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MATRIX INEQUALITIES RELATED TO HÖLDER INEQUALITY

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Abstract. Matrix inequalities of Hölder type are obtained. Among other inequalities, it is shown that if $2 \leq p, q < \infty$ and $r > 1$ with $1/p + 1/q = 1 - 1/r$, then for any $A_i, B_i \in M_n(\mathbb{C})$ and $\alpha_i \in [0,1]$ $(i = 1, 2, \cdots, m)$ with $\sum_{i=1}^{m} \alpha_i = 1$, we have

$$\left| \sum_{i=1}^{m} \alpha_i^{1/r} B_i A_i \right| \leq \left(\sum_{i=1}^{m} |A_i|^p \right)^{1/p}$$

whenever $\sum_{i=1}^{m} |B_i^*|^q \leq I$. Related unitarily invariant norm inequalities are also presented.

1. Introduction

Let $M_n(\mathbb{C})$ denote the algebra of all $n \times n$ complex matrices. A unitarily invariant norm, denoted by $|||.|||$, satisfies the invariance property $|||UAV||| = |||A|||$ for all A and all unitary matrices $U, V \in M_n(\mathbb{C})$. For any matrix $A \in M_n(\mathbb{C})$, the positive semidefinite matrix $|A|$ is defined to be $(A^*A)^{1/2}$ and the usual operator norm is denoted by $||A||$. If A, B are Hermitian matrices whose eigenvalues are in an interval J, then a continuous function $f : J \to \mathbb{R}$ is said to be operator monotone if $A \leq B$ implies $f(A) \leq f(B)$, for all Hermitian matrices A, B whose eigenvalues are in J, while f is said to be operator convex if

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$f(\alpha A+(1-\alpha)B)\leq \alpha f(A)+(1-\alpha)f(B)$. For the theory of unitarily invariant norms and convex functions we refer to [4, 19].

A family of inequalities concerning inner products of vectors and functions began with Cauchy. The extensions and generalizations later led to the inequalities of Schwarz, Minkowski and Hölder. The well known Hölder's inequality is one of the most important inequalities in analysis. Hölder's inequality for sequences of numbers asserts that if $x_i, y_i \in \mathbb{C}$ $(i=1,2,\ldots,m)$, then

$$\sum_{i=1}^{m}|x_i y_i| \leq \left(\sum_{i=1}^{m}|x_i|^p\right)^{1/p}\left(\sum_{i=1}^{m}|y_i|^q\right)^{1/q} \tag{1.1}$$

for all positive real numbers p and q such that $1/p+1/q=1$. Moreover, we have the variational expression

$$\max\left\{\sum_{i=1}^{m}|x_i y_i| : \ \sum_{i=1}^{m}|y_i|^q=1\right\} \leq \left(\sum_{i=1}^{m}|x_i|^p\right)^{1/p}. \tag{1.2}$$

The equality in (1.1) holds if and only if $|x_i|^p=|y_i|^q$, $i=1,2,\ldots,m$. For unitarily invariant norms several Hölder inequalities for matrices and Hilbert space operators have been obtained. These forms can be found in [1, 11, 17]. Cauchy-Schwarz inequalities have been given in [2, 8, 9, 12, 14]. Hölder trace inequalities were given in [18], while matrix Hölder inequalities were given in [3]. Related Minkowski–type inequalities and Q-norm inequalities have been given in [16] and [7], respectively.

The basic Hölder inequality for unitarily invariant norms, see [6], asserts that if $A, B \in M_n(\mathbb{C})$ and p, q are positive real numbers such that $1/p+1/q=1$, then

$$|||AB||| \leq |||\,|A|^p|||^{1/p}\,|||\,|B|^q|||^{1/q}. \tag{1.3}$$

Horn and Zhan [13] proved that if p, q and r are positive real numbers such that $1/p+1/q=1$, then

$$|||\,|AB|^r||| \leq |||\,|A|^{pr}|||^{1/p}\,|||\,|B|^{qr}|||^{1/q}. \tag{1.4}$$

Also they proved an inequality containing an intermidiate matrix, it was shown that if $A, B, X \in M_n(\mathbb{C})$ such that A and B are positive semidefinite, then

$$|||\,|AXB|^r||| \leq |||\,|A^pX|^r|||^{1/p}\,|||\,|XB^q|^r|||^{1/q} \tag{1.5}$$

for all positive real numbers p, q and r such that $1/p+1/q=1$.

Hiai and Zhan [10] proved that if $A, B, C, D \in M_n(\mathbb{C})$, then

$$2^{-\left|\frac{1}{p}-\frac{1}{2}\right|}\,|||\,|C^*A+D^*B|\,||| \leq |||\,|A|^p+|B|^p|||^{1/p}\,|||\,|C|^q+|D|^q|||^{1/q} \tag{1.6}$$

for all positive real numbers p and q such that $1/p+1/q=1$. Inequality (1.6) was generalized to $m-$tuple of matrices, Albadawi [1] proved that if $A_i, B_i \in M_n(\mathbb{C})$ $(i=1,2,\ldots,m)$ and p, q are positive real numbers such that $1/p+1/q=1$, then

$$\left|\left|\left|\sum_{i=1}^{m}A_i^*B_i\right|\right|\right| \leq m^{\left|\frac{1}{p}-\frac{1}{2}\right|}\left|\left|\left|\sum_{i=1}^{m}|A_i|^p\right|\right|\right|^{1/p}\left|\left|\left|\sum_{i=1}^{m}|B_i|^q\right|\right|\right|^{1/q}. \tag{1.7}$$

Moreover, Ando and Hiai proved the most interesting Hölder matrix form, it was shown in [3] that if $2 \leq p, q < \infty$ and $r > 1$ with $1/p + 1/q = 1 - 1/r$, then for any $A, B, C, D \in M_n(\mathbb{C})$ and $\alpha \in [0, 1]$ we have

$$\left|\alpha^{1/r} CA + (1-\alpha)^{1/r} DB\right|^2 \leq \left(|A|^p + |B|^p\right)^{2/p} \tag{1.8}$$

whenever $|C^*|^q + |D^*|^q \leq I$.

In this article, we present Hölder-type matrix inequalities for sums and products of matrices that generalize (1.8). Unitarily invariant norm inequalities of similar type related to (1.4) and (1.7) are also obtained. It should be mentioned that all the results in this article can be extended to the class of bounded linear operators on Hilbert spaces.

2. Main Results

In this section, we will give new matrix Hölder inequalities and unitarily invariant norm inequalities involving $\left(\sum_{i=1}^m |A_i|^p\right)^{1/p}$. These inequalities are based on several lemmas. The first lemma, which can be found in [5], contains an inequality for positive semidefinite block matrices.

Lemma 2.1. *Let $A, B, C \in M_n(\mathbb{C})$ such that $A, B \geq 0$. Then the block matrix $\begin{bmatrix} A & C^* \\ C & B \end{bmatrix}$ is positive semidefinite if and only if $C = B^{1/2} W A^{1/2}$ for some contraction W with $\|W\| = 1$. In particular, $\begin{bmatrix} A & C^* \\ C & I \end{bmatrix} \geq 0$ iff $C^*C \leq A$.*

The next lemma will be useful in this article, it was proved in [13] and can be considered as a Hölder-type inequality for unitarily invariant norms.

Lemma 2.2. *Let $A, B, C \in M_n(\mathbb{C})$ such that $A, B \geq 0$. If the block matrix $\begin{bmatrix} A & C \\ C^* & B \end{bmatrix}$ is positive semidefinite, then*

$$|||\, |C|^r \,||| \leq \left|\left|\left|A^{pr/2}\right|\right|\right|^{1/p} \left|\left|\left|B^{qr/2}\right|\right|\right|^{1/q} \tag{2.1}$$

for all positive real numbers p, q and r such that $1/p + 1/q = 1$ and for every unitarily invariant norm.

We will start by the following inquality.

Lemma 2.3. *Let $A_i, B_i \in M_n(\mathbb{C})$ be positive semidefinite and $\alpha_i \in [0, 1]$ $(1 \leq i \leq n)$ with $\sum_{i=1}^m \alpha_i = 1$. Then for $p \geq 1$*

$$\sum_{i=1}^m \alpha_i^{1-1/p} A_i \leq \left(\sum_{i=1}^m A_i^p\right)^{1/p}. \tag{2.2}$$

Proof. First note that

$$\sum_{i=1}^m A_i^p = \sum_{i=1}^m \alpha_i \alpha_i^{-1} A_i^p = \sum_{i=1}^m \alpha_i \left(\alpha_i^{-1/p} A_i\right)^p,$$

by the operator concavity of $t^{1/p}$ we get

$$\begin{aligned}\left(\sum_{i=1}^{m} A_i^p\right)^{1/p} &= \left(\sum_{i=1}^{m} \alpha_i \left(\alpha_i^{-1/p} A_i\right)^p\right)^{1/p} \\ &\geq \sum_{i=1}^{m} \alpha_i \left(\left(\alpha_i^{-1/p} A_i\right)^p\right)^{1/p} \\ &= \sum_{i=1}^{m} \alpha_i \alpha_i^{-1/p} A_i.\end{aligned}$$

Which completes the proof. □

Our next theorem is a generalization of inequality (1.8).

Theorem 2.4. *Let $2 \leq p, q < \infty$ and $r > 1$ with $1/p + 1/q = 1 - 1/r$. Then for any $A_i, B_i \in M_n(\mathbb{C})$ and $\alpha_i \in [0,1]$ $(i = 1, 2, \cdots, m)$ with $\sum_{i=1}^{m} \alpha_i = 1$, we have*

$$\left|\sum_{i=1}^{m} \alpha_i^{1/r} B_i A_i\right| \leq \left(\sum_{i=1}^{m} |A_i|^p\right)^{1/p} \tag{2.3}$$

whenever $\sum_{i=1}^{m} |B_i^*|^q \leq I$.

Proof. Let $T = \begin{bmatrix} \alpha_1^{1/2-1/p} A_1^* & \alpha_2^{1/2-1/p} A_2^* & \cdots & \alpha_m^{1/2-1/p} A_m^* \\ \alpha_1^{1/2-1/q} B_1 & \alpha_2^{1/2-1/q} B_2 & \cdots & \alpha_m^{1/2-1/q} B_m \end{bmatrix}$. Then

$$0 \leq TT^* = \begin{bmatrix} \sum_{i=1}^{m} \alpha_i^{1-2/p} A_i^* A_i & \sum_{i=1}^{m} \alpha_i^{1/r} A_i^* B_i^* \\ \sum_{i=1}^{m} \alpha_i^{1/r} B_i A_i & \sum_{i=1}^{m} \alpha_i^{1-1/q} B_i B_i^* \end{bmatrix}.$$

Since $p/2 \geq 1$ and $A_i^* A_i \geq 0$, then Lemma 2.3 implies

$$\begin{aligned} 0 &\leq \begin{bmatrix} \left(\sum_{i=1}^{m} |A_i|^p\right)^{2/p} & \sum_{i=1}^{m} \alpha_i^{1/r} A_i^* B_i^* \\ \sum_{i=1}^{m} \alpha_i^{1/r} B_i A_i & \left(\sum_{i=1}^{m} |B_i^*|^q\right)^{2/q} \end{bmatrix} \\ &\leq \begin{bmatrix} \left(\sum_{i=1}^{m} |A_i|^p\right)^{2/p} & \left(\sum_{i=1}^{m} \alpha_i^{1/r} B_i A_i\right)^* \\ \sum_{i=1}^{m} \alpha_i^{1/r} B_i A_i & I \end{bmatrix}. \end{aligned} \tag{2.4}$$

Now Lemma 2.1 implies

$$\left|\sum_{i=1}^{m} \alpha_i^{1/r} B_i A_i\right|^2 \leq \left(\sum_{i=1}^{m} |A_i|^p\right)^{2/p} \tag{2.5}$$

and hence by operator monotonicity of $t^{1/2}$

$$\left|\sum_{i=1}^{m} \alpha_i^{1/r} B_i A_i\right| \leq \left(\sum_{i=1}^{m} |A_i|^p\right)^{1/p}.$$

And the proof is complete. □

As an application of inequality (2.3), we get the following important special cases.

Corollary 2.5. *Let* $1 \leq p, q < \infty$ *with* $1/p + 1/q = 1$ *and* $\alpha_i \in [0,1]$ $(i = 1, 2, \cdots, m)$ *with* $\sum_{i=1}^{m} \alpha_i = 1$. *Then for any* $A_i, B_i \in M_n(\mathbb{C})$ *with* $\sum_{i=1}^{m} |B_i^*|^{2q} \leq I$, *we have*

$$\left|\sum_{i=1}^{m} \sqrt{\alpha_i} B_i A_i\right| \leq \left(\sum_{i=1}^{m} |A_i|^{2p}\right)^{1/2p}. \tag{2.6}$$

In particular,

$$\left|\sum_{i=1}^{m} B_i A_i\right|^2 \leq m \left(\sum_{i=1}^{m} |A_i|^{2p}\right)^{1/p}. \tag{2.7}$$

Proof. Let $r = 2$ and replace p, q by $2p, 2q$ in (2.3) to get inequality (2.6). For the particular case let $r = 2$ and replace p, q by $2p, 2q$ in (2.5) and let $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 1/m$. □

Remark 2.6. The case $p = 1$ and the operator monotonicity of $t^{1/2}$in inequality (2.7) give

$$\left|\sum_{i=1}^{m} B_i A_i\right| \leq \sqrt{m} \left(\sum_{i=1}^{m} |A_i|^2\right)^{1/2} \tag{2.8}$$

whenever $\sum_{i=1}^{m} |B_i^*|^2 \leq I$, which can be considered as a Cauchy matrix inequality. In fact, we can get a stronger version of inequality (2.8) by letting $T = \begin{bmatrix} A_1 & B_1^* \\ A_2 & B_2^* \\ \vdots & \vdots \\ A_m & B_m^* \end{bmatrix}$.

Then

$$0 \leq T^*T = \begin{bmatrix} \sum_{i=1}^{m} A_i^* A_i & \sum_{i=1}^{m} A_i^* B_i^* \\ \sum_{i=1}^{m} B_i A_i & \sum_{i=1}^{m} B_i B_i^* \end{bmatrix}.$$

So, Lemma 2.1 implies

$$\left|\sum_{i=1}^{m} B_i A_i\right|^2 \leq \left(\sum_{i=1}^{m} |A_i|^2\right),$$

whenever $\sum\limits_{i=1}^{m} |B_i^*|^2 \leq I$, and hence

$$\left| \sum_{i=1}^{m} B_i A_i \right| \leq \left(\sum_{i=1}^{m} |A_i|^2 \right)^{1/2}. \tag{2.9}$$

Inequality (2.9) represents a matrix Cauchy Schwarz inequality and clear that it is better than (2.8). In fact, inequality (2.9) is just (2.5) in the case $r = \infty$, i.e. $p = q = 2$.

It is known that limit of $\left\{ \frac{1}{m} \sum\limits_{i=1}^{m} |A_i|^p \right\}^{1/p}$ as $p \to \infty$ exists, and so is $\left\{ \sum\limits_{i=1}^{m} |A_i|^p \right\}^{1/p}$ (see [15]). We write

$$\bigvee_{i=1}^{m} A_i = \lim_{p \longrightarrow \infty} \left(\sum_{i=1}^{m} |A_i|^p \right)^{1/p}. \tag{2.10}$$

The above definition will be used in the following corollary to get new inequalities.

Corollary 2.7. *Let $2 \leq p < \infty$ and $1 < \eta \leq 2$ with $1/p + 1/\eta = 1$. Then for any $A_i, B_i \in M_n(\mathbb{C})$ $(i = 1, 2, \cdots, m)$*

$$\left| \sum_{i=1}^{m} B_i A_i \right|^2 \leq \left(\sum_{i=1}^{m} \|B_i\|^\eta \right)^{2/\eta} \left(\sum_{i=1}^{m} |A_i|^p \right)^{2/p}. \tag{2.11}$$

Moreover,

$$\left| \sum_{i=1}^{m} B_i A_i \right|^2 \leq \left(\sum_{i=1}^{m} \|B_i\| \right)^2 \left(\bigvee_{i=1}^{m} A_i \right)^2. \tag{2.12}$$

Proof. Taking the limit of (2.4) as $q \longrightarrow \infty$ with p fixed (in this case $r \longrightarrow \eta$) to get

$$\begin{bmatrix} \left(\sum\limits_{i=1}^{m} |A_i|^p \right)^{2/p} & \sum\limits_{i=1}^{m} \alpha_i^{1/\eta} A_i^* B_i^* \\ \sum\limits_{i=1}^{m} \alpha_i^{1/\eta} B_i A_i & \lim\limits_{q \longrightarrow \infty} \left(\sum\limits_{i=1}^{m} |B_i^*|^q \right)^{2/q} \end{bmatrix} \geq 0.$$

So,

$$\begin{bmatrix} \left(\sum\limits_{i=1}^{m} |A_i|^p \right)^{2/p} & \sum\limits_{i=1}^{m} \alpha_i^{1/\eta} A_i^* B_i^* \\ \sum\limits_{i=1}^{m} \alpha_i^{1/\eta} B_i A_i & \left(\bigvee\limits_{i=1}^{m} |B_i^*| \right)^2 \end{bmatrix} \geq 0.$$

Note that $\bigvee\limits_{i=1}^{m} |B_i^*| \leq I$ if and only if $|B_i^*| \leq I$ for all $i = 1, 2, \cdots, m$, that is $\|B_i^*\| \leq 1$ for all $i = 1, 2, \cdots, m$, then for every $0 \leq \alpha_i \leq 1$ with $\sum\limits_{i=1}^{m} \alpha_i = 1$ we have

$$\left| \sum_{i=1}^{m} \alpha_i^{1/\eta} B_i A_i \right|^2 \leq \left(\sum_{i=1}^{m} |A_i|^p \right)^{2/p}.$$

By replacing $\alpha_i^{1/\eta} B_i$ by B_i $(i = 1, 2, \cdots, m)$, this means that if $\sum\limits_{i=1}^{m} \|B_i\|^{\eta} \leq I$, then

$$\left|\sum_{i=1}^{m} B_i A_i\right|^2 \leq \left(\sum_{i=1}^{m} |A_i|^p\right)^{2/p},$$

which is equivalent to (2.11). For equation (2.12) take the limit of equation (2.11) as $p \longrightarrow \infty$. □

The next inequality is a generalization of (1.8).

Theorem 2.8. *Let $2 \leq p, q < \infty$ and $r > 1$ with $1/p + 1/q = 1 - 1/r$. Then for $A, B, C, D, X \in M_n(\mathbb{C})$ with $X \geq 0$ and $\alpha \in [0, 1]$ we have*

$$\left|\alpha^{1/r} CXA + (1-\alpha)^{1/r} DXB\right| \leq \left((A^*XA)^{p/2} + (B^*XB)^{p/2}\right)^{1/p} \tag{2.13}$$

whenever $(CXC^*)^{q/2} + (DXD^*)^{q/2} \leq I$.

Proof. Let $T = \begin{bmatrix} \alpha^{1/2-1/p} X^{1/2} A & \alpha^{1/2-1/q} X^{1/2} C^* \\ (1-\alpha)^{1/2-1/p} X^{1/2} B & (1-\alpha)^{1/2-1/q} X^{1/2} D^* \end{bmatrix}$. Then

$$\begin{aligned} & 0 \leq T^*T \\ = & \begin{bmatrix} \alpha^{1-2/p} A^*XA + (1-\alpha)^{1-2/p} B^*XB & \alpha^{1/r} A^*XC^* + (1-\alpha)^{1/r} B^*XD^* \\ \alpha^{1/r} CXA + (1-\alpha)^{1/r} DXB & \alpha^{1-2/q} CXC^* + (1-\alpha)^{1-2/q} DXD^* \end{bmatrix} \end{aligned}$$

Since $p/2 \geq 1$ and $A_i^* X A_i \geq 0$, then Lemma 2.3 implies

$$\begin{aligned} 0 &\leq \begin{bmatrix} \left((A^*XA)^{p/2} + (B^*XB)^{p/2}\right)^{2/p} & \alpha^{1/r} A^*XC^* + (1-\alpha)^{1/r} B^*XD^* \\ \alpha^{1/r} CXA + (1-\alpha)^{1/r} DXB & \left((CXC^*)^{q/2} + (DXD^*)^{q/2}\right)^{2/q} \end{bmatrix} \\ &\leq \begin{bmatrix} \left((A^*XA)^{p/2} + (B^*XB)^{p/2}\right)^{2/p} & \left(\alpha^{1/r} CXA + (1-\alpha)^{1/r} DXB\right)^* \\ \alpha^{1/r} CXA + (1-\alpha)^{1/r} DXB & I \end{bmatrix} \end{aligned}$$

Now lemma 2.1 implies

$$\left|\alpha^{1/r} CXA + (1-\alpha)^{1/r} DXB\right|^2 \leq \left((A^*XA)^{p/2} + (B^*XB)^{p/2}\right)^{2/p},$$

and hence, by operator monotonicity of $t^{1/2}$, we have

$$\left|\alpha^{1/r} CXA + (1-\alpha)^{1/r} DXB\right| \leq \left((A^*XA)^{p/2} + (B^*XB)^{p/2}\right)^{1/p}.$$

Which complete the proof. □

The proof of the following inequality is similar to that of (2.6).

Corollary 2.9. *Let $1 \leq p, q < \infty$ with $1/p + 1/q = 1$. If $A, B, C, D, X \in M_n(\mathbb{C})$ with $X \geq 0$, then*

$$|CXA + DXB| \leq \sqrt{2}\left((A^*XA)^p + (B^*XB)^p\right)^{1/2p} \tag{2.14}$$

whenever $(CXC^*)^q + (DXD^*)^q \leq I$.

Remark 2.10. Inequality (2.13) gives inequality (1.8) directly if $X = I$. Note that inequality (2.14) can be generalized to $m-$tuple of matrices.

Hölder-type inequalities for unitarily invariant norms are included in the following theorem.

Theorem 2.11. *Let $2 \leq p, q < \infty$ and $r > 1$ with $1/p + 1/q = 1 - 1/r$. Then for any $A_i, B_i \in M_n(\mathbb{C})$ and $\alpha_i \in [0,1]$ $(i = 1, 2, \cdots, m)$ with $\sum_{i=1}^{m} \alpha_i = 1$, we have*

$$\left|\left|\left| \, \left| \sum_{i=1}^{m} \alpha_i^{1/r} B_i A_i \right| \, \right|\right|\right| \leq \left|\left|\left| \left(\sum_{i=1}^{m} |A_i|^p \right)^{1/p} \right|\right|\right| \tag{2.15}$$

and

$$\left|\left|\left| \, \left| \sum_{i=1}^{m} \alpha_i^{1/r} B_i A_i \right| \, \right|\right|\right| \leq \left|\left|\left| \sum_{i=1}^{m} |A_i|^p \right|\right|\right|^{1/p} \tag{2.16}$$

whenever $\sum_{i=1}^{m} |B_i^*|^q \leq I$.

Proof. Inequality (2.15) follows directly from (2.3). To prove inequality (2.16) use (2.4) to get

$$\begin{bmatrix} \left(\sum_{i=1}^{m} |A_i|^p \right)^{2/p} & \left(\sum_{i=1}^{m} \alpha_i^{1/r} B_i A_i \right)^* \\ \sum_{i=1}^{m} \alpha_i^{1/r} B_i A_i & I \end{bmatrix} \geq 0.$$

Now Lemma 2.2 in the case $r = 1$ implies

$$\begin{aligned} \left|\left|\left| \, \left| \sum_{i=1}^{m} \alpha_i^{1/r} B_i A_i \right| \, \right|\right|\right| &\leq \left|\left|\left| \left(\left(\sum_{i=1}^{m} |A_i|^p \right)^{2/p} \right)^{p/2} \right|\right|\right|^{1/p} \\ &= \left|\left|\left| \sum_{i=1}^{m} |A_i|^p \right|\right|\right|^{1/p}. \end{aligned}$$

□

We end this article by the following remarks.

Remark 2.12. For a positive semidefinite matrix $T \in M_n(\mathbb{C})$ and a normalized unitarily invariant norm, i.e. $|||diag\,(1, 0, \ldots, 0)||| = 1$, see [19], we have $|||T^r||| \leq |||T|||^r$ for $r \geq 1$ and $|||T|||^r \leq |||T^r|||$ for $0 \leq r \leq 1$. Thus

$$\left|\left|\left| \sum_{i=1}^{m} |A_i|^p \right|\right|\right|^{1/p} \leq \left|\left|\left| \left(\sum_{i=1}^{m} |A_i|^p \right)^{1/p} \right|\right|\right|$$

for every normalized unitarily invariant norm (since $1/p$ is less than 1). So inequality (2.16) is better than inequality (2.15).

Remark 2.13. Inequality (2.7) implies

$$\left|\left|\left| \left|\sum_{i=1}^{m} B_i A_i\right|^2 \right|\right|\right| \le m \left|\left|\left| \left(\sum_{i=1}^{m} |A_i|^{2p}\right)^{1/p} \right|\right|\right|^{1/p}.$$

This inequality is related to

$$\left|\left|\left| \sum_{i=1}^{m} A_i^* B_i \right|\right|\right| \le m^{\left|\frac{1}{p}-\frac{1}{2}\right|} \left|\left|\left| \sum_{i=1}^{m} |A_i|^p \right|\right|\right|^{1/p}$$

which is the variational expression of inequality (1.7).

Remark 2.14. Let $A = U|A|$ and $B = V|B|$ be the polar decompositions with unitaries U, V. If we let $S = CU$ and $T = DV$, then

$$|C^*| = |S^*|, \ \ |D^*| = |T^*|, \ |CA + DB| = |S|A| + T|B| \ |,$$

that means it may assumed without loss of generality that A, B are positive semidefinite.

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