

$$P\left[z_1 \geq \frac{1}{2}\right] = \prod_{1}^{\infty}\left(1 - \frac{1}{2i^{\alpha}}\right) > 0, \tag{30}$$

and therefore,

$$v(\alpha) \geq Ez_1 > 0. \tag{31}$$

Next, for any $A > 0$,

$$\sum_{1}^{\infty} P[n^{\alpha}x_n \leq A] \leq \sum_{1}^{\infty}\frac{A}{n^{\alpha}} < \infty. \tag{32}$$

Hence, by the Borel-Cantelli lemma,

$$P[\lim_{n\to\infty} n^{\alpha}x_n = \infty] = 1. \tag{33}$$

The existence of an optimal t for $\alpha > 1$ now follows from lemma 4 of [1].

4. For $\alpha \geq \frac{3}{2}$, $v(\alpha) = \frac{1}{2}$

We define for $i = 1, 2, \cdots$, and any fixed $\alpha \geq 0$,

$$v_i = \inf_{t\in C_i} E(t^{\alpha}x_t), \tag{34}$$

where C_i denotes the class of all $t \in C$ such that $P[t \geq i] = 1$. Then $v(\alpha) = v_1 \leq v_2 \leq \cdots$. It can be shown [3], although it is not trivial to prove, that in analogy with (8),

$$v_i = E\{\min(i^\alpha x_i, v_{i+1})\} = \int_0^1 \min(i^\alpha x, v_{i+1})\,dx, \qquad (i \geq 1). \tag{35}$$

It follows that

$$v_i \leq \frac{i^\alpha}{2}, \qquad (i \geq 1). \tag{36}$$

From now on in this section we shall assume that $1 < \alpha \leq \frac{3}{2}$. Then

$$v_{i+1} i^{-\alpha} \leq \frac{1}{2}\left(\frac{i+1}{i}\right)^\alpha \leq \frac{1}{2}\left(\frac{3}{2}\right)^\alpha \leq 1, \qquad (i \geq 2). \tag{37}$$

Hence, as in (13),

$$v_i = v_{i+1}\left(1 - \frac{v_{i+1}}{2i^\alpha}\right), \qquad (i \geq 2), \tag{38}$$

and since $v_1 = v(\alpha) > 0$ for $\alpha > 1$ by (31), we have as in (14),

$$\frac{1}{v_i} = \frac{1}{v_{i+1}} + \frac{1}{2i^\alpha - v_{i+1}}, \qquad (i \geq 2). \tag{39}$$

Summing (39) for $i = n, \cdots, m - 1$, we obtain

$$\frac{1}{v_n} = \frac{1}{v_m} + \sum_n^{m-1} \frac{1}{2i^\alpha - v_{i+1}}, \qquad (2 \leq n \leq m). \tag{40}$$

From (29), for any $A > 0$, we have as $m \to \infty$,

$$P[z_m \geq A] = \prod_m^\infty \left(1 - \frac{A}{i^\alpha}\right) \to 1, \tag{41}$$

thus $Ez_m \to \infty$, and since $v_m \geq Ez_m$, it follows that $v_m \to \infty$. Hence from (40),

$$\frac{1}{v_n} = \sum_n^\infty \frac{1}{2i^\alpha - v_{i+1}}, \qquad (n \geq 2). \tag{42}$$

From (42) and (37) we have for $n \geq 1$,

$$\begin{aligned} \frac{1}{(\alpha - 1)n^{\alpha-1}} &\geq \sum_{n+1}^\infty \frac{1}{i^\alpha} \geq \frac{1}{v_{n+1}} \geq \frac{1}{2}\sum_{n+1}^\infty \frac{1}{i^\alpha} \\ &\geq \frac{1}{2}\int_{n+1}^\infty \frac{dt}{t^\alpha} = \frac{1}{2(\alpha - 1)(n + 1)^{\alpha-1}}, \end{aligned} \tag{43}$$

and hence,

$$\frac{\alpha - 1}{n} \leq \frac{v_{n+1}}{n^\alpha} \leq \frac{2(\alpha - 1)}{n + 1}\left(\frac{n + 1}{n}\right)^\alpha, \qquad (n \geq 1). \tag{44}$$

We shall now show that $v_2 > 1$ for $\alpha = \frac{3}{2}$. It will follow from (35) that $v_1 = \frac{1}{2}$ and that $t = 1$ is optimal for $\frac{3}{2}$; the same is true a fortiori for any $\alpha \geq \frac{3}{2}$.

From (38) we obtain

$$v_{i+1} = i^\alpha - \sqrt{i^{2\alpha} - 2i^\alpha v_i}, \qquad (i \geq 2); \tag{45}$$

the + sign being excluded because of (37). Suppose now that $v_2 \le 1$ for $\alpha = \frac{3}{2}$. Then by (45),

$$\begin{aligned} v_3 &\le 2^{3/2} - \sqrt{8 - 2.2^{3/2}} = 1.3, \\ v_4 &\le 3^{3/2} - \sqrt{27 - 2\sqrt{27}(1.3)} = 1.52, \\ v_5 &\le 4^{3/2} - \sqrt{64 - 16(1.52)} = 1.7. \end{aligned} \tag{46}$$

On the other hand, by (44) we have for $\alpha = \frac{3}{2}$,

$$\frac{v_{n+1}}{n^{3/2}} \le \frac{1}{n+1}\left(\frac{n+1}{n}\right)^{3/2} \le \frac{1}{6}\left(\frac{6}{5}\right)^{3/2} \le \frac{11}{50}, \qquad (n \ge 5). \tag{47}$$

Hence, from (42) for $\alpha = \frac{3}{2}$,

$$\begin{aligned} \frac{1}{v_5} &= \sum_5^\infty \frac{1}{2i^\alpha - v_{i+1}} = \sum_5^\infty \frac{1}{2i^\alpha\left(1 - \frac{v_{i+1}}{2i^\alpha}\right)} \le \sum_5^\infty \frac{1}{2i^\alpha\left(1 - \frac{11}{100}\right)} \\ &\le \frac{50}{89}\int_{9/2}^\infty \frac{dt}{t^\alpha} = \frac{50}{89}\frac{1}{\alpha - 1}\sqrt{\frac{2}{9}} = \frac{100}{89}\cdot\frac{\sqrt{2}}{3} < \frac{1}{1.7}, \end{aligned} \tag{48}$$

contradicting (46). Hence $v_2 > 1$ for $\alpha = \frac{3}{2}$.

5. If $1 < \alpha \le 1.4$, then $v(\alpha) < \frac{1}{2}$

By (44) we have for $\alpha = \frac{7}{5}$,

$$v_3 \le \tfrac{4}{5}\cdot 3^{2/5} < \tfrac{5}{4}, \tag{49}$$

and hence by (38), $v_2 < \frac{5}{4}(1 - (5/4.2.2^{7/5})) < 1$. Hence by (35), $v_1 = v(\frac{7}{5}) < \frac{1}{2}$.

For $\alpha > 1$, an optimal t exists by section 3, and from ([3], theorem 2), a minimal optimal t is defined by

$$t = \text{first} \quad n \ge 1 \quad \text{such that} \quad x_n \le \frac{v_{n+1}}{n^\alpha}. \tag{50}$$

Let α be any constant >1 such that $v(\alpha) < \frac{1}{2}$. Then $P[t > 1] > 0$ by (50), and for $\alpha < \frac{3}{2}$ we have from (44) that

$$\frac{v_{n+1}}{n^\alpha} \le \frac{1}{n+1}\left(\frac{n+1}{n}\right)^2 < \frac{n+1}{n^2} < 1, \qquad \text{for} \quad n \ge 2. \tag{51}$$

Hence, $P[t > N] > 0$ for every $N \ge 1$, so t is not bounded. In fact, if $1 < \alpha = (3 - \epsilon)/2$ for some $\epsilon > 0$, then from (44)

$$\frac{v_{n+1}}{n^\alpha} \le (1 - \epsilon)\left(\frac{n+1}{n^2}\right) \le \frac{1}{n}, \qquad \text{for} \quad n \ge \frac{1-\epsilon}{\epsilon}. \tag{52}$$

Hence, if $v(\alpha) < \frac{1}{2}$, so that $P[t > N] > 0$ for every $N \ge 1$, it follows that for $n > N \ge \dfrac{1-\epsilon}{\epsilon}$ and some $K > 0$,

(53) $$P[t > n] \geq K\left(1 - \frac{1}{N}\right)\left(1 - \frac{1}{N+1}\right)\cdots\left(1 - \frac{1}{n}\right) = K \cdot \frac{N-1}{n},$$

so that $Et = \sum_0^\infty P[t > n] = \infty$.

We thus have for $\alpha > 1$: either $0 < v(\alpha) < \frac{1}{2}$ and $Et = \infty$, or $v(\alpha) = \frac{1}{2}$ and $t = 1$, where t is optimal for that α. The least value α^* such that $v(\alpha^*) = \frac{1}{2}$ is not known to us, but by the results of this and the previous section, it lies between 1.4 and 1.5.

6. The identification of optimal rules for $1 < \alpha$

For $N = 1, 2, \cdots$, define t_N by (10). Then $t_N \leq t_{N+1} \leq \cdots$. Let $b_i = \lim_{N\to\infty} v_i^N$. Then from (8),

(54) $$b_i = \int_0^1 \min (i^\alpha x, b_{i+1})\, dx, \qquad (i = 1, 2, \cdots).$$

Define

(55) $$s = \text{first } i \geq 1 \text{ such that } i^\alpha x_i \leq b_{i+1} \text{ if such an } i \text{ exists,}$$
$$= \infty \text{ otherwise.}$$

Then [1] $s = \lim_{N\to\infty} t_N$. Since $v_i^N \geq v_i$ for each N, $b_i \geq v_i$. Therefore $s \leq t$, where t is an optimal rule defined by (50). We shall now show that $s = t$ by showing that $b_i = v_i$ for all $i \geq 1$.

From (54) we have

(56) $$b_i \leq i^\alpha/2, \qquad (i \geq 1),$$

and hence as in (37) and (39), for some $i_0 = i_0(d)$,

$$b_{i+1} i^{-\alpha} \leq 1, \qquad (i \geq i_0),$$

(57) $$\frac{1}{b_i} = \frac{1}{b_{i+1}} + \frac{1}{2i^\alpha - b_{i+1}}, \qquad (i \geq i_0).$$

Since $b_i \geq v_i \to \infty$ as $i \to \infty$, we have, as in (42),

(58) $$\frac{1}{b_n} = \sum_n^\infty \frac{1}{2i^\alpha - b_{i+1}}, \qquad (n \geq i_0).$$

Assume that for some $j \geq 1$, $b_j > v_j$. Then by (35) and (54) this inequality must hold for some $i_1 \geq i_0$ (since if $j < i_0$ and $b_{i_0} \leq v_{i_0}$, then $b_j \leq v_j$), and hence for every $i \geq i_1$. Hence by (42) and (54),

(59) $$\frac{1}{v_{i_1}} = \sum_{i=i_1}^\infty \frac{1}{2i^\alpha - v_{i+1}} < \sum_{i=i_1}^\infty \frac{1}{2i^\alpha - b_{i+1}} = \frac{1}{b_{i_1}},$$

a contradiction. Hence $b_j = v_j$ for all $j \geq 1$.

It follows from the above that for $1 < \alpha$,

(60) $$v(\alpha) = v_1 = b_1 = \lim_{N\to\infty} v_1^N = \lim_{N\to\infty} v^N(\alpha).$$

That this relation holds also for $0 \leq \alpha \leq 1$ has been shown already.

7. Continuity of $v(\alpha)$

From (60), which holds for any $\alpha \geq 0$, given $\epsilon > 0$ we can find $N = N(\alpha, \epsilon)$ so large that

$$v(\alpha) + \frac{\epsilon}{2} \geq v^N(\alpha) = E(t^\alpha x_t) \tag{61}$$

for some t in C^N. Hence for $\alpha' > \alpha$,

$$v(\alpha) \leq v(\alpha') \leq E(t^{\alpha'} x_t) \leq N^{\alpha' - \alpha} E(t^\alpha x_t) \leq N^{\alpha' - \alpha}\left(v(\alpha) + \frac{\epsilon}{2}\right) \leq v(\alpha) + \epsilon, \tag{62}$$

provided that $\alpha' - \alpha$ is sufficiently small. Hence $v(\alpha)$ is continuous on the right for each $\alpha \geq 0$.

Since $v(\alpha)$ is nondecreasing in α for each fixed $i \geq 1$, we have by the bounded or monotone convergence theorem for integrals from (35)

$$\begin{aligned} v_i(\alpha - 0) &= \lim_{\epsilon \to 0} v_i(\alpha - \epsilon) = \lim_{\epsilon \to 0} \int_0^1 \min\,(i^{\alpha - \epsilon}, v_{i+1}(\alpha - \epsilon))\, dx \\ &= \int_0^1 \min\,(i^\alpha, v_{i+1}(\alpha - 0))\, dx \qquad (i \geq 1), \end{aligned} \tag{63}$$

and by the remark preceding (42), $\lim_{n \to \infty} v_n(\alpha - 0) = \infty$ for $\alpha > 1$. Hence, as in the preceding section, (58) holds with b_n replaced by $v_n(\alpha - 0)$, and the argument shows that $v_n(\alpha - 0) = v_n(\alpha)$. In particular, $v_n(\alpha - 0) = v(\alpha)$, which shows that $v(\alpha)$ is continuous on the left for $\alpha > 1$. Since $v(\alpha) = 0$ for $0 \leq \alpha \leq 1$, it follows that $v(\alpha)$ is continuous on the left for each $\alpha \geq 0$.

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