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# ALMOST EVERYWHERE CONVERGENCE OF THE SPHERICAL PARTIAL FOURIER INTEGRALS FOR RADIAL FUNCTIONS

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This paper is dedicated to Professor Lars-Erik Persson

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ABSTRACT. We study new conditions on a radial function f in order to have the almost everywhere convergence of the spherical partial Fourier integrals.

#### 1. Introduction and Preliminaries

Given a function f for which the Fourier transform is well defined, the spherical partial Fourier integral is given by

$$S_R f(x) = \int_{B(0,R)} \hat{f}(\xi) e^{2\pi i x \xi} d\xi$$

and it is an old and difficult open problem to show whether

$$\lim_{R \to \infty} S_R f(x) = f(x), \quad \text{a.e. } x \in \mathbb{R}^n,$$
 (1.1)

whenever  $f \in L^2(\mathbb{R}^n)$  with n > 1. The case n = 1 was solved positively by L. Carleson in [1] (see also [5] for the case  $f \in L^p(\mathbb{R})$ , p > 1).

Looking for conditions on a function f in order to have the convergence (1.1), it was proved in [7] that this is the case if f is a radial function belonging to

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 $L^p(\mathbb{R}^n)$  with

$$\frac{2n}{n+1}$$

To prove this it was shown that, for radial functions,

$$\tilde{S}f(x) = \sup_{R} |S_R f(x)| \le \frac{C(n)}{s^{(n-1)/2}} (M + L + \tilde{H} + \tilde{C})(g)(s)$$
 (1.2)

where s = |x|,  $g(r) = f(r)r^{(n-1)/2}\chi_{(0,\infty)}(r)$ , M is the Hardy–Littlewood maximal operator,  $\tilde{H}$  is the maximal Hilbert transform,  $\tilde{C}$  is the maximal Carleson operator defined by

$$\tilde{C}f(x) = \sup_{y \in \mathbb{R}} \sup_{\varepsilon > 0} \left| \int_{\varepsilon < |x-t|} \frac{e^{-iyt}f(t)}{x - t} dt \right|$$

and L is the Hilbert integral

$$Lf(s) = \int_0^\infty \frac{f(t)}{s+t} dt.$$

Using (1.2) it is proved in [9] and [2] that

$$\tilde{S}: L_{rad}^{p_j,1} \longrightarrow L^{p_j,\infty}, \qquad j = 0, 1$$

is bounded with

$$p_0 = \frac{2n}{n+1}, p_1 = \frac{2n}{n-1}, (1.3)$$

and, for a space X of functions in  $\mathbb{R}^n$ ,

$$X_{rad} = \{ f \in X; \ f \text{ is radial} \}.$$

From this the almost everywhere convergence of  $S_R f(x)$  at the end-point spaces  $L_{rad}^{p_j,1}$  follows.

Again (1.2) is used in [8] to prove that if w is a radial weight such that  $u(s) = w(s)|s|^{(n-1)(1-\frac{p}{2})}$  is in the Muckenhoupt class  $A_p(\mathbb{R})$  (see [6]) then

$$||\tilde{S}f||_{L^p(w)} \le C_{w,p}||f||_{L^p_{res}(w)}.$$
 (1.4)

In fact, from (1.2) we have that, if w is radial,

$$||\tilde{S}f||_{L^p(w)} \lesssim ||Tg||_{L^p\left(\mathbb{R}^+; w(s)s^{(n-1)(1-\frac{p}{2})}\right)} = ||Tg||_{L^p(\mathbb{R};u)}$$

where u(s) is as before and

$$Tg(s) = (M + L + \tilde{H} + \tilde{C})(g)(|s|).$$

Now, if  $u \in A_p(\mathbb{R})$ , all the operators appearing in T are bounded on  $L^p(u)$  and hence (1.4) is obtained.

However, no information is given in [8] about the behavior of the constant  $C_{w,p}$  in (1.4). In the recent paper [3], this constant has been explicitly computed showing that for every 1 and <math>u as before

$$C(w,p) \lesssim \max\left(||u||_{A_p}^{\frac{1}{p-1}}, ||u||_{A_p}, \inf_{r>1} \frac{1}{(r-1)^2} ||u||_{A_p}^{\frac{r}{\frac{r}{p-1}}}\right).$$
 (1.5)

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Using this estimate, it was easy to see, for example, that if w is a radial function such that  $w_0 \in A_1(\mathbb{R})$ , where  $w_0(r) = w(|x|)$  for |x| = r > 0 and  $w_0(r) = w_0(-r)$  for r < 0 and where we recall that  $w_0 \in A_1(\mathbb{R})$  if

$$Mw_0(s) \leq Cw_0(s), \ a.e. \ s \in \mathbb{R}$$

and  $||w_0||_{A_1}$  is the infimum of all the above constants C, then

$$C_{w,p} \lesssim ||w_0||_{A_1(\mathbb{R})} \left(\frac{1}{p-p_0}\right)^3,$$
 (1.6)

for  $p_0 and <math>p_0$  as in (1.3).

**Definition 1.1.** We shall say that a radial weight w defined in  $\mathbb{R}^n$  is in  $A_1(\mathbb{R})$  if  $||w||_{A_1(\mathbb{R})} = ||w_0||_{A_1} < \infty$ . and we shall write

$$w \in A_1(\mathbb{R}).$$

Using (1.4), (1.6) and a Yano's extrapolation argument (see [10]), the following result was obtained in [3].

**Theorem 1.2.** If w is a radial function in  $\mathbb{R}^n$  such that  $w \in A_1(\mathbb{R})$  then (1.1) holds for every radial function f satisfying

$$\int_{\mathbb{R}^{n}} |f(x)|^{p_{0}} \Big( 1 + \log^{+} |f(x)| \Big)^{p_{0}\beta} w(x) dx < \infty$$

with  $\beta > 3$ .

On the other hand, in the other end-point  $p_1$ , the result obtained in [3] reads as follows.

**Theorem 1.3.** If w is a radial function in  $\mathbb{R}^n$  such that  $w(s)|s|^{\frac{-2n}{n-1}} \in A_1(\mathbb{R})$  then (1.1) holds for every radial function f satisfying

$$\int_{\mathbb{R}^n} |f(x)|^{p_1} \left(1 + \log^+ |f(x)|\right)^{p_1\beta} w(x) dx < \infty$$

with  $\beta > 3$ .

It was not completely clear in [3] why the conditions on the weight w differs in  $p_0$  and  $p_1$  and which other condition on a radial weight we can assume in order to have the almost everywhere convergence in a space "near"  $L^p_{rad}(w)$  for other values of 1 . This will be clarified in the present note.

Given two quantities A and B, we shall use the symbol  $A \lesssim B$  to indicate the existence of a positive universal constant C such that  $A \leq CB$ . Also for simplicity, we write

$$\overline{\log} \, x = 1 + \log^+ x$$

with  $\log^+ x = \max(\log x, 0)$ .

### 2. Main results

Let us recall (see [4]) that a weight  $v \in A_p$  if and only if  $v = v_0 v_1^{1-p}$  with  $v_j \in A_1$ , j = 0, 1 and

$$||v||_{A_p} \le ||v_0||_{A_1} ||v_1||_{A_1}^{p-1}.$$

Also, a power weight  $v(x) = |x|^{\alpha} \in A_1(\mathbb{R})$  if and only if  $-1 < \alpha \le 0$  and ([3])

$$||v||_{A_1} \le \frac{2}{1+\alpha}.$$

With these estimates, let us assume now that

$$w(x) = v(x)|x|^{\delta}$$

for some  $\delta \in \mathbb{R}$  and v a radial weight in  $\mathbb{R}^n$  such that  $v \in A_1(\mathbb{R})$ . Then if

$$u(s) = v(s)|s|^{\delta + (n-1)(1-\frac{p}{2})}, \qquad s \in \mathbb{R}$$

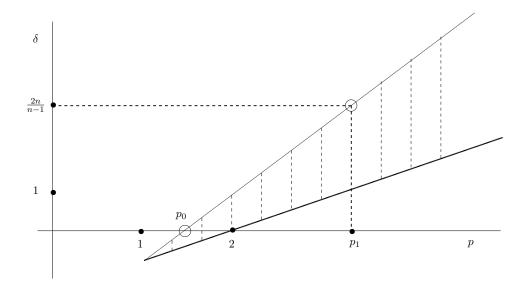
and

$$(n-1)\left(\frac{p}{2}-1\right) \le \delta < \frac{n+1}{2}p-n$$
 (2.1)

we get that  $u \in A_p(\mathbb{R})$ . Moreover,

$$||u||_{A_p} \lesssim ||v||_{A_1} \left(\frac{1}{\delta + n - p^{\frac{n+1}{2}}}\right)^{p-1}$$
 (2.2)

The area inside the cone together with the inferior boundary in the below picture represents the set of pairs  $(p, \delta)$  satisfying (2.1) and will be called the admissible region.



**Theorem 2.1.** If  $(p, \delta)$  belongs to the admissible region and w is a radial function satisfying that

$$w(x)|x|^{-\delta} \in A_1(\mathbb{R})$$

then

$$\tilde{S}: L^p_{rad}(w) \longrightarrow L^p(w)$$

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is bounded. Moreover, for every f radial function,

$$||\tilde{S}f||_{L^p(w)} \lesssim \left(\frac{1}{\delta + n - p^{\frac{n+1}{2}}}\right)^{\max(3, p-1)} ||f||_{L^p(w)}$$
 (2.3)

Consequently, if  $f \in L^p_{rad}(w)$ , (1.1) holds.

*Proof.* By (1.5) and (2.2) we have to compute

$$\inf_{r>1} \frac{1}{(r-1)^2} ||u||_{A_{\frac{p}{r}}}^{\frac{r}{\frac{p}{r}-1}} \lesssim \inf_{r>1} \frac{1}{(r-1)^2} \left(\frac{1}{\delta + n - \frac{p}{r} \frac{n+1}{2}}\right)^r \\ \leq \inf_{r>1} \frac{1}{(r-1)^2} \left(\frac{1}{\delta + n - p \frac{n+1}{2}}\right)^r.$$

Then, taking r such that  $r-1 \approx \delta + n - p \frac{n+1}{2}$  we get that

$$\inf_{r>1} \frac{1}{(r-1)^2} ||u||_{A_{\frac{p}{r}}}^{\frac{r}{\frac{p}{r}-1}} \lesssim \left(\frac{1}{\delta + n - p^{\frac{n+1}{2}}}\right)^3$$

and therefore

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$$C(w,p) \lesssim \left(\frac{1}{\delta + n - p\frac{n+1}{2}}\right)^{\max(3,p-1)}$$

as we wanted to see.

Our purpose now is to use (2.3) and some extrapolation argument in order to obtain the almost everywhere convergence for a radial function in a space "near"  $L_{rad}^{p}(w)$  with

$$w(x)|x|^{n-\frac{n+1}{2}p} \in A_1(\mathbb{R}).$$

Observe that the pair  $(p, \frac{n+1}{2}p - n)$  is in the upper boundary (and hence outside) of the admissible region. Moreover, if  $p = \frac{2n}{n+1}$ , the above condition reads

$$w(x) \in A_1(\mathbb{R})$$

and if  $p = \frac{2n}{n-1}$ 

$$w(x)|x|^{\frac{-2n}{n-1}} \in A_1(\mathbb{R}),$$

which are the conditions in Theorems 1.2 and 1.3 respectively. With the same proof than in those theorems, we now have the following result.

**Theorem 2.2.** Let 1 and let <math>w be a radial weight in  $\mathbb{R}^n$  such that  $w(s)|s|^{n-\frac{n+1}{2}p} \in A_1(\mathbb{R})$  then (1.1) holds for every radial function f satisfying

$$\int_{\mathbb{D}^n} |f(x)|^{p_1} \Big( 1 + \log^+ |f(x)| \Big)^{p_1 \beta} w(x) dx < \infty,$$

with  $\beta > \max(3, p - 1)$ .

Observe that if  $p \leq p_1$ ,  $\max(3, p - 1) = 3$  and the above theorem extends Theorems 1.2 and 1.3.

Remark 2.3. In [3], it was consider the case  $\delta = 0$  and the estimate at the endpoint  $p = p_0$  was done by a Yano's extrapolation argument applying (2.3) with  $p > p_0$ . Also, it was considered the end-point  $p = p_1$  taking  $\delta = \frac{2n}{n-1}$  and  $p > p_1$  which is also inside the admissible region.

Another possibility, which is the one presented in our next theorem is to consider p fixed and move  $\delta$  vertically in such a way that  $(p, \delta)$  is inside the admissible region.

**Theorem 2.4.** Let  $p_n = \frac{n+1}{2}p - n$  and let w be a radial weight in  $\mathbb{R}^n$  such that  $w(x)|x|^{-p_n} \in A_1(\mathbb{R}).$ 

Then, for 1 and <math>f a radial function,

$$\sup_{t>0} \frac{\left\| \min\left(1, \frac{t}{|x|}\right)^{1/p} \tilde{S}f \right\|_{L^{p}(w)}}{(\overline{\log} t)^{\max(p-1,3)}} \lesssim \left( \int_{\{|x| \ge 1/4\}} |f(x)|^{p} w(x) dx \right)^{1/p} + \sum_{i=0}^{\infty} \left( \int_{\{2^{-2^{i+1}} \le |x| < 2^{-2^{i}}\}} |f(x)|^{p} \left( \overline{\log} \frac{1}{|x|} \right)^{p \max(p-1,3)} w(x) dx \right)^{1/p}.$$

Consequently, if f satisfies that the right term is finite, (1.1) holds.

**Proof:** Let us take  $\delta$  in such a way that  $(p, \delta)$  is inside the admissible region. Let us write  $\theta = p_n - \delta$  and take  $\delta$  in such a way that  $0 < \theta < 1$ . It is clear that, for every t > 0,

$$\frac{\min\left(1, \frac{t}{|x|}\right)}{t^{\theta}} \le |x|^{-\theta},$$

and hence, using (2.3) we have that for every radial function f,

$$\left\| \min \left( 1, \frac{t}{|x|} \right)^{1/p} \tilde{S} f \right\|_{L^{p}(w)} \leq t^{\theta/p} ||x|^{-\theta/p} \tilde{S} f||_{L^{p}(w)} = t^{\theta/p} ||\tilde{S} f||_{L^{p}(w(x)|x|^{-\theta})}$$

$$\lesssim \frac{t^{\theta/p}}{\theta^{\max(p-1,3)}} ||f||_{L^{p}(w(x)|x|^{-\theta})}.$$

Using Hölder's inequality we have that, for every t > 0 and every  $0 < \delta < 1$ ,

$$\left\| \min \left( 1, \frac{t}{|x|} \right)^{1/p} \tilde{S}f \right\|_{L^p(w)} \lesssim t^{\theta/p} \theta^{\max(p-1,3)} |||x|^{-1/p} f||_{L^p(w)}^{\theta} ||f||_{L^p(w)}^{1-\theta}$$

and taking the infimum in  $\theta$  in the right hand side, we get that

$$\left\| \min \left( 1, \frac{t}{|x|} \right)^{1/p} \tilde{S}f \right\|_{L^{p}(w)} \lesssim ||f||_{L^{p}(w)} \left( \overline{\log} \, \frac{t^{1/p}||\,|x|^{-1/p}f||_{L^{p}(w)}}{||f||_{L^{p}(w)}} \right)^{\max(3,p-1)}$$

$$\lesssim (\overline{\log} \, t)^{\beta} ||f||_{L^{p}(w)} \left( \overline{\log} \frac{||\,|x|^{-1/p}f||_{L^{p}(w)}}{||f||_{L^{p}(w)}} \right)^{\max(3,p-1)}$$

Let us decompose

$$f = f_0 + \sum_{i=1}^{\infty} f_i$$

with  $f_i = f\chi_{\{2^{-2^{i+1}} \le |x| < 2^{-2^i}\}}$ ,  $i \ge 1$ . Then, by sublinearity,  $\tilde{S}f \le \sum_{i=0}^{\infty} \tilde{S}f_i$  and since  $f_i$  is also radial, we have that

$$\left\| \min \left( 1, \frac{t}{|x|} \right)^{1/p} \tilde{S} f_i \right\|_{L^p(w)}$$

$$\lesssim (\overline{\log} t)^{\max(3, p-1)} ||f_i||_{L^p(w)} \left( \overline{\log} \frac{|||x|^{-1/p} f_i||_{L^p(w)}}{||f_i||_{L^p(w)}} \right)^{\max(3, p-1)}$$

$$\lesssim (\overline{\log} t)^{\max(3, p-1)} ||f_i||_{L^p} 2^{i \max(3, p-1)}.$$

Summing in i we obtain the result.

As an immediate consequence of the previous theorem, we have the following.

Corollary 2.5. Under the condition of Theorem 2.4 we have that if f is a radial function satisfying that

$$\int_{\mathbb{R}^n} |f(x)|^p \left(\overline{\log} \frac{1}{|x|}\right)^{p \max(p-1,3)} \left(\overline{\log} \overline{\log} \frac{1}{|x|}\right)^q w(x) dx < \infty,$$

with q > p - 1, the almost everywhere convergence (1.1) holds.

*Proof.* The proof follows easily by observing that if  $I_i = \{2^{-2^{i+1}} \le |x| < 2^{-2^i}\}$  with  $i \ge 1$ , then  $\overline{\log} \overline{\log} \frac{1}{|x|} \approx 1 + i$  for every  $x \in I_i$  and hence, since q > p - 1,

$$\sum_{i=0}^{\infty} \left( \int_{I_{i}} |f(x)|^{p} \left( \overline{\log} \frac{1}{|x|} \right)^{p \max(p-1,3)} w(x) dx \right)^{1/p}$$

$$\approx \sum_{i=0}^{\infty} \frac{1}{(1+i)^{q/p}} \left( \int_{I_{i}} |f(x)|^{p} \left( \overline{\log} \frac{1}{|x|} \right)^{p \max(p-1,3)} \left( \overline{\log} \overline{\log} \frac{1}{|x|} \right)^{q} w(x) dx \right)^{1/p}$$

$$\lesssim \left( \sum_{i \geq 1} \frac{1}{(1+i)^{q/(p-1)}} \right)^{1/p'}$$

$$\times \left( \int_{\mathbb{R}^{n}} |f(x)|^{p} \left( \overline{\log} \frac{1}{|x|} \right)^{p \max(p-1,3)} \left( \overline{\log} \overline{\log} \frac{1}{|x|} \right)^{q} w(x) dx \right)^{1/p} < \infty.$$

Finally, as a consequence of Theorems 2.2 and 2.4 we can conclude our last result.

**Corollary 2.6.** Let 1 and let <math>w satisfy the hypothesis of Theorem 2.4. Then, for every  $f \in L^p_{rad}(w)$  such that for some constants A, B > 0,

$$\sup_{|x| \le A} |f(x)| \le B,$$

condition (1.1) holds.

*Proof.* The proof reduces to decompose the function in the sum of two functions  $f = f_0 + f_1$  such that  $f_0(x) = f(x)\chi_{B(0,A)}(x)$  and apply Theorem 2.2 to  $f_0$  and Theorem 2.4 to  $f_1$ .

#### References

- 1. L. Carleson, On convergence and growth of partial sums of Fourier series, Acta Math. 116 (1966), 135–157.
- 2. L. Colzani, A. Crespi, G. Travaglini and M. Vignati, Equiconvergence theorems for Fourier-Bessel expansions with applications to the harmonic analysis of radial functions in Euclidean and non-Euclidean spaces, Trans. Amer. Math. Soc. 338 (1993), 43–55
- 3. M. Carro and E. Prestini, Convergence a.e. of the spherical partial Fourier integrals on weighted spaces for radial functions: endpoint estimates, Studia Math. 192 (2009), 173–194.
- 4. J. Duoandikoetxea, *Fourier analysis*, Graduate Studies in Mathematics, 29. American Mathematical Society, Providence, RI, 2001.
- 5. R. Hunt, On the convergence of Fourier series, Proc. Conf., Edwardsville, Ill. 1 (1967), 235–255.
- 6. B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, Trans. Amer. Math. Soc. 165 (1972), 207–226.
- E. Prestini, Almost everywhere convergence of the spherical partial sums for radial functions, Monatsh. Math. 105 (1988), 207–216.
- 8. E. Romera, Weighted bounds for the Carleson maximal operator in  $\mathbb{R}^n$ , Rend. Circ. Mat. Palermo (2)43 (1994), 98–106.
- 9. E. Romera and F. Soria, Endpoint estimates for the maximal operator associated to spherical partial sums on radial functions, Monatsh. Math. 105 (1988), 207–216.
- S. Yano, Notes on Fourier analysis. XXIX. An extrapolation theorem, J. Math. Soc. Japan 3 (1951), 296–305.

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