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Local methods in nonlinear differential equations, by Alexander D. Bruno (Translated by William Hovingh and Courtney S. Coleman), Springer Series in Soviet Mathematics, Springer-Verlag, Berlin, Heidelberg, New York, 1989, x + 348 pp., \$119.00. ISBN 0-387-18926-2

In a lecture some years ago, Henry McKean remarked that the theory of differential equations really amounts to finding an invertible transformation $y = h(x)$ under which the (nonlinear) system

$$\frac{dx}{dt} = f(x) \tag{1}$$

becomes

$$\frac{dx}{dt} = 1, \tag{2a}$$

or (almost as good):

$$\frac{dx}{dt} = x. \tag{2b}$$

Of course, the problems are that the transformation h may not exist and, even when it does, it will be itself nonlinear and can rarely be found explicitly. Hence the great excitement about "soliton" equations: classes of nonlinear partial differential equations for which h can be constructed (more or less) explicitly and which are therefore completely integrable. Generally, one can at best hope to approximate h locally: in the neighborhood of a particular solution of (1), usually a fixed point. Such methods are the topic of this book, a recent translation and expansion of the 1979 Russian original.

To introduce the basic ideas, suppose that (1) can be written

$$\dot{x} = Ax + F(x), \tag{3}$$

where $F = \mathscr{O}(|x|^2)$ is nonlinear and we have assumed that the vector field vanishes at $x = 0$ (a fixed point). We can also assume that A has been diagonalized or put into Jordan form by a similarity transformation. One seeks a near identity transformation

$$x = y + H(y) \tag{4}$$

defined on a neighborhood of 0. Under (4), (3) becomes

$$\dot{y} = [I + DH(y)]^{-1}[A(h + H(y)) + F(y + H(y))].$$

Now suppose that

$$F = \sum_{k=2}^{\infty} F_k(x) \qquad \text{and} \qquad H = \sum_{k=2}^{\infty} H_k(x)$$

can be expressed as (convergent) power series, each term, F_k and H_k being a homogeneous polynomial of order k, and expand (5) to obtain

$$\text{(6)} \qquad \dot{y} = Ay + AH_2(y) - DH_2(y)Ay + F_2(y) + \mathcal{O}(|y|^3).$$

One now tries to choose H_2 so that the $\mathcal{O}(|y|^2)$ terms of (6) all vanish. The Lie bracket of the vector fields $[Ay, H_2(y)]$ determines how completely this may be done, and also determines how the third and higher-order terms are modified. Iterating the procedure, we eventually arrive at the transformed equation

$$\text{(7)} \qquad \dot{z} = Az + \sum_{k=2}^{K} G_k(z) + R(z),$$

where the remainder is "small", $(\mathcal{O}(|z|^{k+1})$. At each stage one removes as many terms as possible. In principle, K can be taken to infinity. The system (7) is said to be in *normal form* and it is simpler to study than the original problem.

Ideally, one wants to remove *all* nonlinear terms $(G_k \equiv 0, \forall k)$, but this is generally impossible even when the linear part is nondegenerate (A has no eigenvalues with zero real apart). However, one can usually choose the H_k so that many of the coefficients in each G_k are zero. One then studies the truncated normal form ((7) with $R(z)$ removed), attempting to characterize its phase portraits, and tries to prove that restoration of $R(z)$ does not change the behavior in some qualitative sense. In this procedure there are two steps: (1) construction of a formal power series for H, and (2) proof that this series converges near 0 and that the truncated system (7) indeed determines behavior. The idea goes back to Poincaré [7] and even earlier, and many people, including the author of this book, have made contributions to the theory.

Construction of the formal series already involves extensive bookkeeping in the computation of coefficients. A key tool is the Newton polygon and the first chapter of Bruno's book introduces this in detail and applies it to the problems of finding zeros and level sets of an analytic function $f(x_1, x_2)$, which are closely related to finding the transformation H of (4). The author then goes on, in chapter two, to the study of planar systems, providing complete descriptions of linearizable problems and characterizing conditions under which certain nonlinear "resonant" terms can and cannot be removed. This is of great importance to the bifurcation theory of vector fields depending upon a parameter, in which resonant terms appear as the linear part passes through a degenerate case (cf. Guckenheimer–Holmes [3]). However, while there is a section on computation of the "Liapunov" coefficients for a degenerate focus, necessary for Hopf bifurcation analysis and in parts of Hilbert's sixteenth problem, and discussions of various degenerate cases, including these for which $A \equiv 0$, the author has not attempted to classify degenerate planar phase portraits in a systematic fashion. Partial classifications can be found in the work of Takens [9–11], Bogdanov [2], and others (cf. Arnold [1], Guckenheimer–Holmes [3]), but many open problems remain, especially for vector fields depending upon three or more parameters.

In chapter three normal forms for $n > 2$ dimensional systems are considered. The discussion proceeds much as in the two-dimensional case, but there is the additional feature here that one can often "uncouple" many nondegenerate directions, in which the linear part dominates. There is a choice here; one can first reduce the dimension to the degenerate directions, appealing to the center manifold theorem (Pliss [6], Kelley [5]) and then apply normal form transformations, or one can transform and uncouple the n-dimensional vector field in one grand operation. Bruno takes the latter viewpoint, which may not always be the easiest computationally: center manifolds are not described explicitly. This chapter also contains a discussion of the method of averaging for systems whose phase spaces are fibered into invariant tori and its relation to normal forms.

Chapter four has a brief account of the Newton polyhedron applied to $n > 2$ dimensional systems, and chapter five gives several applications of normal form theory to celestial and classical mechanical problems such as gyroscopes and the spherical pendulum. This completes the first part of the book.

In part two, the final 70 pages of the book, systems of the form

$$\frac{dz}{dt} = F(z, \mu), \qquad z \in \mathscr{R}^n, \tag{8}$$

where F is an analytic vector field depending on parameters $\mu \in \mathscr{R}^p$ and possessing an invariant k-torus $\mathscr{T}$ at $\mu = 0$, are considered. The main problem is to describe all invariant sets μ near $\mathscr{T}$ existing near $\mu = 0$ and containing $\mathscr{T}$ at $\mu = 0$. The celebrated KAM theorem addresses a part of this problem. Bruno presents a different approach, including results on the continuation of periodic solutions, and examines the analyticity of the resulting normal form transformations.

As a "user" of normal form theory, I found the book somewhat abstract, and I suspect that most applied mathematicians and scientists may find parts of it heavy going. (The expense may also deter them.) Although it contains many recent results, it seems rather classical in spirit, and I was reminded of Ince's [4] famous book on ODEs. (This is high praise.) Like that classic, it contains many examples and material found in no other ODE texts. It is valuable as a reference, especially for its account of Soviet contributions. Not surprisingly, it has nothing on recent implementations of the tedious normal form calculations on computer algebra systems (e.g., Rand–Armbruster [8]), but as a compendium of basic theory it is, as far as I know, unique. I imagine that it could be useful in a second year graduate course or seminar in ODEs.

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Algebras, Lattices, Varieties, vol. 1, by Ralph N. McKenzie, George F. McNulty, and Walter F. Taylor. Wadsworth & Brooks/Cole, Monterey, California, 1987, 361 pp., $44.95. ISBN 0-534-07651-3

Lest the title leave any uncertainty, Volume 1 initiates a comprehensive four-part overview of *universal algebra* as the subject is understood today. It concerns properties of *algebras* that are, by and large, independent of their particular operational type. Special algebras, such as *lattices*, are dealt with from the point of view of their role in the study of universal algebras (nonempty sets augmented with an arbitrary system of finitary operations). *Varieties*, or equational classes of algebras, arise as one of the central themes in universal algebra. This volume presents a thorough and exquisitely executed account of the foundations of universal algebra together with a fine exposition of several sample results that illustrate the depth and the beauty of the subject.

The sheer quantity of new work published in universal algebra makes a strong case for the need for such a series. The Mathematical Reviews' Mathematics Subject Classification encompasses most of the universal algebra in two categories: 06XXX Order, Lattices, Ordered Algebraic Structures; and 08XXX General Mathematical Systems. But the 1970 version, which offered the single letter sections 06AXX and 08AXX, quickly proved to be a poor reflection of the explosion of research that was erupting. Grätzer [3] estimated that about a thousand publications in universal algebra appeared between 1968 and 1979, and it seems likely that an equal