An Averaging Theorem for Ordinary Differential Inclusions

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Abstract

We consider the problem of application of the method of averaging to the asymptotic approximation of solutions of ordinary differential inclusions of the form $\dot{x}(t) \in f(t/\varepsilon, x(t))$. Our result is formulated in both classical mathematics and nonstandard analysis; its proof uses some methods of nonstandard analysis.

1 Introduction and notations

There has recently been a great deal of interest in the field of Bogolubov's type theorems. Many authors [7, 8, 9, 10, 20, 21] have discussed the averaging theorem for differential inclusions.

In the present paper, we prove a theorem on averaging on finite intervals of time for ordinary differential inclusions of the form

$$\dot{x}(t) \in f\left(\frac{t}{\varepsilon}, x(t)\right) \tag{1.1}$$

where f is a multifunction with values that are nonempty compact convex subsets of $\mathbb{R}^d$.

By use of the axiomatic form IST (for *Internal Set Theory*) of Robinson's *nonstandard analysis* (in short NSA) [22], given by Nelson [19], different averaging

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results under weak conditions are proved by the author in references [11] to [16]. This work continues this line of research.

Here, our averaging result for the ordinary differential inclusion (1.1) is formulated in both classical mathematics and nonstandard analysis. Its proof use NSA.

The paper is organized as follows. In Section 2 we state as Theorem 2.1 the main result of this work. A quite efficient tool for the proof of Theorem 2.1, namely, the so-called Stroboscopic method (Theorem 3.8) is presented in Section 3.2 after a short tutorial to NSA in Section 3.1. The nonstandard translate in the language of IST of Theorem 2.1 (Theorem 4.1) is presented in Section 4. Finally, in Section 5 we use Theorem 3.8 to prove Theorem 4.1.

We recall that the stroboscopic method was proposed for the first time in the study of some ordinary differential equations with a small parameter which occur in the theory of nonlinear oscillations [2, 17, 23]. Here, we extend it in the context of ordinary differential inclusions.

We emphasize that none of our proofs needs to be translate into classical form because IST is a conservative extension of ordinary mathematics. This last means that any classical statement which is a theorem of IST was already a theorem of ordinary mathematics.

We finish this section with some definitions and notations.

Let $\mathbb{R}^d$ denotes the d-dimensional space with the euclidean norm $|\cdot|$. $\mathrm{Comp}(\mathbb{R}^d)$ ($\mathrm{Conv}(\mathbb{R}^d)$, respectively) stands for the class of all nonempty compact (nonempty compact and convex, respectively) subsets of $\mathbb{R}^d$. In $\mathrm{Comp}(\mathbb{R}^d)$ the so-called Hausdorff metric is defined by

$$\rho(A,B) := \max\left(\sup_{a\in A}\delta(a,B), \sup_{b\in B}\delta(b,A)\right), \quad \forall A,B\in \mathrm{Comp}(\mathbb{R}^d)$$

where $\delta(x,A) = \inf\{|x-a| : a\in A\}$, for any $x\in\mathbb{R}^d$ and any $A\in\mathrm{Comp}(\mathbb{R}^d)$. The metric space $(\mathrm{Comp}(\mathbb{R}^d),\rho)$ is complete and $\mathrm{Conv}(\mathbb{R}^d)$ is a closed subset of this space.

The norm of any $A\in\mathrm{Comp}(\mathbb{R}^d)$ is given by: $\|A\| = \rho(A,\{0\}) = \sup\{|a| : a\in A\}$.

Let $f:\mathbb{R}_+\times\mathbb{R}^d\to\mathrm{Comp}(\mathbb{R}^d)$ be a multifunction. By a solution of the ordinary differential inclusion $\dot{x}\in f(t,x)$ we understand an absolutely continuous function x defined on some interval and satisfying $\dot{x}(t)\in f(t,x(t))$ for almost every t.

For the usual theory of differential inclusions we refer to the books of Aubin and Cellina [1], Deimling [3], Hu and Papageorgiou [6] and Smirnov [24].

2 The Averaging Method

Let $x_0\in\mathbb{R}^d$ and let $f:\mathbb{R}_+\times\mathbb{R}^d\to\mathrm{Conv}(\mathbb{R}^d)$ be a continuous multifunction. Let $\varepsilon>0$ be a small parameter. We are interested in the initial value problem

$$\dot{x}(t)\in f\left(\frac{t}{\varepsilon}, x(t)\right), \qquad x(0)=x_0 \tag{2.1}$$

on bounded intervals of time $t \in [0, L]$ and in the limit behavior of the motion when the parameter ε tends to zero. Therefore, we apply the averaging method. For this purpose we assume the following:

Assumption 1 The multifunction f is continuous in the second variable uniformly with respect to the first one.

Assumption 2 For all $x \in \mathbb{R}^d$ there exists a limit

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(\tau, x) d\tau := F(x),$$

where the integral is meant in Aumann-Hukuhara's sense.

As a consequence of Assumptions 1 and 2, the average of the multifunction f, that is, the multifunction $F : \mathbb{R}^d \to \mathbb{R}^d$ in Assumption 2, is continuous (see Lemma 5.1).

Consider now problem (2.1) together with the initial value averaged problem

$$\dot{y}(t) \in F(y(t)), \qquad y(0) = x_0 \tag{2.2}$$

on which we make the following assumption:

Assumption 3 The initial value problem (2.2) has a unique solution.

The main theorem of this paper states that the problem (2.2) can be considered as an initial value problem determining the motion when ε goes to zero. It reads as follows.

Theorem 2.1. *Let $f : \mathbb{R}_+ \times \mathbb{R}^d \to Conv(\mathbb{R}^d)$ be a continuous multifunction and let $x_0 \in \mathbb{R}^d$. Suppose that Assumptions 1-3 are fulfilled. Let y be the solution of (2.2) and let $L \in J$, where J is the positive interval of definition of y. Then, for every $\delta > 0$, there exists $\varepsilon_0 = \varepsilon_0(L, \delta) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$, every solution x of (2.1) is defined at least on $[0, L]$ and satisfies $|x(t) - y(t)| < \delta$ for all $t \in [0, L]$.*

3 The Stroboscopic Method

Hereafter we give a short tutorial on Internal Set Theory necessary to the rest of the paper. All of the results mentioned in this section are proved in [19] (see, also, references [4, 5, 18]). Then, we present the fundamental tool on which the proof of Theorem 2.1 is based, namely, the Stroboscopic Method.

3.1 Internal Set Theory: A short tutorial

Internal Set Theory (IST) is a theory extending ordinary mathematics, say ZFC (Zermelo-Fraenkel set theory with the axiom of choice), that axiomatizes (Robinson's) *nonstandard analysis* (NSA). In the IST a new predicate *standard* is added to the system ZFC. The abbreviated form of this predicate is *st*. Formulas of IST without any occurrence of the predicate *st* are called *internal*; otherwise, they are called *external*. Thus internal formulas are the formulas of ZFC. In addition to the usual axioms of ZFC, three others axioms are introduced for handling the new predicate in a relatively *consistent* way. Hence *all theorems* of ZFC *remain valid* in IST. What is new in IST is an addition, not a change.

IST is a *conservative extension* of ZFC, that is, every internal theorem of IST is a theorem of ZFC. There is the so-called *reduction algorithm* to reduce any external formula $\mathcal{F}(x_1, \ldots, x_n)$ of IST, where $x_1, \ldots, x_n$, are all the free variables of $\mathcal{F}$, to an internal formula $\mathcal{F}'(x_1, \ldots, x_n)$ with the same free variables, such that $\mathcal{F} \equiv \mathcal{F}'$, that is, $\mathcal{F}$ is equivalent to $\mathcal{F}'$ for all standard values of $x_1, \ldots, x_n$. In other words, any result which may be formalized within IST by a formula $\mathcal{F}(x_1, \ldots, x_n)$ is equivalent to the classical property $\mathcal{F}'(x_1, \ldots, x_n)$, provided $x_1, \ldots, x_n$ are restricted to standard values. We give the reduction of the frequently occurring formula $\forall x\ (\forall^{st} y\ \mathcal{P}_1 \Longrightarrow \forall^{st} z\ \mathcal{P}_2)$ where $\mathcal{P}_1$ and $\mathcal{P}_2$ are internal formulas

$$\forall x\ (\forall^{st} y\ \mathcal{P}_1 \Longrightarrow \forall^{st} z\ \mathcal{P}_2)\ \equiv\ \forall z\ \exists^{fin} y'\ \forall x\ (\forall y \in y'\ \mathcal{P}_1 \Longrightarrow \mathcal{P}_2). \tag{3.1}$$

The notations $\forall^{st} X$ and $\exists^{fin} X$ stand for $[\forall X, st(X) \Longrightarrow \cdots]$ and $[\exists X, X\text{finite}\ \&\cdots]$, respectively.

Let d be a standard and positive integer and let $\mathbb{R}^d$ be the d-dimensional space with the euclidean norm $|\cdot|$. Let x be in $\mathbb{R}^d$. The element x is said to be *infinitesimal* if $|x| < a$ for all standard and positive $a \in \mathbb{R}^d$ and *limited* if $|x| \leq a$ for some standard and positive $a \in \mathbb{R}^d$. It is said to be *appreciable* if it is both limited and not infinitesimal. If x is not limited it is said to be *unlimited* and denoted by $x \simeq \infty$.

Two points x and y in $\mathbb{R}^d$ are called *infinitely close*, denoted by $x \simeq y$, when $|x - y|$ is infinitesimal. The element x is called *nearstandard in* $\mathbb{R}^d$ when $x \simeq x_0$ for some standard $x_0 \in \mathbb{R}^d$. The element x_0 is called the *standard part* or *shadow* of x. It is unique and is also denoted by ${}^o x$. The *halo* of x, denoted by $\text{hal}(x)$, is the external set of all $y \in \mathbb{R}^d$ such that $x \simeq y$.

Note that each limited element of $\mathbb{R}^d$ is nearstandard (in $\mathbb{R}^d$).

Let A be an internal subset of $\mathbb{R}^d$. The shadow of A, denoted by ${}^o A$, is the standard set whose standard elements are those whose halo intersects A. It is unique and closed. We note that when A is standard, the shadow ${}^o A$ of A coincides with its closure $\overline{A}$.

Theorem 3.1. *Let A be a standard subset of $\mathbb{R}^d$. A is compact if and only if any element of A is nearstandard in A, that is, if and only if any element of A has a shadow in A.*

For any $x \in \mathbb{R}^d$ and any subset A of $\mathbb{R}^d$, the notation $x \simeq A$ means that $\delta(x, A) \simeq 0$, where $\delta(x, A) = \inf\{|x - a| : a \in A\}$.

Two subsets A and A' of $\mathbb{R}^d$ are called *infinitely close*, denoted by $A \simeq A'$, when every element of A is infinitely close to some element of A' and conversely, every element of A' is infinitely close to some element of A.

Let $(\text{Comp}(\mathbb{R}^d), \rho)$ be the metric space of nonempty compact subsets of $\mathbb{R}^d$ with the Hausdorff metric ρ.

Theorem 3.2. *Let A and A' be in Comp$(\mathbb{R}^d)$. $\rho(A, A') \simeq 0$ if and only if A and A' are infinitely close.*

Let $x : I \to \mathbb{R}^d$ be a function, where I is some interval of $\mathbb{R}$. The function x is called *S-continuous at* $t \in I$ provided for all $t' \in I$, $t \simeq t'$ implies $x(t) \simeq x(t')$, and *S-continuous on* I if, for all t and t' nearstandard in I, $t \simeq t'$ implies $x(t) \simeq x(t')$. When x (and then I) and t are standard, the first definition is the same as saying that x is continuous at t, and the uniform continuity of x on I is equivalent to, for all t and t' in I, $t \simeq t'$ implies $x(t) \simeq x(t')$.

We need the following basic result on S-continuous functions.

Theorem 3.3 (Continuous shadow)**.** *Let I be some interval of $\mathbb{R}$ and $x : I \to \mathbb{R}^d$ be a function. Let ${}^sI$ be the standard subset of $\mathbb{R}$ whose standard elements are those of I. There exists a standard and continuous function $x_0 : {}^sI \to \mathbb{R}^d$ such that, for all t nearstandard in I, $x(t) \simeq x_0(t)$ holds, if and only if, x is S-continuous on I and $x(t)$ is nearstandard for all t nearstandard in I.*

The function x_0 in Theorem 3.3 is unique. It is called the *standard part* or *shadow* of the function x and denoted by ${}^o x$.

In ZFC in principle all sets are defined using the only non logical symbol $\in$. In IST there is also the possibility to define collections with the non logical symbol *st*. Those collections which fall outside the range of ZFC are called *external sets*. External sets are often easily recognized: mostly some elementary classical property fails to hold. For instance, the set of infinitesimal real numbers hal(0) must be external, for it constitutes a bounded subset of $\mathbb{R}$ without lower and upper bounds. It happens sometimes in classical mathematics that a property is assumed, or proved, on a certain domain, and that afterwards it is noticed that the character of the property and the nature of the domain are incompatible. So actually the property must be valid on a large domain. In Nonstandard Analysis, statements which affirm that the validity of a property exceeds the domain where it was established in direct way are called *permanence principles*. Many permanence results used in nonstandard analysis are based upon the self-evident statement *"no external set is internal"*. This statement is called the *Cauchy-principle*. It has the following frequently used application.

Lemma 3.4 (Robinson's Lemma)**.** *Let r be an internal real function such that $r(t) \simeq 0$ for all limited $t \geq 0$. There exists $\omega \simeq +\infty$ such that $r(t) \simeq 0$ for all $t \in [0, \omega]$.*

We conclude this section with two others applications of Cauchy's principle which will be used in this work.

Lemma 3.5. *Let a be an appreciable and positive real number and r an internal real function such that $r(t) \simeq 0$ for all standard $t \in [0, a]$. There exists $a_0 \in [0, a]$, $a_0 \simeq a$ such that $r(t) \simeq 0$ for all $t \in [0, a_0]$.*

Lemma 3.6. *Let $\mathcal{P}(.)$ be an internal property such that $\mathcal{P}(a)$ holds for all appreciable and positive real numbers a. There exists $a_0 \simeq 0$ and positive such that $\mathcal{P}(a_0)$ holds.*

3.2 The Stroboscopic Method

Let $x_0 \in \mathbb{R}^d$ be standard and let $F : \mathbb{R}_+ \times \mathbb{R}^d \to \mathrm{Conv}(\mathbb{R}^d)$ be a standard and continuous multifunction. Let I be some subset of $\mathbb{R}$ and let $x : I \to \mathbb{R}^d$ be a function such that $0 \in I$ and $x(0) \simeq x_0$.

Definition 3.7 (Stroboscopic property). *The function x is said to satisfy the stroboscopic property if there exists $\mu > 0$ such that, for all limited $t \geq 0$ in I satisfying $[0, t] \subset I$ and $x(t)$ is limited, there exists $t' \in I$ such that $\mu < t' - t \simeq 0$, $[t, t'] \subset I$, $x(s) \simeq x(t)$ for all $s \in [t, t']$ and $\dfrac{x(t') - x(t)}{t' - t} \simeq F(t, x(t))$.*

Now, we state the main result of this section which asserts that any function satisfying the stroboscopic property can be approximated by a solution of the initial value problem

$$\dot{y}(t) \in F(t, y(t)), \qquad y(0) = x_0. \tag{3.2}$$

Theorem 3.8 (Stroboscopic Lemma). *Suppose that*

(S1) The function x satisfies the stroboscopic property.

(S2) The initial value problem (3.2) has a unique solution y. Let $J = [0, \omega)$, $0 < \omega \leq \infty$, be its maximal positive interval of definition.

Then, for every standard $L \in J$, the interval $[0, L]$ is contained in I and the function x satisfies the approximation $x(t) \simeq y(t)$ for all $t \in [0, L]$.

For the proof of Stroboscopic Lemma we need some preliminaries given in the section below.

3.2.1 Preliminaries

Lemma 3.9. *Let $L > 0$ be limited such that $[0, L] \subset I$. Suppose that*

(i) The function x is limited on $[0, L]$.

(ii) There exist some positive integer N and some infinitesimal partition $\{t_n : n = 0, \ldots, N+1\}$ of the interval $[0, L]$ such that $t_0 = 0$, $t_N \leq L < t_{N+1}$ and, for $n = 0, \ldots, N$, $t_{n+1} \simeq t_n$, $x(t) \simeq x(t_n)$ for all $t \in [t_n, t_{n+1}]$, and $\dfrac{x(t_{n+1}) - x(t_n)}{t_{n+1} - t_n} \simeq F(t_n, x(t_n))$.

Then the function x is S-continuous at each point in $[0, L]$.

Proof. Let $t \in [0, L]$. We will show that x is S-continuous at t.

Let $t' \in [0, L]$ and $p, q \in \{0, \ldots, N\}$ be such that $t \leq t'$, $t \simeq t'$, $t \in [t_p, t_{p+1}]$ and $t' \in [t_q, t_{q+1}]$ with $p < q$.

For each $n \in \{p, \ldots, q-1\}$ let $u_n \in F(t_n, x(t_n))$ such that

$$\left| \frac{x(t_{n+1}) - x(t_n)}{t_{n+1} - t_n} - u_n \right| = \delta \left(\frac{x(t_{n+1}) - x(t_n)}{t_{n+1} - t_n}, F(t_n, x(t_n)) \right).$$

Such an u_n exists (and is unique) since $F(t_n, x(t_n))$ is compact and convex.

From condition (ii) it follows that

$$\frac{x(t_{n+1}) - x(t_n)}{t_{n+1} - t_n} \simeq u_n,$$

that is, for some $\eta_n \simeq 0$,

$$x(t_{n+1}) - x(t_n) = (t_{n+1} - t_n)[u_n + \eta_n].$$

Now, using the fact that the multifunction F is standard and continuous, and that $(t_n, x(t_n))$ is nearstandard since it is limited, we have $F(t_n, x(t_n)) \simeq F({}^o t_n, {}^o(x(t_n)))$. Thus the standard and compact set $F({}^o t_n, {}^o(x(t_n)))$ is the shadow of $F(t_n, x(t_n))$. Hence, $u_n \in F(t_n, x(t_n))$ imply that u_n is nearstandard and then it is limited.

We write

$$x(t_q) - x(t_p) = \sum_{n=p}^{q-1}(x(t_{n+1}) - x(t_n)) = \sum_{n=p}^{q-1}(t_{n+1} - t_n)[u_n + \eta_n]. \tag{3.3}$$

Denote $\eta = \max\{|\eta_n| : n = p, \ldots, q-1\}$ and $m = \max\{|u_n| : n = p, \ldots, q-1\}$. We have $\eta \simeq 0$ and $m = |u_s|$, for some $s \in \{p, \ldots, q-1\}$, is limited. Hence (3.3) leads to the approximation

$$|x(t') - x(t)| \simeq |x(t_q) - x(t_p)| \leq (m + \eta)(t_q - t_p) \simeq 0$$

which proves the S-continuity of x at t and completes the proof. ■

If the real number L in Lemma 3.9 is standard one obtains more precise information about the function x.

Lemma 3.10. *Let $L > 0$ be standard such that $[0, L] \subset I$. Suppose that conditions (i) and (ii) in Lemma 3.9 are satisfied. Then the standard function $y : [0, L] \to \mathbb{R}^d$ defined by $y(t) = {}^o\big(x(t)\big)$, for all standard $t \in [0, L]$, is a solution of (3.2) and satisfies $x(t) \simeq y(t)$, for all $t \in [0, L]$.*

Proof. The proof will be given in two steps.

Step 1. The function y is continuous on $[0, L]$.

Indeed, by Lemma 3.9 the function x is S-continuous on $[0, L]$. Taking hypothesis (i) into account, the claim follows from Theorem 3.3. We have moreover

$$y(t) \simeq y({}^o t) \simeq x({}^o t) \simeq x(t), \quad \forall t \in [0, L].$$

The first part of the proof is complete.

Step 2. There exists a standard and continuous function $\widetilde{y} : [0, L] \to \mathbb{R}^d$ satisfying $\widetilde{y}(t) \in F(t, y(t))$ for all $t \in [0, L]$ ($\widetilde{y}$ is a selection of the standard and continuous multifunction $F(\cdot, y(\cdot)) : [0, L] \to \mathrm{Conv}(\mathbb{R}^d)$) such that

$$y(t) = x_0 + \int_0^t \widetilde{y}(s)ds, \quad \forall t \in [0, L].$$

Let $n \in \{0, \ldots, N\}$. Note that for all $w \in F(t_n, x(t_n))$ we have

$$\begin{aligned} &\delta\left(\frac{x(t_{n+1}) - x(t_n)}{t_{n+1} - t_n}, F(t_n, y(t_n))\right) \\ &\leq \left|\frac{x(t_{n+1}) - x(t_n)}{t_{n+1} - t_n} - w\right| + \rho(F(t_n, x(t_n)), F(t_n, y(t_n))). \end{aligned} \tag{3.4}$$

Let $\overline{w} \in F(t_n, x(t_n))$ be such that

$$\left|\frac{x(t_{n+1}) - x(t_n)}{t_{n+1} - t_n} - \overline{w}\right| = \delta\left(\frac{x(t_{n+1}) - x(t_n)}{t_{n+1} - t_n}, F(t_n, x(t_n))\right).$$

For $w = \overline{w}$, from (3.4) we get

$$\delta\left(\frac{x(t_{n+1}) - x(t_n)}{t_{n+1} - t_n}, F(t_n, y(t_n))\right) \simeq 0;$$

that is

$$\frac{x(t_{n+1}) - x(t_n)}{t_{n+1} - t_n} \simeq F(t_n, y(t_n)),$$

since

$$\delta\left(\frac{x(t_{n+1}) - x(t_n)}{t_{n+1} - t_n}, F(t_n, x(t_n))\right) \simeq 0 \quad \text{and} \quad \rho(F(t_n, x(t_n)), F(t_n, y(t_n)) \simeq 0$$

taking into account condition (ii) and the result of step 1.

Now, if n and p are in $\{0, \ldots, N\}$ then

$$t_n \simeq t_p \Rightarrow \frac{x(t_{n+1}) - x(t_n)}{t_{n+1} - t_n} \simeq F(t_n, y(t_n)) \simeq F(t_p, y(t_p)) \simeq \frac{x(t_{p+1}) - x(t_p)}{t_{p+1} - t_p},$$

since y and F are standard and continuous.

Thus, the standard function z on $[0, L]$ which, for t standard, is given by

$$z(t) = {}^o\left(\frac{x(t_{n+1}) - x(t_n)}{t_{n+1} - t_n}\right),$$

where n is such that $t_n \leq t < t_{n+1}$, is continuous on $[0, L]$, and we have

$$z(t_n) \simeq \frac{x(t_{n+1}) - x(t_n)}{t_{n+1} - t_n}, \quad \forall n \in \{0, \cdots, N\}.$$

By the continuous selection theorem there exists a standard and continuous function $\widetilde{y} : [0, L] \to \mathbb{R}^d$ satisfying, for all $t \in [0, L]$:

1. $\widetilde{y}(t) \in F(t, y(t))$.
2. $|z(t) - \widetilde{y}(t)| = \delta(z(t), F(t, y(t)))$.

Property 2 implies that, for $t = t_n$, $n \in \{0, \ldots, N\}$,

$$\left|\frac{x(t_{n+1}) - x(t_n)}{t_{n+1} - t_n} - \widetilde{y}(t_n)\right| \simeq 0,$$

that is

$$x(t_{n+1}) - x(t_n) = (t_{n+1} - t_n)[\widetilde{y}(t_n) + \eta_n], \qquad \text{with} \quad \eta_n \simeq 0,$$

since, in one hand

$$|z(t_n) - \widetilde{y}(t_n)| \simeq \left|\frac{x(t_{n+1}) - x(t_n)}{t_{n+1} - t_n} - \widetilde{y}(t_n)\right|$$

and, in other hand

$$\delta(z(t_n), F(t_n, y(t_n))) \simeq \delta\left(\frac{x(t_{n+1}) - x(t_n)}{t_{n+1} - t_n}, F(t_n, y(t_n))\right) \simeq 0.$$

Let $t \in [0, L]$ and $n \in \{0, \ldots, N\}$ be such that $t \in [t_n, t_{n+1}]$ with t standard. We have

$$\begin{aligned} y(t) - x_0 &\simeq x(t_n) - x(0) = \sum_{k=0}^{n-1}(x(t_{k+1}) - x(t_k)) = \sum_{k=0}^{n-1}(t_{k+1} - t_k)[\widetilde{y}(t_k) + \eta_k] \\ &\simeq \sum_{k=0}^{n-1}(t_{k+1} - t_k)\widetilde{y}(t_k) \simeq \int_0^t \widetilde{y}(s)ds, \end{aligned}$$

where $\eta_k \simeq 0$ for all $k \in \{0, \ldots, n-1\}$. Thus the approximation

$$y(t) \simeq x_0 + \int_0^t \widetilde{y}(s)ds \tag{3.5}$$

holds for all standard $t \in [0, L]$. Actually (3.5) is an equality since both sides of which are standard. We have thus, for all standard $t \in [0, L]$,

$$y(t) = x_0 + \int_0^t \widetilde{y}(s)ds \tag{3.6}$$

and by transfer (3.6) holds for all $t \in [0, L]$. The proof is complete. ∎

The following statement is a consequence of Lemma 3.10.

Lemma 3.11. *Let $L > 0$ be standard such that $[0, L] \subset I$. Suppose that*

(i) The function x is limited on $[0, L]$.

(ii) There exists $\mu > 0$ such that, for all $t \in [0, L]$, there exists $t' \in I$ such that $\mu < t' - t \simeq 0$, $[t, t'] \subset I$, $x(s) \simeq x(t)$ for all $s \in [t, t']$, and

$$\frac{x(t') - x(t)}{t' - t} \simeq F(t, x(t)).$$

Then the function x is S-continuous on $[0, L]$ and its shadow is a solution y of (3.2). So, we have $x(t) \simeq y(t)$ for all $t \in [0, L]$

Proof. Let A_μ be the subset of $\mathbb{R}$ defined by

$$A_\mu = \{a \in \mathbb{R} \;/\; \forall t \in [0, L]\; \exists t' \in I : \mathcal{P}_\mu(t, t', a)\}$$

where $\mathcal{P}_\mu(t, t', a)$ is the property

$$\mu < t' - t < a,\ [t, t'] \subset I,\ |x(s) - x(t)| < a\ \forall s \in [t, t'], \text{ and}$$

$$\delta\left(\frac{x(t') - x(t)}{t' - t}, F(t, x(t))\right) < a.$$

In virtue of condition (ii) we have $a \in A_\mu$ for all appreciable and positive real numbers a. By Lemma 3.6 there exists also $a_0 \in A_\mu$ with $0 < a_0 \simeq 0$. We have thus for all $t \in [0, L]$ there is $t' \in I$ such that $\mathcal{P}_\mu(t, t', a_0)$ holds. Applying now the axiom of choice to obtain a function $c : [0, L] \to I$ such that $c(t) = t'$, that is, $\mathcal{P}_\mu(t, c(t), a_0)$ holds for all $t \in [0, L]$. Since $c(t) - t > \mu$ for all $t \in [0, L]$, there are a positive integer N and an infinitesimal partition $\{t_n : n = 0, \ldots, N+1\}$ of $[0, L]$ such that $t_0 = 0$, $t_N \leq L < t_{N+1}$ and $t_{n+1} = c(t_n)$. Finally, the conclusion of Lemma 3.11 follows from Lemma 3.10. ∎

Now we can give the proof of the Stroboscopic Lemma (Theorem 3.8).

3.2.2 Proof of Theorem 3.8

Let $L > 0$ be standard in J. Fix $\rho > 0$ to be standard and let W be the (standard) tubular neighborhood around $\Gamma = \{y(t) : t \in [0, L]\}$ given by

$$W = \{z \in \mathbb{R}^d \ / \ \exists t \in [0, L] : |z - y(t)| \leq \rho\}.$$

Let A be the nonempty ($0 \in A$) subset of $[0, L]$ defined by

$$A = \{L_1 \in [0, L] \ / \ [0, L_1] \subset I \text{ and } \{x(t) : t \in [0, L_1]\} \subset W\}.$$

The set A is bounded above by L. Let L_0 be the upper bound of A and let $L_1 \in A$ be such that $L_0 - \mu < L_1 \leq L_0$. Since $\{x(t) : t \in [0, L_1]\} \subset W$, the function x is limited on $[0, L_1]$.

Taking hypothesis (S2) into account, we apply Lemma 3.11 to obtain, for any standard real number T such that $0 < T \leq L_1$,

$$x(t) \simeq y(t), \quad \forall t \in [0, T]. \tag{3.7}$$

By Lemma 3.5 approximation (3.7) still holds for some $T \simeq L_1$. Next, by Lemma 3.9 and the continuity of y we have

$$x(t) \simeq x(T) \quad \text{and} \quad y(t) \simeq y(T), \quad \forall t \in [T, L_1].$$

Combining this with (3.7) yields

$$x(t) \simeq y(t), \quad \forall t \in [0, L_1]. \tag{3.8}$$

Moreover, by hypothesis (S1) there exists $L_1' > L_1 + \mu$ such that $[L_1, L_1'] \subset I$ and

$$x(t) \simeq y(t), \quad \forall t \in [L_1, L_1']. \tag{3.9}$$

Together (3.8) and (3.9) show that $x(t) \simeq y(t)$ for all $t \in [0, L_1']$.

It remains to verify that $L \leq L_1'$. If this is not true, then $[0, L_1'] \subset I$ and $\{x(t) : t \in [0, L_1']\} \subset W$ imply $L_1' \in A$. This contradicts the fact that $L_1' > L_0$. So the proof is complete. ∎

4 Nonstandard Averaging Result

Hereafter we give the nonstandard formulation of Theorem 2.1. Then, by use of the reduction algorithm, we show that the reduction of Theorem 4.1 bellow is Theorem 2.1.

Theorem 4.1. *Let $f : \mathbb{R}_+ \times \mathbb{R}^d \to Conv(\mathbb{R}^d)$ be standard and continuous. Assume that Assumptions 1-3 are fulfilled. Let $x_0 \in \mathbb{R}^d$ be standard. Let y be the solution of (2.2) with $y(0) = x_0$, and let J be its positive interval of definition. Let $\varepsilon > 0$ be infinitesimal. Then for any standard $L > 0$, $L \in J$, any solution x of (2.1) with $x(0) = x_0$ is defined on $[0, L]$ and satisfies $x(t) \simeq y(t)$ for all $t \in [0, L]$.*

The proof of Theorem 4.1 is postponed to Section 5. Theorem 4.1 is an external statement. We show that the reduction of Theorem 4.1 is Theorem 2.1.

Reduction of Theorem 4.1. Let $L \in J$. We suppose that L is standard and positive. Let A be the formula "any solution x of (2.1) with $x(0) = x_0$ is defined on $[0, L]$ and satisfies $x(t) \simeq y(t)$ for all $t \in [0, L]$". The characterization of the conclusion of Theorem 4.1 is

$$\forall \varepsilon \ (\varepsilon \simeq 0 \implies A) \tag{4.1}$$

Let B be the formula "If $\delta > 0$ then any solution x of (2.1) with $x(0) = x_0$ is defined on $[0, L]$ and satisfies the inequality $|x(t) - y(t)| < \delta$ on $t \in [0, L]$". Formula (4.1) is then equivalent to

$$\forall \varepsilon \ (\forall^{st} \eta \ \varepsilon < \eta \implies \forall^{st} \delta \ B). \tag{4.2}$$

In this formula L is standard and ε, η and δ range over the strictly positive real numbers. By (3.1) formula (4.2) is equivalent to

$$\forall \delta \ \exists^{fin} \eta' \ \forall \varepsilon \ (\forall \eta \in \eta' \ \varepsilon < \eta \implies B). \tag{4.3}$$

For finite sets η', property $\forall \eta \in \eta' \ \varepsilon < \eta$ is the same as $\varepsilon < \varepsilon_0$ for $\varepsilon_0 = \min \eta'$, and so formula (4.3) is equivalent to

$$\forall \delta \ \exists \varepsilon_0 \ \forall \varepsilon \ (\varepsilon < \varepsilon_0 \implies B).$$

That is the statement of Theorem 2.1 holds for $L > 0$, L standard. By transfer, it holds for any $L > 0$. ∎

5 Proof of Theorem 4.1

We will prove Theorem 4.1 by applying Stroboscopic Lemma (Theorem 3.8). For this purpose we need first to translate Assumptions 1 and 2 in Section 2 into their external forms and then prove some technical lemmas.

Let $\varepsilon > 0$ be infinitesimal. Let $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathrm{Conv}(\mathbb{R}^d)$ be standard and continuous. The external formulations of Assumptions 1 and 2 are, respectively:

Assumption 1′ $\forall t \in \mathbb{R}_+ \ \forall^{st} x \in \mathbb{R}^d \ \forall x' \in \mathbb{R}^d \ \ (x' \simeq x \Longrightarrow f(t, x') \simeq f(t, x))$.

Assumption 2′ There is a standard multifunction $F : \mathbb{R}^d \to \mathrm{Conv}(\mathbb{R}^d)$ such that

$$F(x) \simeq \frac{1}{R} \int_0^R f(\tau, x) d\tau, \quad \forall^{st} x \in \mathbb{R}^d, \ \forall R \simeq +\infty.$$

5.1 Technical Lemmas

In Lemmas 5.1 and 5.2 below we formulate some properties of the multifunction F defined in Assumption 2′.

Lemma 5.1. *Suppose that the multifunction f satisfies Assumptions 1′ and 2′. Then its average F is continuous and satisfies, for any limited $x \in \mathbb{R}^d$ and any $R \simeq +\infty$,*

$$F(x) \simeq \frac{1}{R}\int_0^R f(t,x)dt.$$

Proof. Let $x, {}^o x \in \mathbb{R}^d$ such that ${}^o x$ is the shadow of x. Fix $\nu > 0$ to be infinitesimal. In virtue of Assumption 2 there exists $R_0 > 0$ such that, for all $R > R_0$

$$\rho\left(F(x), \frac{1}{R}\int_0^R f(t,x)dt\right) < \nu.$$

Hence for some $R \simeq +\infty$ we have

$$F(x) \simeq \frac{1}{R}\int_0^R f(t,x)dt. \tag{5.1}$$

As $f(t,x) \simeq f(t, {}^o x)$ for all $t \in \mathbb{R}_+$, (5.1) implies that

$$F(x) \simeq \frac{1}{R}\int_0^R f(t, {}^o x)dt.$$

Now by Assumption 2′ we deduce that $F(x) \simeq F({}^o x)$. Thus F is continuous. Moreover for $R \simeq +\infty$ we have

$$F(x) \simeq F({}^o x) \simeq \frac{1}{R}\int_0^R f(t, {}^o x)d\tau \simeq \frac{1}{R}\int_0^R f(t,x)d\tau.$$

So, the proof is complete. ∎

Lemma 5.2. *Suppose that the multifunction f satisfies Assumptions 1′ and 2′. Let $\varepsilon > 0$ be infinitesimal. Then, for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^d$, both limited, there exists $\alpha = \alpha(\varepsilon, t, x)$ such that $0 < \alpha \simeq 0$, $\varepsilon/\alpha \simeq 0$ and for all $T \in [0,1]$*

$$\frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon + T\alpha/\varepsilon} f(\tau, x)d\tau \simeq TF(x).$$

Proof. Let $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^d$, both limited. Let $T \in [0,1]$. We consider the following two cases.

Case 1: t/ε is limited. – Let $\alpha > 0$ be such that $\varepsilon/\alpha \simeq 0$. If $T\alpha/\varepsilon$ is limited then we have $T \simeq 0$ and

$$\frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon + T\varepsilon/\alpha} f(\tau, x)d\tau \simeq \{0\} \simeq TF(x)$$

taking into account that $f(\cdot, x)$ is locally integrable and that the shadow of $F(x)$, i.e. $F({}^o x)$, is a standard compact subset of $\mathbb{R}^d$.

If $T\alpha/\varepsilon \simeq +\infty$ we write

$$\frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon+T\alpha/\varepsilon} f(\tau,x)d\tau =$$
$$\left(T+\frac{t}{\alpha}\right)\frac{1}{t/\varepsilon+T\alpha/\varepsilon}\int_0^{t/\varepsilon+T\alpha/\varepsilon} f(\tau,x)d\tau - \frac{\varepsilon}{\alpha}\int_0^{t/\varepsilon} f(\tau,x)d\tau.$$

By Lemma 5.1 we have

$$\frac{1}{t/\varepsilon+T\alpha/\varepsilon}\int_0^{t/\varepsilon+T\alpha/\varepsilon} f(\tau,x)d\tau \simeq F(x)$$

and then

$$\frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon+T\alpha/\varepsilon} f(\tau,x)d\tau \simeq TF(x) \tag{5.2}$$

since

$$\frac{\varepsilon}{\alpha}\int_0^{t/\varepsilon} f(\tau,x)d\tau \simeq \{0\} \quad \text{and} \quad \frac{t}{\alpha} \simeq 0.$$

The approximation (5.2) holds for all $\alpha > 0$ such that $\varepsilon/\alpha \simeq 0$. Choosing α such that $0 < \alpha \simeq 0$ and $\varepsilon/\alpha \simeq 0$ gives the desired result.

Case 2: t/ε is unlimited. – Let $\alpha > 0$. We write

$$\begin{aligned}\frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon+T\alpha/\varepsilon} f(\tau,x)d\tau &= T\frac{1}{t/\varepsilon+T\alpha/\varepsilon}\int_0^{t/\varepsilon+T\alpha/\varepsilon} f(\tau,x)d\tau \\ &+\frac{t}{\varepsilon}\left(\frac{1}{t/\varepsilon+T\alpha/\varepsilon}\int_0^{t/\varepsilon+T\alpha/\varepsilon} f(\tau,x)d\tau - \frac{1}{t/\varepsilon}\int_0^{t/\varepsilon} f(\tau,x)d\tau\right).\end{aligned} \tag{5.3}$$

Define now η by setting, for any $\alpha \geq 0$

$$\eta(\alpha) = \frac{1}{t/\varepsilon+T\alpha/\varepsilon}\int_0^{t/\varepsilon+T\alpha/\varepsilon} f(\tau,x)d\tau - F(x).$$

By Lemma 5.1 we have $\eta(\alpha) \simeq \{0\}$, for all $\alpha \geq 0$.

Return to (5.3) and assume that α is appreciable. Then

$$\begin{aligned}\frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon+T\alpha/\varepsilon} f(\tau,x)d\tau &= TF(x) + T\eta(\alpha) + \frac{t}{\alpha}\left[\eta(\alpha)-\eta(0)\right] \\ &\simeq TF(x),\end{aligned} \tag{5.4}$$

since $T\eta(\alpha) + \frac{t}{\alpha}\left[\eta(\alpha)-\eta(0)\right] \simeq \{0\}$. By Lemma 3.6 approximation (5.4) holds for some $\alpha \simeq 0$ which can be chosen such that $\varepsilon/\alpha \simeq 0$.

This completes the proof of the lemma. ■

Lemma 5.3. *Let $g : \mathbb{R}_+ \times \mathbb{R}^d \to Conv(\mathbb{R}^d)$ and $h : \mathbb{R}_+ \to Conv(\mathbb{R}^d)$ be continuous multifunctions. Let x_0 be limited in $\mathbb{R}^d$. Suppose that*

(i) $g(t,x) \simeq h(t)$ *holds for all* $t \in [0,1]$ *and all limited* $x \in \mathbb{R}^d$,

(ii) $\int_0^t h(s)ds$ *is limited for all* $t \in [0,1]$.

Then any solution x of the initial value problem

$$\dot{x} \in g(t,x),\ t \in [0,1]; \qquad x(0) = x_0 \tag{5.5}$$

is defined and limited on $[0,1]$ *and satisfies*

$$x(t) \simeq y(t), \quad \forall t \in [0,1]$$

where y is a solution of the initial value problem

$$\dot{y} \in h(t),\ t \in [0,1]; \qquad y(0) = x_0$$

Proof. Let x be a maximal solution of (5.5). By Lemma 3.4 there exists $\omega \simeq +\infty$ such that the approximation in hypothesis (i) holds for all $t \in [0,1]$ and all $x \in B(0,\omega)$, where $B(0,\omega) \subset \mathbb{R}^d$ is the ball of center 0 and radius ω. Assume that $x(t') \simeq \infty$ for some $t' \in [0,1]$. Let $t \in [0,1]$ be such that $t \leq t'$, $x(t) \simeq \infty$ and $x(s) \in B(0,\omega)$ for all $s \in [0,t]$. Recall that

$$x(s) = x(0) + \int_0^s \dot{x}(\tau)d\tau, \quad \forall s \in [0,t], \tag{5.6}$$

with

$$\dot{x}(s) \simeq h(s), \quad \text{for a.e. } s \in [0,t], \tag{5.7}$$

since

$$\delta(\dot{x}(s),h(s)) \leq \rho(g(s,x(s)),h(s)) \simeq 0, \quad \text{for a.e. } s \in [0,t].$$

Let now $\widetilde{x} : [0,t] \to \mathbb{R}^d$ be a Lebesgue measurable function (such a function exists by the Lebesgue measurable selection theorem) such that

1. $\widetilde{x}(s) \in h(s)$, for all $s \in [0,t]$,
2. $|\dot{x}(s) - \widetilde{x}(s)| = \delta(\dot{x}(s),h(s))$, for a.e. $s \in [0,t]$.

From (5.7) and property 2, we have $\dot{x}(s) \simeq \widetilde{x}(s)$ for almost all $s \in [0,t]$. Then, it follows that $\int_0^s \dot{x}(\tau)d\tau \simeq \int_0^s \dot{\widetilde{x}}(\tau)d\tau$ for all $s \in [0,t]$. Finally, by (5.6) we obtain the approximation

$$x(s) \simeq x(0) + \int_0^s \widetilde{x}(\tau)d\tau, \quad \forall s \in [0,t]. \tag{5.8}$$

Taking into account hypothesis (ii), the second member of (5.8) is limited since $\int_0^s \widetilde{x}(\tau)d\tau \in \int_0^s h(\tau)d\tau$, and then $x(t)$ is too; this is a contradiction. Therefore $x(t)$ is defined and limited for all $t \in [0,1]$. ∎

5.2 Proof of Theorem 4.1

We are now able to prove our main result (there is not much work left). We assume that the assumptions in Theorem 4.1 are fulfilled. Fix $\varepsilon > 0$ to be infinitesimal and let $x : I \to \mathbb{R}^d$ be a maximal solution of (2.1).

We claim that x satisfies the stroboscopic property.

Indeed, let $t_0 \in I$ such that $t_0 \geq 0$ and limited, and $x(t_0)$ is limited. By Lemma 5.2 there exists $\alpha = \alpha(\varepsilon, t_0, x(t_0))$ such that $0 < \alpha \simeq 0$, $\varepsilon/\alpha \simeq 0$ and

$$\frac{\varepsilon}{\alpha} \int_{t_0/\varepsilon}^{t_0/\varepsilon + T\alpha/\varepsilon} f(t, x(t_0))dt \simeq TF(x(t_0)), \quad \forall T \in [0,1]. \tag{5.9}$$

Consider the function $X : [0,1] \to \mathbb{R}^d$ defined, for any $T \in [0,1]$, by

$$X(T) = \frac{x(t_0 + \alpha T) - x(t_0)}{\alpha}.$$

Differentiating the function X and substituting into (2.1) gives, for any $T \in [0,1]$,

$$\frac{dX}{dT}(T) \in f\left(\frac{t_0}{\varepsilon} + \frac{\alpha}{\varepsilon} T, x(t_0) + \alpha X(T)\right). \tag{5.10}$$

By Assumption 1′ and Lemma 5.3 the function X, as a solution of (5.10), is defined and limited on $[0,1]$ and, for any $T \in [0,1]$,

$$X(T) \simeq \int_0^T f\left(\frac{t_0}{\varepsilon} + \frac{\alpha}{\varepsilon} t, x(t_0)\right) dt = \frac{\varepsilon}{\alpha} \int_{t_0/\varepsilon}^{t_0/\varepsilon + T\alpha/\varepsilon} f(t, x(t_0))dt.$$

Using now (5.9) this leads to the approximation: $X(T) \simeq TF(x(t_0))$ for all $T \in [0,1]$.

Define $t_1 = t_0 + \alpha$ and set $\mu = \varepsilon$. Then $\mu < \alpha = t_1 - t_0 \simeq 0$, $[t_0, t_1] \subset I$ and $x(t_0 + \alpha T) = x(t_0) + \alpha X(T) \simeq x(t_0)$ for all $T \in [0,1]$, that is, $x(t) \simeq x(t_0)$ for all $t \in [t_0, t_1]$, whereas

$$\frac{x(t_1) - x(t_0)}{t_1 - t_0} = X(1) \simeq F(x(t_0)),$$

which shows the claim.

Finally, by Assumption 3 and Theorem 3.8, the solution x is defined at least on $[0, L]$ and satisfies $x(t) \simeq y(t)$ for all $t \in [0, L]$.

The proof of Theorem 4.1 is complete. ∎

Remark 5.4. *It appears clearly throughout the proof of Theorem 4.1 that it is not necessary to consider the whole space $\mathbb{R}_+ \times \mathbb{R}^d$. One can restrict the domain of definition of f to $\mathbb{R}_+ \times D$ for any standard $D \subset \mathbb{R}^d$ with the assumption that y lies on $[0, L]$ in the interior of D.*

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References

[1] J. P. Aubin and A. Cellina, Differential Inclusions, *Springer Verlag, Berlin, Heidelberg, New york*, 1984.

[2] J. L. Callot and T. Sari, *Stroboscopie infinitésimale et moyennisation dans les systèmes d'équations différentielles à solutions rapidement oscillantes*, in Landau I. D., éditeur, Outils et modèles mathématiques pour l'automatique, l'analyse des systèmes et le traitement du signal, tome 3, Editions du CNRS (1983), 345-353.

[3] K. Deimling, Multivalued Differential Equations, *De Gruyter, Berlin*, 1992.

[4] F. Diener and M. Diener (Eds.), Nonstandard Analysis in Practice, *Universitext, Springer Verlag*, 1995.

[5] M. Diener and C. Lobry (Eds.), Analyse Non Standard et Représentation du Réel, *Office des Publications Universitaires, Alger, CNRS, Paris*, 1985.

[6] S. Hu and N. S. Papageorgiou, Handbook of Multivalued Analysis, Volume I: Theory, *Kluwer Academic Publishers, Dordrecht, Boston, London*, 1997.

[7] T. Janiak and E. Luczak-Kumorek, *Method of averaging for integral differential inclusions*, Studia Univ. Babes Bolyai, Math. **39** (2) (1994).

[8] T. Janiak and E. Luczak-Kumorek, *Method of averaging for the system of functional-differential inclusions*, Discus. Math., Diff. Inclu. **16** (1996), 137-151.

[9] T. Janiak and E. Luczak-Kumorek, *Averaging of neutral inclusions when the average value of the right-hand side does not exist*, Acta Math. Viet. **25** (1) (2000), 1-11.

[10] S. S. Klimczuk, *Averaging of differential inclusions with unbounded right-hand side*, Ukr. Math. Zh. **3** (1989), 389-391 (Russian).

[11] M. Lakrib, *On the validity of the averaging method for all time*, Maghreb Math. Rev. **8**/1&2 (1999), 119-128.

[12] M. Lakrib, *The method of averaging and functional differential equations with delay*, Int. J. Math. Math. Sci. **26** (2001), No. 8, 497-511.

[13] M. Lakrib, *On the averaging method for differential equations with delay*, Electron. J. Diff. Eqns. **2002** (2002), No. 65, 1-16.

[14] M. Lakrib, *Time averaging for functional differential equations*, J. Appl. Math. **2003** (2003), no. 1, 1-16.

[15] M. Lakrib and T. Sari, *Averaging results for functional differential equations*, Sibirsk. Mat. Zh. **45** (2004), No. 2, 375-386; *translation in* Siberian Math. J. **45** (2004), No. 2, 311-320.

[16] M. Lakrib, *Sur la moyennisation dans les systèmes à plusieurs fréquences*, Maghreb Math. Rev., in press.

[17] R. Lutz, *L'intrusion de l'Analyse Non Standard dans l'étude des perturbations singulières*, in IIIème Rencontre de Géométrie du Schnepfenried, Vol. 2, Astérisque **107-108** (1983), 101-140.

[18] R. Lutz and M. Goze, Nonstandard Analysis: A Practical Guide with Applications, *Lectures Notes in Math.* 881, *Springer Verlag*, 1982.

[19] E. Nelson, Internal Set Theory: a new approach to Nonstandard Analysis, *Bull. Amer. Math. Soc.*, **83** (1977), 1165-1198.

[20] W. A. Plotnikov, *The averaging method for differential inclusions and its application in control theory*, Diff. Uravn. **15** (8) (1979), 1427-1433 (Russian).

[21] W. A. Plotnikov and W. M. Sawczenko, *Averaging of differential inclusions with unaverage right-hand side*, Ukr. Math. Zh. **48** (1996), 1572-1575.

[22] A. Robinson, Nonstandard Analysis, *American Elsevier, New York*, 1974.

[23] T. Sari, *Nonstandard perturbation theory of differential equations*, presented as an invited talk at the International Research Symposium on Nonstandard Analysis and its Applications, ICMS, Edinburgh, 11-17 August 1996.

[24] G. V. Smirnov, Introduction to the Theory of Differential Inclusions, *Graduate Studies in Math.* 41, *American Math. Soc., Providence*, 2002.

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