THE PROBLEM OF STELLAR EVOLUTION CONSIDERED STATISTICALLY

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The specific problem which I should like to bring to the attention of the statisticians is the following: table I contains a list of 133 pulsating variable stars of a kind described as "RR Lyrae Variables" or "galactic cluster type variables." These stars are characterized by periods of their variation in brightness ranging between about 88 minutes and 24 hours. Column 5 gives for each star the period, expressed in fractions of a day. Columns 2 and 3 give the celestial coordinates, namely, the galactic longitude l measured in degrees along the great circle of the Milky Way, and the galactic latitude b, measuring the angular distance of the star above or below the central plane of the Milky Way.[1] Column 4 gives the median apparent brightness of each star and column 6 the mean radial velocity (corrected for the periodic change produced by the pulsation) as measured by means of the Doppler effect. These data are largely the product of the work of A. H. Joy at the Mount Wilson Observatory. But some additional data were taken from measurements by A. Colacevich and others at the McDonald and Lick Observatories.

If we group the stars according to the period, in column 5, and take mean values of the radial velocity without regard to the sign, we obtain table II. There is a conspicuous progression, indicating an increase in the motions of the stars with period. Next, we plot the velocities against the galactic coordinates, separately for $P < 0.4$ day and $P \geqq 0.4$ day. The result, in figures 1 and 2, confirms what was already known: the velocities indicate a systematic drift with respect to the sun. The direction of this motion is in accordance with the apex determined from about one half of the present material by J. H. Oort [1] in 1939:

$$l_0 = 53°, b_o = +12° (\pm 8°) .$$

The drift motion of the sun against the stars is thus directed approximately at right angles to the direction toward the galactic center in Sagittarius (at $l = 325°$). The stars share in the phenomenon of galactic rotation, but their motion, though directed toward the same point in Cygnus which characterizes the circular motion of the sun's local standard of rest, is smaller by about 100–150 km/sec. The circular velocity of the local standard of rest, as determined from globular clusters and extragalactic nebulae is about 250 km/sec. That of the RR Lyrae variables is, on the average, about 120 km/sec.

[1] These "galactic coordinates" were taken, without interpolation, from the Lund tables by Ohlsson. Their precision is ample for the present purpose.

TABLE I

RADIAL VELOCITIES OF RR LYRAE VARIABLES

Star	*l*	*b*	Med. phot. mag.	*P*	Normal radial velocity in km/sec
SW And	84°	−33°	10.1	0.44	− 32
XX And	97	−23	10.9	.72	− 25
AC And	75	−12	11.4	.53	− 70
SW Aqr	20	−33	11.7	.46	− 5
BS Aqr	53	−66	9.0	.20	+ 50
AA Aql	11	−26	10.4	.36	− 75
X Ari	137	−38	9.2	.65	− 40
RS Boo	16	+66	10.3	.38	− 10
SW Boo	27	+67	11.8	.51	+ 10
TV Boo	44	+66	10.9	.31	− 85
TW Boo	35	+62	10.7	.53	−120
RW Cnc	166	+45	11.3	.55	− 85
SS Cnc	167	+28	12.2	.37	+ 5
W CVn	36	+70	10.3	.55	+ 24
Z CVn	86	+74	9.8	.65	0
RR CVn	115	+82	11.3	.56	− 10
RU CVn	18	+73	11.7	.57	− 50
RZ CVn	25	+76	11.4	.57	− 15
SV CVn	100	+80	12.1	.67	− 25
RV Cap	1	−37	10.3	.45	− 80
RZ Cep	77	+ 5	9.4	.31	0
RR Cet	113	−59	9.8	.55	− 85
S Com	189	+87	11.3	.59	− 35
U Com	180	+89	12.0	.29	0
ST Com	319	+80	11.3	.60	−100
RV CBr	16	+43	11.5	.33	−100
UY Cyg	42	−10	11.1	.56	− 5
XX Cyg	60	+14	12.0	.13	−135
XZ Cyg	55	+16	9.7	.47	−160
DM Cyg	47	−13	11.7	.42	− 25
RW Dra	54	+40	10.9	.44	−110
SU Dra	100	+49	9.9	.66	−180
SW Dra	93	+48	10.5	.57	− 40
XZ Dra	63	+22	10.3	.48	− 25
RX Eri	182	−32	9.2	.59	+ 70
RR Gem	155	+21	11.5	.40	+ 80
SZ Gem	169	+24	11.5	.50	+330
TW Her	23	+23	11.2	.40	− 15
VX Her	2	+38	10.5	.46	−390
VZ Her	26	+34	11.4	.44	−130
AR Her	40	+47	10.4	.47	−335
SV Hya	265	+37	10.8	.48	+100
SZ Hya	208	+27	11.3	.54	+100
RR Leo	177	+54	11.0	.45	+ 65
RX Leo	178	+72	12.0	.65	−115
SS Leo	235	+57	10.6	.63	+150
ST Leo	223	+67	11.1	.48	+180
SZ Leo	213	+59	12.5	.53	+ 90
V LMi	170	+59	11.0	.54	− 85
X LMi	149	+55	12.3	.68	+ 40
U Lep	188	−33	10.8	.58	+120
TV Lib	321	+38	11.5	.27	− 10
RR Lyr	43	+11	7.8	.57	− 69
RZ Lyr	29	+14	11.9	.51	−240
DH Peg	37	−40	9.6	.26	− 70
TU Per	110	− 3	12.1	.61	−380
RU Psc	98	−38	10.6	.39	−115
V 355 Sgr	343	− 3	9.5	.46	+ 10
RU Scl	9	−80	10.2	.49	+ 45
T Sex	205	+41	10.1	.32	+ 10
RV UMa	75	+63	10.5	.47	−180
SX UMa	78	+60	11.0	.31	−135
TU UMa	168	+73	9.8	.56	+105
ST Vir	315	+53	11.5	.41	− 35
UU Vir	251	+61	10.3	.48	− 20
UV Vir	257	+62	11.8	.59	+ 95

TABLE I—*Continued*

Star	*l*	*b*	Med. phot. mag.	*P*	Normal radial velocity in km/sec
BO Aqr	24°	−60°	12.1	0.69	− 55
CY Aqr	38	−48	10.9	.06	− 28
V 341 Aql	13	−24	10.8	.58	−135
RV Ari	118	−39	12.2	.09	+ 35
RW Ari	118	−40	12.4	.26	− 60
ST Boo	24	+54	10.7	.62	+ 10
SV Boo	35	+65	12.9	.58	−160
SZ Boo	9	+65	11.6	.52	− 45
UU Boo	22	+57	11.4	.46	+ 20
UY Boo	322	+67	10.0	.65	+142
TT Cnc	180	+30	11.2	.56	+ 40
RX CVn	51	+71	12.6	.54	+ 5
SS CVn	46	+72	11.8	.48	− 5
ST CVn	12	+74	11.8	.33	− 85
SW CVn	95	+80	12.0	.44	− 55
YZ Cap	3	−41	11.4	.27	− 75
RX Cet	76	−78	11.4	.57	− 79
RY Com	319	+84	11.9	.47	− 31
W Crt	244	+41	10.8	.41	+ 70
X Crt	248	+50	10.5	.73	+ 50
SV Eri	162	−52	9.8	.71	− 22
BB Eri	187	−33	10.5	.57	+233
BC Eri	180	−32	10.2	.26	+ 65
SS For	182	−71	9.2	.50	−115
AF Her	32	+41	12.4	.63	−270
CE Her	6	+21	12.0	1.21	−235
DY Her	355	+35	10.7	.14	− 50
UV Hya	198	+40	11.5	.52	+312
VX Hya	216	+31	10.6	.18	− 15
WZ Hya	223	+35	10.2	.54	+315
HD 73857	185	+30	7.9	.18	+ 25
XX Hya	212	+22	11.2	.34	− 10
TV Leo	232	+50	11.5	.40	− 86
UZ Leo	199	+58	9.4	.31	+ 4
Y Lyr	40	+20	12.8	.50	−110
EZ Lyr	33	+15	11.4	.53	− 75
V 445 Oph	336	+27	10.8	.40	− 15
V 453 Oph	348	+17	11.2	.97	− 95
VV Peg	47	−31	11.7	.49	+ 10
AO Peg	38	−24	12.9	.55	+115
AV Peg	46	−25	10.7	.39	− 85
BH Peg	54	−39	10.6	.64	−260
CG Peg	45	−22	11.2	.47	+ 5
AR Per	123	− 1	10.7	.43	− 10
RY Psc	72	−63	12.2	.53	+ 25
SS Psc	101	−40	11.5	.29	+ 5
V 440 Sgr	343	−21	10.4	.48	− 50
VY Ser	334	+43	10.1	.42	− 5
AN Ser	351	+44	11.2	.52	− 60
AP Ser	340	+51	11.6	.25	− 40
AR Ser	336	+43	11.0	.33	+100
AT Ser	346	+41	12.0	.43	− 70
AV Ser	340	+35	11.2	.33	− 55
XX Vir	307	+49	12.0	.35	− 55
AM Vir	283	+45	11.6	.62	+105
AS Vir	273	+52	11.5	.55	+ 85
AT Vir	275	+57	11.4	.53	+342
AU Vir	287	+55	11.6	.34	+129
AV Vir	297	+70	12.0	.66	+ 45
BB Vir	308	+63	11.0	.47	− 5
BC Vir	293	+66	11.2	.56	+ 5
BN Vul	26	+ 3	11.6	.59	−235
TZ Aur	144	+22	11.6	.39	+ 58
RZ Cet	146	−59	11.5	.51	+ 9
XX Pup	204	+10	11.5	.52	+386
BB Pup	209	+12	10.5	.48	+255
DY Peg	60	−39	10.4	.07	− 25

We next compute for each star the angular distance D from Oort's apex, using the formula:

$$\cos D = \sin b \sin b_0 + \cos b \cos b_0 \cos (l - l_0) .$$

Then the projection of the drift velocity V_0 for each star is

$$V = V_0 \cos D .$$

We shall determine V_0 as a function of P, solving for each star the equation

$$V_0 = \frac{V}{\cos D} ,$$

where V is the quantity given in table I. This results in the data of table III. Since $\cos D$ becomes very small when the stars are nearly 90° from Oort's apex and the influence of the peculiar motions of the stars in determining V_0 becomes

TABLE II

MEAN RADIAL VELOCITIES OF RR LYRAE VARIABLES WITHOUT REGARD FOR SIGN

Range in P in Days	No.	Mean P in Days	Mean $\|V\|$ in km/sec	M.E. in km/sec
<0.20	7	0.12	44	±16
0.20–0.30	9	0.26	42	10
0.30–0.40	17	0.34	62	12
0.40–0.50	35	0.45	78	16
0.50–0.60	43	0.55	110	16
0.60–0.70	16	0.64	120	27
0.70<	6	0.95	80	33

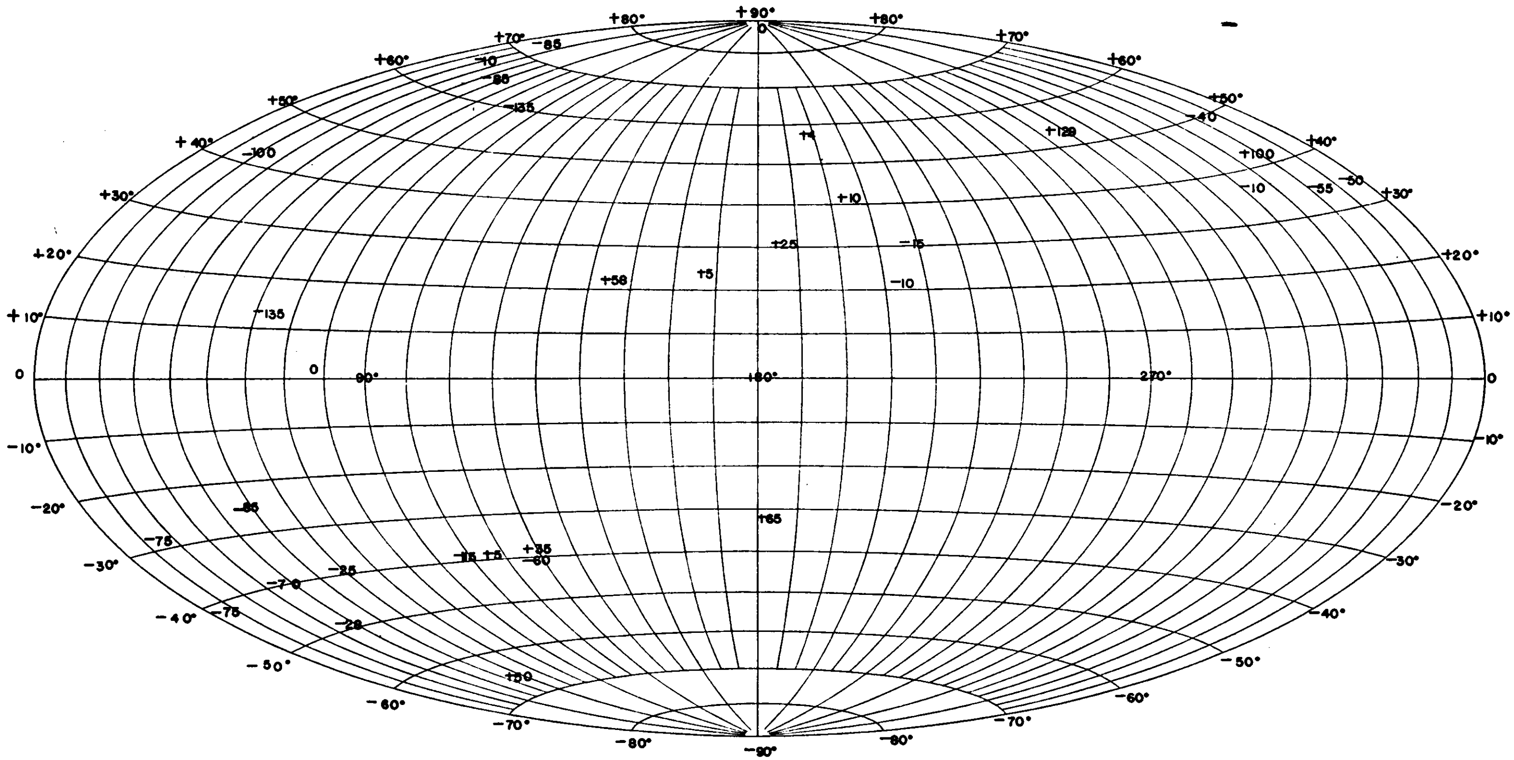
unreasonably large, we have made the computations, first, omitting those stars for which $\cos D < 0.1$ and, next, omitting those for which $\cos D < 0.2$. In both parts of the table we detect an increase in the drift velocity with the period.

It is of interest to list also the dispersions of the peculiar radial velocities remaining after allowing for the drift velocity in each horizontal line of table III. These values are approximately ±97, 223, 266, 289, 318, and 140 km/sec. With the exception of the last group (in which we suspect the admixture of normal Cepheids), this again confirms the well known relation discovered by G. Strömberg for other groups of stars, that the dispersion of the peculiar motions is correlated with the amount of the drift motion, for each group.

The question is this: Are the two relations in tables II and III physically significant? And can they teach us something concerning the origin of the RR Lyrae variables?

In order to enable you to answer my questions, I must give you the astrophysical background that has led to their formulation.

The RR Lyrae variables are exceedingly numerous. We know more than 2000, but most of them are so faint that they cannot be observed with the spectrograph. Hence, we know the radial velocities of only a small fraction. They are also very



P < 0.4 DAY

FIGURE 1

Radial velocities of RR Lyrae variables, plotted against galactic longitude (0° to 360°) and latitude (0° to ±90°)

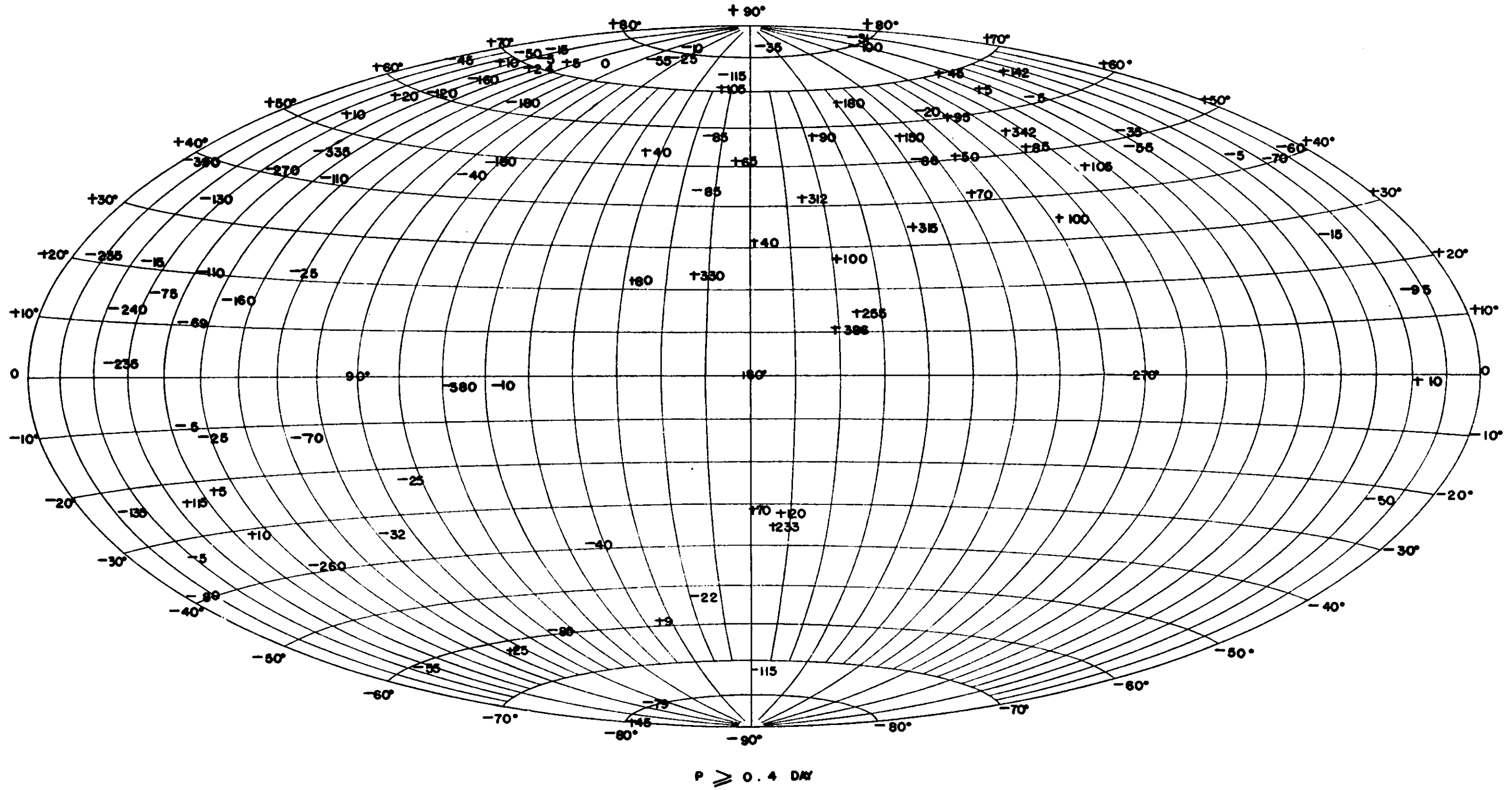


FIGURE 2

Radial velocities of RR Lyrae variables, plotted against galactic longitude (0° to 360°) and latitude (0° to ±90°)

numerous in globular clusters. According to H. Shapley, Mrs. Sawyer Hogg and others, the frequencies of these variables, and their distribution with period, are not the same for all clusters. Shapley, for example, found that the more condensed globular clusters contain fewer variables than the less compact clusters. We cannot now discuss the results of these important investigations. But we should record that M. Schwarzschild [2] found in at least one cluster that these variables occur in a definite, narrow region of the Hertzsprung-Russell diagram (luminosity

TABLE III

DRIFT MOTION OF RR LYRAE VARIABLES WITH RESPECT TO THE SUN

RANGE IN P IN DAYS	MEAN P IN DAYS	OMITTING cos $D < 0.1$			OMITTING cos $D < 0.2$		
		No.	Drift Vel. in km/sec	M.E. in km/sec	No.	Drift Vel. in km/sec	M.E. in km/sec
0.00–0.30	0.20	15	40	±25	12	57	±28
0.31–0.40	0.34	20	126	50	18	57	23
0.41–0.50	0.45	29	193	50	27	156	45
0.51–0.60	0.55	37	184	47	34	200	48
0.61–0.70	0.64	12	144	91	10	286	85
>0.70	0.95	5	129	56	5	129	56

TABLE IV

DRIFT MOTION OF LONG PERIOD VARIABLES

Range in P in Days	No.	Velocity in km/sec	Mean Error in km/sec
150–199	27	129	±26
200–299 (and less than 150)	124	42	7
300–349	75	14	8
Greater than 349	79	7	3

of a star plotted against spectral type, or temperature), and that, moreover, "stars which can pulsate do pulsate." In other words, there are no nonvariables in the region of the diagram occupied by the variables.

The relations found in tables II and III are not unique. There is another group of variables with very long periods. They, too, exhibit a systematic drift motion with respect to the local standard of rest, and the amount of the drift velocity is correlated with the period as is shown in table IV. The apex is, however, nearly the same as that of the RR Lyrae variables. This relation, which has been known for some years from the work of P. W. Merrill and R. E. Wilson [3], of J. H. Oort and J. J. M. van Tulder [4] and of P. G. Kulikovsky [5], has the opposite trend to that which we have found in the case of the RR Lyrae stars: the drift velocity increases as the period becomes shorter.

The dispersion of the individual motions of the long period variables is also correlated with the values of V_0, as in the case of the RR Lyrae variables: the larger is V_0, the larger is also the dispersion.

Both types of variables, those of the RR Lyrae and of the long period classes, are believed to be pulsating stars which obey the fundamental relation $P\sqrt{\rho} =$ constant, where ρ is the mean density of the star. Observationally, this law is equivalent to the famous period luminosity relation which Shapley and others have used for the determination of stellar distances.

There is another group of pulsating objects, the Cepheid variables, which also obey the same relation, and which have periods between $P = 1$ day and $P = 45$ days. These stars have almost no systematic motion with respect to the local standard of rest, except for a small number having peculiar spectra of the W Virginis class, for which the drift velocity is probably similar to that of the average RR Lyrae star.

A very simple interpretation of the drift motions of different systems of stars with respect to the sun was first given by K. F. Bottlinger [6]. It has been used extensively by other workers, most recently by E. T. R. Williams and A. Vyssotsky in their work on the dynamics of our galaxy [7], and by M. Lohmann [8] in the study of fast moving subdwarfs. Figure 3 is due to Bottlinger. The galactic center, assumed to be a point mass, is at the bottom, outside the limits of the drawing. We consider the point 0 as the center of our coordinates and plot the vector OP representing the circular velocity of the local standard of rest, 250 km/sec, around the galactic center. The vector PS represents the motion of the sun with respect to the local standard. The diagram is in the plane of the Milky Way. Any other velocity than the circular one defines a vector in the diagram, starting from 0. For example, the average of the RR Lyrae variables would give us a vector in the same direction as OP (galactic longitude $l = 53°$) but about 130 km/sec shorter. Any such vector represents a noncircular motion. Assuming that the orbits are Keplerian ellipses, we can draw a net of curves: the circles correspond to equal semimajor axes, and the ovals correspond to constant eccentricities. For the RR Lyrae stars we find, roughly $a = 0.6R_{\odot}$ and $e = 0.7$. These orbits are very eccentric. When we observe these stars in our vicinity they are near apogalacticon. At perigalacticon they dip into the nucleus of the Milky Way, where the approximation of a point mass no longer holds. This suggests that all objects having such highly eccentric orbits, including the globular clusters and some long period variables, subdwarfs, etc., may be temporary visitors from the inner regions of the galaxy to the neighborhood of the sun which an astronomer has aptly described as our "local swimming hole." It should be remembered, however, that these transients spend most of their lives in the most distant parts of their orbits from the nucleus and not in their supposed home.

We need one more characterization: the RR Lyrae stars are scattered all over the sky: they show little galactic concentration. The same is true of the long period variables, and of the W Virginis Cepheids. But the normal Cepheids occur only in and near the Milky Way; their galactic concentration is very conspicuous.

This remarkable diversity in galactic concentration strongly suggests that despite the similarity of the physical process of pulsation, these three groups of variable stars are really different objects in regard to their origin and evolution. The

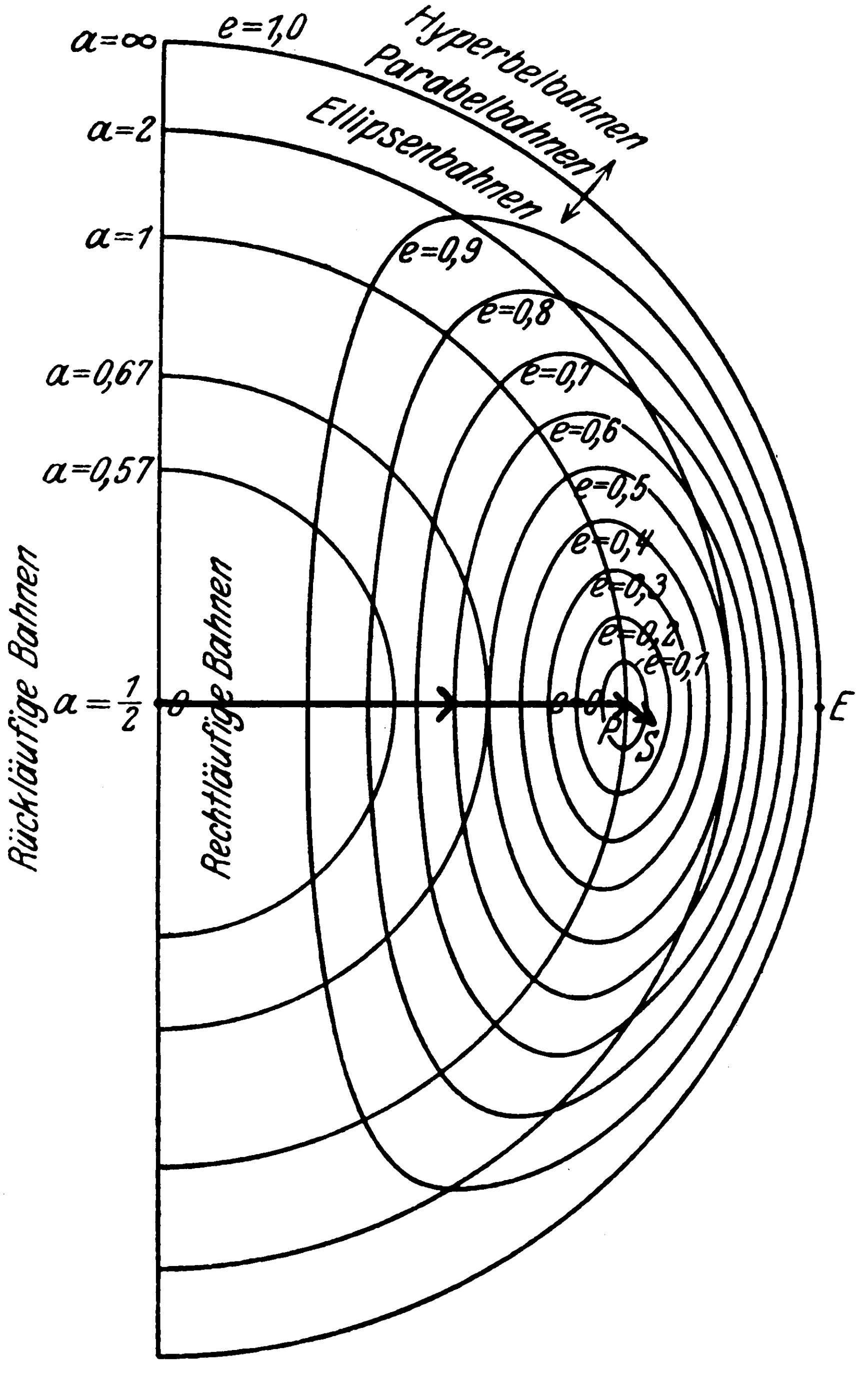

a=∞
e=1,0
Hyperbelbahnen
Parabelbahnen
Ellipsenbahnen
a=2
a=1
e=0,9
e=0,8
a=0,67
e=0,7
e=0,6
a=0,57
e=0,5
e=0,4
e=0,3
e=0,2
e=0,1
Rückläufige Bahnen
Rechtläufige Bahnen
a=1/2
0
e=0
P
S
E

Figure 3

normal Cepheids belong to the highly flattened, rapidly rotating system of stars to which belong also our sun and the stars near us which define the local standard of rest. The RR Lyrae stars belong to an approximately spherical system, which rotates slowly and which resembles the great spherical system of globular clusters investigated by Shapley about 30 years ago. The long period variables form an intermediate system, less flattened than that of the Cepheids, but not nearly as spherical as the system of the RR Lyrae variables.

We associate the different systems of our galaxy with their age. W. Baade has introduced the concept of stellar populations: Population I represents the stars of the flattened systems, population II those of the spherical systems. There are various indications that the stars of population I are relatively young, those of population II may be old. Very faint variables of the RR Lyrae class have been observed by Baade, even at great distances from the galactic plane. For example, a variable found by him in position $l = 311°$, $b = +48°$ has a median magnitude of 18, and is therefore about 34,000 parsecs, or 100,000 light years, above the plane of the Milky Way! The work of Shapley and others at Harvard has demonstrated that the density of these stars increases toward the galactic center and that it decreases, more or less uniformly, in all directions from the center. These results have been confirmed in a recent statistical discussion of all available data by B. V. Kukarkin [9]. In fact, the latter has revived the interesting hypothesis that the RR Lyrae variables have strayed away from the globular clusters and that they once were members of these closely packed groups. Kukarkin exempted only those variables whose periods are between 0.425 and 0.435 day. These periods are rare in globular clusters, but they are fairly frequent in our galaxy. The stars appear to form a flat system (from 17 objects) and their spectra may differ from those of other RR Lyrae variables.

Our discussion throws no new light upon Kukarkin's group of variables. But it raises the question whether the hypothesis of the origin of other RR Lyrae stars in globular clusters can be maintained in view of the relations found in tables II and III. As far as we now know, the globular clusters form a single system whose drift motion with respect to the sun is 190 km/sec, according to P. P. Parenago [10], who made extensive use of the radial velocities of globular clusters determined by N. U. Mayall at the Lick Observatory [11].

It is customary to describe as "young" the very luminous, blue stars of high surface temperature which occur in the Milky Way and are sometimes also called "helium stars." On the contrary, we regard as "old" those stars which, like the sun, must have existed, essentially in their present condition, for several billion years. The "young" stars radiate energy prodigiously: while the sun emits into space approximately 10^{41} ergs per year, a very luminous star of the helium class, like Rigel in Orion, emits 10^{46} ergs per year. The available nuclear source of this energy output, namely the conversion of hydrogen into helium, provides a total store of energy amounting to about 10^{52} ergs in the case of the sun, and 10^{53} ergs in the case of the more massive star, Rigel. Hence, by the process of division we find that the sun could continue radiating at its present rate for about 10^{11} years—which is a long interval even in astronomy! But Rigel could exist as a star of its present luminosity during an interval of only ten million years.

The details of the computations may not be exact, but the general nature of the argument is probably correct: the blue stars of the Milky Way were formed at a time when the fossils of our museums were walking through the swamps of the North American continent. We must be careful to realize that we are speaking of these stars as they are now. They could not have existed as hot, blue stars earlier than about 10^7 years ago, but they might very well have existed for a long time as solar type dwarfs before they became helium stars.

Now, the problem which I shall try to formulate, but not to solve, is this: The limitations of our telescopes and the opacity of the heavy smoke clouds of the Milky Way prevent us from studying the stars at great distances from the sun. For practical purposes we may assume that we can penetrate to distances of the order of 5000 light years. But the distance of the center of our galaxy is about 30,000 light years and the outer edge of the Milky Way, in the direction opposite to that of the center, is about 15,000 light years. Within the relatively small spherical volume surrounding the sun, which we are able to explore, the "young" and the "old" stars occur in a certain mixture, described by their ratio, N_y/N_0. This ratio is not constant: it is largest in the plane of the Milky Way, where our sun happens to be located, and especially in the outer regions of the lens shaped system of stars which has become familiar to us through the work of Shapley, Plaskett, Lindblad, Oort, and others. By analogy with other galaxies we can say that the young stars abound in the spiral arms, and are rare, or absent in the spaces between the arms, in the galactic nucleus, and at great distances above or below the galactic plane. These young stars are always closely associated with clouds of smoke and gas, and there is some reason to think that even now they are being formed out of condensations in the diffuse medium.

But the origin of the old stars poses a more complicated problem. What we would like to find is an answer to these questions: Are all old stars generically similar, or may they not consist of several groups having different places and times of origin? Can we definitely isolate any such groups and determine when and where they were formed? Is it possible, for example, as v. Weizsäcker has recently suggested, that all stars were formed approximately at the same time, in the very distant past, when the galaxy consisted only of gas and when "starlight" did not inhibit the process of initial condensation of "protostars," as it now does; according to this view the young, blue stars are really old stars which are being rejuvenated by the process of accretion of interstellar material—a process which is reasonably rapid only when the stellar nucleus that is being accreted already possesses a considerable amount of mass.

Astronomers have always been tempted to consider all pulsating variables as representing a sequence of stars having similar properties and therefore probably also a similar origin, constitution and evolution. The simplicity of this view was destroyed when it become known that the RR Lyrae variables are members of Baade's population II, while the ordinary Cepheids belong to population I. We then were inclined to consider the RR Lyrae stars as a uniform group, and the Cepheids as another group. But among the Cepheids we encounter occasionally members of population II—the W Virginis stars—and among the RR Lyrae variables we now find a wide diversity of systematic motions with regard to the local standard of rest.

One conclusion forces itself upon us: variation in light as produced by pulsation must be an attribute of all stars which happen to have the appropriate internal constitution, irrespective of whether they are members of one population or of another. If two stars have similar periods, but belong to different populations, we may investigate whether their spectra and light curves show significant differences. If they do, we have a reliable clue to real differences in internal constitution, and thus could perhaps attack the problem of the hydrogen abundances in "old" and "young" systems. This approach suggests itself in regard to the W Virginis variables as contrasted to the ordinary Cepheids.

In the case of the RR Lyrae variables the best procedure will be to study the degree of their spectroscopic anomalies (weakness of hydrogen absorption lines) as a function of P. We do not yet know whether these anomalies are the same in all RR Lyrae stars. Present indications are that they are not. It is possible, then, that among the variables having $0.4 \leqq P < 1$ day, the great majority, perhaps all, are members of population II and are really "old" stars in the sense advocated by Kuiper, namely, that they have already converted much of their original hydrogen supply into helium. These stars fall in the Hertzsprung-Russell diagram in the so called Hertzsprung gap, where stars of population I do not occur. But variables with $P < 0.4$ day approach that region in the diagram where the horizontal branch of population II (crossing the Hertzsprung gap) joins the normal main sequence of population I. Perhaps we have here a mixture of stars of both populations. The fact that they all pulsate means only that they are approximately similar in internal constitution, and has not necessarily any bearing upon their origin. The sharp distinction between stars of populations I and II is now less conspicuous than it seemed to be a few years ago. Similarly, there is no clearcut separation into "old" and "young" stars.

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