# On the mathematical work of Jean Schmets

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**Abstract**

In the introductory Section 0. some of the most important points in the professional curriculum vitae of Jean Schmets are given. Then Section 1 is devoted to present (part of) the work for which Schmets was very well known until 1990: He has been the leading specialist in the world for locally convex spaces $C(X)$ of continuous functions with various topologies and for the corresponding spaces $C(X, E)$ of vector valued continuous functions. Finally, in Section 2 some results which Schmets obtained in cooperation with Manuel Valdivia since 1990 are reviewed: work on domains of real analytic existence, continuous linear right inverses for $C^\infty$- functions and Whitney extensions for non quasianalytic functions.

## 0 Introduction : Curriculum Vitae of Jean Schmets

*a) Some data*

Jean François Hubert Schmets was born in Rocourt, Belgium on September 22, 1940. After getting the "Diplôme d'humanités latines-mathématiques" at the "Athénée Royal de Liège" in 1958 and after spending one year at the Jefferson High School in Cedar Rapids, Iowa, USA, he studied mathematics at the Université de Liège. He received the "Deuxième Licence en Sciences Mathématiques" and the "Agrégation de l'Enseignement Secondaire Supérieur en Sciences Mathématiques" in 1963 and the "Doctorat en Sciences, groupe des Sciences Mathématiques" in 1965, adviser: Professor H.G. Garnir. The title of the dissertation was "Sur quelques

2000 *Mathematics Subject Classification :* primary 01A70, secondary 26E05, 26E10, 46A08, 46A63, 46E10, 46E25, 46E40, 46G20, 46M18.

*Key words and phrases :* spaces of (vector valued) continuous functions, (quasi)barrelled and (ultra)bornological spaces associated with a locally convex space, domains of real analytic existence, Borel theorem, Whitney jets, continuous linear right inverses for restriction maps, Whitney extensions for non quasianalytic functions, real analytic extensions.

points d'analyse fonctionnelle". In 1976, he was admitted to the "Agrégation de l'Enseignement Supérieur"; the title of the dissertation (or of the "Habilitationsschrift", as one would say in Germany) was "Espaces de fonctions continues".

From 1963 to 1973 Schmets was assistant in the group of Garnir. Concerning Garnir and his school in Liège, see the obituary [BuVa] by P.L. Butzer and J. Vaillant, where among other things, it is written: "Essentially, Garnir was a man whose charisma, stability and true goodness were immensely beneficial to all who had the luck to have known him." Schmets became "chef de travaux" in 1973, "agrégé à l'Université" in 1976, professor in 1978 and "Professeur ordinaire" in October 1988. Since 1986, Jean Schmets was "Directeur de l'Unité de Documentation Mathématique" and since 1987 "Représentant de la Faculté des Sciences" at the "Conseil Scientifique des Bibliothèques". At the Université de Liège, he was "Vice-doyen de la Faculté des Sciences" from October 1, 2000 until the end of September 2005 and President of the mathematical department from 2001 to 2005. He retired at the end of September 2005.

Schmets had six Ph.D. students: J. Zafarani (1975), J.-L. Lieutenant (1982), J.-P. Schneiders (1986), F. Bastin (1989), E. Ngimbi Ngembo (1994) and M. Mauer (2000). The authors and Susanne Dierolf were members of the committee during the oral exam of Françoise Bastin, which took place on a very historic day: In the late evening of this day, the Berlin wall was opened.

*b) Honors, services*

Schmets was a visiting professor at, among other places, the Universities of Bonn, Düsseldorf, Jena, Kiel, Mainz, Oldenburg, Paderborn, Saarbrücken, Trier and Tübingen (Germany), at the Technical University and the University of Valencia (Spain), Krakow (Poland), Paris XIII (France), Napoli (Italy), Budapest (Hungary), Bujumbura (Burundi), and the University of Maryland at College Park (USA).

Among the honors with which Schmets was decorated, let us only mention that he received the "Prix des Amis de l'Université de Liège" in 1976 and twice the "Prix Jacques Deruyts" (1980–84 and 1996–2000) of the Royal Belgian Academy. He became a foreign corresponding member of the "Real Academia de Ciencias Exactas, Físicas y Naturales" of Madrid, Spain in 1998.

Jean served as President of the Belgian Mathematical Society, was a member, vice-secretary, secretary, vice-president and now is President of the "Belgian National Committee for Mathematics". He has served as member of the mathematical commission of the "Fonds voor Wetenschappelijk Onderzoek Vlaanderen", Belgium and as Treasurer of the "Société Royale des Sciences de Liège".

Schmets was member of the Editorial Board of the journal "Simon Stevin", later "Bulletin of the Belgian Math. Soc. – Simon Stevin", until he became President of the Belgian Math. Soc. He is still member of the Editorial Board of the journal "Results in Mathematics".

*c) Talks, organization*

Jean Schmets gave over 170 colloquium lectures and invited lectures at various conferences all over the world. His talks were always very interesting and very well organized. In particular, the authors remember Schmets' excellent talk on the recent work of Manuel Valdivia during the "International Functional Analysis Meeting on the Occasion of the 70th Birthday of Professor Valdivia" in Valencia in 2000.

But Schmets has also been very well known as organizer or co-organizer of more than 30 international meetings. A whole series of meetings on functional analysis and partial differential equations was organized in Esneux and Han-sur-Lesse, since 1979. In 2000 the authors, Maestre and Schmets co-organized the international functional analysis meeting at the Technical University of Valencia and edited the Proceedings volume [FA] of this conference. In 2001 Jean and the first named author were co-organizers of the joint meeting of the Belgian and German math. societies in Liège. During this meeting, Bastin, Bierstedt, Laubin, Meise, and Schmets co-organized the Special Session "Functional Analysis and Functional Analytic Methods in Partial Differential Equations" (24 talks).

*d) Publications*

According to his list of publications in mid-2006, Schmets now has 68 articles published in scientific journals, among them exactly one joint article with the authors. This article [47] appeared in Note Mat. in the special volume dedicated to the memory of Professor Köthe.

Schmets is, together with H.G. Garnir and M. De Wilde, co-author of the three volumes "Analyse Fonctionnelle – Théorie constructive des espaces linéaires à semi-normes" ([AF1] 1968, [AF2] 1972, and [AF3] 1973) of Birkhäuser. In these books the authors do not make use of the uncountable axiom of choice or of Zorn's lemma. The books were reviewed for Math. Reviews by W.A.J. Luxemburg who wrote concerning volume I: "All in all the book seems to be well suited for those who want to acquaint themselves with the topological theory of functional analysis. A number of useful problems are scattered through the text." For volume II he wrote: "The book contains a number of exercises with hints which makes it more suitable for a textbook. This volume is written in the same clear style of volume I and the printing is excellent." And finally, concerning volume III it is said: "As with the previous volumes, this one is also well written. The reviewer can see its main function as a useful reference text."

Schmets published two Springer Lecture Notes in Mathematics volumes, "Espaces de fonctions continues" ([FC] 1976) and "Spaces of vector-valued continuous functions" ([VCF] 1983). More will be said about the contents of these lecture notes in Chapter I below. All in all, Schmets is the (co-) author of 14 books and co-editor of one Proceedings volume.

*e) Disclaimer, organization of the paper*

Of course, it will not be possible here to mention all the books and articles published by Schmets, and for the purposes of this article we decided to review in Section 1 the results for which Schmets has primarily been known for a long time: on spaces of continuous and vector valued continuous functions, and we will mainly treat the scalar case here in more detail. This covers most of Jean's work up to 1990, but it should be mentioned that during this time Schmets also contributed to measure and integration theory ([2], [4], [8]) as well as to distribution theory ([10]), derived a generalization of Lyapunov's theorem ([5]), and a version of the bang-bang principle ([9]).

A second section will be devoted to one of most successful scientific cooperations of the last 15 years; viz., the joint work of Schmets with Manuel Valdivia, starting around 1990. From 1994 on Schmets and Valdivia received support by "Actions

concertées CGRI-MEC" (Belgium/Spain) and FEDER for their joint research and for the visits at each other's universities.

**Note for the reader:** In this article, references are given to items in three different lists. For $n \in \mathbb{N}$, clearly $[n]$ refers to article $n$ in the list of articles of Jean Schmets, while e.g. [AF1] (only capital letters) refers to an item in the list of his books. On the other hand, abbreviations like [La0] (containing at least one lower case letter) always refer to items in the list of references to other people's work.

## 1 Spaces of continuous functions

*a) Notation, some definitions*

In the sequel, $X$ will always denote a completely regular Hausdorff space, and $C(X)$ will be the algebra of all real or complex valued continuous functions on $X$. $C_s(X)$ denotes $C(X)$ with the topology of pointwise convergence on $X$, and $C_c(X)$ stands for $C(X)$ endowed with the topology of uniform convergence on the compact subsets of $X$. In the vector valued case we use the similar abbreviations $C_s(X, E)$ and $C_c(X, E)$.

A *character* on $C(X)$ is a multiplicative linear functional $\neq 0$ on $C(X)$. The set $\upsilon X$ of all characters on $C(X)$ is endowed with the weak topology and the uniform structure induced by the algebraic dual of $C(X)$, and $X$ is identified canonically with a dense subspace of $\upsilon X$. $\upsilon X$ has the weakest uniform structure which makes all $f \in C(X)$ uniformly continuous; it will be called the *repletion* of $X$ – in the literature one sometimes also finds the term "realcompactification". $\upsilon X$ is a topological subspace of the Stone-Čech compactification $\beta X$ of $X$. As a uniform space, $\upsilon X$ is complete, and each function $f \in C(X)$ has a unique continuous extension $\tau f$ to $\upsilon X$, defined by $(\tau f)(u) = u(f)$ for each $u \in \upsilon X$. The mapping $\tau$, defined in this way, is an algebra isomorphism of $C(X)$ onto $C(\upsilon X)$. The space $X$ is said to be *replete* if $X = \upsilon X$. Each Lindelöf space is replete. The notations $C_s(\upsilon X)$, $C_c(\upsilon X)$ etc. should be self explanatory by now.

A subset $B$ of $X$ is said to be *bounding* if $f(B)$ is bounded for each $f \in C(X)$. Recall that $X$ is said to be a *pseudocompact* space if and only if $X$ is a bounding subset of itself. Each bounding subset of $\upsilon X$, hence also of $X$, is relatively compact in $\upsilon X$. The *$\mu$-space $\mu X$ associated with a completely regular Hausdorff space $X$* is the smallest subspace of $\upsilon X$ which contains $X$ and in which each bounding subset is relatively compact. $X$ is said to be a *$\mu$-space* if $X = \mu X$. Of course, each replete space is a $\mu$-space, but also each paracompact space is a $\mu$-space.

*b) Prehistory and an open problem*

In the beginning of the theory of locally convex spaces, there were obvious reasons (viz., the generalization of classical results to a setting more general than Banach and Fréchet spaces) to introduce the classes of bornological spaces and of barrelled spaces. While it was easy to find examples of bornological spaces which were not barrelled, the first example of a barrelled space which was not bornological was given by use of the famous Nachbin-Shirota theorems and of an example due to Gillman and Henriksen of a $\mu$-space which is not replete.

**Theorem (L. Nachbin, T. Shirota 1954, [Nach0], [Shir]).** *For a completely regular Hausdorff space $X$, $C_c(X)$ is bornological if and only if $X$ is replete, and this space is barrelled if and only if $X$ is a $\mu$-space.*

On the other hand, it follows from the Nachbin-Shirota theorems that each bornological $C_c(X)$ must be barrelled.

In his work [Nach] in approximation theory (concerning Stone-Weierstrass type theorems for spaces of continuous functions on completely regular Hausdorff spaces $X$), Leopoldo Nachbin developed a framework of spaces of continuous functions with 0- or o-weight conditions with respect to a system $V$ of weights on $X$; viz., he introduced the weighted spaces $CV(X)$ and $CV_0(X)$. In this context, however, no general Nachbin-Shirota type theorems have been proved so far. – Incidentally, an example of a space $CV_0(X)$ is the space of all continuous and bounded functions on $X$, equipped with the strict topology of R.C. Buck, a space on properties of which Jean Schmets has also done some interesting research, see [18], [26].

Some years later than Nachbin and Shirota, a fairly complete study of the locally convex properties of the space $C_c(X)$ was given in a seminal paper by Seth Warner. Among other things he proved:

**Theorem (S. Warner 1958, [War]).** *$C_c(X)$ is quasibarrelled if and only if each set $B \subset X$ on which each positive lower semicontinuous function on $X$ which is bounded on each compact subset of $X$ is also bounded, must already be relatively compact.*

*c) The first contributions of Jean Schmets, scalar case*

In 1971, De Wilde and Schmets [12] showed that in the Nachbin-Shirota theorem "bornological" can be replaced by "ultrabornological". In the next two years, H. Buchwalter and J. Schmets were led to study the space $C_s(X)$. Here is one of their results.

**Theorem (H. Buchwalter, J. Schmets 1972/73, [17], [20]).** *The space $C_s(X)$ is always quasibarrelled, and it is barrelled if and only if each bounding subset of $X$ is finite.*

Instead of "only" characterizing when $C_s(X)$ resp. $C_c(X)$ is (ultra)bornological resp. (quasi)barrelled, it is also interesting to characterize the *associated spaces.*

Let $R$ be a property which a locally convex space can have and which is stable under separated inductive limits and satisfied by each linear space equipped with the system of all seminorms. Ultrabornological, bornological, barrelled and quasi-barrelled are examples of such properties. Then the *associated R-space* to a locally convex space $E$ is $E$, equipped with the weakest locally convex topology which is stronger than the original topology of $E$ and satisfies property $R$.

For example, the bornological space $E_b$ associated with a locally convex space $E$ is the inductive limit of the spaces $E_B$ where $B$ runs through all absolutely convex bounded sets and $E_B$ is the normed space associated with $E$ and $B$; i.e., the linear span of $B$ in $E$ with the Minkowski functional of $B$ as norm. A similar construction works for the associated ultrabornological space by replacing the absolutely convex and bounded sets by the Banach discs.

In the case of the associated (quasi)barrelled space, however, one must use a transfinite construction. For a locally convex topology $\tau$, let $\tau_1$ be the stronger topology which one gets by taking all $\tau$-barrels as neighborhoods of 0. $\tau_1$ need not be barrelled, but one can continue this procedure by transfinite induction: For an ordinal number $\alpha$, let $\tau_\alpha = (\tau_{\alpha-1})_1$ if $\alpha$ has a predecessor, and let $\tau_\alpha$ be the topology given by the union of all 0-neighborhoods of $\tau_\beta$ for $\beta < \alpha$ if this not the case.

There must be an ordinal $d$ for which $\tau_d = \tau_{d+1}$, and this is the associated barrelled topology. The associated quasibarrelled topology is obtained in the same way by replacing barrels by bornivorous barrels.

In the same two papers as above, the following results were also proved:

**Theorem (H. Buchwalter, J. Schmets 1972/73, [17], [20]).** *Let $X$ be a completely regular Hausdorff space.*

*1. The space $C_s(\upsilon X)$ is always bornological; it is the bornological space associated with $C_s(X)$. $C_s(X)$ is bornological if and only if $X$ is replete.*

*2. The space $C_c(\upsilon X)$ is always ultrabornological; it is the ultrabornological space associated with $C_s(X)$. $C_s(X)$ is ultrabornological if and only if $X$ is replete and each compact subset of $X$ is finite.*

*3. The space $C_c(\mu X)$ is the barrelled space associated with $C_s(X)$.*

Incidentally, the reviewer (which happens to be the first named author of the present article) for Zbl. MATH of the 1973 article of Buchwalter and Schmets wrote: "The problem of characterizing the bornological space associated with $C_c(X)$... had been left open and is solved in the last part of the present well and carefully written paper."

*d) Still scalar case, but a more general setting*

From this point on, a general setting to treat spaces of continuous functions with the topology of uniform convergence on a system of relatively compact subsets of $\upsilon X$ emerged, which we will now report on, following the first Springer Lecture Notes volume [FC] of Schmets.

Let $X$ be a completely regular Hausdorff space and $\mathcal{P}$ a system of bounding subsets of $\upsilon X$ such that $Y_{\mathcal{P}} = \cup\{B;\ B \in \mathcal{P}\}$ is dense in $\upsilon X$ and such that for $B_1, B_2 \in \mathcal{P}$ there is $B \in \mathcal{P}$ with $B_1 \cup B_2$ contained in the closure of $B$ in $\upsilon X$. Then the topology $\tau_{\mathcal{P}}$ of uniform convergence on the system $\mathcal{P}$ of subsets of $\upsilon X$ is given by the seminorms $||.||_B$, $B \in \mathcal{P}$, where $||f||_B = \sup_B |\tau f|$ for $f \in C(X)$. Without loss of generality one can assume that the system $\mathcal{P}$ contains the closure in $Y_{\mathcal{P}}$ of each of its elements as well as each subset of its elements.

Let $Y$ be a dense subspace of $\upsilon X$. Then $\upsilon_Y X$ denotes the subspace of $\upsilon X$ of all elements $x \in \upsilon X$ on which each Banach disc of $C(X)$ equipped with the topology of pointwise convergence on $Y$ is bounded. That is, $\upsilon_Y X$ consists of all characters on $C(X)$ which are continuous on the ultrabornological space associated with $C(X)$ equipped with pointwise convergence on $Y$. One has $\upsilon_X X = \upsilon X$, and if $X$ is metrizable, then $\upsilon_Y X = \upsilon X$ holds for each dense subspace $Y$ of $X$. In fact, $\upsilon_Y X = \upsilon X$ also holds for each dense subspace $Y$ of $X$ if $X$ is locally compact or pseudocompact.

**Theorem (J. Schmets 1975, [27]).** *Let $Y$ be a dense subspace of $\upsilon X$ with the property that $\upsilon_Y X = \upsilon X$. Then $C_c(\upsilon X)$ is the ultrabornological space associated with $C(X)$, endowed with any locally convex topology between the topology of pointwise convergence on $Y$ and the topology of uniform convergence on the compact subsets of $\upsilon X$.*

Buchwalter wrote as a reviewer of [27]: "L'originalité du papier, par rapport aux articles antérieurs sur la question, réside précisément dans le fait que des différences substantielles apparaissent lorsqu'on a $Y \neq X$; par exemple la topologie sur $C(X)$ de la convergence simple sur $Y$ peut être tonnelée sans que celle de la convergence simple sur $X$ le soit."

For a dense subspace $Y$ of $\upsilon X$ Schmets [27] defined another subspace $\mu_Y X$ of $\upsilon X$ by transfinite induction, as follows. Let $X_Y^0 = Y$, and for an ordinal $\alpha$, let $X_Y^{\alpha+1}$ be the union of all bounding subsets of $\upsilon X$ which are contained in $X_Y^\alpha$, and let $X_Y^\alpha$ be the union of all $X_Y^\beta$ for $\beta < \alpha$ if $\alpha$ is a limit ordinal. Then there must be an ordinal $d$ for which $X_Y^{d+1} = X_Y^d$, and this is the space $\mu_Y X$. Note that $\mu_X X = \mu X$ holds and that if $Y$ is a dense subspace of $\upsilon X$ which is contained in $\mu X$, then $\mu_Y X$ is a subspace of $\mu X$.

**Theorem (H. Buchwalter, J. Schmets 1970–75, see [27]).** *Let $Y$ be a dense subspace of $\upsilon X$. Then $C_c(\mu_Y X)$ is the barrelled space associated with $C(X)$, endowed with any locally convex topology between pointwise convergence on $Y$ and uniform convergence on the compact subsets of $\mu_Y X$.*

In the first Lecture Notes volume [FC] of Schmets one can also find characterizations of the associated bornological and quasibarrelled spaces to $(C(X), \tau_{\mathcal{P}})$, $\mathcal{P}$ as above, but since this is a bit more technical and requires additional constructions, we will not give details here.

Furthermore, the book [FC] contains a chapter on separability conditions and on weak compactness in $C(X)$. In fact, everything is also done for the space of all continuous and bounded functions, and besides barrelledness and quasibarrelledness also "d-tonnelé, $\sigma$-tonnelé, d-évaluable, $\sigma$-évaluable" are considered.

This is perhaps the right place to point out that there is some recent work [KaST1], [KaST2] of Kąkol, Saxon and Todd on weak barrelledness properties of $C_c(X)$ and $C_p(X)$. In particular, a question of Buchwalter and Schmets [20] from 1973 is finally solved in [KaST1].

*e) Some remarks on the vector valued case*

The first Springer Lecture Notes book [FC] of Schmets closes with a chapter on the vector valued case. The following results give a flavor of what is done here:

**Theorem (J. Schmets 1976, see [FC]).** *Let $X$ be a completely regular Hausdorff space and $E$ a locally convex space.*

*1. The space $C_s(X, E)$ of all continuous $E$-valued functions on $X$ is quasibarrelled if and only if $E$ is quasibarrelled. The quasibarrelled space associated with this space is $C_s(X, E_{qb})$, where $E_{qb}$ is the quasibarrelled space associated with $E$.*

*2. $C_s(X, E)$ is barrelled if and only if both $C_s(X)$ and $E$ are barrelled. If $C_s(X)$ is barrelled, then the barrelled space associated with $C_s(X, E)$ is $C_s(X, E_{ba})$, where $E_{ba}$ denotes the barrelled space associated with $E$.*

*3. If $C_s(X, E)$ is bornological, then both $C_s(X)$ and $E$ must be bornological. The converse holds if each point in $X$ has a countable basis of neighborhoods. If $C_s(X)$ is bornological and if each point in $X$ has a countable basis of neighborhoods, then $C_s(X, E_b)$ is the bornological space associated with $C_s(X, E)$, where $E_b$ is the bornological space associated with $E$.*

The last chapter of the first Lecture Notes volume of Schmets set the stage for the second Lecture Notes volume [VCF] in which the space $C(X, E)$ is treated with the topology $\tau_{\mathcal{P}}$ of uniform convergence on a system $\mathcal{P}$ of (relatively) compact subsets of $\upsilon X$ as before (so that, in particular, $Y_{\mathcal{P}}$ is a dense subspace of $\upsilon X$). In the proofs the main ingredients are the properties of the support of an absolutely convex nonvoid subset of $C(X, E)$ and Singer's [Sin] representation theorem of the dual of $C_c(X, E)$ as space of $E'$-valued measures, which allows to define a support for elements of $C(X,E)'$.

During the first visit of the second named author in Paderborn, in the summer term of 1983, Jean Schmets was also visiting Paderborn and gave lectures on the contents of [VCF]. In the copy of [VCF] which the first named author got from Jean at that time, one can find the following dedication: "En souvenir d'un séjour formidable à Paderborn. Bien amicalement, Jean".

Clearly, if $(C(X,E),\tau_{\mathcal{P}})$ is (ultra)bornological or (quasi)barrelled, then $(C(X),\tau_{\mathcal{P}})$ must be of the same type, as well as $E$ if the union of the closures of the sets in $\mathcal{P}$ contains a point of $X$. In the other direction, however, there are counterexamples due to S. Dierolf (see [37]) and A. Marquina, J.M. Sanz Serna [MarS]. The case of $C(X,E)$ with the topology of uniform convergence on the compact subsets of $X$ was treated in 1980–83 by J. Mendoza (Casas), see [Men1], [Men2]. By use of a theorem of J. Mujica [Muj], some of his results could be generalized (see part 2. of the next theorem). Also, by tensor product methods certain results for this case were obtained by A. Defant and W. Govaerts in 1984–86, see [DeGo1], [DeGo2]. Here is a sample of results in Jean Schmets' second Lecture Notes volume:

**Theorem.** *1.* **(J. Schmets 1977, see [31], [32])** *If the space $C_c(\upsilon X,E)$ is bornological – which is the case whenever $E$ is metrizable, then $(C(X,E),\tau_{\mathcal{P}})$ is bornological if and only if $(C(X),\tau_{\mathcal{P}})$ is bornological.*

*2. Let $X$ be replete and locally compact and let $E=\operatorname{ind}_n E_n$ be a compactly regular (countable) locally convex inductive limit such that $C_c(\beta X,E_n)$ is (ultra)bornological for each $n$. Then $C_c(X,E)$ is (ultra)bornological. Also, in the bornological case, $(C(X,E),\tau_{\mathcal{P}})$ is then bornological if and only if $(C(X),\tau_{\mathcal{P}})$ is bornological.*

*3.* **(A. Defant, W. Govaerts 1986, [DeGo2])** *Let $K$ be compact and $(F_\alpha)_\alpha$ be an inductive spectrum of locally convex spaces such that $C_c(K,F_\alpha)$ is bornological for each $\alpha$ and such that each $f\in C(K,\operatorname{ind}_\alpha F_\alpha)$ canonically comes from an element of $C(K,F_\alpha)$. Then $C_c(K,\operatorname{ind}_\alpha F_\alpha)$ is bornological if and only if the strong dual of $\operatorname{ind}_\alpha F_\alpha$ has property (B) of Pietsch.*

*4.* **(J. Schmets 1977, [31])** *If $X$ is replete and if $E$ is a Fréchet space, then $C_c(X,E)$ is ultrabornological.*

*5.* **(J. Mendoza Casas 1980, [Men1])** *If $X$ is replete and locally compact and if $C_c(\beta X,E)$ is ultrabornological, then $C_c(X,E)$ is ultrabornological.*

Note that every metrizable locally convex space and each strong dual of a metrizable space has property (B).

**Theorem (J. Mendoza 1983, [Men2]).** *1. If $\mathcal{P}$ consists of compact subsets of $X$, if $(C(X),\tau_{\mathcal{P}})$ and $C_c(\beta X,E)$ are (quasi)barrelled, then $(C(X,E),\tau_{\mathcal{P}})$ is (quasi)barrelled.*

*2. $E$ is (quasi)barrelled and its strong dual has property (B) if and only if each $C_c(K,E)$, $K$ compact, is (quasi)barrelled.*

*3. If every compact subset of $X$ is finite, then $C_c(X,E)$ is quasibarrelled if and only if $E$ is quasibarrelled. If every compact subset of $X$ is finite, then $C_c(X,E)$ is barrelled if and only if both $C_c(X)$ and $E$ are barrelled. If $X$ does not have this property, then $C_c(X,E)$ is (quasi)barrelled if and only if both $C_c(X)$ and $E$ are (quasi)barrelled and the strong dual of $E$ has property (B).*

*f) Another open problem*

While in the setting explained in subsection d) all results are final and nearly all questions are solved, this definitively does not apply to the vector valued case. The following is only one example for a problem which has remained open for quite a while.

In 1977, Schmets and Bierstedt (see [32]) asked if for a compact space $K$ and an (LB)-space $E = \operatorname{ind}_n E_n$ the space $C_c(K, E)$ must be bornological. This problem remains open. But S. Dierolf and P. Domański (1993-95) (see [DiD1], [DiD2]) made some interesting studies in this direction and, among other things, proved that the space $c_0(E)$ is bornological if $E$ is the strong dual of an (FM)-space or if $E$ is the inductive dual of a Köthe echelon space. They also showed that the problem is related to the last open problem of Grothendieck in functional analysis; viz., to the question whether a regular (LF)- (here even (LB)-) space must be complete.

## 2 The cooperation of Schmets and Valdivia

*a) Introduction, acknowledgment*

At some point, Jean Schmets finished his research on spaces of continuous vector valued functions. Most of the papers from number [48] in the list of publications of Jean Schmets until the last number [68] (as of mid-2006) are joint articles with Valdivia.

Many of these articles were surveyed in Section 4 of Jean's article [60] on the mathematical works of Manuel Valdivia in the Proceedings volume of the 2000 Valencia meeting (in which Jean never mentioned his own name when joint research of Schmets and Valdivia was concerned), and in what follows we owe much to this survey.

The first joint article [48] of Schmets and Valdivia dealt with the extent of the (non) quasi analytic classes on the real line. But from this point on the first main field of interest of both authors together were domains of real analytic existence ([50], [51]).

*b) Domains of real analytic existence*

Let $X$ be a real normed space. A domain $\Omega$ in $X$ is a real analytic domain if, for every domain $\Omega_1$ of $X$ verifying $\Omega_1 \not\subset \Omega \not\subset X \setminus \Omega_1$ and every connected component $\Omega_0$ of $\Omega \cap \Omega_1$, there is a real analytic function $f$ on $\Omega$ such that the restriction $f|_{\Omega_0}$ has no real analytic extension to $\Omega_1$. And $\Omega$ is a domain of real analytic existence if there is a real analytic function $f$ on $\Omega$ such that, for every domain $\Omega_1$ of $X$ with $\Omega_1 \not\subset \Omega \not\subset X \setminus \Omega_1$ and every connected component $\Omega_0$ of $\Omega \cap \Omega_1$, $f|_{\Omega_0}$ has no real analytic extension to $\Omega_1$. Of course each domain of real analytic existence is a real analytic domain.

**Theorem (J. Schmets, M. Valdivia 1993, [50], [51]).** *1. Every nonvoid domain $\Omega$ of a separable real normed space is a domain of real analytic existence.*

*2. If $A$ is an uncountable set, then the open unit ball of $c_0(A, \mathbb{R})$ is a real analytic domain, but not a domain of real analytic existence.*

The research on domains of real analytic existence was continued in the Ph.D. thesis of M. Mauer, which was supervised by Jean.

*c) The Borel theorem, Whitney jets, and continuous linear right inverses for $C^\infty$-functions*

In 1895, E. Borel proved that for every sequence $(c_n)_n$ of complex numbers there is a $C^\infty$-function $f$ on $\mathbb{R}$ such that $f^{(n)}(0) = c_n$ for all $n \in \mathbb{N}_0$. There were several improvements of this result due to Bernstein, Ritt and others, in particular concerning the real analyticity of the function $f$ outside the origin. In 1934, H. Whitney generalized these results from the origin to an arbitrary closed set $F \subset \mathbb{R}^N$. He characterized the jets $\varphi = (\varphi_\alpha)_{\alpha \in \mathbb{N}_0^N}$ on such a set $F$ which come from a function $f \in C^\infty(\mathbb{R}^N)$; i.e., such that $\varphi_\alpha = D^\alpha f|_F$ for every $\alpha$. For these jets Whitney proved that the function $f$ can be taken real analytic on $\mathbb{R}^N \setminus F$, and indeed holomorphic on some open subset of $\mathbb{C}^N$ containing $\mathbb{R}^N \setminus F$. Whitney introduced the space $\mathcal{E}(F)$ of Whitney jets on $F$, endowed it with a Fréchet topology and asked the following question: When does the continuous linear and surjective restriction map $R : C^\infty(\mathbb{R}^N) \to \mathcal{E}(F)$ have a continuous linear right inverse; i.e., when is there a continuous linear extension map from $F$ to $\mathbb{R}^N$ in the $C^\infty$-setting?

B. Mityagin showed in 1961 that there is no such extension map from 0 to $\mathbb{R}$, but that there is one from $[-1, 1]$ to $\mathbb{R}$. Several positive and negative examples were given by Seeley [Seeley], Stein [Stein], Bierstone [Bierstone] and many others. Finally, in 1979, Tidten [Tidten] gave the first characterization for the existence of an extension operator in the case of a compact set $F$ in terms of the condition (DN) of D. Vogt.

It was only in a paper of Schmets and Valdivia that the analyticity property appears again in connection with the existence of continuous linear extension operators.

**Theorem (J. Schmets, M. Valdivia 1997, [56]).** *If there exists a continuous linear extension operator $\mathcal{E}(K) \to C^\infty(\mathbb{R}^N)$ for $K$ compact, then there exists such an operator for which the extensions are holomorphic on*

$$(\mathbb{R}^N \setminus K)^* := \{x + iy \in \mathbb{C}^N;\ x \in \mathbb{R}^N, |y| < d(x, K)\}.$$

Two years later, M. Langenbruch [La1] even gave an explicit formula to obtain this result. Another three years later, in 2002, L. Frerick, D. Vogt [FrV] characterized the closed sets $F$ for which there exists a continuous linear extension operator $\mathcal{E}(F) \to C^\infty(\mathbb{R}^N)$ such that the extension of each jet is real analytic in $\mathbb{R}^N \setminus F$ as follows: For each $r > 0$ the boundary of the union of the components of $\mathbb{R}^N \setminus F$ which have nonempty intersection with the ball of center 0 and radius $r$ is bounded. Thus they solved an open problem of Schmets and Valdivia [56].

In the one variable case, M. Langenbruch [La2] proved in 2003 that every Whitney jet $\varphi$ on a closed set $F \subset \mathbb{R}$ can be extended to a $C^\infty$ function which is analytic in $\mathbb{C} \setminus F$ – hence the domain where the extensions can be chosen analytic is as large as possible. Note that such a result in several variables is not possible due to Hartogs' theorem. The final step was taken by C. Boonen and L. Frerick [BoFR] in a paper which appeared recently. They showed that a necessary and sufficient condition for a closed subset $F$ of $\mathbb{R}$ such that if the space $\mathcal{E}(F)$ admits a continuous linear extension operator, then this extension operator can be chosen with values holomorphic in $\mathbb{C} \setminus F$, is that the boundary of $F$ is compact.

*d) Whitney extensions for non quasianalytic functions*

Among various ways to define ultradifferentiable functions the following two are most frequently used. The older one goes back to the work of Gevrey and measures the behavior of the derivatives of these functions in terms of a weight sequence $(M_p)_{p\in\mathbb{N}_0}$, which is $((p!)^s)_{p\in\mathbb{N}_0}$, $s \geq 1$, in the Gevrey case, and which satisfies certain technical conditions in the general case. One speaks of ultradifferentiable functions of Roumieu type in this case; we refer to Komatsu [Ko] for a systematic treatment. Later Beurling, see Björck [Bj], pointed out that one can also use weight functions $\omega$ to measure the smoothness of $C^\infty$-functions with compact support by the decay properties of their Fourier transform. This method was modified by Braun, Meise, and Taylor [BrMT] who showed that also these classes can be defined by the decay behavior of their derivatives if one uses the Young conjugate of the function $t \mapsto \omega(e^t)$. For an open set $G$ in $\mathbb{R}^n$, the spaces of ultradifferentiable functions of Beurling type are denoted by $\mathcal{E}_{(M_p)}(G)$ and $\mathcal{E}_{(\omega)}(G)$, which are Fréchet spaces, while the spaces of ultradifferentiable functions of Roumieu type are denoted by $\mathcal{E}_{\{M_p\}}(G)$ and $\mathcal{E}_{\{\omega\}}(G)$; they have a much more complicated structure: they are countable projective limits of countable inductive limits of Banach spaces. We write $\mathcal{E}_*$ when we refer to any of these classes.

Several authors, like Carleson, Ehrenpreis or Komatsu investigated conditions to extend the Borel and Whitney theorems to the ultradifferentiable setting. A systematic study was initiated in the late 80's by Meise and Taylor [MeT1]. Bruna [Bruna] presented an extension of Whitney's theorem for classes of non quasianalytic functions defined by Komatsu, with some extra conditions on the sequence $(M_p)_p$. A full extension of Whitney's theorem was presented by Bonet, Braun, Meise and Taylor in [BoBMT] in the case of $\mathcal{E}_{(\omega)}$ and $\mathcal{E}_{\{\omega\}}$, which covered Bruna's result. Petzsche [PeZ] obtained interesting characterizations for spaces of type $\mathcal{E}_{(M_p)}$ and $\mathcal{E}_{\{M_p\}}$. The extension of Whitney's extension theorem in this setting was investigated by Chaumat and Chollet [ChCh].

The existence of continuous linear extension operators for classes of non quasianalytic functions was first considered by Meise and Taylor [MeT0], see also [MeT2]. The existence of continuous linear extension operators in the case of Beurling spaces $\mathcal{E}_{(M_p)}$ from the point and from a closed interval was considered by Petzsche [PeZ] and Langenbruch [La00]. Meise and Taylor [MeT2] had characterized those weights $\omega$ such that there is a continuous and linear extension operator from the point in the setting of Beurling classes. This research was completed by Franken [Fr1], [Fr2]. The case of functions of Roumieu type is more difficult. Langenbruch [La0] showed that there is no continuous linear extension operator for ultradifferentiable functions of Gevrey type from an interval. His proof uses the category of tame Fréchet spaces and an appropriate variant of the property (DN) of Vogt.

As Schmets wrote in his 2000 report on Valdivia's work: "In this vast literature, the real analyticity part of the Borel-Ritt theorem or of the Whitney theorem had not been considered before Valdivia's work in this direction." Here is the main result he got.

**Theorem (M. Valdivia 1996, [Va1],[Va2]).** *Let $K$ be a compact subset of $\mathbb{R}^N$.*

*1. If the jet $\varphi \in \mathcal{E}_*(K)$ comes from an $\mathcal{E}_*(\mathbb{R}^N)$-function, then it also comes from an element of the same space which moreover is real analytic on $\mathbb{R}^N \setminus K$, indeed holomorphic on an open subset of $\mathbb{C}^N$ containing $\mathbb{R}^N \setminus K$.*

*2. If there is a continuous linear extension operator $\mathcal{E}_*(K) \to \mathcal{E}_*(\mathbb{R}^N)$, then there is such an extension map $E$ such that $E(\varphi)$ is real analytic on $\mathbb{R}^N \setminus K$ for each $\varphi \in \mathcal{E}_*(K)$.*

Part 1. of this theorem was extended to closed subsets of $\mathbb{R}^N$ in articles by Schmets and Valdivia [55], [57], [58] in 1997-1999 . The method is basically the same, but it requires quite refined arguments. In 2001 Schmets and Valdivia, using the result of Frerick and Vogt mentioned in subsection c), obtained results about the existence of continuous linear extension operators from the space of Whitney jets on a closed set $F \subset \mathbb{R}^N$ into a space of holomorphic functions on an open set $D \subset \mathbb{C}^N$ which intersects $\mathbb{R}^N$ in $\mathbb{R}^N \setminus F$, [61], and in the same year, in another joint paper [62], this was carried over to classes of non quasianalytic functions.

Schmets and Valdivia continued their detailed study of extension problems for ultradifferentiable functions. In [59] they obtained an improvement of Ritt's type of the results of Petzsche [PeZ], and in [64] they investigated the range of the Borel map when it is not surjective. This work is closely related to Bonet, Meise, Taylor [BoMTV]. The articles [65] and [67] extend results of a paper of Chaumat and Chollet [ChCh2] and of the Thesis of Beaugendre [Beau] about countable intersections of non quasianalytic classes. They use methods and results of their paper [64]. The theorems in [65] constitute a proper extension of the Beurling and the Roumieu cases. In the theorems about analytic extension, they need methods developed in [52].

*e) Disclaimer and evaluation*

As was the case in Section 1, also in the present section not all the results in the joint work of Schmets and Valdivia could be mentioned here. The authors also extended Borel's theorem to real Banach spaces [53], showed the validity of the Zahorski theorem, in the form generalized by J. Siciak [54], in Gevrey classes and the existence of holomorphic functions having prescribed asymptotic expansions [52].

To sum up, the cooperation of Jean Schmets and Manuel Valdivia has been very fruitful and has produced many nice results. In a review of a joint paper of Schmets and Valdivia, Plesniak wrote: "This beautiful result is obtained due to subtle and very ingenious estimates of the derivatives..." In fact, the joint articles of Schmets and Valdivia are in places quite subtle or technical, but all of them are written very carefully. We hope and are convinced that this collaboration will continue in the future.

**Acknowledgments.** First of all, we thank Jean Schmets for establishing the series of meetings in Esneux and Han-sur-Lesse. They have been a welcome forum to present some of our research, to learn about recent research of others and to discuss new developments. We would also like to thank Jean and his amiable wife Josée for all the hospitality during our various visits in Liège.

Moreover, we thank Jean Schmets for sending various items which helped us very much to write this article: his curriculum vitae, list of publications, lists of meetings (co-) organized by him and of all his visits and talks.

The authors also thank their friend Reinhold Meise for helpful conversations and a number of references concerning Section 2 of this article.

This work of the authors was partially supported by FEDER and MEC, Project MTM 2004-02262.

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