Sliding Vector Fields via Slow–Fast Systems

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Dedicated to Freddy Dumortier for his 60–th birthday.

Abstract

This paper concerns differential equation systems on $\mathbb{R}^n$ with discontinuous right–hand sides. We deal with non-smooth vector fields in $\mathbb{R}^n$ having a codimension-one submanifold M as its discontinuity set. After a regularization of a such system and a global blow–up we are able to bring out some results that bridge the space between discontinuous systems and singularly perturbed smooth systems.

1 Introduction and statement of the main results

In this paper a discussion is focused to study the phase portraits of certain non–smooth vector fields defined in $\mathbb{R}^n$ having a codimension–one submanifold M as its discontinuity set. We present some results in the framework developed by Sotomayor and Teixeira in [9] and establish a bridge between those systems and the fundamental role played by the Geometric Singular Perturbation Theory (GSPT). This transition was introduced in [1] and [7], in dimensions 2 and 3, respectively. Needless to say that in this area very good surveys are available (see [2, 3, 5, 6] for instance).

Let $U \subseteq \mathbb{R}^n$ be an open set. We suppose that $M = F^{-1}(0)$ where $F : U \to \mathbb{R}$ is a smooth function and $0 \in \mathbb{R}$ is a regular value of F. Clearly M is the separating boundary of the regions $M_+ = \{q \in U | F(q) \geq 0\}$ and $M_- = \{q \in U | F(q) \leq 0\}$. We denote by $\mathcal{C}^r(U, \mathbb{R}^n)$ the set of all vector fields of class $\mathcal{C}^r$ defined on U, with $r \geqslant 1$, endowed with the C^r–topology.

Denote by $\Omega^r(U)$ the set of all vector fields X on U such that

$$X(q) = \begin{cases} X_1(q) & \text{if} \quad q \in M_+, \\ X_2(q) & \text{if} \quad q \in M_-, \end{cases} \tag{1}$$

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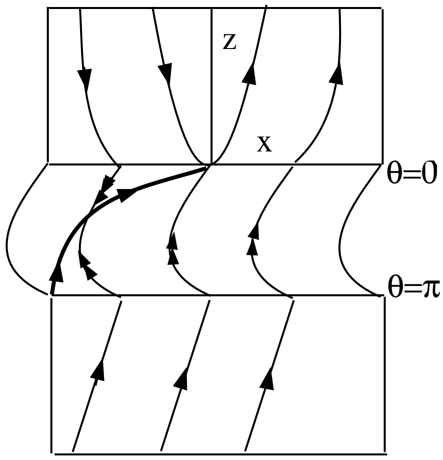

(A)

Figure 1: Fast and slow dynamics, case fold–regular.

where $X_i \in \mathcal{C}^r(U, \mathbb{R}^n)$, $i = 1, 2$. The vector field X can be multivalued at the points of M. We will denote $X = (X_1, X_2) \in \Omega^r(U)$.

The notions and definitions which appear in this introduction are given with all the details in Section 2.

According to Filippov's convention [4] there may exist generically a *Sliding Region* in M (*SR(X)*) associated to $X \in \Omega^r(U)$ such that any orbit which meets the *SR(X)* remains tangent to M for positive time. The vector field on *SR(X)* determined in this way is called the *Sliding Vector Field* and it will be denote X^M.

Sotomayor and Teixeira [9] and Llibre and Teixeira [8] in dimensions 2 and 3, respectively, introduced a regularization process to study discontinuous vector fields. Using this process we get a one–parameter family of vector fields $X_\varepsilon \in \mathcal{C}^r(U, \mathbb{R}^n)$ such that for each $\varepsilon_0 > 0$ fixed we have

(i) X_{ε_0} is equal to X_1 in all points of M_+ whose distance to M is bigger than ε_0;

(ii) X_{ε_0} is equal to X_2 in all points of M_- whose distance to M is bigger than ε_0.

We assume that M is represented, locally around a point p, by the function $F(x_1, ..., x_n) = x_1$. Moreover we denote the vector fields X_1 and X_2 by $X_1 = (f_1, ..., f_n)$ and $X_2 = (g_1, ..., g_n)$. Applying the regularization process (see Definition 2.2) we have that the trajectories of the regularized vector field X_ε are the solutions of the differential system

$$\dot{x}_i = \frac{f_i + g_i}{2} + \varphi\left(\frac{x_1}{\varepsilon}\right)\frac{f_i - g_i}{2}, i = 1, ..., n; \quad \dot{\varepsilon} = 0. \tag{2}$$

As in the papers [1, 7] we transform system (2) into a singular perturbation problem by considering $x_1 = r\cos\theta$ and $\varepsilon = r\sin\theta$, with $r \geq 0$ and $\theta \in [0, \pi]$. We remark that a blow up involving a parameter ε and a variable x_1 is usually called a *global blow up*.

Our main result is the following.

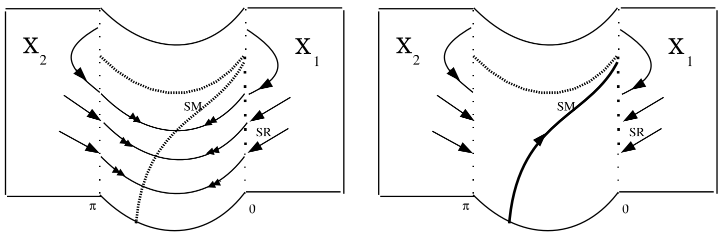

Figure 2: Fast and slow dynamics with slow manifold with horizontal part.

Theorem 1.1. *Let $X = (X_1, X_2) \in \Omega^r(U)$ with $0 \in U \subseteq \mathbb{R}^n$ and M represented by $F(x_1, ..., x_n) = x_1$. If for any $q \in M$ we have that $X_1F(q) \neq 0$ or $X_2F(q) \neq 0$ then there exists a singular perturbation problem*

$$\theta' = \alpha(r, \theta, \rho), \quad \rho' = r\beta(r, \theta, \rho), \tag{3}$$

with $r \geq 0$, $\theta \in (0, \pi)$, $\rho \in M$ and α and β of class C^r such that the following statements hold.

(a) The sliding region $SR(X)$ is homeomorphic to the slow manifold $\alpha(0, \theta, \rho) = 0$ of (3)*.*

(b) The sliding vector field X^M and the reduced problem of (3) *(see below) are topologically equivalent.*

For a singular perturbation problem the equation $\alpha(0, \theta, \rho) = 0$ defines a set composed by singular points when $r = 0$ and it is called *slow manifold.* After a convenient time–rescaling system (3) can be written as

$$r\dot{\theta} = \alpha(r, \theta, \rho), \quad \dot{\rho} = \beta(r, \theta, \rho),$$

and we have a differential system defined on the slow manifold when $r = 0$. This differential system is called the *reduced problem.*

We remark that the slow manifold of system (3) can have a piece defined by $\rho =$constant and $\theta \in (0, \pi)$, which is formed by singular points of the reduced flow if there are points $q \in M$ such that $X_1F(q) = X_2F(q) = 0$. See figure 1.

Let X be a smooth vector field on $\mathbb{R}^n$ and M be a codimension one submanifold with $0 \in M$. We say that 0 is an M–singularity of X if $X(0)$ belongs to the tangent space T_pM.

Consider $X \in \Omega^r(U)$ and suppose that $p \in M$ is not an M–singular point of X_1 and X_2. Note that Theorem 1.1 says that locally the slow manifold $SM(X)$ and the sliding region $SR(X)$ are homeomorphic.

We define a geometric object composed by the union of M_+, M_- and $M_0 = \{(\theta, \rho); \theta \in (0, \pi), \rho = (x_2, ..., x_n) \in M\}$. We denote $\mathcal{M} = M_+ \cup M_0 \cup M_-$ and remark that the set $\{(0, \rho); \rho \in M\}$ has two distinct copies: ∂M_+ and ∂M_-.

Let $X \in \Omega^r(U)$. First of all we suppose that $X_i \equiv 0, i = 1, 2$ on M. In fact if it does not happen, we multiply by the non negative function $|x_1|$ and the phase portrait of X remains unchanged outside $x_1 = 0$. The blow up process induces a smooth vector field $\widehat{X}$ on $\mathcal{M}$ whose trajectories coincide with those ones of X_1 on M_+, X_2 on M_- and of system (3) on M_0.

Next we introduce the rules for defining the singular orbits of $\widehat{X}$. If a point of $M_+ \cup M_-$ moving on an orbit of X_i, $i = 1$ or $i = 2$, falls onto the sewing region M_1 (see Section 2 for a precise definition) then it crosses $M_1 \subseteq M$ over to another part by a fast orbit of system (3). If it falls onto $M_2 \cup M_3$ then it follows firstly a fast orbit of system (3) over to the slow manifold and then it follows a slow orbit.

The major part of notions and terminology used in this paper can be found in [9, 10, 13]. We refer to such references for additional details and informations.

For technical reasons we define the notion of an S–singularity of $X = (X_1, X_2) \in \Omega^r(U)$ with $U \subseteq \mathbb{R}^3$ only in Section 2. In Section 6 we prove next result.

Theorem 1.2. *Let $X = (X_1, X_2) \in \Omega^r(U)$ with $0 \in U \subseteq \mathbb{R}^3$ and M represented by $F(x_1, x_2, x_3) = x_1$. If $0 \in U$ is an S–singularity of X and γ_q is a singular orbit of $\widehat{X}$ on $\mathcal{M}$ by $q \in \mathcal{M}$ then γ_q converges to a singular point of the reduced problem*

$$0 = \alpha(0, \theta, \rho), \quad \dot{\rho} = \beta(0, \theta, \rho).$$

The paper is organized as follows. In Section 2 we provide some basic results on discontinuous vector fields. Moreover we show how the analysis of the dynamics of X^M can be applied to study the asymptotic stability of a special singularity. In Section 3 we prove Theorem 1.1. In Section 4 we show the possible boundaries of the simply connected slow manifolds obtained from the regularization process of discontinuous vector fields for $n = 2$. In Section 5 we describe the slow manifolds derived from the codimensions 0 and 1 sliding vector fields. In Section 6 we prove Theorem 1.2.

2 Preliminaries

In this section we present some basic facts of discontinuous vector fields.

Let $U \subseteq \mathbb{R}^n$ be an open set and $X = (X_1, X_2) \in \Omega^r(U)$ with discontinuity set M.

We distinguish the following objects on M :

(i) The *sewing region* is $M_1 = \{p \in M : X_1F(p).X_2F(p) > 0\}$;

(ii) The *escaping region* is $M_2 = \{p \in M : X_1F(p) > 0, X_2F(p) < 0\}$;

(iii) The *sliding region* is $M_3 = \{p \in M : X_1F(p) < 0, X_2F(p) > 0\}$;

(iv) A *M–regular point* of X_i is a point $p \in M$ such that $X_iF(p) \neq 0$.

(v) A *$M-$singular point* of the vector field X_i for $i = 1$ or $i = 2$ is a point $p \in M$ such that $X_iF(p) = 0$.

The sliding vector field associated to X is the vector field X^s tangent to M and defined at $q \in M_3$ by $X^s(q) = m - q$ with m being the point where the segment joining $q + X_1(q)$ and $q + X_2(q)$ is tangent to M. It is clear that if $q \in M_3$ then

$q \in M_2$ for $-X$. Then we can define the *escaping vector field* on M associated to X by $X^e = -(-X)^s$. Here we use for both cases the notation X^M. Moreover, for a discontinuous vector field $X = (X_1, X_2) \in \Omega^r(U)$ on $U \subseteq \mathbb{R}^n$, with discontinuous set M given by F, we denote by *SR(X)* the union $M_2 \cup M_3$.

According to the rules due to Gantmaher and Filippov if a point of the phase space which is moving in an orbit of $X = (X_1, X_2)$ falls onto M_1 then it crosses M_1 over to another part of the space and the solutions of $X = (X_1, X_2)$ through points of $M_2 \cup M_3$ follow the orbit of X^M.

2.1 Structural stability in $\Omega^r(U), U \subset \mathbb{R}^3$

Let $U, V \subseteq \mathbb{R}^3$ be open sets. We say that $X = (X_1, X_2) \in \Omega^r(U)$ and $Y = (Y_1, Y_2) \in \Omega^r(V)$ are C^0-equivalent if there exists a M-invariant homeomorphism $h : U \to V$ which sends orbits of X in orbits of Y. From this definition the concept of structural stability in $\Omega^r(U)$ is naturally obtained. We denote by $\Sigma_0(p)$ the set of the structurally stable discontinuous vector fields at $p \in U$.

We say that a $M-$*singular point* $p \in M$ for the vector field X_i for $i = 1$ or $i = 2$ is a *fold point* if $X_i^2 F(p) \neq 0$. We say that a $M-$*singular point* $p \in M$ for the vector field X_i for $i = 1$ or $i = 2$ is a *cusp point* if $X_i^2 F(p) = 0$ and $\{\nabla F(p), \nabla(X_i F)(p), \nabla(X_i^2 F)(p)\}$ is a linearly independent set. We denote $S_{X_i} = \{q \in M : (X_i F)(q) = 0\}$.

We say that $p \in M$ is a *generic singularity* (or a *codimension zero singularity*) of $X = (X_1, X_2) \in \Omega^r(U)$ if one of the following conditions is satisfied:

(i) $p \in SR(X)$ and it is a hyperbolic singular point of X^M;

(ii) $p \in M$ is a cusp point of X_1 (resp.X_2) and a regular point of X_2 (resp.X_1);

(iii) $p \in M$ is a fold point of X_i for $i = 1$ and for $i = 2$. Moreover we assume that $S_{X_1} \pitchfork S_{X_2}$ and that the C^1–extension of X^M has a hyperbolic singular point at p with eigenvectors transversal to S_{X_1} and S_{X_2}. In this case the curves S_{X_1} and S_{X_2} determine four quadrants: $Q_1(M_3)$, $Q_2(M_1^+$ with the orbits of X pointing M_+),Q_3 (M_2), $Q_4(M_1^-$ with the orbits of X pointing M_-).

We denote by $\mathcal{A}(U) \subset \Omega^r(U)$ the set of discontinuous vector fields with $M_2 \cup M_3 \neq \emptyset$ and by $\mathcal{A}_0(U) \subset \mathcal{A}(U)$ the set of the discontinuous vector fields for which all singular points of X^M and all M–singular points are generic singularities.

We denote by $\mathcal{A}_1(U) \subset \mathcal{A}(U)$ the set of the discontinuous vector fields for which all singular points of X^M and all $M-$singular points are of codimension 1, according with the definition in [13], and for $p \in U$ we denote $\Sigma_1(p) = \mathcal{A}_1(U) \cap (\Sigma_0(p))^C$.

We say that $p \in M$ is a *U–singularity* of $X = (X_1, X_2)$ if

(a) p is a generic singularity of X;

(b) p is an M–singular point of both vector fields X_1 and X_2;

(c) $X_1^2 F(p) < 0, X_2^2 F(p) > 0$ and $S_{X_1} \pitchfork S_{X_2}$.

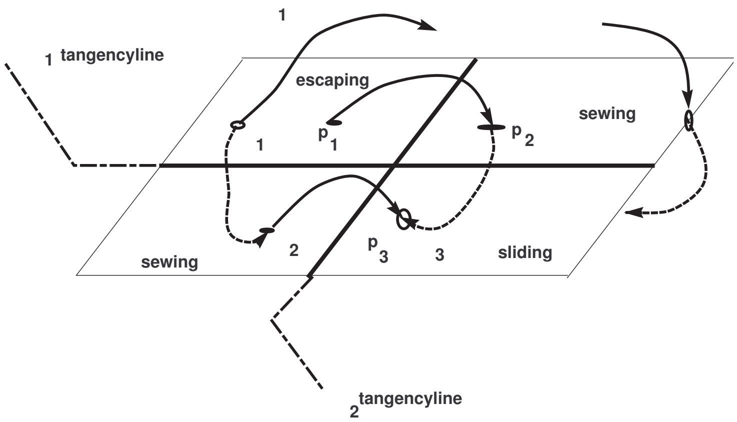

Figure 3: On the S–singularity.

The open set $U_p \subset \Omega^r(U)$ is the set of all vector fields $X \in \Omega^r(U)$ having p as a U–singularity. We observe that $\mathcal{A}_0(U) \bigcap U_p \neq \emptyset$.

We may define around a U–singularity $p \in M$ a first return mapping $\phi_X : M \to M$ associated with X which belongs to the same class of differentiability as X_1 and X_2. Moreover this mapping is an area preserving diffemorphism and so $\det J\phi_X(p) = 1$. Observe that the orbits the orbits of X are spirals around S_{X_1} and S_{X_2}.

We say that ϕ_X is hyperbolic at p if the eigenvalues of $J\phi_X(p)$ are real and different from ± 1. We may separate U_p into two distinct connected components, $U_p = H_p \cup L_p$ with $X \in H_p$ provided that p is a hyperbolic fixed point of ϕ_X and $L_p = U_p / H_p$.

We say that $p \in M$ is an S*–singularity* of X (see Figure 3) if

(a) $X \in L_p$;

(b) The rotation number associated to ϕ_X is irrational;

(c) The eigenvalues of the linear part of X^M at p are real, negative and distinct. Moreover the eigenvectors are transversal to S_{X_1} and S_{X_2}, the eigenspace associated with the eigenvalue of small absolute value does meet M_3 and the other eigenspace does not meet M_3.

The following results are proved in [12].

(A) If $X \in \Sigma_0(p)$ then p is either a M–regular point of X_i for $i = 1, 2$ or a generic singularity of X.

(B) $X \in \Sigma_0(p)$ provided that p is a M–regular point of X_i for $i = 1, 2$ or $X \in \mathcal{A}_0(U) - U_p$.

(C) If ϕ_X or X^M is not hyperbolic at p then X is not structurally stable at p.

(D) $X = (X_1, X_2) \in \Omega^r(U)$ is asymptotically stable at p provided that $p \in M_3$ and X^M is asymptotically stable at p or p is an S–singularity of X.

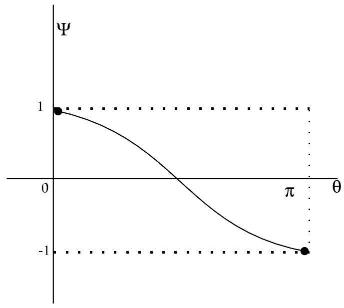

Figure 4: The function $\psi(\theta) = \varphi(\cot\theta)$.

It seems clear that a U–singularity deserves a special attention since on it we are pushed to analyze simultaneously the behavior of the first return mapping and the sliding vector field (as well as the boundary of the sliding region). In Section 6 we define the first return mapping and we analyze the asymptotic stability in M_3.

2.2 Regularization

(a) A C^∞ function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ is a *transition function* if $\varphi(s) = -1$ for $s \leqslant -1$, $\varphi(s) = 1$ if $s \geqslant 1$ and $\varphi'(s) > 0$ if $s \in (-1, 1)$.

(b) The *φ–regularization* of $X = (X_1, X_2) \in \Omega^r(U)$, $U \subset \mathbb{R}^n$, is the one–parameter family $X_\varepsilon \in \mathcal{C}^r(U, \mathbb{R}^n)$ given by

$$X_\varepsilon(q) = \left(\frac{1}{2} + \frac{\varphi_\varepsilon(F(q))}{2}\right) X_1(q) + \left(\frac{1}{2} - \frac{\varphi_\varepsilon(F(q))}{2}\right) X_2(q), \tag{4}$$

with $\varphi_\varepsilon(s) = \varphi(s/\varepsilon)$, for $\varepsilon > 0$.

3 Proof of Theorem 1.1.

In this section we prove Theorem 1.1.

Lemma 3.1. *Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ be a transition function. Then $\psi : [0, \pi] \longrightarrow \mathbb{R}$ given by $\psi(\theta) = \varphi(\cot\theta)$ for $0 < \theta < \pi$, $\psi(0) = 1$ and $\psi(\pi) = -1$ is a non increasing smooth function.*

Proof: We compute the derivative

$$\frac{d}{d\theta}\psi(\theta) = \varphi'(\cot\theta)\frac{d}{d\theta}(\cot\theta) = -\frac{\varphi'(\cot\theta)}{\sin^2\theta},$$

and we have that $\varphi'(s) \geq 0$ for any $s \in \mathbb{R}$. ■

Proof of Theorem 1.1: Let $X = (X_1, X_2) \in \Omega^r(U)$ be a discontinuous vector field on $U \subseteq \mathbb{R}^n$ with discontinuous set M given by $F(x_1, ..., x_n) = x_1$. First of all we apply the regularization process (see Definition 2.2) and then we have that the trajectories of the regularized vector field X_ε are the solutions of the differential system (2). Next we consider the polar blow up coordinates given by $x_1 = r\cos\theta$

and $\varepsilon = r\sin\theta$, with $r \geq 0$ and $\theta \in [0,\pi]$. Using these coordinates the parameter value $\varepsilon = 0$ is represented by $r = 0$ and the blow up induces the vector field on $[0,+\infty) \times [0,\pi] \times M$ given by

$$\begin{aligned} r' &= r\cos\theta\left(\frac{f_1+g_1}{2}+\varphi(\cot\theta)\frac{f_1-g_1}{2}\right), \\ \theta' &= -\sin\theta\left(\frac{f_1+g_1}{2}+\varphi(\cot\theta)\frac{f_1-g_1}{2}\right), \\ x_i' &= r\left(\frac{f_i+g_i}{2}+\varphi(\cot\theta)\frac{f_i-g_i}{2}\right), \quad i=2,\dots,n. \end{aligned} \tag{5}$$

Denote

$$\begin{aligned} a(r,\theta,\rho) &= \frac{f_1(r\cos\theta,\rho)+g_1(r\cos\theta,\rho)}{2}, \\ b(r,\theta,\rho) &= \frac{f_1(r\cos\theta,\rho)-g_1(r\cos\theta,\rho)}{2}, \end{aligned}$$

where $\rho = (x_2,\dots,x_n)$.

Take $\alpha(r,\theta,\rho) = -\sin\theta\,[a(r,\theta,\rho)+\varphi(\cot\theta)\,b(r,\theta,\rho)]$ and

$$\beta(r,\theta,\rho) = \left(\frac{f_2+g_2}{2}+\varphi(\cot\theta)\frac{f_2-g_2}{2},\dots,\frac{f_n+g_n}{2}+\varphi(\cot\theta)\frac{f_n-g_n}{2}\right).$$

The system is described bellow.

(i) On the region $(r,\theta,\rho) \in \{0\} \times (0,\pi) \times M$:

$$\theta' = -\sin\theta\,[a(r,\theta,\rho)+\varphi(\cot\theta)\,b(r,\theta,\rho)], \quad \rho' = 0. \tag{6}$$

(ii) On the region $(r,\theta,\rho) \in \{0\} \times \{0,\pi\} \times M$:

$$\theta' = 0, \quad \rho' = 0. \tag{7}$$

The discontinuity set M is now represented by $S^1_+ \times M = \{(\cos\theta,\sin\theta,\rho)|\theta \in [0,\pi], \rho \in M\}$. The manifolds $\theta = 0$ and $\theta = \pi$ are composed by singular points. The fast flow on $S^1_+ \times M$ is given by the solutions of system (6) and the slow flow is given by the solutions of the reduced problem represented by

$$\begin{aligned} 0 &= -\sin\theta\,[a(0,\theta,\rho)+\varphi(\cot\theta)\,b(0,\theta,\rho)], \\ \dot{x}_i &= \frac{f_i+g_i}{2}+\varphi(\cot\theta)\frac{f_i-g_i}{2}, \quad i=2,\dots,n. \end{aligned} \tag{8}$$

The slow manifold for $\theta \in (0,\pi)$ is implicitly determined by the equation

$$a(0,\theta,\rho)+\varphi(\cot\theta)\,b(0,\theta,\rho) = 0. \tag{9}$$

We have that

$$b(0,\theta,\rho) = 0 \Leftrightarrow f_1(0,\rho) = g_1(0,\rho).$$

Since $X_1F(0,\rho) = f_1(0,\rho), X_2F(0,\rho) = g_1(0,\rho)$, and $X_1F(0,\rho).X_2F(0,\rho) < 0$, for any $(0,\rho) \in SR(X)$ we have that $b(0,\theta,\rho) \neq 0$ for any $\theta \in (0,\pi), (0,\rho) \in SR(X)$.

Moreover

$$\frac{a(0,\theta,\rho)}{b(0,\theta,\rho)} = \frac{f_1(0,\rho)+g_1(0,\rho)}{f_1(0,\rho)-g_1(0,\rho)},$$

implies that

$$-1 \le -\frac{a(0,\theta,\rho)}{b(0,\theta,\rho)} \le 1,$$

for all $(0,\rho) \in SR(X)$. Since ψ^{-1} is increasing on $(-1,1)$, the equation $a(0,\theta,\rho) + \varphi(\cot\theta)b(0,\theta,\rho) = 0$ defines a continuous graphic. Then the statement (a) holds.

According with the definition of X^M we have that $X^M = X_1 + \lambda(X_2 - X_1)$ with $\lambda \in \mathbb{R}$ such that $X_1(x_1,...,x_n) + \lambda(X_2 - X_1)(x_1,...,x_n) = (0, y_2,...,y_n)$ for some $y_i, i = 2,...,n$. Thus it is easy to see that the X^M is given by

$$X^M = \left(0, \frac{f_1g_2 - f_2g_1}{f_1 - g_1}, ..., \frac{f_1g_n - f_ng_1}{f_1 - g_1}\right). \tag{10}$$

The reduced problem is represented by

$$\dot{x}_i = \frac{f_i + g_i}{2} + \varphi(\cot\theta)\frac{f_i - g_i}{2}, \quad i = 2,..n,$$

under the restriction given by $\varphi(\cot\theta) = -\dfrac{a(0,\theta,\rho)}{b(0,\theta,\rho)} = -\dfrac{f_1 + g_1}{f_1 - g_1}$. Then we must have

$$\dot{x}_i = \frac{f_1g_i - f_ig_1}{f_1 - g_1}, \quad i = 2,..,n. \tag{11}$$

From (10) and (11) it follows immediately that the flows of X^M and the reduced problem are equivalent. ■

4 Simply connected slow manifolds for $n = 2$

In this section we discuss the topological structure of the slow manifold obtained from the regularization process of systems defined in an open subset $U \subset \mathbb{R}^2$.

Proposition 4.1. *Let $X = (X_1, X_2) \in \Omega^r(U)$ with $0 \in U \subseteq \mathbb{R}^2$ and M represented by $F(x_1,x_2) = x_1$. If $SR(X) \subset M$ is a non–empty simply connected subset, then $SR(X)$ is homeomorphic to a non–degenerated real interval. Moreover if I is open in its left (resp. right) endpoint, then this endpoint is an M–singular point of X_1 or X_2.*

Proof: According to the topological classification of the simply connected open subsets of $\mathbb{R}$ we know that $SR(X)$ must be an interval. Moreover, the fact that $SR(X)$ is non empty implies that there exists $(0,q) \in SR(X)$ and thus there is a neighborhood of $(0,q)$ contained in $SR(X)$. Therefore $SR(X)$ is homeomorphic to an interval different from a point. The last statement follows of the fact that if $(0,q) \in SR(X)$ is a boundary point then $\nabla F(0,q) \cdot X_i(0,q) \neq 0$ for $i = 1$ or $i = 2$. ■

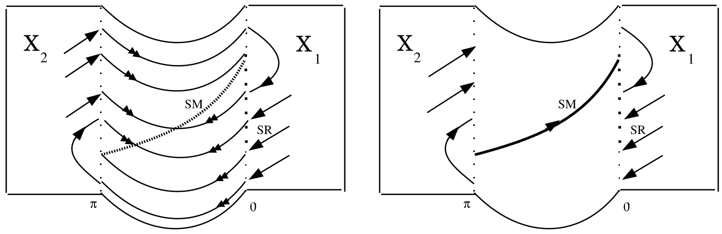

Figure 5: Fast and slow dynamics with slow manifold connecting two folds.

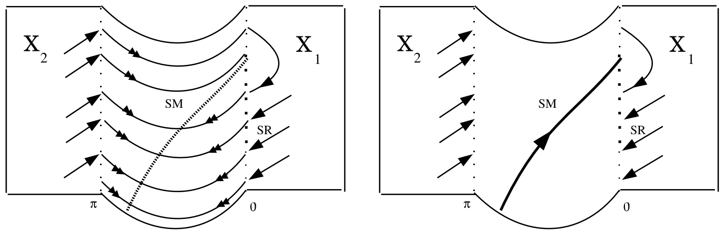

Figure 6: Fast and slow dynamics with slow manifold connecting a fold with a point $(0, \theta_0, x_2)$ with $0 < \theta_0 < \pi$.

According with Proposition 4.1 we now discuss the following situations.

Case 1: Sliding region with boundary formed by $M-$singular points. In this case the slow manifold connects two M-singular points. We have $M = \{(x_1, x_2) \in \mathbb{R}^2; x_1 = 0, x_2 \in [q_1, q_2]\}$, $f_1(0, q_2) = g_1(0, q_1) = 0, f_1(0, q_1) \neq 0, g_1(0, q_2) \neq 0$

and $f_1(0, q) < 0$ and $g_1(0, q) > 0$ for any $q \in (q_1, q_2)$. Observe that the hypothesis on f_1 and g_1 implies that the sliding region is $SR(X) = M \setminus \{(0, q_1), (0, q_2)\}$. Moreover the case $q_1 = q_2$ is not excluded. Applying Lemma 3.1 and taking the limit when θ tends to 0 in the equation (9) we have that $a(0, 0, x_2) + b(0, 0, x_2) = 0$. So we get $f_1(0, x_2) = 0$. It means that one of the endpoints of the slow manifold is $(0, 0, q_2)$. Next we apply the limit when θ tends to π and with a similar argument we conclude that the other endpoint of the slow manifold is $(0, \pi, q_1)$. A model of this case is illustrated in figure 4. We remark that the dynamics represented in figure 4 is not the only possible in this case. In fact, the X^M, X and consequently the reduced problem can have singular points.

Case 2: Bounded sliding region with boundary with only one $M-$singular point. We have $M = \{(x_1, x_2) \in \mathbb{R}^2; x_1 = 0, x_2 \in [q_1, q_2]\}$, $f_1(0, q_2) = 0, g_1(0, q_2) \neq 0$ and $f_1(0, q) < 0$ and $g_1(0, q) > 0$ for any $q \in [q_1, q_2)$. The slow manifold is connecting $(0, 0, q_2)$ and $(0, \theta_0, q_1)$ where θ_0 is such that $\varphi(\cot \theta_0) = -\dfrac{a(0, 0, q_1)}{b(0, 0, q_1)}$. See figure 4.

Case 3: Unbounded sliding region with boundary with only one $M-$singular point. In this case the slow manifold is connecting a $M-$ singular point with the infinity. We have $M = \{(x_1, x_2) \in \mathbb{R}^2; x_1 = 0, x_2 \in (-\infty, q_2]\}$,

$f_1(0, q_2) = 0, g_1(0, q_2) \neq 0$ and $f_1(0, q) < 0$ and $g_1(0, q) > 0$ for any $q \in (-\infty, q_2)$. See figure 4.

Case 4: Unbounded sliding region. In this case the slow manifold will connect the $-\infty$ with the ∞. We have $M = \{(x_1, x_2) \in \mathbb{R}^2; x_1 = 0, x_2 \in \mathbb{R}\}$, $f_1(0, q) < 0$ and $g_1(0, q) > 0$ for any $q \in \mathbb{R}$.

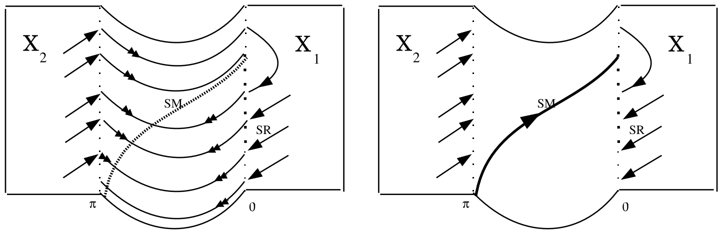

Figure 7: Fast and slow dynamics with slow manifold connecting a fold with the infinity.

Case 5: Examples. In what follows $X = (X_1, X_2) \in \Omega^\omega(\mathbb{R}^2)$.

1) Consider $X_1(x, y) = (1, 1)$ and $X_2(x, y) = (-1, 1)$. The slow manifold

$$\{(\theta, y) \in (0, \pi) \times \mathbb{R}; \theta = \theta_0, \varphi(\cot \theta_0) = 0\}$$

is of kind 4.4. The reduced flow goes in the positive direction of the y-axis.

2) Consider $X_1(x, y) = (x+1, -y)$ and $X_2(x, y) = (x-1, -y)$ The slow manifold $\{(\theta, y) \in (0, \pi) \times \mathbb{R}; \theta = \theta_0, \varphi(\cot \theta_0) = 0\}$ is of the kind 4.4. The reduced flow has an attracting singular point at $(0, 0)$.

3) Consider $X_1(x, y) = (y, 1)$ and $X_2(x, y) = (1, 1)$. The slow manifold

$$y\,(\varphi(\cot \theta) + 1) = (\varphi(\cot \theta) - 1)$$

is of kind 4.3. It is the graphic of a decreasing function which is 0 for $\theta = 0$ and tends to $-\infty$ when $\theta \to \pi$. The reduced flow goes in the positive direction of the y-axis.

4) Consider $X_1(x, y) = (-1, -y^2)$ and $X_2(x, y) = (1, 0)$. The slow manifold $\{(\theta, y) \in (0, \pi) \times \mathbb{R}; \theta = \theta_0, \varphi(\cot \theta_0) = 0\}$ is of kind 4.4. The reduced flow on the slow manifold goes in the negative direction and there exists a singular point at $(\theta, y) = (\theta_0, 0)$.

5) Consider $X_1(x, y) = (-y, 1)$ and $X_2(x, y) = (y, 1)$. The slow manifold has a horizontal part given by $(\theta, y) = (\theta, 0)$ and the part given by $(\theta, y) = (\frac{\pi}{2}, y)$ which is the union of two simply connected parts of kind 4.3. The reduced flow, on $\frac{\pi}{2}$, follows the positive direction of the y–axis because $\dot{y} = 1 > 0$ and on $y = 0$ it is composed by singular points.

6) Consider $X_1(x, y) = (y, 1)$ and $X_2(x, y) = (2y, -1)$. The slow manifold has only horizontal part given by $y = 0$. The reduced flow is composed by singular points.

7) Consider $X_1(x, y) = (-y, -1)$ and $X_2(x, y) = (2y, 1)$. The slow manifold has a horizontal part given by $y = 0$ and the other part given by $\theta = \theta_0, \varphi(\cot \theta_0) = 1/3$ is the union of two simply connected parts of kind 4.3. The reduced flow is composed by singular points if $y = 0$ and it goes in the negative direction of the y-axis if $\theta = \theta_0$.

8) Consider $X_1(x, y) = (3y^2 - y - 2, 1)$ and $X_2(x, y) = (-3y^2 - y + 2, -1)$. The slow manifold is given implicitly by $\varphi(\cot \theta) = \dfrac{y}{3y^2 - 2}$ which defines two functions

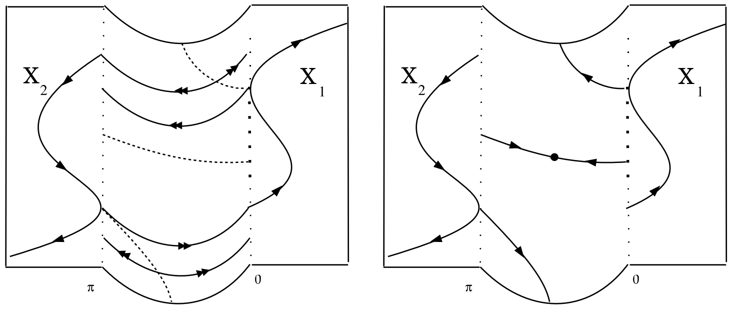

Figure 8: Fast and slow dynamics with slow manifold with three connected parts.

$y_1(\theta) = \dfrac{1+\sqrt{1+24\varphi^2(\cot\theta)}}{6\varphi(\cot\theta)}$ and $y_2(\theta) = \dfrac{1-\sqrt{1+24\varphi^2(\cot\theta)}}{6\varphi(\cot\theta)}$. The function $y_1(\theta)$ is increasing, $y_1(0) = 1$, $\lim\limits_{\theta\to\frac{\pi}{2}^-} y_1(\theta) = +\infty$, $\lim\limits_{\theta\to\frac{\pi}{2}^+} y_1(\theta) = -\infty$ and $y_1(\pi) = -1$. The function $y_2(\theta)$ is increasing, $y_2(0) = -\dfrac{2}{3}$, $\lim\limits_{\theta\to\frac{\pi}{2}} y_2(\theta) = 0$ and $y_2(\pi) = \dfrac{2}{3}$. The slow manifold is the union of three simply connected pieces, one of them in the case 1 and the others in the case 3. See figure 4.

5 Sliding vector fields and slow manifolds in dimensions 3

Let $X = (X_1, X_2) \in \Omega^r(U)$ with $0 \in U \subseteq \mathbb{R}^3$ and M represented by $F(x_1, x_2, x_3) = x_1$. We will describe the slow manifolds of $X \in \Sigma_0(0) \cup \Sigma_1(0)$. Assuming that $0 \in M$ we have to distinguish the following cases:

(i) 0 is a regular point;

(ii) 0 is an M–singular point of X_1 or of X_2.

If $0 \in U$ is a regular point then for $i = 1$ or 2 or 3 we have that $0 \in M_i$. Moreover there exists $0 \in V \subset U$ such that $(V \cap M) \subset M_i$.

If $V \cap M \subset M_1$ then $f_1(q).g_1(q) > 0$ for any $q \in V$. The equation $(f_1(q)+g_1(q)) + \varphi(\cot\theta)(f_1(q) - g_1(q)) = 0$ has no solution $(q, \theta) \in V \times [0, \pi]$. In fact it is easy to see that $|\dfrac{f_1+g_1}{f_1-g_1}| > 1$. If $V \cap M \subset M_3$ then $f_1(q) < 0$ and $g_1(q) > 0$ for any $q \in V$. In this case $|\dfrac{f_1+g_1}{f_1-g_1}| < 1$. It means that the slow manifold is a 2–dimensional surface. The same conclusion is achieved when $V \cap M \subset M_2$. Normal forms of these cases are listed in Table 1 and they can be found in [7]. In Table 2 the sliding regions and the equations of the corresponding slow manifolds are listed.

We observe that if $0 \in M_3$ is a hyperbolic critical point of X^M then the normal form is $X_1(x, y, z) = (0, 0, \delta)$, $X_2(x, y, z) = (ax+by+z\alpha_1, cx+dy+z\alpha_2, e+z\alpha_3)+h.o.t$ with $\delta = \pm1, \delta e < 0, \alpha_i(0, 0, 0) = 0$ for $i = 1, 2$ and the roots of $\lambda^2-(a+d)\lambda+(ad-bc)$ have nonzero real parts.

If $0 \in M$ is an M–singular point then generically we have that either 0 is a cusp–regular or a fold–regular or a fold–fold point, see [11] for precise definitions. The

Normal Forms	X_1	X_2	X^M
Sewing	$(0,0,1)$	$(0,0,1)$	$-$
Escaping	$(0,0,1)$	$(0,1,-1)$	$(0,1,0)$
Sliding	$(0,0,-1)$	$(0,1,1)$	$(0,-1,0)$

Table 1: $X = (X_1, X_2) \in \Sigma_0(0)$, and 0 is a M–regular point.

Normal Forms	SR	SM
Sewing	$\emptyset$	$\emptyset$
Escaping	$\{(0, x_2, x_3); x_2, x_3 \in \mathbb{R}\}$	$\theta = \frac{\pi}{2}$
Sliding	$\{(0, x_2, x_3); x_2, x_3 \in \mathbb{R}\}$	$\theta = \frac{\pi}{2}$

Table 2: Sliding regions and slow manifolds of $X = (X_1, X_2) \in \Sigma_0(0)$, and 0 is a M–regular point.

normal forms are given in Table 3 and in Table 4 are listed the sliding regions and the equations of the corresponding slow manifolds. In figure 5 one can see the slow manifold of the singular problems corresponding to the normal forms fold–regular and fold–fold. In figure 5 one can see the slow manifold of the singular problems corresponding to the normal form cusp–regular.

The normal forms for the unfoldings of codimension 1 singularities are given in Table 5 and in Table 6. In Table 7 are listed the sliding regions and the corresponding slow manifolds. They involve the descriptions of the generic one–parameter family $X_\lambda = (X_{1_\lambda}, X_{2_\lambda})$ around $p = (0,0,0)$, with local form of M given by $x_1 = 0$. We refer to [13] for more details.

6 The first return mapping

Let $X = (X_1, X_2) \in \Omega^\infty(U)$ be defined in an open set $U \subseteq \mathbb{R}^n$ with $0 \in U, X_1(0) \neq 0$ and $X_2(0) \neq 0$. As before we assume that the discontinuous set M is represented, locally around 0, by a function F. Assume that $X_1F(0) = 0$. Choose coordinates $y = (y_1, y_2, y_3)$ around $0 \in \mathbb{R}^3$ such that $X_1(y_1, y_2, y_3) = (0,0,1)$. Let $y_1 = g(y_2, y_3)$ be a C^∞ solution of $F(y_1, y_2, y_3) = 0$ with $g(0,0) = 0$. Fix $N = \{y_3 = 0\}$ as the section transverse to X at 0. One sees that the projection $G_{X_1} : M \to N$ of M, along the orbits of X_1, onto N is given by $G_{X_1}(g(y_2, y_3), y_2, y_3) = (g(y_2, y_3), y_2, 0)$.

Normal Forms	X_1	X_2	X^M
Fold–regular	$(x_3, 0, \delta), \delta = \pm 1$	$(1,0,0)$	$(0,0,-\delta)$
Cusp–regular	$(x_3^2 - x_2, 0, -1)$	$(1,0,0)$	$(0,0,1)$
Fold–Fold	$(x_3, 0, \delta), \delta = \pm 1$	$(x_2, -1, 0)$	$(0, -x_3, -x_2)$

Table 3: $X = (X_1, X_2) \in \Sigma_0(0)$, and 0 is an M–singular point.

Normal Forms	SR	SM
Fold–regular	$\{(0, x_2, x_3); x_3 < 0\}$	$\psi(\theta) = -\dfrac{x_3 + 1}{x_3 - 1}$
Cusp–regular	$\{(0, x_2, x_3); x_2 > x_3^2\}$	$\psi(\theta) = -\dfrac{x_3^2 - x_2 + 1}{x_3^2 - x_2 - 1}$
Fold–Fold	$\{(0, x_2, x_3); x_2 x_3 < 0\}$	$\psi(\theta) = -\dfrac{x_3 + x_2}{x_3 - x_2}$

Table 4: Sliding regions and slow manifolds of $X = (X_1, X_2) \in \Sigma_0(0)$, and 0 is an M–singular point.

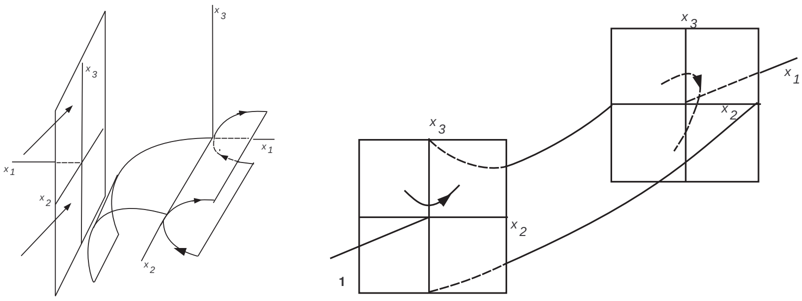

Figure 9: Fold-regular with $\delta = 1$ and Fold-fold with $\delta = -1$.

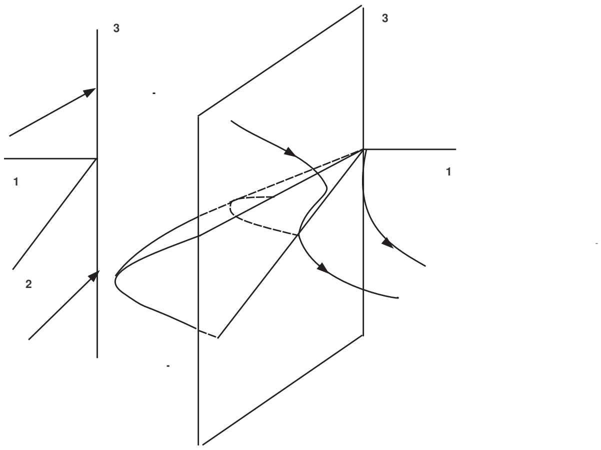

Figure 10: Cusp-Regular.

Normal Forms	X_1	X_2
Q1- Bec to bec	$(x_2^2 - x_3^2 + \lambda, 1, 0)$	$(1, 0, 0)$
Q1- Lips	$(x_2^2 + x_3^2 + \lambda, 1, 0)$	$(1, 0, 0)$
Q1- Dove's tail	$(-x_2^3 - x_3 - \lambda x_2, 1, 0)$	$(1, 0, 0)$
Q2	$(x_3 + \lambda, 0, 1)$	$(x_3 + x_2^2, 0, 1)$
Q3	$(x_2, 1, 1)$	$(x_3, 1, \lambda + x_2)$
Q4	$(x_3, ax_2 + bx_3, cx_2 + dx_3 + \lambda)$	$(1, 0, 0)$
Q5- Case (i)	$(x_2, 1 + \lambda + \frac{x_2+x_3}{2}, 1)$	$(x_3, -1 - \lambda + \frac{x_2+x_3}{2}, 1)$
Q5- Case (ii)	$(x_2 - x_3, x_2 + \lambda, 1)$	$(2x_2 - x_3, x_2 + \lambda, 2)$
Q6	$(x_2 - x_3, 1, \lambda + 1)$	$(x_2 + x_3, 0, 1)$

Table 5: $X = (X_1, X_2) \in \Sigma_1(0)$.

Normal Forms	X^M
Q1- Bec to bec	$(0,-1,0)$
Q1- Lips	$(0,-1,0)$
Q1- Dove's tail	$(0,-1,0)$
Q2	$(0,0,\lambda-x_2^2)$
Q3	$(0,x_2-x_3,\lambda x_2+x_2^2-x_3)$
Q4	$(0,-ax_2-bx_3,-cx_2-dx_3-\lambda)$
Q5- Case (i)	$(0,-(1+\lambda)(x_2+x_3)+\frac{x_2^2}{2}-\frac{x_3^2}{2},x_2-x_3)$
Q5- Case (ii)	$(0,-x_2^2-\lambda x_2,-x_3)$
Q6	$(0,-x_2-x_3,-2x_3-\lambda x_2-\lambda x_3)$

Table 6: Sliding vector fields of $X=(X_1,X_2)\in\Sigma_1(0)$.

Normal Forms	SR	SM	
Q1- Bec to bec ($\lambda=0$)	$x_2^2-x_3^2<0$	$\psi(\theta)=-\frac{x_2^2-x_3^2+1}{x_2^2-x_3^2-1}$	Fig. 11
Q1- Lips ($\lambda=-1$)	$x_2^2+x_3^2<1$	$\psi(\theta)=-\frac{x_2^2+x_3^2}{x_2^2+x_3^2-2}$	Fig. 11
Q1- Dove's tail ($\lambda=0$)	$x_3>-x_2^3$	$\psi(\theta)=\frac{-x_2^3-x_3+1}{x_2^3+x_3+1}$	Fig. 12
Q2($\lambda=0$)	$-x_2^2<x_3<0$	$\psi(\theta)=\frac{2x_3+x_2^2}{x_2^2}$	Fig. 12
Q3	$x_2.x_3<0$	$\psi(\theta)=-\frac{x_2+x_3}{x_2-x_3}$	Fig. 13
Q4	$x_3<0$	$\psi(\theta)=-\frac{x_3+1}{x_3-1}$	Fig. 13
Q5- Case (i)	$x_2.x_3<0$	$\psi(\theta)=-\frac{x_2+x_3}{x_2-x_3}$	Fig. 13
Q5- Case (ii)	$x_3<x_2<\frac{x_3}{2}$ ou $\frac{x_3}{2}<x_2<x_3$	$\psi(\theta)=\frac{3x_2-2x_3}{x_2}$	Fig. 14
Q6	$(x_2-x_3)(x_2+x_3)<0$	$\psi(\theta)=\frac{x_2}{x_3}$	Fig. 14

Table 7: Sliding regions and slow manifolds of $X=(X_1,X_2)\in\Sigma_1(0)$.

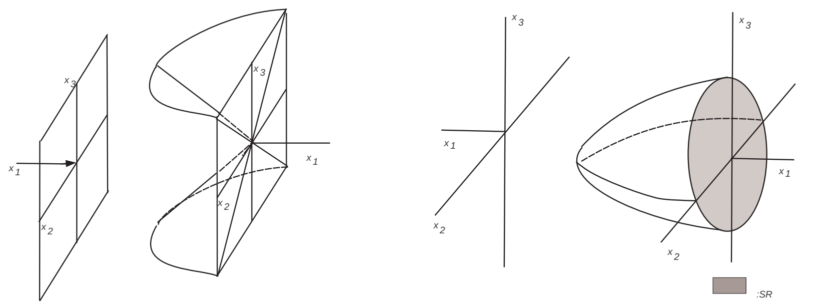

Figure 11: Bec to bec and Lips.

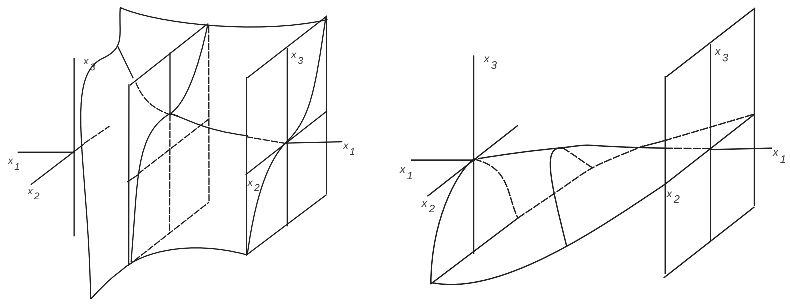

Figure 12: Dove's tail and Q2.

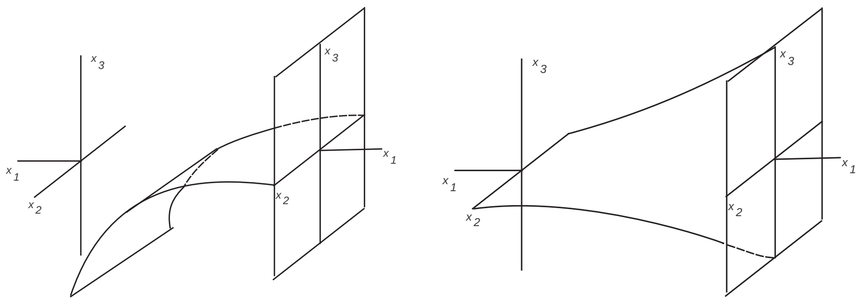

Figure 13: Q4, Q51(=Q3).

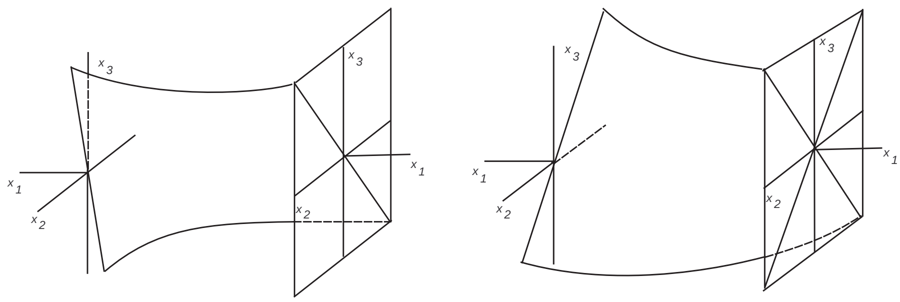

Figure 14: Q52 and Q6.

Moreover, we have that $X_1F = \dfrac{\partial g}{\partial y_3}, X_1^2F = \dfrac{\partial^2 g}{\partial y_3^2}$, and $X_1^3F = \dfrac{\partial^3 g}{\partial y_3^3}$.

When 0 is a fold singularity of G_{X_1} then there exists a C^∞–diffeomorphism $\phi_{X_1} : M \to M$, called the symmetric associated with G_{X_1}. The mapping ϕ_{X_1} satisfies $\phi_{X_1}(0) = 0, G_{X_1} \circ \phi_{X_1} = G_{X_1}$, and $\phi_{X_1}^2 = Id$. We observe that ϕ_{X_1} is C^∞ conjugate to $\phi_0(y_2, y_3) = (-y_2, y_3)$ and $S_{X_1} = Fix(\phi_{X_1})$. Moreover, if $q \notin S_{X_1}$ then $\phi_{X_1}(q)$ is the point where the trajectory of X passing through q meets M, in positive or negative time.

Assume now that $0 \in M$ is a U–singularity of $X = (X_1, X_2)$. The diffeomorphism $\phi = \phi_{X_1} \circ \phi_{X_2}$ works as a first return mapping of X at 0, with $\phi(0) = 0$.

Given ϕ_{X_1} and ϕ_{X_2} as above we may choose coordinates around 0 on M such that $\phi_{X_1}(y_2, y_3) = (-y_2, y_3)$. Then for some $a, b, c \in \mathbb{R}$ we have

$$J\phi_{X_2}(0) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

with $a^2+bc = 1$. It is proved in [14] that 0 is a hyperbolic fixed point of $\phi = \phi_{X_1} \circ \phi_{X_2}$ if and only if $a^2 > 1$. The eigenvalues of $J\phi(0)$ are $\lambda = a \pm \sqrt{a^2 - 1}$. If $a^2 > 1$ then then p is a saddle point of ϕ and if $a^2 < 1$ then the eigenvalues of $J\phi(0)$ have nonzero imaginary parts. If 0 is a hyperbolic fixed point of ϕ we refer to it as a hyperbolic $U-$singularity of X.

As 0 is a generic singularity we may assume without loss of generality that $X_1F = x_3$ and $X_2F = x_2$. Thus we have that $f_1(x_1, x_2, x_3) = x_3$ and $g_1(x_1, x_2, x_3) = x_2$. The sliding region is given by $x_2.x_3 < 0$. We have:

$$X^M = (0, \frac{x_3g_2 - x_2f_2}{x_3 - x_2}, \frac{x_3g_3 - x_2f_3}{x_3 - x_2}),$$

with

$$\begin{aligned} f_i(x_1, x_2, x_3) &= a_{i-1} + A_{i-1}(x_1, x_2, x_3), \\ g_i(x_1, x_2, x_3) &= b_{i-1} + B_{i-1}(x_1, x_2, x_3), \end{aligned}$$

where a_{i-1}, b_{i-1} are constants and $A_{i-1}(0, 0, 0) = 0, B_{i-1}(0, 0, 0) = 0, i = 2, 3$. We remark that $a_2 < 0$ and $b_1 > 0$ provide that 0 is a U–singularity. So the orbits of X^M in a neighborhood of 0 coincide with the orbits of the fields

$$G(x_2, x_3) = \begin{pmatrix} -a_1 & b_1 \\ -a_2 & b_2 \end{pmatrix} . \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} + H(x_2, x_3)$$

with $H(0, 0) = (0, 0)$ and $H(x_2, x_3) = O(|x_2, x_3|^2)$.

We give in the above coordinates the expression of the first return mapping $\phi = \phi_{X_1} \circ \phi_{X_2}$. The solutions of X_1 and X_2 passing through $(0, y, z)$ are respectively

$$\left\{ \begin{aligned} x_1 &= zt + \tfrac{a_2}{2}t^2 + r_1 \\ x_2 &= y + a_1t + r_2 \\ x_3 &= z + a_2t + r_3 \end{aligned} \right. , \qquad \left\{ \begin{aligned} x_1 &= yt + \tfrac{b_2}{2}t^2 + s_1 \\ x_2 &= y + b_2t + s_2 \\ x_3 &= z + b_1t + s_3 \end{aligned} \right. ,$$

with $r_i, s_i, i = 1, 2, 3$, being higher order terms. Thus we have that

$$\phi(x_2, x_3) = (0, -x_2 + \frac{2a_1}{a_2}x_3, -x_3 - \frac{2b_1}{b_2}(x_2 - \frac{2a_1}{a_2}x_3)) + O(|x_2, x_3|^2).$$

Proof Theorem 1.2. As before we assume that $f_1 \equiv x_3$, $g_1 \equiv x_2$ and that the sliding region is given by $x_2x_3 < 0$. The reduced flow (8) is given by

$$\begin{array}{rcl} \dot{x}_i & = & \dfrac{f_i + g_i}{2} + \varphi(\cot\theta)\dfrac{f_i - g_i}{2}, \quad i = 2, 3, \\ 0 & = & -\sin\theta\left(\dfrac{x_3 + x_2}{2} + \varphi(\cot\theta)\dfrac{x_3 - x_2}{2}\right), \end{array}$$

with $\theta \in (0, \dfrac{\pi}{2})$, and $x_2, x_3 \in \mathbb{R}$. Observe that the straight line $L = \{x_2 = x_3 = 0\}$ is composed by singular points of the reduced flow. Let $q \in M = \{(0, x_2, x_3)\}$. If the singular orbit of $\widehat{X}$ falls at $q \in M_3 = \{x_2.x_3 < 0\}$ then it follows firstly a fast orbit of

$$\begin{array}{rcl} x_i' & = & \dfrac{f_i + g_i}{2} + \varphi(\cot\theta)\dfrac{f_i - g_i}{2}, \quad i = 2, 3, \\ \theta' & = & -0, \end{array}$$

over to the slow manifold

$$\frac{x_3 + x_2}{2} + \varphi(\cot\theta)\frac{x_3 - x_2}{2} = 0$$

and then it stays there. Moreover this part of the singular orbit is just the projection of the trajectory of the sliding vector field X^M over q onto the slow manifold. So it converges to a point on L. If $q \in M_1 \cup M_2$ then the positive trajectory of X^M is governed by the first return mapping ϕ or by ϕ^{-1}. Since by hypothesis the rotation angle of the diffeomorphism is irrational there will be a positive integer k such that $\phi^k(q) \in M_3$ and so the singular orbit converges to a point on L. ■

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