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COMPOSITION OPERATORS BETWEEN GENERALLY WEIGHTED BLOCH SPACE AND $Q^q_{\log}$ SPACE

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ABSTRACT. Let φ be a holomorphic self-map of the open unit disk D on the complex plane and $p,\ q > 0$. In this paper, the boundedness and compactness of composition operator C_φ from generally weighted Bloch space $B^p_{\log}$ to $Q^q_{\log}$ are investigated.

1. INTRODUCTION AND PRELIMINARIES

Suppose that D is the unit disc on the complex plane, ∂D its boundary and φ a holomorphic self-map of D. We denote by $H(D)$ the space of all holomorphic functions on D, denote by $dm(z)$ the normalized Lebesgue area measure and define the composition operator C_φ on $H(D)$ by $C_\varphi f = f \circ \varphi$.

For $0 < p \le \infty$, the Hardy space H^p is the Banach space of analytic functions on D such that

$$\|f\|^p_{H^p} = \sup_{r\in[0,1)} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty, \quad 0 < p < \infty,$$

and

$$\|f\|_{H^\infty} = \sup_{z\in D} |f(z)| < \infty.$$

For more details see [15] and [16].

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We say that $f \in H(D)$ belongs to $BMOA_{\log}$ if $f \in H^2$ and has weighted bounded mean oscillation, i.e.

$$\|f\|_{BMOA_{\log}} = \sup_{I \subseteq \partial D} \frac{(\log \frac{2}{|I|})^2}{|I|} \int_{S(I)} |f'(z)|^2 \log \frac{1}{|z|} dm(z) < \infty,$$

where

$$S(I) = \{z \in D: \ 1 - |I| \le |z| < 1, \ \frac{z}{|z|} \in I\}$$

is the Carleson square of the arc I and $|I|$ its length.

By definition it is immediate that $BMOA_{\log}$ is exactly $Q^1_{\log}$. In [10], the above relation helped to describe the pointwise multipliers of the Mӧbius invariant Banach spaces Q_q, $q \in [0, 1]$, consisting of $f \in H(D)$, such that

$$\|f\|_{Q_q} = |f(0)| + \sup_{\alpha \in D} \int_D |f'(z)|^2 g^q(z, \alpha) dm(z) < \infty,$$

where $g(z, \alpha) = \log \frac{1}{|\phi_\alpha(z)|}$ is the Green's function and $\phi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha} z}$. For more details on these spaces see for example [2] and the two monographs [11]and [12].

The space of analytic functions on D such that

$$\|f\|_{B_{\log}} = |f(0)| + \sup_{z \in D} |f'(z)|(1 - |z|^2) \log \frac{2}{1 - |z|^2} < \infty$$

is called weighted Bloch space $B_{\log}$.

$B_{\log}$ and $BMOA_{\log}$ first appeared in the study of boundedness of the Hankel operators on the Bergman space

$$A^1 = \{f \in H(D): \ \int_D |f(z)| dm(z) < \infty\}$$

and the Hardy space H^1, respectively. $BMOA_{\log}$ also appeared in the study of a Volterra type operator. For more details [1], [3], [8] and [9].

In [13], Yoneda studied the composition operators from $B_{\log}$ to $BMOA_{\log}$. He found one sufficient and a different necessary condition for the boundedness of the composition operators from $B_{\log}$ to $BMOA_{\log}$. So it is natural to ask for the approximate conditions that characterize boundedness and compactness of the composition operators $C_\varphi: \ B^p_{\log} \to BMOA_{\log}$.

In [6], we introduced the space $B^p_{\log}$. The space of analytic functions on D such that

$$\|f\|_{B^p_{\log}} = |f(0)| + \sup_{z \in D} |f'(z)|(1 - |z|^2)^p \log \frac{2}{1 - |z|^2} < \infty$$

is called generally weighted Bloch space $B^p_{\log}$. When $p = 1$, the space $B^p_{\log}$ is just the weighted Bloch space $B_{\log}$.

In [5], Petros Galanopoulos considered the space $Q^q_{\log}$, $q > 0$, the spaces of analytic functions on the unit disc such that

$$\|f\|_* = \sup_{I \subseteq \partial D} \frac{(\log \frac{2}{|I|})^2}{|I|^q} \int_{S(I)} |f'(z)|^2 (\log \frac{1}{|z|})^q dm(z) < \infty.$$

In this paper, we consider composition operator C_φ from generally weighted Bloch space $B^p_{\log}(D)$ to $Q^q_{\log}(D)$. We find a necessary and sufficient condition for

Taylor coefficients of a function in $B^p_{\log}$. Using the results for the Hadamard gap series and following a technique used before in the Bloch space in [7], we construct two functions $f, g \in B^p_{\log}$ such that for each $z \in D$,

$$|f'(z)| + |g'(z)| \geq \frac{C}{(1-|z|)^p \log \frac{2}{1-|z|}},$$

where C is a positive constant. Using this fact we prove the following theorems:

Theorem 1.1. *Let $p,\ q > 0$. If φ is an analytic self-map of the unit disc, then the induced composition operator $C_\varphi : B^p_{\log} \to Q^q_{\log}$ is bounded if and only if*

$$\sup_{\alpha\in D}(\log\frac{2}{1-|\alpha|^2})^2 \int_D |\varphi'(z)|^2 \frac{(1-|\phi_\alpha(z)|^2)^q}{(1-|\varphi(z)|^2)^{2p}(\log\frac{2}{1-|\varphi(z)|^2})^2} dm(z) < \infty.$$

Theorem 1.2. *Let $p,\ q > 0$. If φ is an analytic self-map of the unit disc, then the induced composition operator $C_\varphi : B^p_{\log} \to Q^q_{\log}$ is compact if and only if $\varphi \in Q^q_{\log}$ and*

$$\lim_{r\to 1}\sup_{\alpha\in D}(\log\frac{2}{1-|\alpha|^2})^2 \int_{\{|\varphi(z)|>r\}} |\varphi'(z)|^2 \frac{(1-|\phi_\alpha(z)|^2)^q}{(1-|\varphi(z)|^2)^{2p}(\log\frac{2}{1-|\varphi(z)|^2})^2} dm(z) = 0.$$

By the definition of $B^p_{\log}$, we can easily obtain the following corollaries.

Corollary 1.3. *Let $q > 0$. If φ is an analytic self-map of the unit disc, then the induced composition operator $C_\varphi : B_{\log} \to Q^q_{\log}$ is bounded if and only if*

$$\sup_{\alpha\in D}(\log\frac{2}{1-|\alpha|^2})^2 \int_D |\varphi'(z)|^2 \frac{(1-|\phi_\alpha(z)|^2)^q}{(1-|\varphi(z)|^2)^{2}(\log\frac{2}{1-|\varphi(z)|^2})^2} dm(z) < \infty.$$

Corollary 1.4. *Let $q > 0$. If φ is an analytic self-map of the unit disc, then the induced composition operator $C_\varphi : B_{\log} \to Q^q_{\log}$ is compact if and only if $\varphi \in Q^q_{\log}$ and*

$$\lim_{r\to 1}\sup_{\alpha\in D}(\log\frac{2}{1-|\alpha|^2})^2 \int_{\{|\varphi(z)|>r\}} |\varphi'(z)|^2 \frac{(1-|\phi_\alpha(z)|^2)^q}{(1-|\varphi(z)|^2)^{2}(\log\frac{2}{1-|\varphi(z)|^2})^2} dm(z) = 0.$$

Throughout the remainder of this paper C will denote a positive constant, the exact value of which will vary from one appearance to the next.

2. Main results

Let f be a holomorphic function in D with the gap series expansion

$$f(z) = \sum_{k=1}^{\infty} a_k z^{n_k},\ z \in D, \tag{a}$$

where for a constant $\lambda > 1$, the natural numbers n_k satisfy

$$\frac{n_{k+1}}{n_k} \geq \lambda,\ k \geq 1. \tag{b}$$

Lemma 2.1. *Let f be a holomorphic function in D with (a) and (b). Then for $p>0$, $f \in B^p_{\log}$ if and only if*

$$\limsup_{k\to\infty} |a_k|\cdot n_k^{1-p}\cdot \log n_k < \infty.$$

Proof. Let f be a holomorphic function in D, $f(z)=\sum_{k\geq 0} a_k z^k \in B^p_{\log}$.

Since $a_k = \frac{1}{2k\pi}\int_0^{2\pi} f'(re^{i\theta})r^{1-k}e^{i(1-k)\theta}d\theta$, then

$$\begin{aligned}|a_k| &\leq \frac{1}{2k\pi}\int_0^{2\pi} f'(re^{i\theta})r^{1-k}d\theta\\ &\leq \frac{\|f\|_{B^p_{\log}}\cdot r^{1-k}}{k(1-r)^p\log\frac{1}{1-r}}.\end{aligned}$$

Let $r=1-\frac{1}{k}$, then

$$|a_k| \leq \frac{\|f\|_{B^p_{\log}}(1-\frac{1}{k})^{1-k}}{k^{1-p}\log k} = \frac{\|f\|_{B^p_{\log}}(1+\frac{1}{-k})^{-k}(1-\frac{1}{k})}{k^{1-p}\log k},$$

then

$$\limsup_{k\to\infty} |a_k|\cdot k^{1-p}\cdot\log k \leq e\cdot\|f\|_{B^p_{\log}} < \infty.$$

Conversely, Since $f(z)=\sum_{k\geq 0} a_k z^{n_k}$, then

$$|zf'(z)| \leq \sum_{k\geq 0}|a_k|n_k|z|^{n_k} \leq C\sum_{k\geq 0}\frac{n_k^p}{\log n_k}|z|^{n_k},$$

$$\frac{n_{k+1}^p\log n_k}{n_k^p\log n_{k+1}} = (\frac{n_{k+1}}{n_k})^p(\frac{\log n_{k+1}}{\log n_k})^{-1} = (\frac{n_{k+1}}{n_k})^p(1+\frac{\log\frac{n_{k+1}}{n_k}}{\log n_k})^{-1} = \lambda^p(1+\frac{\log\lambda}{\log n_k})^{-1}.$$

Then for each $\varepsilon\in(0,1)$, there exists k_0 such that when $k\geq k_0$ we have

$$\frac{n_{k+1}^p\log n_k}{n_k^p\log n_{k+1}} \geq (1-\varepsilon)\lambda^p \tag{2.1}$$

thus

$$\frac{n_k^p}{\log n_k} \leq \frac{1}{(1-\varepsilon)\lambda^p}\cdot\frac{n_{k+1}^p}{\log n_{k+1}}.$$

$$\begin{aligned}\frac{|zf'(z)|\log\frac{1}{1-|z|}}{1-|z|} &\leq C(\sum_{k\geq 0}\frac{n_k^p}{\log n_k}|z|^{n_k})(\sum_{n\geq 0}|z|^n)|z|\sum_{n\geq 0}\frac{|z|^n}{n+1}\\ &\leq C\,'(\sum_{n\geq n_0}(\sum_{n_k\leq n}\frac{n_k^p}{\log n_k})|z|^n)\sum_{n\geq 0}\frac{|z|^n}{n+1}.\end{aligned}$$

Let k' be a positive integer number such that $n_{k'}\leq n\leq n_{k'+1}$, we fix $(1-\varepsilon)\lambda^p > 1,\ \varepsilon>0$, then we get an index k_0 such that (2.1)) holds.

If $k' \geq k_0$, then

$$\begin{aligned}
\sum_{n_k \leq n} \frac{n_k^p}{\log n_k} &= \sum_{k \leq k_0} \frac{n_k^p}{\log n_k} + \sum_{k' > k > k_0} \frac{n_k^p}{\log n_k} \\
&\leq C\frac{n^p}{\log n} + \frac{n^p}{\log n} \cdot \sum_{k'>k>k_0} \frac{1}{[\lambda^p(1-\varepsilon)]^{k'-k}} \\
&\leq C\frac{n^p}{\log n} + \frac{n^p}{\log n} \cdot \frac{\frac{1}{[\lambda^p(1-\varepsilon)]^{k'-(k_0+1)}}(1-[\lambda^p(1-\varepsilon)]^{k'-k_0})}{1-\lambda^p(1-\varepsilon)} \\
&= C\frac{n^p}{\log n} + \frac{n^p}{\log n} \cdot \frac{\lambda^p(1-\varepsilon) - \frac{1}{[\lambda^p(1-\varepsilon)]^{k'-(k_0+1)}}}{\lambda^p(1-\varepsilon)-1} \\
&\leq (C+1)\frac{n^p}{\log n} + \frac{n^p}{\log n} \cdot \frac{1}{\lambda^p(1-\varepsilon)-1}.
\end{aligned}$$

Since

$$\sum_{n=0}^{\infty}(n+1)^p|z|^n \leq \frac{C}{(1-|z|)^{1+p}}, \quad z \in D,$$

thus

$$\begin{aligned}
\frac{|zf'(z)|\log\frac{1}{1-|z|}}{1-|z|} &\leq C(\sum_{n\geq 3}\frac{n^p}{\log n}|z|^n)(\sum_{n\geq 0}\frac{|z|^n}{n+1}) \\
&\leq C\sum_{n\geq 3} n^p|z|^n \\
&= C|z|\sum_{n\geq 2}(n+1)^p|z|^n \\
&\leq C\frac{|z|}{(1-|z|)^{1+p}}.
\end{aligned}$$

□

Lemma 2.2. *There exist $f,\ g \in B^p_{\log}$ such that*

$$|f'(z)| + |g'(z)| \geq \frac{C}{(1-|z|)^p \log\frac{2}{1-|z|}}.$$

Proof. We consider the function

$$f(z) = Kz + \sum_{j\geq 1}\frac{q^{(j+k_0)(p-1)+\frac{p}{2}}}{\log q^{j+k_0}} z^{q^{j+k_0}}$$

for q an appropriately large integer, K a properly small chosen positive constant and k_0 the index for which (2.1) holds for the sequence n_j such that $n_j = q^{j+k_0}$. So this function is a member of the $B^p_{\log}$ space.

$$1 - q^{-(k+k_0)} \leq |z| < 1 - q^{-(k+k_0+\frac{1}{2})} \quad (k \geq 1),$$

$$\begin{aligned}
|f'(z)| &= |K+\sum_{j\geq 1}\frac{q^{(j+k_0)p+\frac{p}{2}}}{\log q^{j+k_0}}z^{q^{(j+k_0)-1}}| \\
&= |K+\sum_{j=1}^{k-1}\frac{q^{(j+k_0)p+\frac{p}{2}}}{\log q^{j+k_0}}z^{q^{(j+k_0)-1}} \\
&\quad +\frac{q^{(k+k_0)p+\frac{p}{2}}}{\log q^{k+k_0}}z^{q^{(k+k_0)-1}}\sum_{j=k+1}^{\infty}\frac{q^{(j+k_0)p+\frac{p}{2}}}{\log q^{j+k_0}}z^{q^{(j+k_0)-1}}| \\
&\geq \frac{q^{(k+k_0)p+\frac{p}{2}}}{\log q^{k+k_0}}|z|^{q^{k+k_0}}-(K+\sum_{j=1}^{k-1}\frac{q^{(j+k_0)p+\frac{p}{2}}}{\log q^{j+k_0}}|z|^{q^{j+k_0}}) \\
&\quad -\sum_{j=k+1}^{\infty}\frac{q^{(j+k_0)p+\frac{p}{2}}}{\log q^{j+k_0}}|z|^{q^{j+k_0}} \\
&= I_1-I_2-I_3.
\end{aligned}$$

Since

$$1-q^{-(k+k_0)}\leq |z|<1-q^{-(k+k_0+\frac{1}{2})}.$$

Thus

$$(1-q^{-(k+k_0)})^{q^{k+k_0}}\leq |z|^{q^{k+k_0}}<(1-q^{-(k+k_0+\frac{1}{2})})^{q^{k+k_0}}.$$

Then

$$\frac{1}{3}\leq |z|^{q^{k+k_0}}<(\frac{1}{2})^{q^{-\frac{1}{2}}}.$$

$$\begin{aligned}
I_1 &= \frac{q^{(k+k_0)p+\frac{p}{2}}}{\log q^{k+k_0}}|z|^{q^{(k+k_0)}} \\
&\geq \frac{1}{3}\frac{q^{(k+k_0)p+\frac{p}{2}}}{\log q^{k+k_0}}.
\end{aligned}$$

$$\begin{aligned}
I_2 &= K+\sum_{j=1}^{k-1}\frac{q^{(j+k_0)p+\frac{p}{2}}}{\log q^{j+k_0}}|z|^{q^{(j+k_0)}} \\
&\leq K\cdot\frac{q^{(k+k_0)p+\frac{p}{2}}}{\log q^{k+k_0}}(1-\frac{1}{q^{k+k_0+\frac{1}{2}}})+\frac{q^{(k+k_0)p+\frac{p}{2}}}{\log q^{k+k_0}}\cdot\sum_{j=1}^{k-1}\frac{1}{((1-\varepsilon)q^p)^{k-j}} \\
&\leq \frac{q^{(k+k_0)p+\frac{p}{2}}}{\log q^{k+k_0}}\cdot\frac{1}{(1-\varepsilon)q^p-1}+K\cdot\frac{q^{(k+k_0)p+\frac{p}{2}}}{\log q^{k+k_0}}.
\end{aligned}$$

$$\begin{aligned}
I_3 &= \sum_{j=k+1}^{\infty} \frac{q^{(j+k_0)p+\frac{p}{2}}}{\log q^{j+k_0}} |z|^{q^{j+k_0}} \\
&= \sum_{j=0}^{\infty} \frac{q^{(j+k+1+k_0)p+\frac{p}{2}}}{\log q^{j+k+1+k_0}} |z|^{q^{j+k+1+k_0}} \\
&= q^{(k+1+k_0)p+\frac{p}{2}} |z|^{q^{k+1+k_0}} \sum_{j=0}^{\infty} \frac{q^{jp}}{\log q^{j+k+1+k_0}} |z|^{q^j} \\
&\le \frac{q^{(k+1+k_0)p+\frac{p}{2}}}{\log q^{k+k_0}} |z|^{q^{k+1+k_0}} \sum_{j=0}^{\infty} q^{jp} |z|^{q^j} \\
&\le \frac{q^{(k+1+k_0)p+\frac{p}{2}}}{\log q^{k+k_0}} |z|^{q^{k+1+k_0}} \sum_{j=0}^{\infty} (q^p |z|^{q^{(k+2)}-q^{(k+1)}})^j \\
&= \frac{q^{(k+1+k_0)p+\frac{p}{2}}}{\log q^{k+k_0}} \frac{|z|^{q^{k+1+k_0}}}{1-q^p|z|^{q^{(k+2)}-q^{(k+1)}}} \\
&= \frac{q^{(k+k_0)p+\frac{p}{2}}}{\log q^{k+k_0}} \frac{q^p(|z|^{q^{k+k_0}})^q}{1-q^p(|z|^{q^k})^{(q^2-q)}} \\
&\le \frac{q^{(k+k_0)p+\frac{p}{2}}}{\log q^{k+k_0}} \frac{q^p(\frac{1}{2})^{q^{\frac{1}{2}}}}{1-q^p(\frac{1}{2})^{(q^{\frac{3}{2}}-q^{\frac{1}{2}})}}.
\end{aligned}$$

Thus

$$|f'(z)| \ge \frac{q^{(k+k_0)p+\frac{p}{2}}}{\log q^{k+k_0}} \left(\frac{1}{3} - \frac{1}{(1-\varepsilon)q^p-1} - K - \frac{q^p(\frac{1}{2})^{q^{\frac{1}{2}}}}{1-q^p(\frac{1}{2})^{(q^{\frac{3}{2}}-q^{\frac{1}{2}})}}\right).$$

If K is so small that

$$\frac{1}{3} - \frac{1}{(1-\varepsilon)q^p-1} - K - \frac{q^p(\frac{1}{2})^{q^{\frac{1}{2}}}}{1-q^p(\frac{1}{2})^{(q^{\frac{3}{2}}-q^{\frac{1}{2}})}} > 0,$$

then we have

$$|f'(z)| \ge C\frac{q^{(k+k_0)p+\frac{p}{2}}}{\log q^{k+k_0}} \ge \frac{C}{(1-|z|)^p \log\frac{2}{1-|z|}}.$$

Now with a similar argument for the function

$$g(z) = \sum_{j\ge 1} \frac{q^{(j+k_0)(p-1)+\frac{p}{2}}}{\log q^{j+k_0+\frac{1}{2}}} |z|^{q^{j+k_0+\frac{1}{2}}},$$

where $n_j = q^{j+k_0+\frac{1}{2}}$, for q a large positive integer, $k = 1, 2, \dots,$

$$1 - q^{-(k+k_0+\frac{1}{2})} \le |z| < 1 - q^{-(k+k_0+1)},$$

we get

$$|g'(z)| \ge \frac{C}{(1-|z|)^p \log\frac{2}{1-|z|}}.$$

In the case where f', g' have common zeros ($\neq 0$) in $\{|z| < 1 - q^{-(k+k_0+1)}\}$, we consider instead of $g(z)$ the function $g(e^{i\theta}z)$ for suitable θ. $\square$

In order to understand better the $Q^q_{\log}$, we need the following definition introduced in [14].

Definition 2.3. A positive Borel measure on D is called an s-logarithmic q-Carleson measure $(q, s > 0)$ if

$$\sup_{I\subseteq\partial D} \frac{\mu(S(I))(\log\frac{2}{|I|})^s}{|I|^q} < \infty.$$

In [14], the sufficient and necessary condition of the measure is given as follows.

Lemma 2.4. *μ is an s-logarithmic q-Carleson measure on D if and only if*

$$\sup_{\alpha\in D}(\log\frac{2}{1-|\alpha|^2})^s \int_D |\phi'_\alpha(z)|^q d\mu(z) < \infty.$$

Using techniques well known to mathematics and by Lemma 2.4 we can prove the following proposition.

Proposition 2.5. *Let $q > 0$. Then the following are equivalent:*

(i) $\quad f \in Q^q_{\log}$;

(ii) $\quad \sup_{\alpha\in D}(\log\frac{2}{1-|\alpha|^2})^2 \int_D |f'(z)|^2(1-|\phi_\alpha(z)|^2)^q dm(z) < \infty$;

(iii) $\quad \sup_{\alpha\in D}(\log\frac{2}{1-|\alpha|^2})^2 \int_D |f'(z)|^2 g^q(z,\alpha) dm(z) < \infty$.

Theorem 2.6. *Let $p,\ q > 0$. If φ is an analytic self-map of the unit disc, then the induced composition operator $C_\varphi : B^p_{\log} \to Q^q_{\log}$ is bounded if and only if*

$$\sup_{\alpha\in D}(\log\frac{2}{1-|\alpha|^2})^2 \int_D |\varphi'(z)|^2 \frac{(1-|\phi_\alpha(z)|^2)^q}{(1-|\varphi(z)|^2)^{2p}(\log\frac{2}{1-|\varphi(z)|^2})^2} dm(z) < \infty. \tag{2.2}$$

Proof. Firstly we assume that (2.2) holds, by Proposition 2.5, then for $f \in B^p_{\log}$,

$$\begin{aligned} & \sup_{\alpha\in D}(\log\frac{2}{1-|\alpha|^2})^2 \int_D |(f\circ\varphi)'(z)|^2(1-|\phi_\alpha(z)|^2)^q dm(z) \\ = & \sup_{\alpha\in D}(\log\frac{2}{1-|\alpha|^2})^2 \int_D |f'(\varphi(z))|^2|\varphi'(z)|^2(1-|\phi_\alpha(z)|^2)^q dm(z) \\ \leq & \sup_{\alpha\in D}(\log\frac{2}{1-|\alpha|^2})^2 \int_D |\varphi'(z)|^2 \frac{(1-|\phi_\alpha(z)|^2)^q}{(1-|\varphi(z)|^2)^{2p}(\log\frac{2}{1-|\varphi(z)|^2})^2} dm(z)\cdot \|f\|^2_{B^p_{\log}}. \end{aligned}$$

By (2.2), then $C_\varphi f \in Q^q_{\log}$, thus $C_\varphi : B^p_{\log} \to Q^q_{\log}$ is bounded.

Conversely, we assume that $C_\varphi : B^p_{\log} \to Q^q_{\log}$ is bounded, for $f \in B^p_{\log}$, $C_\varphi f \in Q^q_{\log}$, by Lemma 2.2, there exist $f, g \in B^p_{\log}$ such that

$$|f'(z)| + |g'(z)| \geq \frac{C}{(1-|z|)^p \log\frac{2}{1-|z|}}.$$

Then

$$\begin{aligned}\infty &> \sup_{\alpha\in D}(\log\frac{2}{1-|\alpha|^2})^2\int_D 2[|(f\circ\varphi)'(z)|^2+(g\circ\varphi)'(z)|^2](1-|\phi_\alpha(z)|^2)^q dm(z)\\ &\geq \sup_{\alpha\in D}(\log\frac{2}{1-|\alpha|^2})^2\int_D [|(f\circ\varphi)'(z)|+|(g\circ\varphi)'(z)|]^2(1-|\phi_\alpha(z)|^2)^q dm(z)\\ &= \sup_{\alpha\in D}(\log\frac{2}{1-|\alpha|^2})^2\int_D [|f'(\varphi(z))|+|g'(\varphi(z))|]^2|\varphi'(z)|^2(1-|\phi_\alpha(z)|^2)^q dm(z)\\ &\geq C\sup_{\alpha\in D}(\log\frac{2}{1-|\alpha|^2})^2\int_D |\varphi'(z)|^2\frac{(1-|\phi_\alpha(z)|^2)^q}{(1-|\varphi(z)|^2)^{2p}(\log\frac{2}{1-|\varphi(z)|^2})^2}dm(z).\end{aligned}$$

□

Remark 2.7. Since every element of $Q^q_{\log}$ satisfies the following radial growth condition:

$$|f(z)-f(0)|\leq C\log(\log\frac{1}{1-|z|})\|f\|_{Q^q_{\log}},\quad C>0,$$

then $C_\varphi: B^p_{\log}\to Q^q_{\log}$ is compact if and only if for every sequence $\{f_n\}_{n\in N}\subseteq Q^q_{\log}$, bounded in norm and $f_n\to 0$ as $n\to\infty$, uniformly on compact subsets of the unit disk, then $\|C_\varphi(f_n)\|_{Q^q_{\log}}\to 0$ as $n\to\infty$.

This is similar to [4].

We give the characterization of compactness.

Theorem 2.8. *Let $p,\ q>0$. If φ is an analytic self-map of the unit disc, then the induced composition operator $C_\varphi: B^p_{\log}\to Q^q_{\log}$ is compact if and only if $\varphi\in Q^q_{\log}$ and*

$$\lim_{r\to 1}\sup_{\alpha\in D}(\log\frac{2}{1-|\alpha|^2})^2\int_{\{|\varphi(z)|>r\}}|\varphi'(z)|^2\frac{(1-|\phi_\alpha(z)|^2)^q}{(1-|\varphi(z)|^2)^{2p}(\log\frac{2}{1-|\varphi(z)|^2})^2}dm(z)$$
$$=0. \quad (2.3)$$

Proof. Firstly we assume that $C_\varphi: B^p_{\log}\to Q^q_{\log}$ is compact, let $f(z)=z$, then $C_\varphi(f(z))=\varphi(z)\in Q^q_{\log}$. Since $\|\frac{z^n}{n}\|_{B^p_{\log}}\leq C$(in fact $C=\frac{2^p}{pe}$) and $\frac{z^n}{n}\to 0$ as $n\to\infty$, locally uniformly on the unit disc, then by the compactness of C_φ, $\|C_\varphi(z^n)\|_{Q^q_{\log}}\to 0$ as $n\to\infty$. This means that for each $r\in(0,1)$ and each $\varepsilon>0$, there exists $n_0\in N$ such that

$$r^{2(n_0-1)}\sup_{\alpha\in D}(\log\frac{2}{1-|\alpha|^2})^2\int_{\{|\varphi(z)|>r\}}|\varphi'(z)|^2(1-|\phi_\alpha(z)|^2)^q dm(z)<\varepsilon.$$

If we choose $r\geq 2^{-\frac{1}{2(n_0-1)}}$, then

$$\sup_{\alpha\in D}(\log\frac{2}{1-|\alpha|^2})^2\int_{\{|\varphi(z)|>r\}}|\varphi'(z)|^2(1-|\phi_\alpha(z)|^2)^q dm(z)<2\varepsilon. \quad (2.4)$$

Let now f with $\|f\|_{B^p_{\log}}<1$. We consider the functions $f_t(z)=f(tz),\ t\in(0,1)$. By the compactness of C_φ we get that for each $\varepsilon>0$, there exists $t_0\in(0,1)$ such that for all $t > t_0$,

$$\sup_{\alpha\in D}(\log\frac{2}{1-|\alpha|^2})^2\int_D |(f\circ\varphi)'(z)-(f_t\circ\varphi)'(z)|^2(1-|\phi_\alpha(z)|^2)^q dm(z) < \varepsilon.$$

Then we fix t, by (2.4)

$$\begin{aligned}
&\sup_{\alpha\in D}(\log\frac{2}{1-|\alpha|^2})^2\int_{\{|\varphi(z)|>r\}} |(f\circ\varphi)'(z)|^2(1-|\phi_\alpha(z)|^2)^q dm(z)\\
\le\ & 2\sup_{\alpha\in D}(\log\frac{2}{1-|\alpha|^2})^2\int_{\{|\varphi(z)|>r\}} |(f\circ\varphi)'(z)-(f_t\circ\varphi)'(z)|^2(1-|\phi_\alpha(z)|^2)^q dm(z)\\
&+2\sup_{\alpha\in D}(\log\frac{2}{1-|\alpha|^2})^2\int_{\{|\varphi(z)|>r\}} |(f_t\circ\varphi)'(z)|^2(1-|\phi_\alpha(z)|^2)^q dm(z)\\
\le\ & 2\varepsilon+2\|f'\|^2_{H^\infty}\sup_{\alpha\in D}(\log\frac{2}{1-|\alpha|^2})^2\int_{\{|\varphi(z)|>r\}} |\varphi'(z)|^2(1-|\phi_\alpha(z)|^2)^q dm(z)\\
\le\ & 4\varepsilon(1+\|f'\|^2_{H^\infty}). \qquad (2.5)
\end{aligned}$$

Having in mind (2.4) and (2.5) we conclude that for each $\|f\|_{B^p_{\log}}\le 1$ and $\varepsilon>0$, there is δ depending on f,ε, such that for $r\in[\delta,1)$,

$$\sup_{\alpha\in D}(\log\frac{2}{1-|\alpha|^2})^2\int_{\{|\varphi(z)|>r\}} |(f\circ\varphi)'(z)|^2(1-|\phi_\alpha(z)|^2)^q dm(z) < \varepsilon. \qquad (2.6)$$

Since C_φ is compact, it maps the unit ball of $B^p_{\log}$ to a relative compact subset of $Q^q_{\log}$. Thus for each $\varepsilon>0$, there exists a finite collection of functions $f_1, f_2, ..., f_N$ in the unit ball of $B^p_{\log}$, such that for each $\|f\|_{B^p_{\log}}\le 1$ there is a $k\in\{1,2,...,N\}$ with

$$\sup_{\alpha\in D}(\log\frac{2}{1-|\alpha|^2})^2\int_D |(f\circ\varphi)'(z)-(f_k\circ\varphi)'(z)|^2(1-|\phi_\alpha(z)|^2)^q dm(z) < \varepsilon.$$

By (2.6), we get that for $\delta=\max_{1\le k\le N}\delta(f_k,\varepsilon)$ and $r\in[\delta,1)$,

$$\sup_{\alpha\in D}(\log\frac{2}{1-|\alpha|^2})^2\int_{\{|\varphi(z)|>r\}} |(f_k\circ\varphi)'(z)|^2(1-|\phi_\alpha(z)|^2)^q dm(z) < \varepsilon.$$

Thus we get that

$$\sup_{\|f\|_{B^p_{\log}}\le 1}\sup_{\alpha\in D}(\log\frac{2}{1-|\alpha|^2})^2\int_{\{|\varphi(z)|>r\}} |(f_k\circ\varphi)'(z)|^2(1-|\phi_\alpha(z)|^2)^q dm(z) < 2\varepsilon.$$

By Lemma 2.2, (2.3) holds.

Conversely, we assume that $\varphi\in Q^q_{\log}$ and (2.3) holds. Let $\{f_n\}_{n\in N}$ be a sequence of functions in the unit ball of $B^p_{\log}$, such that $f_n\to 0$ as $n\to\infty$, uniformly on the compact subsets of the unit disc.

Let $r \in (0,1)$, then

$$\begin{aligned}
&\|f_n \circ \varphi\|^2_{Q^q_{\log}} \\
\leq\ & 2|f_n(\varphi(0))|^2 \\
&+2\sup_{\alpha \in D}(\log \frac{2}{1-|\alpha|^2})^2 \int_{\{|\varphi(z)|\leq r\}} |(f_n \circ \varphi)'(z)|^2 (1-|\phi_\alpha(z)|^2)^q dm(z) \\
&+2\sup_{\alpha \in D}(\log \frac{2}{1-|\alpha|^2})^2 \int_{\{|\varphi(z)|> r\}} |(f_n \circ \varphi)'(z)|^2 (1-|\phi_\alpha(z)|^2)^q dm(z) \\
=\ & 2I_1 + 2I_2 + 2I_3.
\end{aligned}$$

Since $f_n \to 0$ as $n \to \infty$, uniformly on D, then $I_1 \to 0$ as $n \to \infty$ and for each $\varepsilon > 0$ there is $n_0 \in N$ such that for each $n > n_0$, $I_2 \leq \varepsilon \|\varphi\|^2_{Q^q_{\log}}$,

$$I_3 \leq \sup_{\alpha \in D}(\log \frac{2}{1-|\alpha|^2})^2 \int_{\{|\varphi(z)|> r\}} |\varphi'(z)|^2 \frac{(1-|\phi_\alpha(z)|^2)^q}{(1-|\varphi(z)|^2)^{2p}(\log \frac{2}{1-|\varphi(z)|^2})^2} dm(z).$$

By (2.3), then for every n, that means for every $n > n_0$ and for every $\varepsilon > 0$, there exists r_0 such that for every $r > r_0$, $I_3 < \varepsilon$. Thus $\|C_\varphi(f_n)\|_{Q^q_{\log}} \to 0$ as $n \to \infty$. □

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